Lecture 5
The System-Optimized (S-O) Problem

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Some Background

User-optimization or decentralized decision-making behavior is relevant to congested urban transportation networks as well as the Internet and electric power generation and distribution networks as well as certain financial networks.

System-optimization or centralized decision-making behavior is relevant to networks, including transportation and logistical ones, in which there is a central controller who seeks to route the traffic/flows so that the total cost in the network is minimized.

Examples of transportation networks in which system-optimization is the governing behavioral principle include freight networks – truck, rail and waterway.

Supply chains in the case of humanitarian operations and logistics would operate under the system-optimization principle.

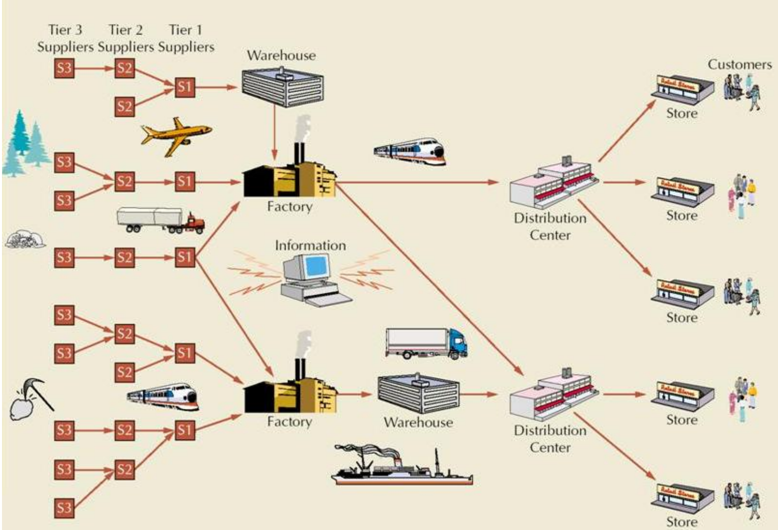
Sea Freight

www.scanlogistics.com

Rail Freight

www.galleywinter.com

A General Supply Chain



Healthcare Supply Chains

Humanitarian Relief

A System-Optimized Transportation Network

A central authority can route traffic according to his/her will (users can't make their own choices).

Criterion or Objective: To minimize total cost in the network.

The total cost on a link $a = \hat{c}_a(f_a) = c_a(f_a) \times f_a$,

where $c_a(f_a)$ is the user link cost on a and f_a is the flow on link a.

Recall that the total cost on a network can be expressed as:

$$\sum_{a \in L} \hat{c}_a(f_a).$$

The System-Optimization Problem

The system optimization (S-O) problem can, hence, be expressed as:

Minimize $\sum_{a \in L} \hat{c}_a(f_a)$ [the total cost]

subject to the constraints:

$$d_w = \sum_{p \in P_w} F_p, \text{ for all O/D pairs } w \in W,$$

$$f_a = \sum_{p \in P} F_p \delta_{ap}, \text{ for all links } a \in L,$$

$$F_p \geq 0, \text{ for all } p \in P.$$

Note: The constraints are identical to those in the U-O problem.

The S-O Conditions

Theorem

A link flow pattern f, induced by a path flow pattern F, is system-optimizing, if and only if, there exists an ordering of paths: $p_1, ..., p_{s'}, p_{s'+1}, ..., p_{n_w}$ that connect each O/D pair $w \in W$, such that

$$\hat{C}'_{p_1}(f) = ... = \hat{C}'_{p_{s'}}(f) = \mu_w \leq \hat{C}'_{p_{s'+1}}(f) \leq ... \leq \hat{C}'_{p_{n_w}}(f)$$

$$F_{p_r} > 0; r = 1, ..., s';$$

$$F_{p_r} = 0; r = s' + 1, ..., n_w,$$

where

$$\hat{C}'_p(f) = \sum_{a \in L} \hat{c}'_a(f_a)\delta_{ap} = \sum_{a \in L} \frac{\partial \hat{c}_a(f_a)}{\partial f_a}\delta_{ap}.$$

$\hat{C'}_p(f)$: the marginal of the total cost on path p,

$\hat{c}'_a(f_a)$: the marginal of the total cost on link a.

The above conditions must be satisfied for every O/D pair in the network.

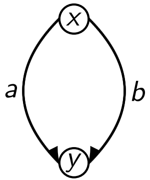
The System-Optimizing Conditions

These conditions are equivalent to the Kuhn-Tucker conditions of the nonlinear optimization problem since the feasible set is convex and the user link cost functions are assumed to be continuous.

If the marginal cost functions are **strictly** increasing functions of the flows on the links, we are guaranteed a **unique** S-O link flow pattern since the objective function is strictly convex.

An Example

Determine the U-O and the S-O flow pattern for the following network:



The demand is $d_{xy} = 100$ and the user link cost functions are:

$$c_a(f_a) = 3f_a + 1000, \quad c_b(f_b) = 2f_b + 1500.$$

Recall that a flow pattern f^* would be U-O if: $c_a = c_b$ and $f_a^* > 0$, $f_b^* > 0$, or $c_a \geq c_b$ and $f_b^* > 0$, $f_a^* = 0$, or $c_a \leq c_b$ and $f_a^* > 0$, $f_b^* = 0$.

Let's apply the U-O EEA:

- Sort the h_as: $1000 < 1500$.
- Compute

$$\lambda_w^1 = \frac{d_w + \frac{h_{a_1}}{g_{a_1}}}{\frac{1}{g_{a_1}}} = \frac{100 + \frac{1000}{3}}{\frac{1}{3}} = 1300.$$

- Check: Is: $1000 < 1300 \leq 1500$; Yes!

The critical $s = 1$, so

$$f_a^* = F_{p_1}^* = \frac{\lambda_w^1 - h_{a_1}}{g_{a_1}} = \frac{1300 - 1000}{3} = 100, \qquad f_b^* = F_{p_2}^* = 0.$$

A flow pattern would be S-O if:

$\hat{c}'_a = \hat{c}'_b$ and $f_a > 0, f_b > 0$,

or $\hat{c}'_a \geq \hat{c}'_b$ and $f_b > 0, f_a = 0$,

or $\hat{c}'_a \leq \hat{c}'_b$ and $f_a > 0, f_b = 0$.

For this example, let's see if both paths (which are links here) can be used.

Then we must have: $\hat{c}'_a = \hat{c}'_b$.

How do we construct these?

The total costs on links a and b are:

$$\hat{c}_a(f_a) = (3f_a + 1000) \times f_a = 3f_a^2 + 1000f_a,$$

$$\hat{c}_b(f_b) = (2f_b + 1500) \times f_b = 2f_b^2 + 1500f_b.$$

Then the marginal total costs on these links are:

$$\hat{c}_a' = \frac{\partial \hat{c}_a}{\partial f_a} = 6f_a + 1000, \quad \hat{c}_b' = \frac{\partial \hat{c}_b}{\partial f_b} = 4f_b + 1500.$$

What else do we know?

$f_a + f_b = d_w = 100 \Rightarrow f_b = 100 - f_a$

A System-Optimized Transportation Network

Hence,

$\hat{c}'_a(f_a) = \hat{c}'_b(f_b)$ means that:

$$6f_a + 1000 = 4(100 - f_a) + 1500,$$

$$6f_a + 1000 = 1900 - 4f_a,$$

$$10f_a = 900, \; f_a = 90; \; f_b = 10.$$

∗ **The S-O pattern is distinct from the U-O pattern (except for certain very specific networks or a special cost structure).**

The total cost under the **S-O pattern** is
$c_a = 1270$; $c_b = 1520$.

<u>The total cost</u> $= \hat{c}_a + \hat{c}_b = c_a \times f_a + c_b \times f_b =$
$= (1270)90 + (1520)10 = 129,500.$

The total cost under the **U-O pattern** is:
$c_a = 1300$; $c_b = 1500$

<u>The total cost</u> $= \hat{c}_a + \hat{c}_b = c_a \times f_a + c_b \times f_b =$
$= 1300(100) + 1500(0) = 130,000.$

Note: The total cost under the S-O pattern is less than the total cost under the U-O pattern.

$$129{,}500 < 130{,}000!$$

The System-Optimizing (S-O) Exact Equilibration Algorithm

- Step 1: Sort the h_{a_i}s in non-descending order and relabel accordingly. Set $r = 1$ and $h_{a_{m+1}} = \infty$.
- Step 2: Compute

$$\mu_w^r = \frac{d_w + \sum_{i=1}^{r} \frac{h_{a_i}}{2g_{a_i}}}{\sum_{i=1}^{r} \frac{1}{2g_{a_i}}}.$$

System-Optimizing EEA (continued)

- Step 3: Check

  If

  $$h_{a_r} < \mu_w^r \leq h_{a_{r+1}} \quad ,$$

  then STOP.

  Set the critical $s' = r$;

  $F_{p_r} = \frac{\mu_w^r - h_{a_r}}{2g_{a_r}}$; $r = 1, ..., s'$;

  $F_{p_r} = 0$; $r = s' + 1, ..., m$.

  Else, set $r = r + 1$ and goto Step 2.

The U-O Conditions

For each O/D pair w, there exists an ordering of the paths:

$$C_{p_1}(f^*) = ... = C_{p_s}(f^*) = \lambda_w \leq C_{p_{s+1}}(f^*) \leq ... \leq C_{p_{n_w}}(f^*)$$

$F^*_{p_r} > 0; \;\; r = 1, ..., s;$
$F^*_{p_r} = 0; \;\; r = s+1, ..., n_w.$

Here user costs on used paths are “equilibrated or equal.”

The S-O Conditions

For each O/D pair w, there exists an ordering of the paths:

$$\hat{C}'_{p_1}(f) = ... = \hat{C}'_{p_{s'}}(f) = \mu_w \leq \hat{C}'_{p_{s'+1}}(f) \leq ... \leq \hat{C}'_{p_{m'}}(f)$$

$F_{p_r} > 0; \;\; r = 1, ..., s';$
$F_{p_r} = 0; \;\; r = s' + 1, ..., m'.$

Here the marginal total costs on used paths are "equilibrated" or equal.

Definition: U-O or Network Equilibrium – Fixed Demands
A path flow pattern F^*, of nonnegative path flows, with O/D pair demand satisfaction, is said to be U-O or in equilibrium, if the following condition holds for each O/D pair $w \in W$ and each path $p \in P_w$:

$$C_p(F^*) \begin{cases} = \lambda_w, & \text{if} \quad F_p^* > 0, \\ \geq \lambda_w, & \text{if} \quad F_p^* = 0. \end{cases}$$

Definition: S-O Conditions – Fixed Demands
A path flow pattern F, of nonnegative path flows, with O/D pair demand satisfaction, is said to be S-O, if the following condition holds for each O/D pair $w \in W$ and each path $p \in P_w$:

$$\hat{C}'_p(F) \begin{cases} = \mu_w, & \text{if} \quad F_p > 0, \\ \geq \mu_w, & \text{if} \quad F_p = 0, \end{cases}$$

where $\hat{C}'_p(F) = \sum_{a \in L} \frac{\partial \hat{c}_a(f_a)}{\partial f_a} \delta_{ap}$.

Question: **When does the U-O solution coincide with the S-O solution?**

Answer: **In a general network, when the user link cost functions are given by:**

$$c_a(f_a) = c_a^0 f_a^{\beta},$$

for all links, with $c_a^0 \geq 0$, and $\beta \geq 0$.

In particular, if $c_a(f_a) = c_a^0$, that is, in the case of ***uncongested networks*****, this result always holds.**

In a subsequent lecture we will show how policies, in the forms of tolls, that, once imposed, will guarantee that the S-O solution is, at the same time, U-O!

References

⇒ S. C. Dafermos and F. T. Sparrow (1969) The Traffic Assignment Problem for a General Network. *Journal of Research of the National Bureau of Standards*, Vol. 73B, No. 2, 1969, pp 91-118.

For more advanced formulations and associated theory, see Professor Nagurney's Fulbright Network Economics lectures.

http://supernet.som.umass.edu/austria_lectures/fulmain.html