Lecture 2
Generalized Representation of a Transportation Network

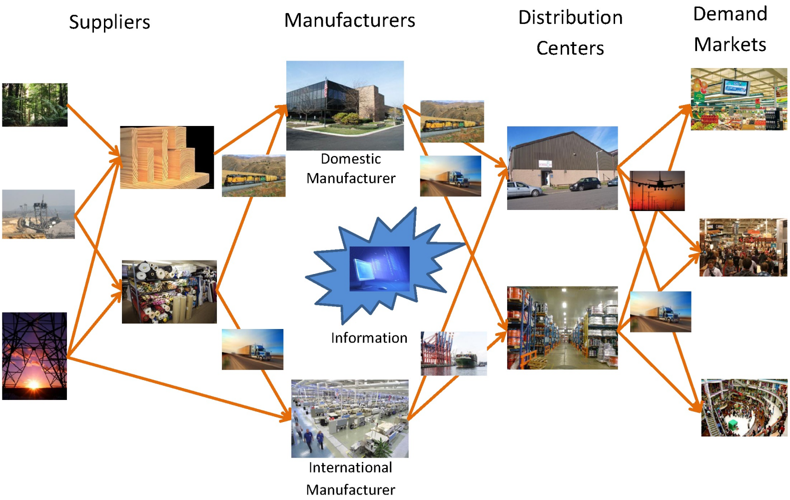
Professor Anna Nagurney

John F. Smith Memorial Professor
and
Director – Virtual Center for Supernetworks
Isenberg School of Management
University of Massachusetts
Amherst, Massachusetts 01003

Characteristics of Networks Today

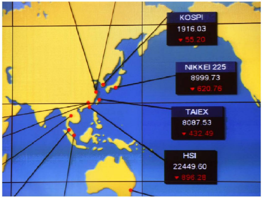
- **large-scale nature** and complexity of network topology;
- **congestion**, which leads to nonlinearities;
- **alternative behavior of users of the networks**, which may lead to paradoxical phenomena;
- **interactions among networks themselves**, such as the Internet with electric power networks, financial networks, and transportation and logistical networks;
- recognition of **the fragility and vulnerability of network systems**;
- policies surrounding networks today may have major impacts not only economically, but also **socially, politically, and security-wise**.

Complex Logistical Networks



Electric Power Generation and Distribution Networks

Financial Networks



Social Networks

Generalized Representation of a Transportation Network

The topology is represented by a mathematical graph consisting of nodes and links:

1. finite set of nodes, N;

2. set of directed links (arcs, branches, edges), L, respresented by arrows.

Examples

In a **road network,** nodes are where traffic is generated or attracted to, or intermediate points. Links are the roads.

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In an **airline network,** nodes are the airports, links are the air routes.
In a **railroad freight network,** nodes are loading/unloading points and switching points. Links are made up of tracks.

An Airline Example

Freight Trains

Origin/Destination Pairs of Nodes and Travel Demands

In addition to the set of nodes N and the set of links L used to represent the topology of a transportation network, we also denote the set of origin/destination (O/D) pairs by W and the set of all paths connecting all O/D pairs by P. An origin/destination pair of nodes represents where traffic originates and is destined to.

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We denote the individual O/D pairs by w_1, w_2, etc., for a particular network, and we associate a travel demand d_w with each O/D pair w. We let P_w denote the set of all paths connecting O/D pair w.

Fixed Demand Transportation Network Models

In the first type of transportation network models that we will be studying we assume that the travel demands are known and fixed over the time horizon of interest, such as the morning or evening commuting period.

Paths / Routes

Path: A path (or route) is a sequence of links connecting an O/D pair $w = (x, y)$ of nodes. It can be represented by linking all the distinct links from the origin to the destination.

We exclude all cycles or loops.

We assume that the travel demand (rate) is constant in time over the time horizon under analysis (such as the commuting period).

Hence, the flows are constant in time. We are focusing on steady-state phenomena.

Some Real-World Network Sizes

In Chicago's Regional Transportation Network, there are **12,982 nodes, 39,018 links, and 2,297,945 origin/destination (O/D) pairs**, whereas in the Southern California Association of Governments model there are **3,217 origins and/or destinations, 25,428 nodes, and 99,240 links, plus 6 distinct classes of users**.

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In the case of the Internet, in 2010, there were **1.8 billion users**

Components of Common Physical Networks

Network System	Nodes	Links	Flows
Transportation	Intersections, Homes, Workplaces, Airports, Railyards	Roads, Airline Routes, Railroad Track	Automobiles, Trains, and Planes,
Manufacturing and logistics	Workstations, Distribution Points	Processing, Shipment	Components, Finished Goods
Communication	Computers, Satellites, Telephone Exchanges	Fiber Optic Cables Radio Links	Voice, Data, Video
Energy	Pumping Stations, Plants	Pipelines, Transmission Lines	Water, Gas, Oil, Electricity

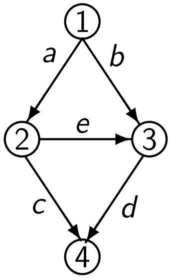
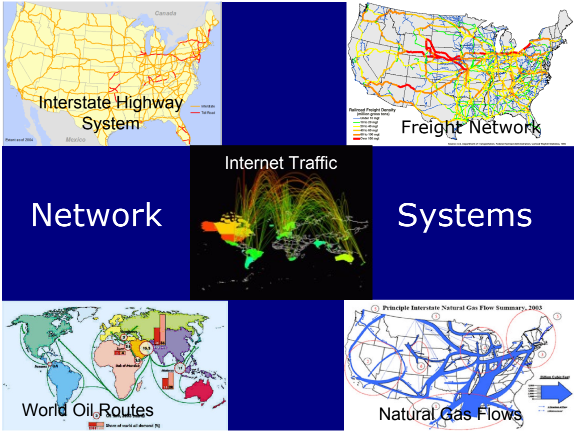


Figure: Network Example 1

Nodes: $1, 2, 3, 4$, Links: a, b, c, d, e, O/D pair $w_1 = (1, 4)$
Links have a direction and may be uniquely expressed as follows: $a = (1, 2)$, $b = (1, 3)$, *etc*.
Travelers/commuters enter the network at origin node 1 and wish to get to destination node 4. **There are 3 paths/routes in this network as options for the travelers/commuters.**

Flows Matter

In transportation networks, logistical networks, as well as energy and numerous network systems, it is not only the topology that is relevant but also the actual use of the network as measured by the flow patterns!



In order to properly formulate, study, and solve transportation network problems we need to also consider the flows on the networks and we distinguish between path flows and link flows.

Path and Link Flows

Notation:

F_p: flow on path p (measured in units/hrs.,users/unit time).

$\Rightarrow$ *Path flows are always assumed to be nonnegative.*

f_a: flow on link a (measured in users/unit time).

Since the path flows are nonnegative, the link flows will be as well and this makes sense since we are dealing with traffic flows (vehicles, freight, messages, energy, etc.).

The Conservation of Flow Equations

The general expression relating the travel demands and path flows:

$$d_w = \sum_{p \in P_w} F_p, \quad \forall w \in W,$$

that is, the travel demand for each O/D pair must be equal to the sum of the flows on paths that connect that O/D pair.

The general expression relating link flows and path flows:

$$f_a = \sum_p F_p \delta_{ap}, \quad \forall a \in L,$$

where

$$\delta_{ap} = \begin{cases} 1, & \text{if link } a \text{ is contained in path } p; \\ 0, & \text{otherwise.} \end{cases}$$

In other words, the flow on a link is equal to the sum on flows on paths that use / share that link.

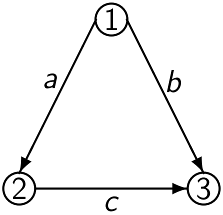
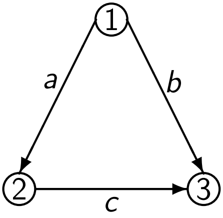


Figure: Network Example 2

Nodes: $1, 2, 3$; Links: a, b, c; O/D pair: $w_1 = (1, 3)$ with $d_{w_1} = 300$ vehicles/hr

P_{w_1} denotes the set of paths connecting O/D pair w_1, where: $P_{w_1} = \{p_1, p_2\}$, with $p_1 = (a, c)$ and $p_2 = (b)$.
Hence, $f_a = F_{p_1}$, $f_b = F_{p_2}$, and $f_c = F_{p_1}$.



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Hence, $f_a = F_{p_1}$, $f_b = F_{p_2}$, and $f_c = F_{p_1}$.
If there are 100 cars per hour using path p_1 the volume of traffic on link a and link c is also 100. If there are 200 cars per hour using path p_2 then the traffic on link b is 200.

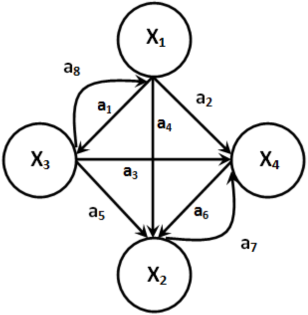

Figure: Network Example 3

The set of nodes $N = \{x_1, x_2, x_3, x_4\}$.

The set of links $L = \{a_1, a_2, ..., a_8\}$.

The set of O/D pairs $W = \{w_1, w_2\}$, where $w_1 = (x_1, x_2)$ and $w_2 = (x_3, x_4)$.

Sets of paths connecting the O/D pairs

$P_{w_1} = \{p_1, p_2, p_3, p_4\}$, where

$$p_1 = (a_4), p_2 = (a_1, a_5),$$

$$p_3 = (a_2, a_6), p_4 = (a_1, a_3, a_6).$$

$P_{w_2} = \{p_5, p_6, p_7, p_8\}$, where

$$p_5 = (a_3), p_6 = (a_5, a_7),$$

$$p_7 = (a_8, a_2), p_8 = (a_8, a_4, a_7).$$

A link can be contained in 2 (or more) paths connecting an O/D pair. It can be contained in paths connecting O/D pairs; or both of the above.

An Interesting Problem

Suppose that we know the link flows. Can we solve for the path flows uniquely?

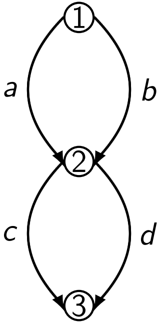

Figure: Network Example 4

The travel demand for O/D pair $w_1 = (1,3)$, d_{w_1}, is 200/hr.

The set of paths P_{w_1} connecting O/D w_1 : $\begin{cases} \text{path } p_1 = (a, c) \\ \text{path } p_2 = (b, d) \\ \text{path } p_3 = (a, d) \\ \text{path } p_4 = (b, c). \end{cases}$

We must have, from the conservation of flow equations, that the demand must be satisfied, that is:

$$d_{w_1} = F_{p_1} + F_{p_2} + F_{p_3} + F_{p_4}.$$

Also, the following expressions must hold, since link flows are related to the path flows:

$$f_a = F_{p_1} + F_{p_3},$$
$$f_b = F_{p_2} + F_{p_4},$$
$$f_c = F_{p_1} + F_{p_4},$$
$$f_d = F_{p_2} + F_{p_3}.$$

An Interesting Problem

Given link flows f_as, find the path flows F_ps for the network in Example 4.

Suppose that the link flow pattern for this network is:

$$f_a = f_b = f_c = f_d = 100.$$

How many different path flow patterns induce this link flow pattern (we can allow for fractional flows)?

Case I:

100 travelers take path p_1 and 100 travelers take path p_2. No one uses paths p_3 and p_4. This means that $F_{p_1} = F_{p_2} = 100$ and $F_{p_3} = F_{p_4} = 0$. Clearly, the demand $d_{w_1} = 200$.

The induced link flow pattern is:

$$f_a = 100, f_b = 100, f_c = 100, f_d = 100.$$

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The induced link flow pattern is:

$$f_a = 100, f_b = 100, f_c = 100, f_d = 100.$$

Case II:

We may also have the case that 100 travelers use path p_3 and 100 use path p_4, with no-one using either path p_1 or p_2, that is, $F_{p_3} = F_{p_4} = 100$, and $F_{p_1} = F_{p_2} = 0$.
This path flow pattern will also yield the same link flow pattern:

$$f_a = 100, f_b = 100, f_c = 100, f_d = 100.$$

This example illustrated how different path flow patterns may induce the identical link flow pattern!

Networks with Special Structure

In many applications, we will see that the computed link flow patterns tend to be unique, whereas the path flow patterns are not, except for rather stylized networks (such as those, clearly, when the paths consist of single links; see below).

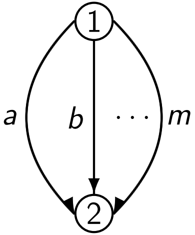

Figure: An example of a network in which paths coincide with links

Some Background Reading

Supernetworks: Paradoxes, Challenges, and New Opportunities, Anna Nagurney, in **Transforming Enterprise**, William H. Dutton, Brian Kahin, Ramon O'Callaghan, and Andrew W. Wyckoff, Editors, MIT Press, Cambridge, MA, 2004, pp 229-254; http://supernet.isenberg.umass.edu/articles/transform.pdf