A Paradox on Traffic Networks

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Bochum

- Historical remarks. The detection of the paradox is also counterintuitive
- Is the "mathematical paradox" consistent with the psychological behavior of the car drivers?
- People knowing the paradox make conclusions forgetting that there are counterintuitive effects.
- Quantitative estimates of the deterioration

The Traffic Assignment Problems

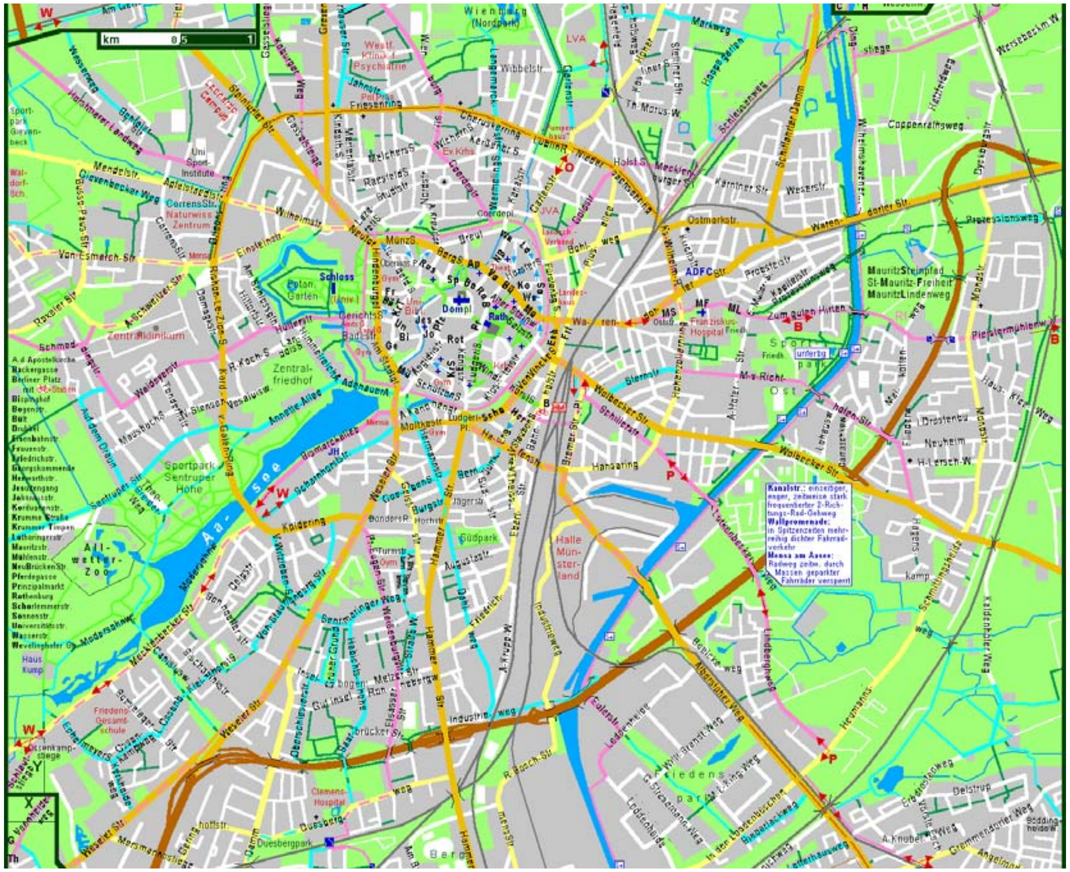
The point of departure is the following question:

How is the traffic flow distributed in a transportation network when the demand for each origin/destination pair is given.

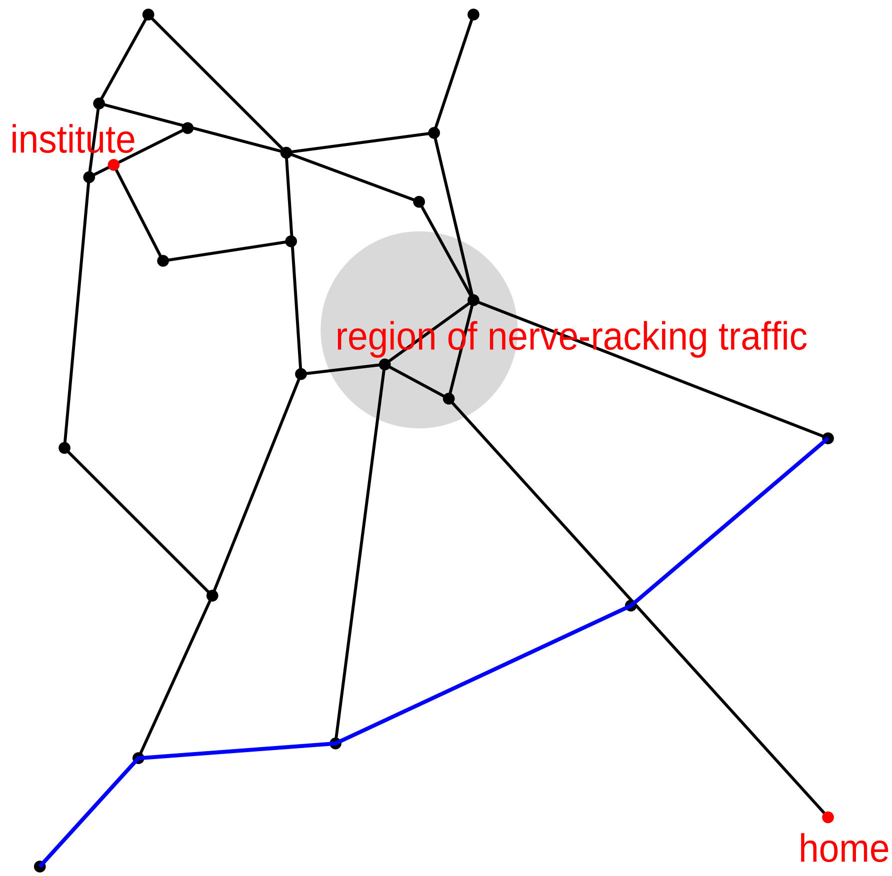
The travel time on each road depends on the amount of the traffic on that road.
The time becomes larger if the traffic increases.

Otherwise the problem would be easy.
Each car chooses the shortest path.
It is easily computed.

An Example



Münster



Münster

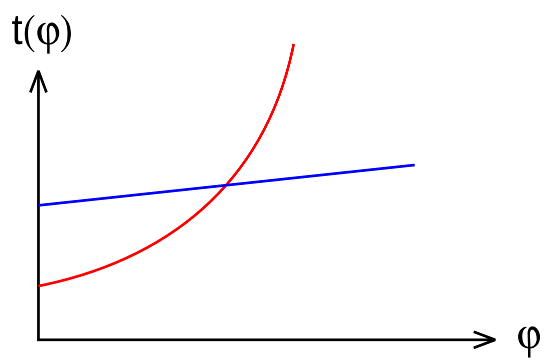
Notation

u_α – link in a road map – arc in a graph (the graph is directed)
φ_α – traffic flow on u_α
$t_\alpha = t_\alpha(\varphi)$ – travel time for passing u_α

U_β – path from the origin to the destination
Φ_β – flow along the path U_β
$T_\beta(\Phi)$ – travel time for the path U_β
$T_\beta(\Phi) = \sum_\alpha t_\alpha(\varphi_\alpha)$
the sum runs over α *with* $u_\alpha \in U_\beta$

Properties of the time function:

$t_\alpha(0) \geq 0$,
$t_\alpha(\varphi)$ is monotonously increasing
$t_\alpha(\varphi)$ is continuous
or semicontinuous from the left



Time function of a broad and a narrow street

Wardrop's two principles

System-optimal traffic:
A central controller routes the traffic in an optimal way.

User-optimal traffic:
The travellers select their routes
independently according to their knowledge.
The travel times of all routes actually used
for one origin/destination pair are equal,
The result is a Nash equilibrium.

An additional road can change the flow in a counterintuitive way!

Historical Remarks.

- I was not aware of Wardrop's research
  and the book by Beckmann, McGuire and Winston. – Lucky for me!

- Knödel has given a talk in Münster, and I wanted to study his algorithm.
  There was the need of emphasizing the differences.

- I had just moved from Theoretical Physics to Mathematics.
  A counterintuitive symmetry breaking argument was essential.
  A uniqueness result in the literature could not be true.

- Counterintuitive effects – good for demonstrating differences.

- Counterintuitive effects in numerical methods are dangerous
  (Here even for people who know the paradox)

System-Optimal Traffic

Criterium that is democratic and fits to the comparison with the equilibrium.
The travel time of the latest car is to be as small as possible.

Optimal Flow: Given Φ_{tot}, determine Φ_β for all routes U_β subject to

$$\sum_\beta \Phi_\beta = \Phi_{tot}$$

such that

$$\max_{\substack{\beta \\ \Phi_\beta > 0}} T_\beta(\Phi) \longrightarrow \min!$$

User-Optimal Traffic – Equilibrium

Definition. A traffic flow Φ is in an equilibrium state if there exists a (resulting) time $T^* = T_{eq}$ such that

$$\begin{array}{rcll} T_\beta(\Phi) & = & T^* & \text{for all used routes,} \\ T_\beta(\Phi) & \geq & T^* & \text{for all routes not used.} \end{array}$$

In the more general semicontinuous case the conditions are

$$\begin{array}{rcll} T_\beta(\Phi) & \leq & T^* & \text{for all used routes,} \\ T_\beta(\Phi+) & \geq & T^* & \text{for all routes,} \\ & & & \Phi+ := \Phi + \text{ one more car on } U_\beta. \end{array}$$

The equilibrium property holds also for
all nodes between origin and destination.

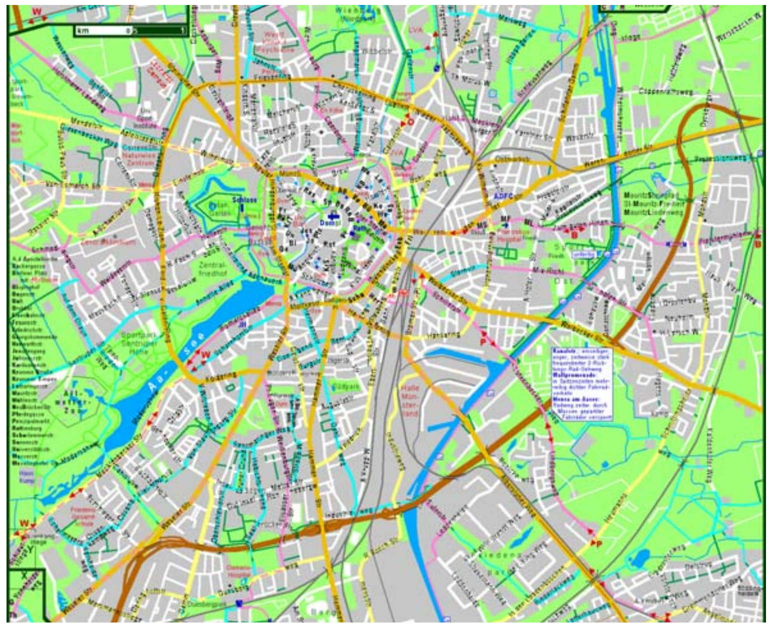
Methods of proof of existence:
Convex programming and
fixed-point theorems for set-valued mappings.
[Beckmann et al, Dafermos et al]

Special case: one origin/destination pair.
Set $f_\alpha(\varphi) := \int^\varphi t_\alpha(\psi) d\psi$.
Find an admissible flow such that

$$\sum_\alpha f_\alpha(\varphi_\alpha) \longrightarrow \min!$$

Although we have an optimization with a slightly different cost function,
the behavior is different!

Psychological Effects



The route through the center looks more time-consuming than it is
– for car drivers who like to go fast.
Some people rather drive 5 minutes than wait 5 minutes.

Another psychological argument:

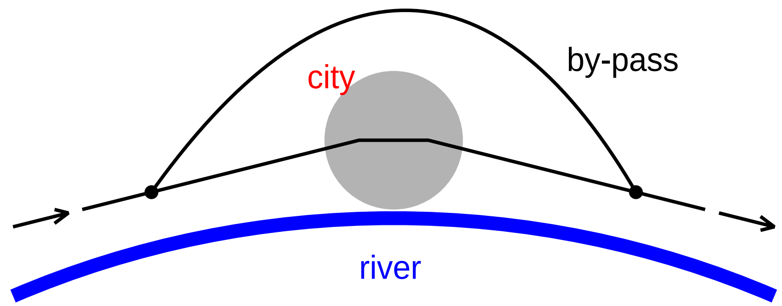
1st route: First part slow, second part fast,
2nd route: First part fast, second part slow.
total time is the same.

Many drivers consider the second route as better.

No problem for the mathematical treatment:
Weight slow links at the beginning of a tour with a factor > 1.

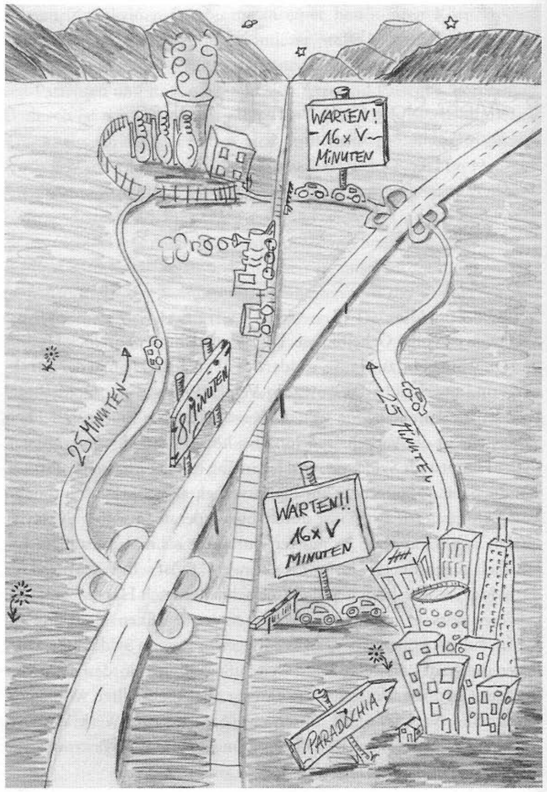
A Conflict: User-Optimization – System-Optimization

Old German city: 1st route through the city: short and narrow
2nd route on the by-pass: longer, insensitive to the load



Night: The road through the city is faster in spite of lower speed.
Day: About 10% of the global traffic would produce congestion.

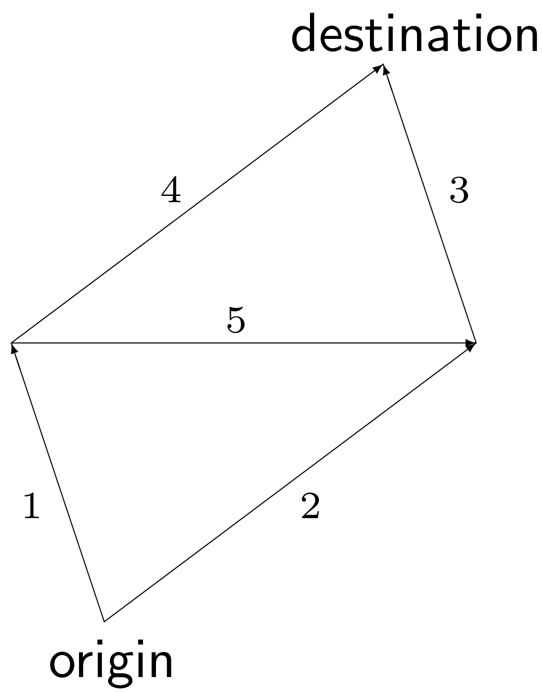
Travel time in the equilibrium = time for the by-pass.
Remedy: Forbid global traffic through the city.
Improvement of the road through the city is no help.



Mehr Straßen, mehr Stau! Das Braess'sche Paradoxon plagt den syldavischen Straßenverkehr und den gesunden Menschenverstand. Aber Verkehrsministerin Imma Schnälla findet eine pragmatische Lösung.

Paradox in Dubben, Beck–Bornholdt

Paradox



$$
\begin{aligned}
t_1(\varphi) = t_3(\varphi) &= 10\varphi, \\
t_2(\varphi) = t_4(\varphi) &= 50 + \varphi, \\
t_5(\varphi) &= 10 + \varphi.
\end{aligned}
$$

$$\Phi_{tot} = 6$$

$$
\begin{aligned}
t_1(\varphi) = t_3(\varphi) &= 10\varphi, \\
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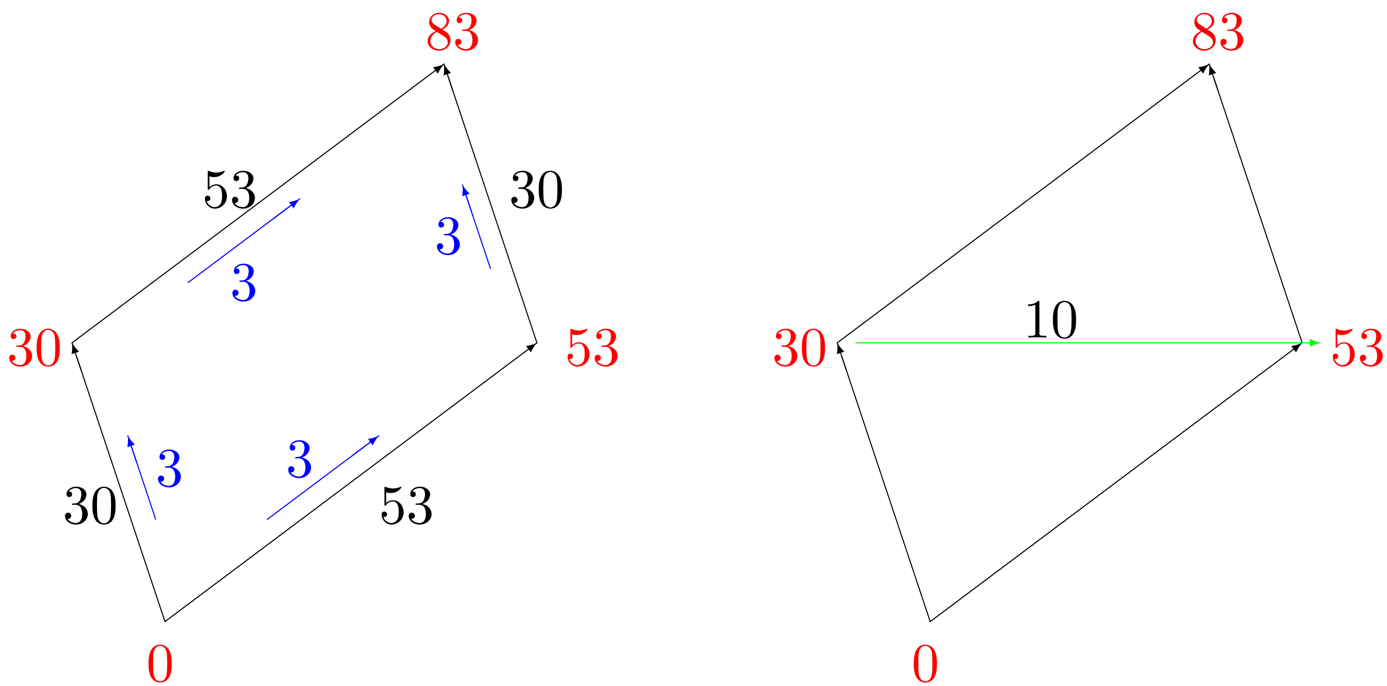
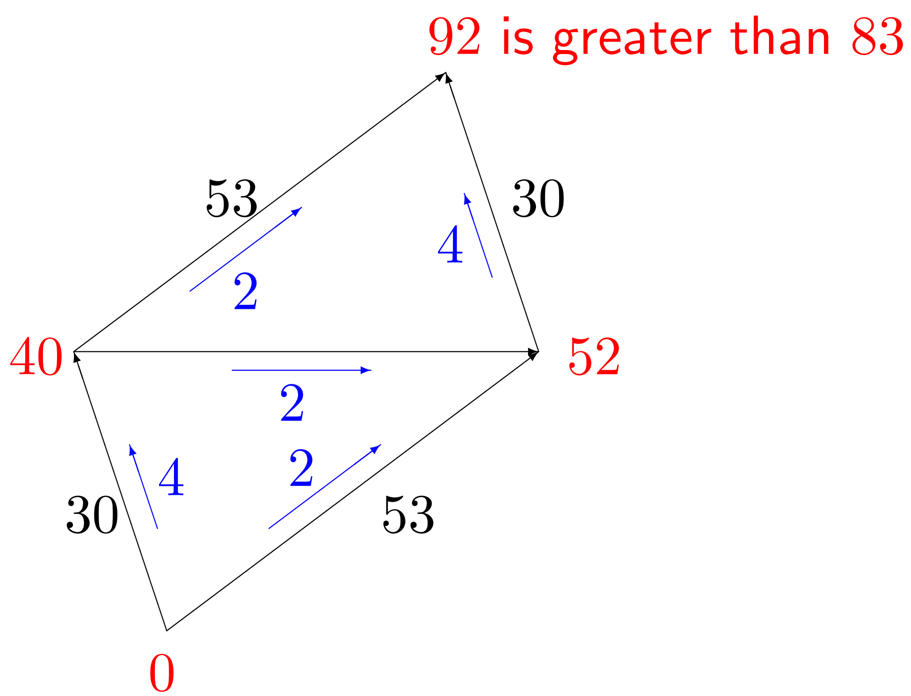


Equilibrium without arc no. 5.

New Equilibrium with Arc no. 5



When arc no. 5 is opened, it looks very favorable.
You save 13. – Later, however!

Amount of Deterioration

How large can

$$\frac{T_{eq}}{T_{opt}}$$

be? – Question studied by Roughgarden, Tardos and Kameda et al.

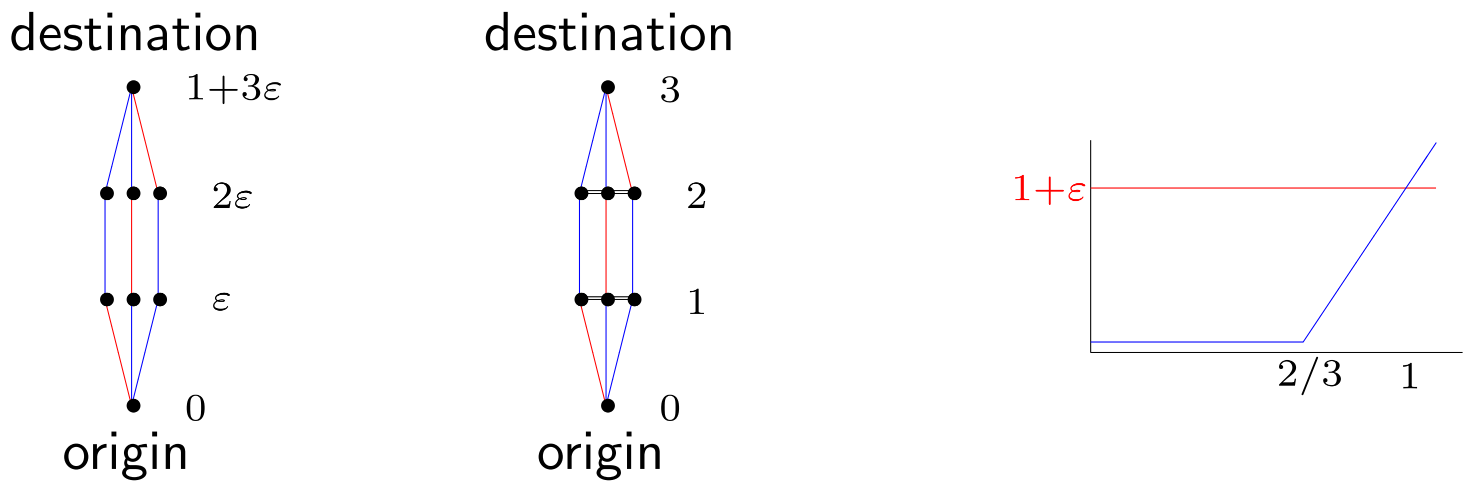
Theorem. If all t_α are arbitrary monotonous functions of φ and if all used paths in the optimal distribution have at most m links, then

$$T_{eq} \leq m T_{opt}.$$

Theorem. If all functions $t_\alpha(\varphi)$ are linear (as in internet traffic), then

$$T_{eq} \leq \frac{4}{3} T_{opt}.$$

Deterioration and Paradoxial Behavior



Number of levels: $m = 3$. Total flow: $\Phi_{tot} = m - 1 = 2$.

Left: $\Phi = m/(m+1) = 2/3$ on every route.
Right: No flow on red arcs.

Modification of study by *Roughgarden and Tardos*

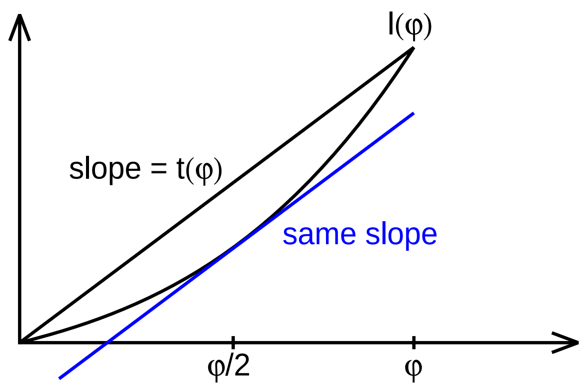
Idea of Roughgarden's Proof

The load function

$$\ell_\alpha(\varphi) := \varphi\, t_\alpha(\varphi)$$

is a quadratic polynomial if t_α is linear, and

$$\ell(\frac{1}{2}\varphi) \geq \frac{1}{4}\ell(\varphi), \qquad \ell'(\frac{1}{2}\varphi) = t(\varphi).$$



Consider Taylor's expansion at $\frac{1}{2}\varphi_{eq}$ and estimate the global costs/load.

Consider the total load

$$C(\Phi) \quad := \quad \sum_{\beta} T_{\beta}(\Phi)\Phi_{\beta} = \sum_{\alpha} t_{\alpha}(\varphi_{\alpha})\varphi_{\alpha} = \sum_{\alpha} \ell_{\alpha}(\varphi_{\alpha})$$

The convex function is expanded at $\frac{1}{2}\Phi_{eq}$.

$$\begin{aligned} C(\Phi) \quad &\geq \quad C(\frac{1}{2}\Phi_{eq}) + \sum_{\alpha} \frac{\partial \ell_{\alpha}}{\partial \varphi}(\frac{1}{2}\varphi_{eq})[\varphi_{\alpha} - \frac{1}{2}\varphi_{eq,\alpha}] \\ &= \quad C(\frac{1}{2}\Phi_{eq}) + \sum_{\beta} T_{eq,\beta}[\Phi_{\beta} - \frac{1}{2}\Phi_{eq,\beta}] \\ &\geq \quad \frac{1}{4}C(\Phi_{eq}) + T_{eq}\sum_{\beta} \Phi_{\beta} - \frac{1}{2}C(\Phi_{eq}) = \frac{3}{4}C(\Phi_{eq}) \end{aligned}$$

Apply the inequality to the optimum.

When do we encounterr paradoxial behavior in real-life traffic?

There is one route with the tendency of congestion
and a parallel one with heavy traffic.

What happens if the "congested" road is made faster?

Answer (Mathematics only):
The time T_{eq} will be nearly the same as before
since a portion of the traffic will be shifted.

Answer (Mathematics and Psychology):
The situation will be worse on the "congested" road
since the psychological barrier is reduced by the improvement of the road.