Quantifying the Inefficiency of Wardrop Equilibria

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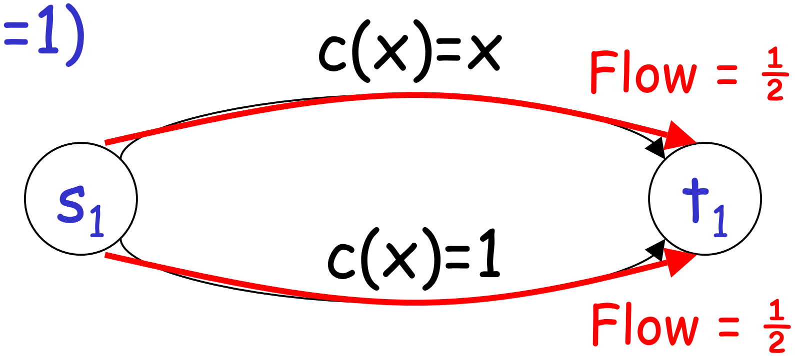
Traffic Equilibria (Inelastic Demand)

- a directed graph $G = (V,E)$
- k origin-destination pairs $(s_1, t_1), \ldots, (s_k, t_k)$
- fixed amount d_i of traffic from s_i to t_i
- for each edge e, a cost function $c_e(\cdot)$
  - assumed continuous, nonnegative, nondecreasing

Example: $(k,r=1)$



Wardrop Equilibria

Defn [Wardrop 52]: a traffic flow is a Wardrop equilibrium if all flow routed on min-cost paths (given current congestion).

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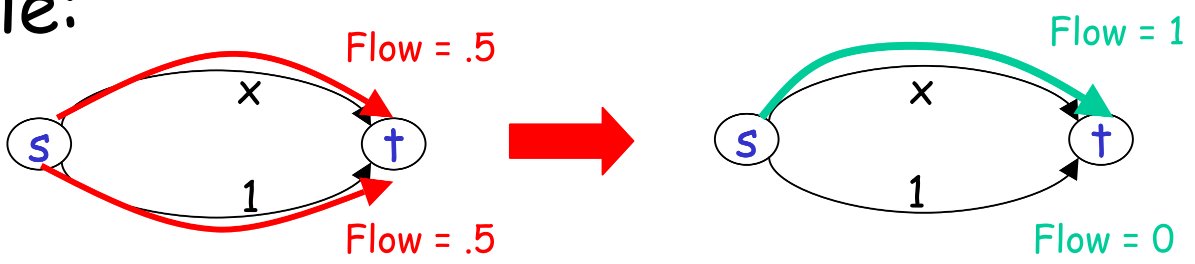

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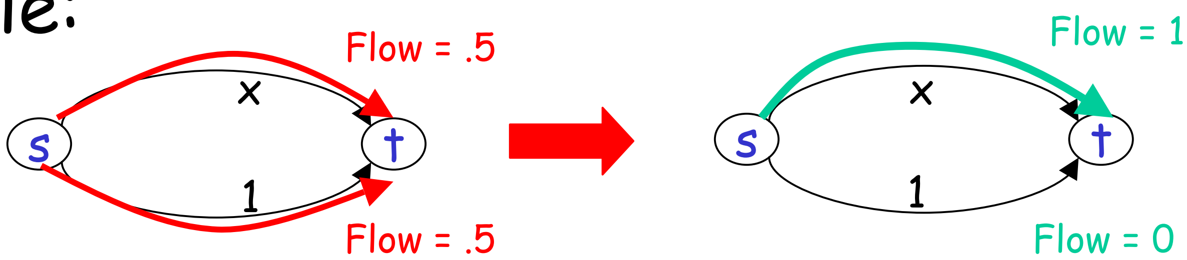

Question [Ch 3, Beckmann/McGuire/Winsten 56]: "Will there always be a well determined equilibrium[...]?"

The BMW Potential Function

Answer [Beckmann/McGuire/Winsten 56]: Yes.

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Proof: Consider the "potential function":

$$\square(f) = \Sigma_e \int_0^{f_e} c_e(x)dx$$

- defined so that first-order optimality condition = defn of Wardrop equilibrium
- apply Weierstrauss's Theorem

QED. (also get uniqueness, etc.)

Potential Functions in Game Theory

Did you know?: Potential functions now standard tool in game theory for proving the existence of a pure-strategy Nash eq.

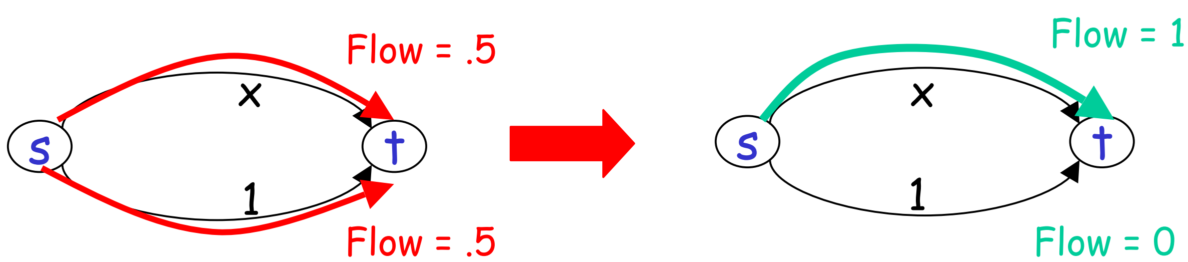
- define function Φ s.t. whenever player i switches strategies, $\Delta\Phi = \Delta u_i$
  - local optima of Φ = pure-strategy Nash equilibria
  - [Rosenthal 73]: traffic eq w/ discrete population
  - [Monderer/Shapley 96]: general "potential games"

Inefficiency of Wardrop Eq

Motivation [Ch 4, BMW 56]:

- "An economic approach to traffic analysis should [...] provide criteria by which to judge the performance of the system."

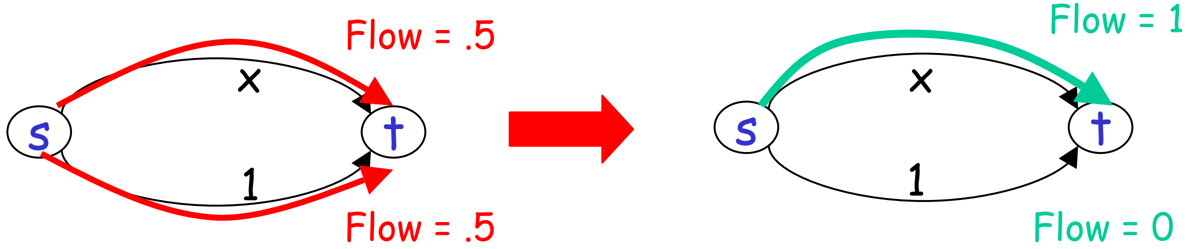
Pigou's example [Pigou 1920]:



(WE not Pareto optimal)

Quantifying Inefficiency

Goal: quantify inefficiency of WE.

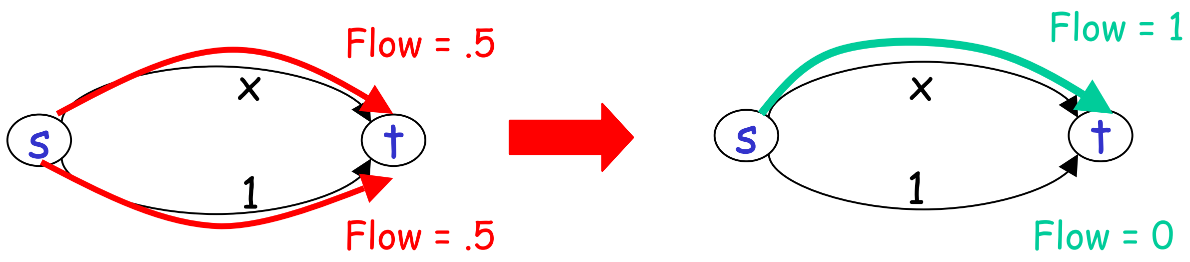


Quantifying Inefficiency

Goal: quantify inefficiency of WE.

Ingredient #1: objective function.

- will use average travel time (standard)



Quantifying Inefficiency

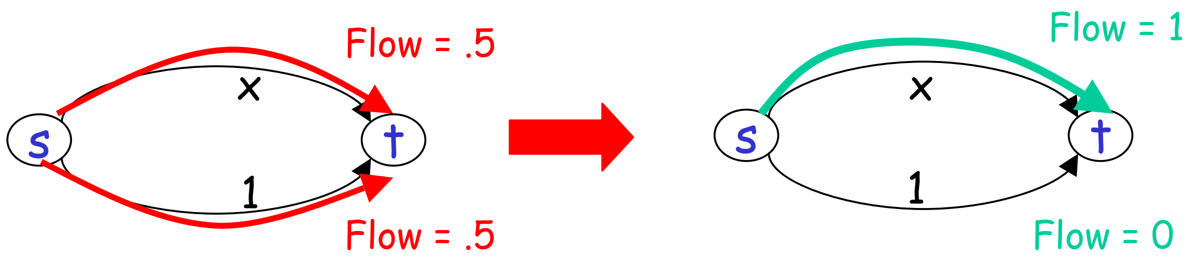
Goal: quantify inefficiency of WE.

Ingredient #1: objective function.

- will use average travel time (standard)

Ingredient #2: measure of approximation.

- will use ratio of obj fn values of WE, system opt (standard in theoretical CS)



Quantifying Inefficiency

Defn:

$$\text{inefficiency ratio} = \frac{\text{average travel time in WE}}{\text{average travel time in sys opt}}$$

- = 4/3 in Pigou's example (33% loss)
- the closer to 1 the better
- aka "coordination ratio", "price of anarchy" [Kousoupias/Papadimitriou 99,01]
- first studied for WE by [Roughgarden/Tardos 00]

Potential Fns & Inefficiency

Assume: each cost fn is affine: $c_e(x) = a_e x + b_e$

Claim: BMW potential fn a good approximation of true objective function (avg travel time).

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Potential: $\Phi(f) = \Sigma_e \int_0^{f_e} c_e(x)\,dx = \Sigma_e\, [\tfrac{1}{2} a_e f_e + b_e] f_e$

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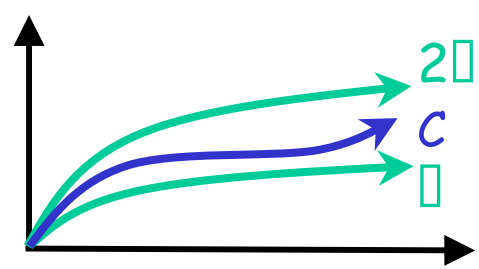
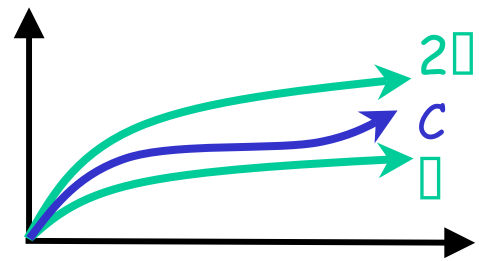


Potential Fns & Inefficiency

So: □(f) ≤ C(f) ≤ 2□(f)

- (affine cost functions)



Potential Fns & Inefficiency

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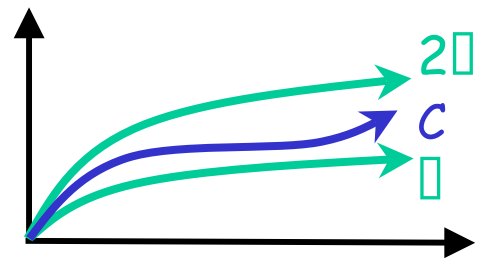

Consequence: inefficiency ratio ≤ 2

- proof: $C(WE) \le 2□(WE) \le 2□(OPT) \le 2C(OPT)$

Potential Fns & Inefficiency

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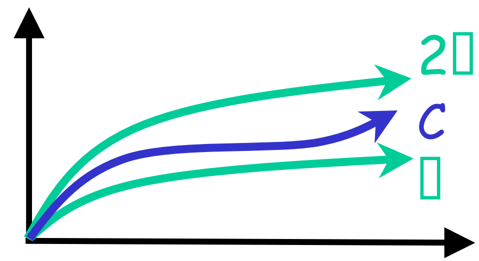

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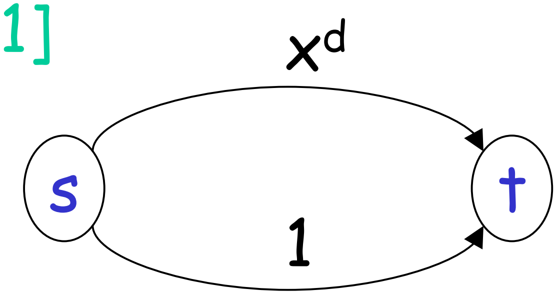
In fact: [RT00] more detailed argument ⇒ inefficiency ratio ≤ 4/3

- Pigou's example the worst! (among all networks, traffic matrices)

More General Cost Fns?

General Cost Functions: worst inefficiency ratio grows slowly w/"steepness"

- e.g., degree-d bounded polynomials (w/nonnegative coefficients) [Roughgarden 01]
- naive argument: ratio ≤ d+1
- optimal bound: ≈ d/ln d
- worst network = analogue of Pigou's example
- for d = 4: ≈ 2.15



Epilogue

- potential function introduced in [Beckmann/McGuire/Winsten 56] to prove existence of Wardrop equilibria
- now standard tool in game theory to prove existence of pure Nash equilibria
- now standard tool in theoretical CS + OR to bound inefficiency of equilibria