Quantifying the Inefficiency of Wardrop Equilibria

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Traffic Equilibria (Inelastic Demand)

- a directed graph \( G = (V,E) \)
- \( k \) origin-destination pairs \( (s_1,t_1), \ldots, (s_k,t_k) \)
- fixed amount \( d_i \) of traffic from \( s_i \) to \( t_i \)
- for each edge \( e \), a cost function \( c_e(\cdot) \)
  - assumed continuous, nonnegative, nondecreasing

Example: \( (k,r=1) \)

\[ c(x) = x \]

\( s_1 \) \hspace{2cm} \( t_1 \)

Flow = \( \frac{1}{2} \)

\( c(x) = 1 \)

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Wardrop Equilibria

**Defn [Wardrop 52]:** a traffic flow is a Wardrop equilibrium if all flow routed on min-cost paths (given current congestion).

**Example:**

![Diagram showing traffic flow with various flow values](image)
**Wardrop Equilibria**

**Defn [Wardrop 52]:** a traffic flow is a Wardrop equilibrium if all flow routed on min-cost paths (given current congestion).

**Example:**

```
\[\text{Flow} = 0.5 \quad \text{Flow} = 1\]
```

**Question [Ch 3, Beckmann/McGuire/Winsten 56]:** "Will there always be a well determined equilibrium[...]?"
The BMW Potential Function

Answer [Beckmann/McGuire/Winsten 56]: Yes.
The BMW Potential Function

Answer [Beckmann/McGuire/Winsten 56]: Yes.

Proof: Consider the "potential function":

$$(f) = \sum_e \int_0^f c_e(x)dx$$

- defined so that first-order optimality condition = defn of Wardrop equilibrium
- apply Weierstrass's Theorem

QED. (also get uniqueness, etc.)
Potential Functions in Game Theory

Did you know?: Potential functions now standard tool in game theory for proving the existence of a pure-strategy Nash eq.

• define function s.t. whenever player $i$ switches strategies, $\Delta = \Delta u_i$
  - local optima of $\Delta$ = pure-strategy Nash equilibria
  - [Rosenthal 73]: traffic eq w/ discrete population
  - [Monderer/Shapley 96]: general "potential games"
Inefficiency of Wardrop Eq

Motivation [Ch 4, BMW 56]:
• "An economic approach to traffic analysis should [...] provide criteria by which to judge the performance of the system."

Pigou's example [Pigou 1920]:

(WE not Pareto optimal)
Quantifying Inefficiency

*Goal:* quantify inefficiency of WE.
Quantifying Inefficiency

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**Ingredient #1:** objective function.
- will use average travel time (standard)
Quantifying Inefficiency

Goal: quantify inefficiency of WE.

Ingredient #1: objective function.
- will use average travel time (standard)

Ingredient #2: measure of approximation.
- will use ratio of obj fn values of WE, system opt (standard in theoretical CS)
Quantifying Inefficiency

Defn:

\[
\text{inefficiency ratio} = \frac{\text{average travel time in WE}}{\text{average travel time in sys opt}}
\]

- \(= 4/3\) in Pigou's example (33% loss)
- the closer to 1 the better
- aka "coordination ratio", "price of anarchy"
  [Kousoupias/Papadimitriou 99,01]
- first studied for WE by [Roughgarden/Tardos 00]
Potential Fns & Inefficiency

Assume: each cost fn is affine: $c_e(x) = a_ex+b_e$

Claim: BMW potential fn a good approximation of true objective function (avg travel time).
Potential Fns & Inefficiency

**Assume:** each cost fn is affine: \( c_e(x) = a_e x + b_e \)

**Claim:** BMW potential fn a good approximation of true objective function (avg travel time).

**Objective:** \( C(f) = \sum_e c_e(f_e) f_e = \sum_e [a_e f_e + b_e] f_e \)

**Potential:** \( (f) = \sum_e \int_0^f c_e(x) dx = \sum_e [\frac{1}{2} a_e f_e + b_e] f_e \)
Potential Fns & Inefficiency

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Potential: $(f) = \sum_e \int_0^f c_e(x)dx = \sum_e \left[ \frac{1}{2} a_e f_e + b_e \right]f_e$

So: $(f) \leq C(f) \leq 2 (f)$
Potential Fns & Inefficiency

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- (affine cost functions)
Potential Fns & Inefficiency

So: \( f \leq C(f) \leq 2 \cdot f \)

- (affine cost functions)

Consequence: inefficiency ratio \( \leq 2 \)
- proof: \( C(\text{WE}) \leq 2 \cdot (\text{WE}) \leq 2 \cdot (\text{OPT}) \leq 2C(\text{OPT}) \)
Potential Fns & Inefficiency

So: \((f) \leq C(f) \leq 2 \ (f)\)
- (affine cost functions)

**Consequence:** inefficiency ratio \(\leq 2\)
- proof: \(C(WE) \leq 2 \ (WE) \leq 2 \ (OPT) \leq 2C(OPT)\)

**In fact:** [RTO00] more detailed argument \(\Rightarrow\)
inefficiency ratio \(\leq 4/3\)
- Pigou's example the worst! (among all networks, traffic matrices)
More General Cost Fns?

**General Cost Functions**: worst inefficiency ratio grows slowly w/"steepness"

- e.g., degree-d bounded polynomials (w/nonnegative coefficients) [Roughgarden 01]
- naive argument: ratio ≤ d+1
- optimal bound: ≈ d/ln d
- worst network = analogue of Pigou's example
- for d = 4: ≈ 2.15
Epilogue

• potential function introduced in [Beckmann/McGuire/Winsten 56] to prove existence of Wardrop equilibria

• now standard tool in game theory to prove existence of pure Nash equilibria

• now standard tool in theoretical CS + OR to bound inefficiency of equilibria