Recent Results on Congestion Toll Pricing of Traffic Networks

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Outline

• Introduction
• First-Best Toll Pricing – a summary
  – Toll Sets
  – Toll Pricing Framework
  – Numerical Issues
• 2nd Best Toll Pricing Problem
  – Equivalent Formulations
  – Properties of 2nd Best Tolls
  – Cutting Constraint Algorithm
• Current Research
  – Tolling in multi-mode networks
  – Periodic DTA for dynamic tolling
Traffic Congestion
Electronic Toll Collection Facilities
• Traffic congestion has become part of everyday life in major metropolitan areas.
  – An article in *The Economist*, April 27, 2002, discusses the congestion in Britain.
Introduction

• “The real costs of motoring (in Britain) have been falling for decades.”

![Graph showing the real costs of motoring in Britain](image)
Introduction

• “Nearly all the recent road studies the government has commissioned have supported the use of road tolls.”

• “A big road-building programme without pricing is as ludicrous as giving a heroine addict a last fix.” [David Begg, Chairman, Commission for Integrated Transport.]

• “The capital’s mayor, Ken Livingstone, is committed to introducing a £5 daily fee on cars entering the city centre from next January. London is the first big city in the world to try this, . . .”

• Toll is now £8
Introduction

• Two types of models in traffic assignment
  – **User equilibrium** (or optimum) models assume that at equilibrium, no traveler has any incentive to change his or her route.
    • An example of Nash equilibrium.
  – **System optimum** models choose routes that minimize total system cost when demand is fixed, or maximize Net User Benefit ($NUB$) when demand is elastic.
    • Implicitly assume that it is possible to control travelers’ behavior.

• These models generally distribute travel demands to routes in the network differently.
Braess’ Paradox

User Cost = 83

User Cost = 92

In either case, System Cost = 6 x 83
Traffic Assignment Models: Fixed Demand

- $V^{FD} = \{ v: v = \sum_k x^k, Ax^k = b_k, x^k \geq 0, \forall k \in K \}$

- **User Equilibrium**: Find $v^U \in V^{FD}$ such that
  \[
  s(v^U)^T(v - v^U) \geq 0, \quad \forall v \in V^{FD}
  \]

- **Tolled User Equilibrium**: Find $v[\beta] \in V^{FD}$ such that
  \[
  (s(v[\beta]) + \beta)^T(v - v[\beta]) \geq 0, \quad \forall v \in V^{FD}
  \]
  where $\beta$ is a given toll vector.

- **System Optimum**:
  \[
  v^S = \text{argmin}\{s(v)^Tv: v \in V^{FD}\}\]
Marginal Social Cost Pricing

- Toll Pricing Problem: Find $\beta$ so that $v[\beta] = v^S$.

- An optimality condition for the system problem:

$$ (s(v^S) + \nabla s(v^S)' v^S)' (u - v^S) \geq 0, \forall u \in V^{FD} $$

- Marginal Social Cost Pricing tolls

$$ \beta = \nabla s(v^S)' v^S $$
Braess’ Paradox with MSCP Tolls

Note:
• User Cost = 83 + 33 = 116
• Four out of five arcs are tolled.
First-Best Toll Pricing

- **Stockholm Network – Fixed Demand**
  - Morning Rush 278,873 trips
  - Nodes/Links/Centroids = 417/963/46

- **Results per Vehicle**
  - Travel Time 42.96 minutes
  - MSCP tolls 128.53 minutes (88.86 Kr), 914 toll booths
  - MINREV tolls 9.4 minutes (8.125 Kr), 192 toll booths
First-Best Toll Set

- The system flow, $v^S$, is in a tolled user equilibrium with $\beta$ being the toll vector if and only if there is $\rho$ such that $(\beta, \rho)$ satisfies the following:

$$s(v^S) + \beta \geq A^T \rho^k \quad \forall k$$

$$(s(v^S) + \beta)^T v^S = \sum_{k \in K} b_k^T \rho^k$$

First-Best Toll Pricing Framework

• Solve $\nu^S = \text{argmin} \{ s(\nu)^T v : \nu \in V^{FD} \}$

• Solve a toll selection problem:

$$\min \ f(\beta)$$

$$\text{s.t.} \quad s(\nu^S) + \beta \geq A^T \rho_k \quad \forall k$$

$$(s(\nu^S) + \beta)^T \nu^S = \sum_{k \in K} b_k^T \rho_k$$

$$\beta \geq 0$$

where, e.g.,

$$f(\beta) = s(\nu^S)^T \beta, \quad \text{(MINREV)}$$

$$f(\beta) = \sum_{a : \beta_a > 0} 1, \text{ or} \quad \text{(MINTB)}$$

$$f(\beta) = \max \{ \beta_a : a \notin Y \} \quad \text{(MINMAX)}$$

Numerical Results for MINREV

- Customized DWD versus CPLEX

<table>
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<tr>
<th>Network</th>
<th>Nodes</th>
<th>Arcs</th>
<th>OD</th>
<th>Dantzig-Wolfe</th>
<th>CPLEX 7.0</th>
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### Numerical Results for MINTB

- **Dynamic Slope Scaling Procedure (Modified)**

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<th>Test Set</th>
<th>Nodes (Ave.)</th>
<th>Arcs (Ave.)</th>
<th>OD Pairs (Ave.)</th>
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<th>Modified DSSP</th>
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Numerical Results for MINTB (cont.)

- Real Networks

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<th>OD Pairs</th>
<th>Original DSSP</th>
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Traffic Assignment Models: Elastic Demand

- \( V^{ED} = \{ v: v = \sum_k x^k, Ax^k = t_k E_k, x^k \geq 0, t_k \geq 0, \forall k \in K \} \)

- **User Equil.**: Find \((v^U, t^U) \in V^{ED}\) such that
  \[
s(v^U)^T(v - v^U) - w(t^U)^T(d - t^U) \geq 0, \forall (u, d) \in V^{ED}
  \]
  where \(w(t) = \text{inverse demand function} \).

- **Tolled User Equil.**: Find \((v[\beta], t[\beta]) \in V^{ED}\) such that
  \[
  (s(v[\beta]) + \beta)^T(u - v[\beta]) - w(t[\beta])^T(d - t[\beta]) \geq 0, \forall (u, d) \in V^{ED}
  \]
Traffic Assignment Models: Elastic Demand (cont)

• **System Optimum (Maximize NUB):**

\[
(v^S, t^S) = \arg \max \left\{ \sum \int_0^{t_k} w_k(z)dz - s(v)^T v : (v, t) \in V^{ED} \right\}
\]

• **Toll Set:**

\[
s(v^S) + \beta \geq A^T \rho^k \quad \forall k
\]

\[
w(t^S)^T t^S \leq E_k^T \rho^k \quad \forall k
\]

\[
(s(v^S) + \beta)^T v^S - w(t^S)^T t^S = 0
\]


Numerical Issue: Approximate System Solution

• Existing algorithms for TA (e.g., Frank-Wolfe, PARTAN, RSD) only provide an approximate SOPT solution. This resulted in empty (nonegative) toll sets for Hull, Winnipeg and Stockholm even with $10^{-6}$ optimality gap.

• In general, feasible flows (even near optimal) may not have nonegative toll sets. See Bai et al., “Relaxed Toll Sets for Congestion Pricing Problems,” in Mathematical and Computational Models for Congestion Pricing, S. Lawphongpanich, D. W. Hearn and M. J. Smith (eds.), forthcoming, Springer-Verlag, 2005/06.
Relaxed Toll Sets

For a given feasible flow vector \( \hat{u} \in V^{FD} \) and \( \varepsilon > 0 \), the relaxed toll set at \( \hat{u} \), \( T^+ (\hat{u}, \varepsilon) \), is the set of all \( \beta \) for which there exists a corresponding \( \rho \) satisfying the following conditions:

\[
(s(\hat{u}) + \beta) \geq A^T \rho^k \quad \forall k \in K
\]

\[
(s(\hat{u}) + \beta)^T \hat{u} \leq \sum_{k \in K} b^T_k \rho^k + \varepsilon
\]

\( \beta \geq 0 \)
Relaxed Toll Sets – Primary result

- Let
  \[ \varepsilon_{mscp} = -\min \{(s(\hat{u}) + (\nabla s(\hat{u})\hat{u})^T (v - \hat{u}) : v \in V^{FD}\} \]

- Then,
  - \( \varepsilon_{mscp} > 0 \)
  - If \( \nabla s(\hat{u}) > 0 \), then \( T^+(\hat{u}, \varepsilon_{mscp}) \) is nonempty.

- Theorem: Let \( s(.) \) be strongly monotone with modulus \( \alpha > 0 \). For any \( \eta > 0 \), there exists a \( \delta > 0 \) such that \( \|v^\beta - v^s\| \leq \eta \) whenever \( \beta \in T^+(\hat{u}, \varepsilon_{mscp}) \) and \( \|\hat{u} - v^s\| \leq \delta \).
  - A toll vector from a “good” relaxed toll set induces a user equilibrium that is approximately system optimal.
First-Best Toll Pricing - Results

- Characterized toll sets as polyhedra
- Toll Pricing Framework allows secondary objectives:
  - MINREV, MINTB, MINMAX, and ROBINHOOD
- Decomposition techniques for MINREV
  - Cutting Plane Algorithm
  - Dantzig-Wolfe Decomposition
- Modified DSSP algorithm for MINTB
- Extended results to all variable demand models
  - Elastic Demand
  - Combined Distribution-Assignment
- Relaxed toll sets
• For political reasons or otherwise, there are some roads that are not tollable.
  – The second-best problem belongs to a harder class of problems – Mathematical Programs with Equilibrium Constraints (MPECs).
  – Problems of current interest such as pricing of cordon, HOT (High Occupancy Toll) and FAIR (Fast and Intertwined Regular) Lanes.
• In the FAIR LANE concept, lanes in a designated highway are separated into two sections, fast and regular lanes.
  – Fast lanes would be electronically tolled and users of the regular lanes would receive credits that can be used as toll payments on days when they choose to use the fast lanes.

Acceptable tolls:
\[ \beta_b, \beta_c, \beta_d, \beta_e = 0, \beta_f \geq 0, \beta_r \leq 0 \]
\[ v_f \beta_f + v_r \beta_r \geq 0 \]
Cordon Pricing

- Cordon Pricing is a system that collects tolls from vehicles that pass through certain roads or points in a traffic network. Typically, these points form a loop around a defined area, e.g., a city center or a historical area, where traffic needs to be restricted.

![Centroids and Road Intersections]

- Centroids
- Road intersections

Arrows in the cordon form a cut-set separating centroids in the cordoned area from those on the perimeter.
\[
\text{max} \quad \sum_{k}^{t_k} \int_{0}^{w_k(z)} dz - s(v)^T v \\
\text{s.t.} \quad (v, t) \in V^{ED} \\
(s(v) + \beta)^T (u - v) - w(t)^T (d - t) \geq 0 \quad \forall (u, d) \in V^{ED} \\
\beta \geq 0 \quad \forall a \not\in Y \\
\beta_a = 0 \quad \forall a \in Y
\]
Equivalent Formulation – ED-KKT

• The sequentially bounded constraint qualification (SBCQ) holds for ED-VI.

\[
\max \sum_k \int_0^{t_k} w_k(z)dz - s(v)^T v \\
\text{s.t.} \quad (v, t) \in V^{ED} \\
s(v) + \beta \geq A^T \rho^k \quad \forall k \\
w(t)^T t \leq E_k^T \rho^k \quad \forall k \\
(s(v) + \beta)^T v = w(t)^T t \\
\beta_a \geq 0 \quad \forall a \not\in Y \\
\beta_a = 0 \quad \forall a \in Y
\]
Properties of 2nd Best Tolls – NUB bounds

- **Theorem**: Assume that the user and system problems have solutions \((v^U,t^U)\) and \((v^S,t^S)\). Further, assume that \(s(v)\) and \(-w(t)\) are monotonic and continuous and \((\nabla s(v), \nabla w(t))\) exists and is continuous. Then, ED-VI has a global optimal solution with objective value in the interval \([NUB(v^U,t^U), NUB(v^S,t^S)]\).
Properties of 2\textsuperscript{nd} Best Tolls – Constant Toll Revenue* 

- Let $(\bar{v}, \bar{t}, \bar{\beta})$ be an optimal solution to ED - VI.
- Any $\beta$ such that $(\beta, \rho)$ satisfies the following system of equations is a valid toll.

$$s(\bar{v}) + \beta \geq A^T \rho^k, \quad \forall k$$

$$w_k(\bar{t}_k) \leq E_k^T \rho^k, \quad \forall k$$

$$(s(\bar{v}) + \beta)^T \bar{v} = w(\bar{t})^T \bar{t} \quad (***)$$

$$\beta_a \geq 0 \quad \forall a \notin Y$$

$$\beta_a = 0 \quad \forall a \in Y$$
To motivate another property, consider the following two-arc problem where Arc 1 is tollable and Arc 2 is not.

where $s_1(v_1) = v_1$, $s_2(v_2) = v_2 + 2$, and $w(t) = 9 - t/2$
Properties of 2nd Best Tolls – Formulas?

• In the literature (see, e.g., McDonald, 1995, and Verhoef, 2000), the optimal toll for Arc 1 is

\[
\overline{\beta}_1 = s'_1(\overline{v}_1)\overline{v}_1 + \frac{w'(\bar{t})}{s'_2(\overline{v}_2) - w'(\bar{t})} s'_2(\overline{v}_2)\overline{v}_2
\]

\[
= 3.3636 - \frac{0.5}{(1 + 0.5)} 3.5455 = 2.1818
\]

• In this expression, the optimal toll includes a portion of MSCP from the non-tollable arc.
  – Are there similar formulas for general networks?
Properties of 2\textsuperscript{nd} Best Tolls – Formulas?

• Results related to the previous question:
  – When the KKT multipliers exist, the second-best tolls can always be written as an expression involving marginal social cost pricing (MSCP) terms.
  – The KKT conditions associated with ED-KKT yields the following expression of an optimal toll vector.

\[
\bar{\beta} = \frac{1}{\theta} \left[ \delta^k - (1 + \theta)[s(\bar{v}) + \nabla s(\bar{v})^T \bar{v}] - A^T \lambda^k + \nabla s(\bar{v})^T \sum_{k \in K} \psi^k \right]
\]

– An interpretation:
  • An optimal 2\textsuperscript{nd} best toll on a link involves its own MSCP as well as those from non-tollable arcs via the KKT multipliers.
Properties of 2nd Best Tolls – Formulas?

- ED-KKT for the two arc example

\[
\begin{align*}
\text{max} & \quad v_1^2 + v_2^2 + 2v_2 - 9t + (t^2/4) \\
\text{s.t.} & \quad v_1 + v_2 - t = 0, \\
& \quad \rho - v_1 - \beta_1 \leq 0, \\
& \quad \rho - v_2 - 2 \leq 0, \\
& \quad -\rho + (9 - \frac{t}{2}) \leq 0, \\
& \quad (v_1 + \beta_1)v_1 + (v_2 + 2)v_2 - (9 - \frac{t}{2})t = 0, \\
& \quad v_1, v_2, t, \beta_1 \geq 0.
\end{align*}
\]
Properties of 2nd Best Tolls – Formulas?

• An optimal solution

The net user benefit = 19.2727

\((\bar{v}_1, \bar{v}_2, \bar{t}) = (3.3636, 3.5455, 6.9091), (\bar{\beta}_1, \bar{\rho}) = (2.1818, 5.5455)\)

KKT Mult.

\((\lambda, \psi_1, \psi_2, \xi, \theta) = (-12.2727, 3.3636, 5.9091, 9.2727, 1.0)\)

• Using the expression,

\[
\begin{pmatrix}
\bar{\beta}_1 \\
\bar{\beta}_2 \\
\end{pmatrix} = \frac{1}{\theta} \left\{ \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} - (1 + \theta) \begin{pmatrix} s_1(\bar{v}_1) + s'_1(\bar{v}_1)^T \bar{v}_1 \\ s_2(\bar{v}_2) + s'_2(\bar{v}_2)^T \bar{v}_2 \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} s'_1(\bar{v}_1)\psi_1 \\ s'_2(\bar{v}_2)\psi_2 \end{pmatrix} \right\}
\]

\[
= \begin{pmatrix} 0 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 6.7272 \\ 9.0909 \end{pmatrix} + 12.2727 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3.3636 \\ 5.9091 \end{pmatrix} = \begin{pmatrix} 2.1819 \\ 0 \end{pmatrix}.
\]
• The above result assumes that the multipliers exist.
  – Scheel and Scholtes [2000] show that MFCQ is violated at every feasible solution of ED-KKT.
  – However, the multipliers exist when the strong stationarity conditions hold at an optimal solution to ED-KKT.
  – A similar expression for the tolls can be obtained using the ‘tightened’ NLP associated with ED-KKT.
• The multipliers for this problem exist, e.g., when $s(v)$ and $w(t)$ are linear.
The set $V^{ED}$ can be expressed as a convex combination of its extreme points, $(u^i, d^i)$, $i = 1,\ldots, n$.

$$\max_{(v,t,\beta)} \sum_{k \in K} \int_0^{t_k} w_k(z)dz - s(v)^T v$$

subject to:

$$\beta_a \geq 0 \quad \forall a \notin Y$$

$$\beta_a = 0 \quad \forall a \in Y$$

$$(s(v) + \beta)^T (u^i - v) - w(t)^T (d^i - t) \geq 0 \quad \forall i = 1,\ldots, n$$
Let \((u^1, d^1)\) be a system optimal solution. Set \(r = 1\).

Solve the following master problem:

\[
(v^r, t^r, \beta^r) = \arg \max_{(v, t, \beta)} \sum_{k \in K} \int_0^{t_k} w_k(z)dz - s(v)^T v \\
\text{s.t.} \quad (v, t) \in V^{ED}; \\
(s(v) + \beta)^T (u^i - v) - w(t)^T (d^i - t) \geq 0 \quad i = 1, \ldots, r \\
\beta_a \geq 0, \forall a \notin Y; \beta_a = 0, \forall a \in Y
\]

Solve the subproblem:

\[
(u^{r+1}, d^{r+1}) = \arg \min \{(s(v^r + \beta^r)^T u - w(t^r)^T d : (u, d) \in V^{ED}\}
\]

If \((s(v^r + \beta^r)^T (u^{r+1} - u^r) - w(t^r)^T (d^{r+1} - d^r) \geq 0, \text{stop.}\)
Otherwise, set \(r = r + 1\) and go to 1.
Cutting constraint algorithm for ED-EX

• The solutions to the problem in Step 3 are distinct.
  – Because the number of extreme points of $V^{ED}$ is finite, the algorithm must stop after a finite number of iterations.

• In Step 2, the master problem is generally nonconvex and may not satisfy MFCQ.
  – Unless we obtain global solutions, the sequence of objective values for the master problem may not decrease monotonically.
  – In our implementation, MINOS is able to solve the master problem when the cutting constraints are relaxed, i.e.,

$$\begin{align*}
(s(v) + \beta)^T (u^i - v) - w(t)^T (d^i - t) &\geq -\epsilon, \quad i = 1, \cdots, r
\end{align*}$$
Example: Elastic Demand

Network

Inverse Demand Function: $w_k(t) = a_k + b_k t$

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<th>$b_k$</th>
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<tr>
<td>(2, 4)</td>
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Example: Elastic Demand

Travel Cost function: $s_a(v) = T_a(1+0.15(v_a/C_a))$

<table>
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<tr>
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<th>$T_a$</th>
<th>$C_a$</th>
<th>Arcs</th>
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<td>48</td>
<td>(9, 8)</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>
### Example: Elastic Demand

<table>
<thead>
<tr>
<th>It.</th>
<th>Objective Value</th>
<th>% Equil. Gap</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Master</td>
<td>Subprob</td>
</tr>
<tr>
<td>1</td>
<td>1468.11</td>
<td>-740.15</td>
</tr>
<tr>
<td>2</td>
<td>1466.31</td>
<td>-289.68</td>
</tr>
<tr>
<td>3</td>
<td>1446.59</td>
<td>-91.22</td>
</tr>
<tr>
<td>4</td>
<td>1441.52</td>
<td>-35.64</td>
</tr>
<tr>
<td>5</td>
<td>1431.88</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Tollable arcs: (5,7), (7,4), and (8,4).
Example: NUB versus Arcs Tolled

Tollable arcs:
Numerical Results: Fixed Demand

- Use GAMS
  - CPLEX to solve the subproblem in Step 3.
  - MINOS to solve the master problem in Step 2.
- Two networks from the literature
  - Sioux Falls: 76 links, 24 nodes, 528 OD pairs.
  - Hull: 798 links, 501 nodes, 158 OD pairs.
- Tollable arc selection
  - An arc is tollable if its user equilibrium flow exceeds its system optimum flow by a given percentage (‘excess’ percentage).
### Numerical Results: Fixed Demand

- **Sioux Falls:**

<table>
<thead>
<tr>
<th>Excess %</th>
<th># of Tollable Arcs</th>
<th>Total Delay</th>
<th>Relative Gap (%)</th>
<th>Iterations Required</th>
<th>Master Problem (sec)</th>
<th>Sub-problem (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>18</td>
<td>72.1036</td>
<td>0.9354</td>
<td>49</td>
<td>1731.61</td>
<td>11.77</td>
</tr>
<tr>
<td>10%</td>
<td>12</td>
<td>72.1861</td>
<td>0.9024</td>
<td>36</td>
<td>879.26</td>
<td>8.3</td>
</tr>
<tr>
<td>15%</td>
<td>4</td>
<td>73.0681</td>
<td>0.7764</td>
<td>14</td>
<td>182.13</td>
<td>2.65</td>
</tr>
<tr>
<td>25%</td>
<td>2</td>
<td>73.4916</td>
<td>0.4992</td>
<td>10</td>
<td>107.73</td>
<td>1.9</td>
</tr>
</tbody>
</table>

- Total delay at SOPT = 71.9426
- Total delay at UOPT = 74.8023
## Numerical Results: Fixed Demand

- **Hull**

<table>
<thead>
<tr>
<th>Excess %</th>
<th>Tollable Arcs</th>
<th>Total Delay</th>
<th>Relative Gap (%)</th>
<th>Iterations Required</th>
<th>Master Problem (sec)</th>
<th>Sub-problem (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>179</td>
<td>179117</td>
<td>&lt; 0.0001</td>
<td>16</td>
<td>6251.09</td>
<td>133.23</td>
</tr>
<tr>
<td>10%</td>
<td>135</td>
<td>179420</td>
<td>&lt; 0.0001</td>
<td>8</td>
<td>1081.88</td>
<td>59.76</td>
</tr>
<tr>
<td>15%</td>
<td>93</td>
<td>179988</td>
<td>&lt; 0.0001</td>
<td>7</td>
<td>1671.44</td>
<td>55.06</td>
</tr>
<tr>
<td>25%</td>
<td>58</td>
<td>180629</td>
<td>&lt; 0.0001</td>
<td>7</td>
<td>2007.45</td>
<td>53.14</td>
</tr>
<tr>
<td>50%</td>
<td>21</td>
<td>181092</td>
<td>&lt; 0.0001</td>
<td>10</td>
<td>3947.43</td>
<td>66.82</td>
</tr>
<tr>
<td>75%</td>
<td>12</td>
<td>181315</td>
<td>&lt; 0.0001</td>
<td>8</td>
<td>3293.28</td>
<td>46.84</td>
</tr>
<tr>
<td>100%</td>
<td>10</td>
<td>181326</td>
<td>&lt; 0.0001</td>
<td>11</td>
<td>5270.45</td>
<td>63.39</td>
</tr>
<tr>
<td>200%</td>
<td>8</td>
<td>181329</td>
<td>&lt; 0.0001</td>
<td>11</td>
<td>5620.71</td>
<td>61.49</td>
</tr>
</tbody>
</table>

- Total delay at SOPT = 179063
- Total delay at UOPT = 186720
1. Solve ED - VI or one of the equivalent problems to obtain $(\bar{v}, \bar{t}, \bar{\beta})$.

- Solve the following toll selection problem:

$$\begin{align*}
\min & \quad f(\beta) \\
\text{s.t.} & \quad s(\bar{v}) + \beta \geq A^T \rho^k \quad \forall k \\
& \quad w_k (\bar{t}_k) \leq E_k^T \rho^k \quad \forall k \\
& \quad (s(\bar{v}) + \beta)^T \bar{v} = w(\bar{t})^T \bar{t} \\
& \quad \beta_a \geq 0 \quad \forall a \not\in Y \\
& \quad \beta_a = 0 \quad \forall a \in Y
\end{align*}$$

where, e.g., $f(\beta) = \sum_{a: \beta_a > 0} 1$ or $\max\{\beta_a : a \not\in Y\}$.
## Numerical Example

<table>
<thead>
<tr>
<th>Arcs</th>
<th>ED-VI</th>
<th>Min Max</th>
<th>MINTB</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Flow</td>
<td>Toll</td>
<td>Toll</td>
</tr>
<tr>
<td>(1,5)</td>
<td>5.12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1,6)</td>
<td>2.82</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2,5)</td>
<td>36.35</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2,6)</td>
<td>10.56</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(5,7)</td>
<td>41.42</td>
<td>3.56</td>
<td>3.23</td>
</tr>
<tr>
<td>(5,9)</td>
<td>0.05</td>
<td>0.05</td>
<td>0.73</td>
</tr>
<tr>
<td>(6,8)</td>
<td>13.37</td>
<td>2.69</td>
<td>1.69</td>
</tr>
<tr>
<td>(6,9)</td>
<td>0.61</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(7,3)</td>
<td>21.24</td>
<td>1.59</td>
<td>1.92</td>
</tr>
<tr>
<td>(7,4)</td>
<td>12.91</td>
<td>2.9</td>
<td>3.23</td>
</tr>
<tr>
<td>(7,8)</td>
<td>7.28</td>
<td>3.9</td>
<td>3.23</td>
</tr>
<tr>
<td>(8,4)</td>
<td>20.7</td>
<td>1</td>
<td>2.69</td>
</tr>
<tr>
<td>(9,8)</td>
<td>0.05</td>
<td>4.92</td>
<td>3.23</td>
</tr>
<tr>
<td>Tot Rev.</td>
<td>283.46</td>
<td>283.46</td>
<td>283.46</td>
</tr>
<tr>
<td>Max Toll</td>
<td>4.92</td>
<td>3.23</td>
<td>4.78</td>
</tr>
<tr>
<td># booths</td>
<td>8</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>NUB</td>
<td>1453.27</td>
<td>1453.27</td>
<td>1453.27</td>
</tr>
</tbody>
</table>

Note: Nonessential arcs are not listed.
2nd Best Tolls - Conclusions

- Three equivalent formulations (ED-VI, ED-KKT, and ED-EX) for the 2nd best toll pricing problem
- Properties of the 2nd best tolls
  - Via the KKT multipliers, optimal 2nd best tolls involves MSCP tolls on individual arcs as well as those from non-tollable arcs.
  - Toll revenue is constant.
- Cutting constraint algorithm for ED-EX
  - Converges finitely
  - Relaxed version can be implemented using existing software for LP and NLP
  - Can potentially solve large problems
- Toll pricing framework
  - Find a 2nd best toll vector that optimizes a (secondary) objective.

Current Research

- **Dynamic tolls** to vary with time and traffic conditions
  - Ph. D. research of Artyom Nahapetyan
  - Period DTA – tomorrow TSL session on DTA (I) at 3:30

- Toll pricing for systems with multiple modes of transportation, e.g., tolls on the roads and fares on the transit network. Partial support from Volvo Research Foundation. PIs are Toi Lawphongpanich (UF), Younes Hammdouch (UAE), Agachai Sumalee (U. of Leeds)
Traffic Assignment Problem with Multiple Modes

• Consider two travel options:
  – Automobile only (auto-only)
  – Mixed modes are

Walk-Metro: Travelers walk to metro stations, use metro lines to reach the final metro stations, and walk from there to their destinations.

Auto-Metro: Travelers drive to metro stations, use metro lines to reach the final stations, and walk from there to their destinations.

Y. Hamdouch, M. Florian, D. W. Hearn, and S. Lawphongpanich, "Congestion Pricing for Multi-Modal Transportation Systems," accepted and under revision for Transportation Research B.
Underlying Networks

• There is an underlying network for each travel option.
  – Auto only option

![Graph with nodes and arcs]

Nodes = road intersections
Arcs = roads and highways
Underlying Networks: Walk-Metro Option
Underlying Networks: Auto-Metro Option
Multi-mode Pricing - Summary

• Developed a system problem that leads to the toll pricing framework
  – SOPT maximizes partial (metro benefit) NUB
  – UOPT problem is a VI
• Show that 2\textsuperscript{nd} best pricing is not needed for zero tolls on walk, embark, disembark links
• Propose secondary toll selection problems unique to the multi-mode case.
  – For example, auto tolls are used to reduce transit fares
End