Variational Inequalities

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Background

Equilibrium is a central concept in numerous disciplines including economics, management science/operations research, and engineering.

Methodologies that have been applied to the formulation, qualitative analysis, and computation of equilibria have included

- systems of equations,
- optimization theory,
- complementarity theory, and
- fixed point theory.

Variational inequality theory is a powerful unifying methodology for the study of equilibrium problems.

Variational inequality theory was introduced by Hartman and Stampacchia (1966) as a tool for the study of partial differential equations with applications principally drawn from mechanics. Such variational inequalities were *infinite-dimensional* rather than *finite-dimensional* as we will be studying here.

The breakthrough in finite-dimensional theory occurred in 1980 when Dafermos recognized that the traffic network equilibrium conditions as stated by Smith (1979) had a structure of a variational inequality.

This unveiled this methodology for the study of problems in economics, management science/operations research, and also in engineering, with a focus on transportation.

To-date problems which have been formulated and studied as variational inequality problems include:

- traffic network equilibrium problems
- spatial price equilibrium problems
- oligopolistic market equilibrium problems
- financial equilibrium problems
- migration equilibrium problems, as well as
- environmental network problems, and
- knowledge network problems.

Variational Inequality Theory

Variational inequality theory provides us with a tool for:

formulating a variety of equilibrium problems;

qualitatively analyzing the problems in terms of existence and uniqueness of solutions, stability and sensitivity analysis, and

providing us with algorithms with accompanying convergence analysis for computational purposes.

It contains, as special cases, such well-known problems in mathematical programming as: systems of nonlinear equations, optimization problems, complementarity problems, and is also related to fixed point problems.

The Variational Inequality Problem

Definition 1 (Variational Inequality Problem)

The finite - dimensional variational inequality problem, VI(F,K), *is to determine a vector* $x^* \in K \subset R^n$, *such that*

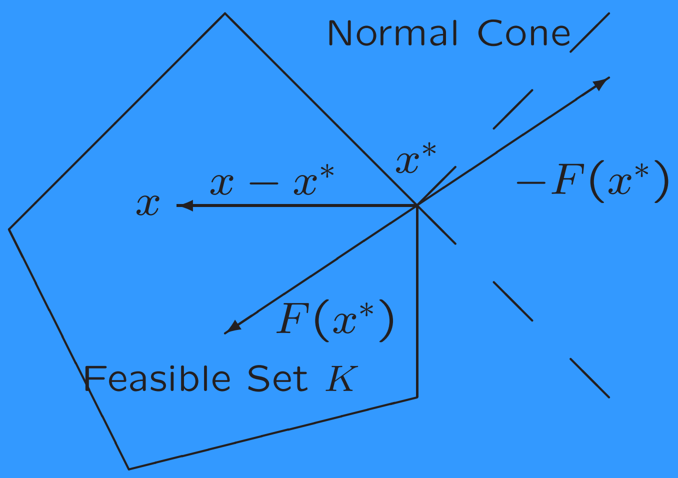
$$F(x^*)^T \cdot (x - x^*) \geq 0, \quad \forall x \in K,$$

or, equivalently,

$$\langle F(x^*)^T, x - x^* \rangle \geq 0, \quad \forall x \in K \tag{1}$$

where F *is a given continuous function from* K *to* R^n, K *is a given closed convex set, and* $\langle \cdot, \cdot \rangle$ *denotes the inner product in* n *dimensional Euclidean space.*

In geometric terms, the variational inequality (1) states that $F(x^*)^T$ is "orthogonal" to the feasible set K at the point x^*. This formulation, as shall be demonstrated, is particularly convenient because it allows for a unified treatment of equilibrium problems and optimization problems.



Geometric interpretation
of the variational inequality problem

Indeed, many mathematical problems can be formulated as variational inequality problems, and several examples applicable to equilibrium analysis follow.

Systems of Equations

Many classical economic equilibrium problems have been formulated as systems of equations, since market clearing conditions necessarily equate the total supply with the total demand. In terms of a variational inequality problem, the formulation of a system of equations is as follows.

Proposition 1

Let $K = R^n$ and let $F : R^n \mapsto R^n$ be a given function. A vector $x^* \in R^n$ solves VI(F, R^n) *if and only if $F(x^*) = 0$.*

Proof: If $F(x^*) = 0$, then inequality (1) holds with equality. Conversely, if x^* satisfies (1), let $x = x^* - F(x^*)$, which implies that

$$F(x^*)^T \cdot (-F(x^*)) \geq 0, \quad \text{or} \quad -\|F(x^*)\|^2 \geq 0 \tag{2}$$

and, therefore, $F(x^*) = 0$.

Note that systems of equations, however, preclude the introduction of inequalities, which may be needed, for example, in the case of nonnegativity assumptions on certain variables such as prices.

Optimization Problems

An optimization problem is characterized by its specific objective function that is to be maximized or minimized, depending upon the problem and, in the case of a constrained problem, a given set of constraints. Possible objective functions include expressions representing profits, costs, market share, portfolio risk, etc. Possible constraints include those that represent limited budgets or resources, nonnegativity constraints on the variables, conservation equations, etc. Typically, an optimization problem consists of a single objective function.

Both unconstrained and constrained optimization problems can be formulated as variational inequality problems. The subsequent two propositions and theorem identify the relationship between an optimization problem and a variational inequality problem.

Proposition 2

Let x^* be a solution to the optimization problem:

$$\textit{Minimize} \quad f(x) \tag{3}$$

$$\textit{subject to:} \quad x \in K,$$

where f is continuously differentiable and K is closed and convex. Then x^* is a solution of the variational inequality problem:

$$\nabla f(x^*)^T \cdot (x - x^*) \geq 0, \quad \forall x \in K. \tag{4}$$

Proof: Let $\phi(t) = f(x^* + t(x - x^*))$, for $t \in [0, 1]$. Since $\phi(t)$ achieves its minimum at $t = 0$, $0 \leq \phi'(0) = \nabla f(x^*)^T \cdot (x - x^*)$, that is, x^* is a solution of (4).

Proposition 3

If $f(x)$ is a convex function and x^* is a solution to VI$(\nabla f, K)$, *then x^* is a solution to the optimization problem (3).*

Proof: Since $f(x)$ is convex,

$$f(x) \geq f(x^*) + \nabla f(x^*)^T \cdot (x - x^*), \quad \forall x \in K. \tag{5}$$

But $\nabla f(x^*)^T \cdot (x - x^*) \geq 0$, since x^* is a solution to VI$(\nabla f, K)$. Therefore, from (5) one concludes that

$$f(x) \geq f(x^*), \quad \forall x \in K,$$

that is, x^* is a minimum point of the mathematical programming problem (3).

If the feasible set $K = R^n$, then the unconstrained optimization problem is also a variational inequality problem.

On the other hand, in the case where a certain symmetry condition holds, the variational inequality problem can be reformulated as an optimization problem. In other words, in the case that the variational inequality formulation of the equilibrium conditions underlying a specific problem is characterized by a function with a symmetric Jacobian, then the solution of the equilibrium conditions and the solution of a particular optimization problem are one and the same. We first introduce the following definition and then fix this relationship in a theorem.

Definition 2

An $n \times n$ *matrix* $M(x)$*, whose elements* $m_{ij}(x)$*;* $i = 1, \dots, n$*;* $j = 1, \dots, n$*, are functions defined on the set* $S \subset R^n$*, is said to be positive semidefinite on* S *if*

$$v^T M(x) v \geq 0, \quad \forall v \in R^n, x \in S.$$

It is said to be positive definite on S *if*

$$v^T M(x) v > 0, \quad \forall v \neq 0, v \in R^n, x \in S.$$

It is said to be strongly positive definite on S *if*

$$v^T M(x) v \geq \alpha \|v\|^2, \quad \textit{for some} \quad \alpha > 0, \quad \forall v \in R^n, x \in S.$$

Note that if $\gamma(x)$ is the smallest eigenvalue, which is necessarily real, of the symmetric part of $M(x)$, that is, $\frac{1}{2}\left[M(x) + M(x)^T\right]$, then it follows that (i). $M(x)$ is positive semidefinite on S if and only if $\gamma(x) \geq 0$, for all $x \in S$; (ii). $M(x)$ is positive definite on S if and only if $\gamma(x) > 0$, for all $x \in S$; and (iii). $M(x)$ is strongly positive definite on S if and only if $\gamma(x) \geq \alpha > 0$, for all $x \in S$.

Theorem 1

Assume that $F(x)$ is continuously differentiable on K and that the Jacobian matrix

$$\nabla F(x) = \left[\begin{array}{ccc} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{array} \right]$$

is symmetric and positive semidefinite. Then there is a real-valued convex function $f : K \mapsto R^1$ satisfying

$$\nabla f(x) = F(x)$$

with x^* the solution of $VI(F,K)$ also being the solution of the mathematical programming problem:

$$\textit{Minimize} \quad f(x) \tag{6}$$

$$\textit{subject to:} \quad x \in K.$$

Proof: Under the symmetry assumption it follows from Green's Theorem that

$$f(x) = \int F(x)^T dx, \tag{7}$$

where $\int$ is a line integral. The conclusion follows from Proposition 3.

Hence, although the variational inequality problem encompasses the optimization problem, a variational inequality problem can be reformulated as a convex optimization problem, *only when the symmetry condition and the positive semidefiniteness condition hold*.

The variational inequality, therefore, is the more general problem in that it can also handle a function $F(x)$ with an asymmetric Jacobian.

Complementarity Problems

The variational inequality problem also contains the complementarity problem as a special case. Complementarity problems are defined on the nonnegative orthant.

Let R^n_+ denote the nonnegative orthant in R^n, and let $F : R^n \mapsto R^n$. The nonlinear complementarity problem over R^n_+ is a system of equations and inequalities stated as:

Find $x^* \geq 0$ such that

$$F(x^*) \geq 0 \quad \text{and} \quad F(x^*)^T \cdot x^* = 0. \tag{8}$$

Whenever the mapping F is affine, that is, whenever $F(x) = Mx + b$, where M is an $n \times n$ matrix and b an $n \times 1$ vector, problem (8) is then known as the linear complementarity problem.

The relationship between the complementarity problem and the variational inequality problem is as follows.

Proposition 4

VI(F, R^n_+) *and (8) have precisely the same solutions, if any.*

Proof: First, it is established that if x^* satisfies VI(F, R^n_+), then it also satisfies the complementarity problem (8). Substituting $x = x^* + e_i$ into $VI(F, R^n_+)$, where e_i denotes the n-dimensional vector with 1 in the i-th location and 0, elsewhere, one concludes that $F_i(x^*) \geq 0$, and $F(x^*) \geq 0$.

Substituting now $x = 2x^*$ into the variational inequality, one obtains

$$F(x^*)^T \cdot (x^*) \geq 0. \tag{9}$$

Substituting then $x = 0$ into the variational inequality, one obtains

$$F(x^*)^T \cdot (-x^*) \geq 0. \tag{10}$$

(9) and (10) together imply that $F(x^*)^T \cdot x^* = 0$.

Conversely, if x^* satisfies the complementarity problem, then

$$F(x^*)^T \cdot (x - x^*) \geq 0$$

since $x \in R^n_+$ and $F(x^*) \geq 0$.

Fixed Point Problems

Fixed point theory has been used to formulate, analyze, and compute solutions to economic equilibrium problems. The relationship between the variational inequality problem and a fixed point problem can be made through the use of a projection operator. First, the projection operator is defined.

Lemma 1

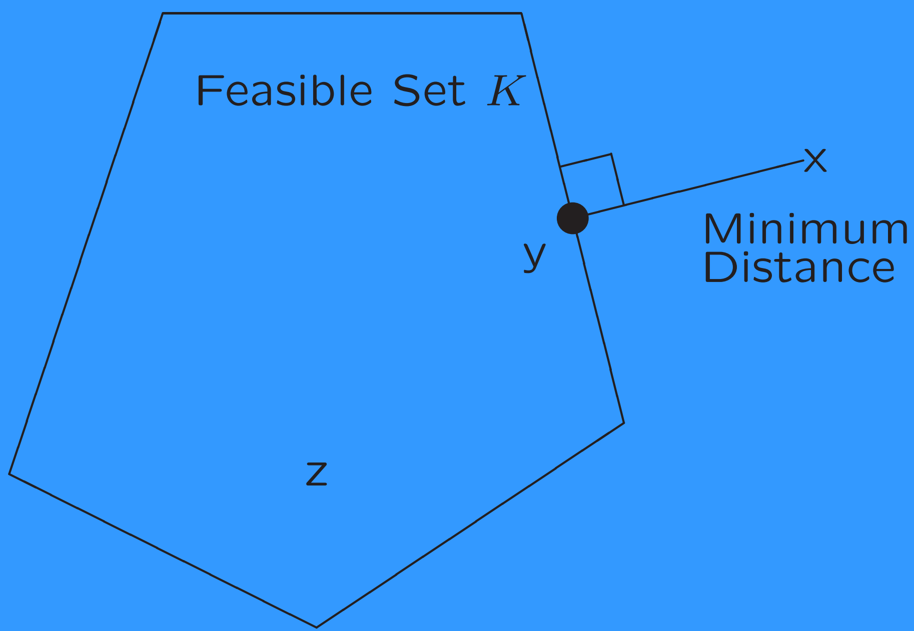
Let K be a closed convex set in R^n. Then for each $x \in R^n$, there is a unique point $y \in K$, such that

$$\|x-y\| \leq \|x-z\|, \quad \forall z \in K, \tag{11}$$

and y is known as the orthogonal projection of x on the set K with respect to the Euclidean norm, that is,

$$y = P_K x = \arg\min_{z \in K} \|x-z\|.$$

Proof: Let x be fixed and let $w \in K$. Minimizing $\|x-z\|$ over all $z \in K$ is equivalent to minimizing the same function over all $z \in K$ such that $\|x-z\| \leq \|x-w\|$, which is a compact set. The function g defined by $g(z) = \|x-z\|^2$ is continuous. Existence of a minimizing y follows because a continuous function on a compact set always attains its minimum. To prove that y is unique, observe that the square of the Euclidean norm is a strictly convex function. Hence, g is strictly convex and its minimum is unique.



The projection y of x on the set K

Theorem 2

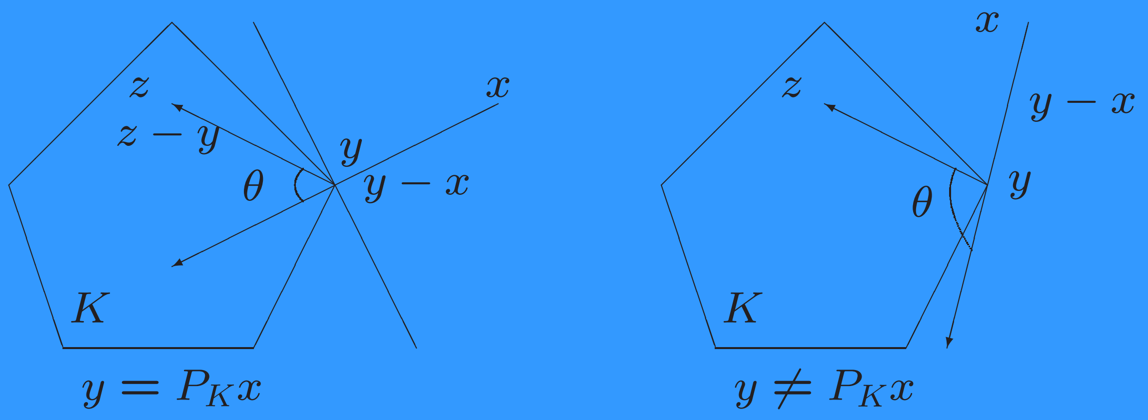
Let K be a closed convex set. Then $y = P_K x$ if and only if

$$y^T \cdot (z - y) \geq x^T \cdot (z - y), \quad \forall z \in K$$

or

$$(y - x)^T \cdot (z - y) \geq 0, \quad \forall z \in K. \tag{12}$$

Proof: Note that $y = P_K x$ is the minimizer of $g(z)$ over all $z \in K$. Since $\nabla g(z) = 2(z - x)$, the result follows from the optimality conditions for constrained optimization problems.



Geometric interpretation of $\langle (y-x)^T, z-y \rangle \geq 0$, for $y = P_K x$ and $y \neq P_K x$

A property of the projection operator which is useful both in qualitative analysis of equilibria and their computation is now presented.

Corollary 1

Let K be a closed convex set. Then the projection operator P_K is nonexpansive, that is,

$$\|P_K x - P_K x'\| \leq \|x - x'\|, \quad \forall x, x' \in R^n. \tag{13}$$

Proof: Given $x, x' \in R^n$, let $y = P_K x$ and $y' = P_K x'$. Then from Theorem 2 note that

$$\text{for} \quad y \in K : y^T \cdot (z - y) \geq x^T \cdot (z - y), \quad \forall z \in K, \tag{14}$$

$$\text{for} \quad y' \in K : y'^T \cdot (z - y') \geq x'^T \cdot (z - y'), \quad \forall z \in K. \tag{15}$$

Setting $z = y'$ in (14) and $z = y$ in (15) and adding the resultant inequalities, one obtains:

$$\|y - y'\|^2 = (y - y')^T \cdot (y - y') \leq (x - x')^T \cdot (y - y')$$

$$\leq \|x - x'\| \cdot \|y - y'\|$$

by an application of the Schwarz inequality. Hence,

$$\|y - y'\| \leq \|x - x'\|.$$

The relationship between a variational inequality and a fixed point problem is as follows.

Theorem 3

Assume that K is closed and convex. Then $x^* \in K$ is a solution of the variational inequality problem $VI(F, K)$ if and only if for any $\gamma > 0$, x^* is a fixed point of the map

$$P_K(I - \gamma F) : K \mapsto K,$$

that is,

$$x^* = P_K(x^* - \gamma F(x^*)). \tag{16}$$

Proof: Suppose that x^* is a solution of the variational inequality, i.e.,

$$F(x^*)^T \cdot (x - x^*) \geq 0, \quad \forall x \in K.$$

Multiplying the above inequality by $-\gamma < 0$, and adding $x^{*T} \cdot (x - x^*)$ to both sides of the resulting inequality, one obtains

$$x^{*T} \cdot (x - x^*) \geq [x^* - \gamma F(x^*)]^T \cdot (x - x^*), \quad \forall x \in K. \tag{17}$$

From Theorem 2 one concludes that

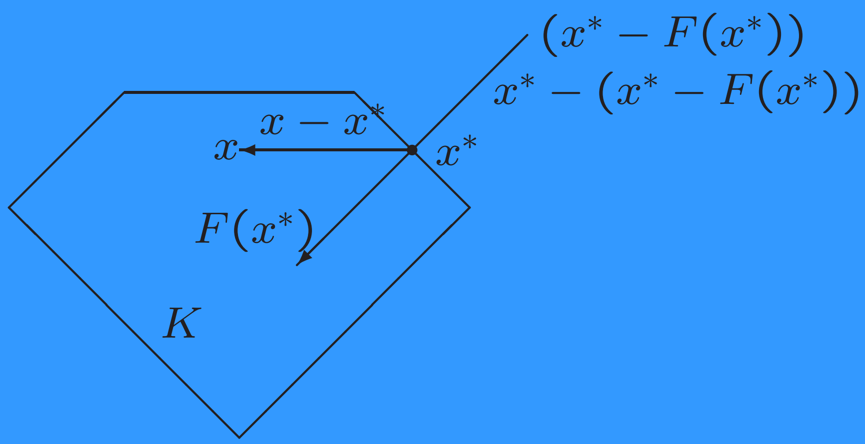
$$x^* = P_K(x^* - \gamma F(x^*)).$$

Conversely, if $x^* = P_K(x^* - \gamma F(x^*))$, for $\gamma > 0$, then

$$x^{*T} \cdot (x - x^*) \geq (x^* - \gamma F(x^*))^T \cdot (x - x^*), \quad \forall x \in K,$$

and, therefore,

$$F(x^*)^T \cdot (y - x^*) \geq 0, \quad \forall y \in K.$$



Geometric depiction of the variational inequality problem and its fixed point equivalence (with $\gamma = 1$)

Basic Existence and Uniqueness Results

Variational inequality theory is also a powerful tool in the qualitative analysis of equilibria. We now provide conditions for existence and uniqueness of solutions to VI(F, K) are provided.

Existence of a solution to a variational inequality problem follows from continuity of the function F entering the variational inequality, provided that the feasible set K is compact. Indeed, we have the following:

Theorem 4 (Existence Under Compactness and Continuity)

If K is a compact convex set and $F(x)$ is continuous on K, then the variational inequality problem admits at least one solution x^*.

Proof: According to Brouwer's Fixed Point Theorem, given a map $P : K \mapsto K$, with P continuous, there is at least one $x^* \in K$, such that $x^* = Px^*$. Observe that since P_K and $(I - \gamma F)$ are each continuous, $P_K(I - \gamma F)$ is also continuous. The conclusion follows from compactness of K and Theorem 3.

In the case of an unbounded feasible set K, Brouwer's Fixed Point Theorem is no longer applicable; the existence of a solution to a variational inequality problem can, nevertheless, be established under the subsequent condition.

Let $B_R(0)$ denote a closed ball with radius R centered at 0 and let $K_R = K \cap B_R(0)$. K_R is then bounded. Let VI_R denote the variational inequality problem:

Determine $x^*_R \in K_R$, such that

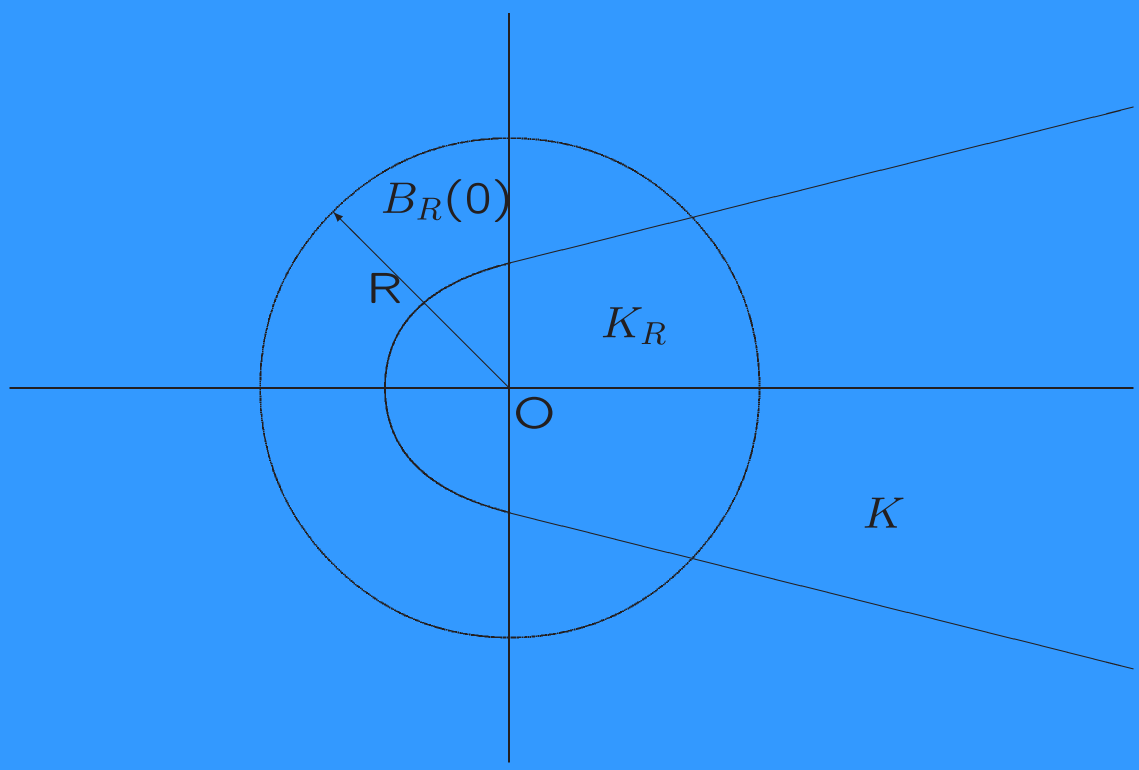
$$F(x^*_R)^T \cdot (y - x^*_R) \geq 0, \quad \forall y \in K_R. \tag{18}$$

We now state:

Theorem 5

$\mathrm{VI}(F, K)$ *admits a solution if and only if there exists an* $R > 0$ *and a solution of* $\mathrm{VI_R}$, x^*_R, *such that* $\|x^*_R\| < R$.

Although $\|x^*_R\| < R$ may be difficult to check, one may be able to identify an appropriate R based on the particular application.



Depiction of bounded set K_R

Existence of a solution to a variational inequality problem may also be established under the coercivity condition, as in the subsequent corollary.

Corollary 2 (Existence Under Coercivity)

Suppose that $F(x)$ satisfies the coercivity condition

$$\frac{(F(x) - F(x_0))^T \cdot (x - x_0)}{\|x - x_0\|} \to \infty \tag{19}$$

as $\|x\| \to \infty$ for $x \in K$ and for some $x_0 \in K$. Then VI(F, K) *always has a solution.*

Corollary 3

Suppose that x^* is a solution of VI (F, K) *and $x^* \in K^0$, the interior of K. Then $F(x^*) = 0$.*

Qualitative properties of existence and uniqueness become easily obtainable under certain monotonicity conditions. First we outline the definitions and then present the results.

Definition 3 (Monotonicity)

$F(x)$ *is monotone on* K *if*

$$\left[F(x^1) - F(x^2)\right]^T \cdot (x^1 - x^2) \geq 0, \quad \forall x^1, x^2 \in K.$$

Definition 4 (Strict Monotonicity)

$F(x)$ *is strictly monotone on* K *if*

$$\left[F(x^1) - F(x^2)\right]^T \cdot (x^1 - x^2) > 0, \quad \forall x^1, x^2 \in K,\ x^1 \neq x^2.$$

Definition 5 (Strong Monotonicity)

$F(x)$ *is strongly monotone on* K *if for some* $\alpha > 0$

$$\left[F(x^1) - F(x^2)\right]^T \cdot (x^1 - x^2) \geq \alpha \|x^1 - x^2\|^2, \quad \forall x^1, x^2 \in K.$$

Definition 6 (Lipschitz Continuity)

$F(x)$ is Lipschitz continous on K if there exists an $L > 0$, such that

$$\|F(x^1) - F(x^2)\| \leq L\|x^1 - x^2\|, \quad \forall x^1, x^2 \in K.$$

A uniqueness result is presented in the subsequent theorem.

Theorem 6 (Uniqueness)

Suppose that $F(x)$ is strictly monotone on K. Then the solution is unique, if one exists.

Proof: Suppose that x^1 and x^* are both solutions and $x^1 \neq x^*$. Then since both x^1 and x^* are solutions, they must satisfy:

$$F(x^1)^T \cdot (x' - x^1) \geq 0, \quad \forall x' \in K \tag{25}$$

$$F(x^*)^T \cdot (x' - x^*) \geq 0, \quad \forall x' \in K. \tag{26}$$

After substituting x^* for x' in (25) and x^1 for x' in (26) and adding the resulting inequalities, one obtains:

$$\left(F(x^1) - F(x^*)\right)^T \cdot (x^* - x^1) \geq 0. \tag{27}$$

But inequality (27) is in contradiction to the definition of strict monotonicity. Hence, $x^1 = x^*$.

Monotonicity is closely related to positive definiteness.

Theorem 7

Suppose that $F(x)$ is continuously differentiable on K and the Jacobian matrix

$$\nabla F(x) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix},$$

which need not be symmetric, is positive semidefinite (positive definite). Then $F(x)$ is monotone (strictly monotone).

Proposition 5

Assume that $F(x)$ is continuously differentiable on K and that $\nabla F(x)$ is strongly positive definite. Then $F(x)$ is strongly monotone.

One obtains a stronger result in the special case where $F(x)$ is linear.

Corollary 4

Suppose that $F(x) = Mx + b$, where M is an $n \times n$ matrix and b is a constant vector in R^n. The function F is monotone if and only if M is positive semidefinite. F is strongly monotone if and only if M is positive definite.

Proposition 6

Assume that $F : K \mapsto R^n$ is continuously differentiable at $\bar{x}$. Then $F(x)$ is locally strictly (strongly) monotone at $\bar{x}$ if $\nabla F(\bar{x})$ is positive definite (strongly positive definite), that is,

$$v^T F(\bar{x}) v > 0, \quad \forall v \in R^n, v \neq 0,$$

$$v^T \nabla F(\bar{x}) v \geq \alpha \|v\|^2, \quad \textit{for some} \quad \alpha > 0, \quad \forall v \in R^n.$$

The following theorem provides a condition under which both existence and uniqueness of the solution to the variational inequality problem are guaranteed. Here no assumption on the compactness of the feasible set K is made.

Theorem 8 (Existence and Uniqueness)

Assume that $F(x)$ is strongly monotone. Then there exists precisely one solution x^* to VI(F, K).

Proof: Existence follows from the fact that strong monotonicity implies coercivity, whereas uniqueness follows from the fact that strong monotonicity implies strict monotonicity.

Hence, in the case of an unbounded feasible set K, strong monotonicity of the function F guarantees both existence and uniqueness. If K is compact, then existence is guaranteed if F is continuous, and only the strict monotonicity condition needs to hold for uniqueness to be guaranteed.

Assume now that $F(x)$ is both strongly monotone and Lipschitz continuous. Then the projection $P_K[x - \gamma F(x)]$ is a contraction with respect to x, that is, we have the following:

Theorem 9

Fix $0 < \gamma \leq \frac{\alpha}{L^2}$ *where* α *and* L *are the constants appearing, respectively, in the strong monotonicity and the Lipschitz continuity condition definitions. Then*

$$\|P_K(x - \gamma F(x)) - P_K(y - \gamma F(y))\| \leq \beta \|x - y\| \tag{31}$$

for all $x, y \in K$, *where*

$$(1 - \gamma\alpha)^{\frac{1}{2}} \leq \beta < 1.$$

An immediate consequence of Theorem 9 and the Banach Fixed Point Theorem is:

Corollary 5

The operator $P_K(x - \gamma F(x))$ *has a unique fixed point* x^*.

Stability and Sensitivity Analysis

Important issues in the qualitative analysis of equilibrium patterns are the stability and sensitivity of solutions when the problem is subjected to perturbations in the data.

Stability

The following theorem establishes that a small change in the function F entering the variational inequality induces a small change in the resulting solution pattern. Denote the original function by F with solution x to VI(F, K) and the perturbed function by F^* with solution x^* to VI(F^*, K).

Assume that the strong monotonicity condition on F holds. Then one has:

Theorem 14

Let α be the positive constant in the definition of strong monotonicity. Then

$$\|x^* - x\| \leq \frac{1}{\alpha}\|F^*(x^*) - F(x^*)\|. \tag{37}$$

Proof: The vectors x and x^* must satisfy the variational inequalities

$$F(x)^T \cdot (x' - x) \geq 0, \quad \forall x' \in K \tag{38}$$

$$F^*(x^*)^T \cdot (x' - x^*) \geq 0, \quad \forall x' \in K. \tag{39}$$

Rewriting (38) for $x' = x^*$ and (39) for $x' = x$, and then adding the resulting inequalities, one obtains

$$[F^*(x^*) - F(x)]^T \cdot [x^* - x] \leq 0 \tag{40}$$

or

$$[F^*(x^*) - F(x) + F(x^*) - F(x^*)]^T \cdot [x^* - x] \leq 0. \tag{41}$$

Using then the monotonicity condition, (41) yields

$$[F^*(x^*) - F(x^*)]^T \cdot [x - x^*] \geq [F(x) - F(x^*)]^T \cdot [x - x^*]$$

$$\geq \alpha \|x - x^*\|^2. \tag{42}$$

By virtue of the Schwarz inequality, (42) gives

$$\alpha \|x^* - x\|^2 \leq \|F^*(x^*) - F(x^*)\| \cdot \|x^* - x\|, \tag{43}$$

from which (37) follows.

Below we give the citations referenced in the lecture as well as other relevant ones.

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