Sustainable System-Optimized Networks

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Introduction

In this lecture the focus is on policies for sustainable transportation networks, but from a *system-optimized* perspective, rather than from a user-optimized viewpoint, which was the subject of earlier lectures.

Recall that in a system-optimized network, the total cost associated with traveling on the network is minimized.

This concept is useful not only in the context of congested urban transportation networks but, in fact, in other networks, including freight networks, in which there exists a central controller who can route traffic on the network in an optimal fashion; optimality here, unlike in the case of user-optimized networks, is represented by the solution of an optimization problem, even in the case of general user link travel cost functions.

With this lecture, I add another dimension to the modeling and analysis of sustainable transportation networks in that I explicitly now handle not only the total emissions generated and guarantee that they do not exceed the imposed environmental quality standard, but also include an explicit objective function whose minimization reflects the minimization of total cost in the network as reflected by the total congestion.

I assume here that the networks are viable. Furthermore, I consider traffic networks in which the travel demands are fixed.

Some of the policy instruments that will be developed in this course for sustainable system-optimized networks are given in the table.

Modeling of Sustainable System-Optimized Networks

Now, I develop models for sustainable transportation networks in which the underlying behavior is that of system-optimization and the policy is that of emission tolls. I first present a simple model, which is, subsequently, generalized.

In the next Table, the relationship between a viable, system-optimized transportation network in the presence of emission/congestion tolls, as discussed in this lecture, is highlighted.

Summary of policy instruments for sustainable S-O networks

this lecture
Emission/congestion pricing – Fixed demand networks Emission/congestion tolls for U-O – Fixed demand networks
next lecture
Tradable pollution permits – Fixed demand networks Tradable pollution permits/tolls for U-O – Elastic demand networks

A Simple Model

As in other lectures, consider a traffic network consisting of the graph $G = [N, L]$, where N denotes the set of nodes and L the set of links. Moreover, assume, as given, a vector of travel demands d associated with the origin/destination pairs.

Here I consider the "classical" form of the total link travel cost functions, due to Beckmann, McGuire, and Winsten (1956), in which the total cost on a link, denoted by $\hat{c}_a$ is equal to the product of the user link travel cost, which is assumed to be a separable function of the link load, times the total load on the link, that is:

$$\hat{c}_a = c_a(f_a) \times f_a, \quad \forall a \in L, \tag{1}$$

in which it is assumed that the user link travel cost function c_a is increasing in the flow for each link in the network.

The total travel cost on a path p, hence, is equal to the sum of the total travel costs on links that comprise that path, that is:

$$\hat{C}_p = \sum_{a \in L} \hat{c}_a \delta_{ap}, \quad \forall a \in L. \tag{2}$$

The marginal of the total cost on a path p, in turn, which is denoted by $\hat{C}'_p$, for all $p \in P$, is given by the sum of the marginals of the total costs on the links that comprise the path, that is,

$$\hat{C}'_p = \sum_{a \in L} \hat{c}'_a \delta_{ap} = \sum_{a \in L} \frac{\partial \hat{c}_a(f_a)}{\partial f_a} \delta_{ap}. \tag{3}$$

Recall that in the system-optimization problem one seeks to minimize the total cost in the network, where the objective function is given by:

$$\text{Minimize } \sum_{a \in L} \hat{c}_a(f_a) = \text{Minimize } \sum_{a \in L} c_a(f_a) \times f_a \tag{4}$$

or, equivalently, in path flows by:

$$\text{Minimize } \sum_{p \in P} \hat{C}_p = \text{Minimize } \sum_{p \in P} C_p(x) \times x_p. \tag{5}$$

I am now ready to state the system-optimization problem, whose solution will guarantee that the transportation network is sustainable. I utilize the path flow form of the objective function given by (5).

Hence, I retain the objective function in the classical traffic network system-optimization model due to Beckmann, McGuire, and Winsten (1956), as well as the constraints, but now I add the environmental quality constraint, as was also done in the user-optimized models.

The system-optimization problem can, therefore, be expressed as:

$$\text{Minimize} \quad S(x) = \sum_{p \in P} \hat{C}_p \tag{6}$$

subject to:

$$\sum_{p \in P_w} x_p = d_w, \quad \forall w \in W, \tag{7}$$

$$\sum_{a \in L} h_a \sum_{p \in P} x_p \delta_{ap} \leq \bar{Q}, \tag{8}$$

$$x_p \geq 0, \quad \forall p \in P. \tag{9}$$

Observe that conditions (7)–(9) correspond precisely to Linear System 1, the existence of a solution to which guarantees viability of a transportation network with given O/D pairs and travel demands.

Optimality Conditions

I now derive the optimality conditions for the system-optimization problem given by (6)–(9). These are optimality conditions, rather than equilibrium conditions, as is the case where the travelers behave in a user-optimizing manner, since here one now has an explicit objective function, which represents the total cost or congestion to be minimized.

Note that, since the user travel cost functions are increasing functions of the flow, the objective function is convex and the constraints, which are linear, are also convex. Thus, the Kuhn-Tucker optimality conditions can be stated as follows: $x^* \in R_+^{n_P}$ is an optimal solution if it satisfies the travel demands, and satisfies the following system of equalities and inequalities: For each O/D pair $w \in W$, and each path $p \in P_w$:

$$\bar{\hat{C}}'_p(x^*, \tau^*) = \hat{C}'_p(x^*) + \tau^* \sum_{a \in L} h_a \delta_{ap} \begin{cases} = & \mu_w, \quad \text{if } x_p^* > 0 \\ \geq & \mu_w, \quad \text{if } x_p^* = 0, \end{cases} \tag{10}$$

where τ^* is the Lagrange multiplier associated with the environmental quality constraint (8) with τ^* having the interpretation here as being the marginal cost of emission abatement, as in the case of the user-optimized models.

In addition, one must have that:

$$\bar{Q} - \sum_{a \in L} h_a \sum_{p \in P} x_p^* \delta_{ap} \begin{cases} = & 0, \quad \text{if } \tau^* > 0 \\ \geq & 0, \quad \text{if } \tau^* = 0, \end{cases} \tag{11}$$

Observe that $\hat{C}'_p = \frac{\partial S(x)}{\partial x_p}$, $\forall p \in P$. Note that $\bar{\hat{C}}'_p$ in (10) denotes the *generalized* marginal cost associated with traveling now on path p, where the term: $\tau^* \sum_{a \in L} h_a \delta_{ap}$ represents the, de facto, marginal of the total cost on path p due to the emissions generated by traveling on path p. Hence, this term may also be interpreted as a "price" which is associated with traveling on the path. There exists a similar interpretation of the term arising in the case of the user-optimized model. Consequently, the higher the emission factors on a utilized path, the higher this term for travelers on that path.

Note that, in the case of system-optimization, it is the generalized marginal travel costs on used paths for each O/D pair that get equalized (or equilibrated), rather than the generalized user travel costs.

Pricing Policies

As already stated, in the case of system-optimized networks, the central authority controls the flows, be they flows which correspond to motor vehicles, freight, or others. Hence, the solution to the sustainable system-optimization problem is implemented by the authority or central controller.

Nevertheless, one may interpret the costs associated with emissions so that one may assign the following price policies which reflect the cost of emitting. Specifically, one can construct pricing policies, in links, and in paths, respectively, which satisfy optimality conditions (10) and (11), and which are analogous to those for the user-optimized problem described in an earlier lecture, but which are interpreted there as tolls, since the users in that case do not control the system and act independently.

Here, however, the total cost in the network is also minimized simultaneously.

Relationship between system-optimization and emission/congestion pricing and sustainability

$$\boxed{\begin{array}{c} \textbf{System-Optimized Viable Network} \\ + \\ \textbf{Emission/Congestion Pricing} \end{array} \Rightarrow \textbf{Sustainability}}$$

The following emission pricing policies satisfy the conditions (10) and (11).

Link Pricing Policy

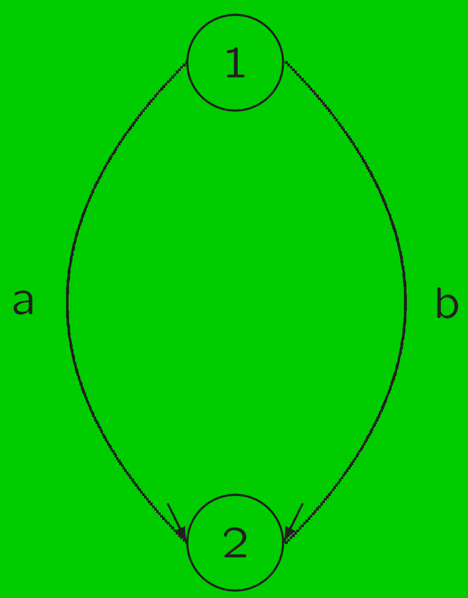
The link pricing policy given by $t_a = \tau^* h_a$, $\forall a \in L$, where τ^* is the equilibrium marginal cost of emission abatement and t_a denotes the payment on link a, guarantees that the transportation network is sustainable.

Path Pricing Policy

The path pricing policy given by $t_p = \tau^* \sum_{a \in L} h_a \delta_{ap}$ for all $p \in P$, where t_p denotes the payment on path p, guarantees that the network will be sustainable.

Note that here τ^* does not correspond to the marginal cost of emission abatement resulting from the solution of the policy problem in the case of user-optimization. The similar notation is used simply for convenience and is clear from the context of the problem under study.

I now present an example, for which the sustainable system-optimized flow pattern can be solved for explicitly.



Network topology for Example 1

Example 1

Consider the network depicted in the Figure, consisting of two nodes: 1 and 2, and two links: a and b.

There is a single O/D pair $w_1 = (1, 2)$. Let path $p_1 = a$ and path $p_2 = b$.

The travel demand $d_{w_1} = 10$. The user link travel cost functions are:

$$c_a(f_a) = f_a + 7, \quad c_b(f_b) = 2f_b + 4.$$

Hence, the total cost functions on the links are:

$$\hat{c}_a(f_a) = f_a^2 + 7f_a, \quad \hat{c}_b(f_b) = 2f_b^2 + 4f_b.$$

The marginals of the total costs on the links, in turn, are given by the expressions:

$$\hat{c}'_a(f_a) = 2f_a + 7, \quad \hat{c}'_b(f_b) = 4f_b + 4,$$

which also correspond to the path marginals of the total costs, since each path consists of a single link:

$$\hat{C}'_{p_1} = 2x_{p_1} + 7, \quad \hat{C}'_{p_2} = 4x_{p_2} + 4.$$

Suppose that the emission factors are $h_a = 0.1$ and $h_b = 0.5$, and the environmental quality standard $\bar{Q} = 3$.

It is easy to verify that the network is viable (for example, one may simply let $x_{p_1} = 10$ and $x_{p_2} = 0$ and the Linear System 1 is satisfied).

Due to the simplicity of the example, one can solve explicitly for the solution.

Indeed, note that the path flow pattern $x^*_{p_1} = 6\frac{1}{6}$, $x^*_{p_2} = 3\frac{5}{6}$ with $\tau^* = 0$, yields:

$$\hat{C}'_{p_1} = \hat{C}'_{p_2} = 19\frac{1}{3}.$$

Hence, there are no tolls needed in this transportation network example, and the system-optimized solution is also sustainable, with the total emissions generated by the flow pattern equal to 2.533, which is less than $\bar{Q} = 3$.

Suppose now that one tightens the environmental quality standard so that the new $\bar{Q} = 1.5$. The preceding pattern violates the environmental quality condition and, hence, a new sustainable system-optimized flow pattern needs to be determined. The new system-optimal flow pattern is:

$$x^*_{p_1} = 8.75, \quad x^*_{p_2} = 1.25,$$

and the equilibrium marginal cost of emission abatement $\tau^* = 36.25$, yielding a link pricing policy (which also corresponds to a path pricing policy in this case) given by:

$$t_a = .1(36.25) = 3.625, \quad t_b = .5(36.25) = 18.125,$$

and with generalized marginals of the total costs on paths equal to:

$$\bar{\hat{C}}'_{p_1} = \bar{\hat{C}}'_{p_2} = 28.125.$$

Indeed, the environmental quality standard is precisely met by this pattern.

Remark 1

One might be tempted to construct an optimization problem which has as its objective function to minimize both the total cost in the network as well as the total emissions. For example, one may consider the following problem:

$$\text{Minimize} \quad \sum_{p\in P} C_p(x)x_p + \sum_{a\in L} h_a \sum_{p\in P} x_p$$

subject to:

$$\sum_{p\in P_w} x_p = d_w, \quad \forall w \in W,$$

$$x_p \geq 0, \quad \forall p \in P.$$

Note that the Kuhn-Tucker conditions for this problem can be stated as, assuming that the flow pattern is feasible: For all O/D pairs $w \in W$ and all paths $p \in P_w$:

$$\bar{\hat{C}}'_p(x^*) = \hat{C}'_p(x^*) + \sum_{a\in L} h_a \delta_{ap} \begin{cases} = & \mu_w, & \text{if } x^*_p > 0 \\ \geq & \mu_w, & \text{if } x^*_p = 0. \end{cases} \tag{12}$$

However, although the objective function includes the expression for the total emissions, the environmental quality standard may not be met by the solution to this problem and, consequently, the resulting flow pattern may not correspond to a sustainable one.

Indeed, consider the following example.

Example 2

The network topology is given in the Figure and consists of two nodes: 1 and 2 and two links: a and b, and a single O/D pair $w_1 = (1, 2)$. Let $p_1 = a$ and $p_2 = b$.

Assume that the emission factors are $h_a = 0.1$ and $h_b = 0.5$.

The travel demand $d_{w_1} = 10$ and the user link cost functions are:

$$c_a(f_a) = f_a + 5, \quad c_b(f_b) = f_b + 5,$$

so the total cost functions are:

$$\hat{c}_a(f_a) = f_a^2 + 5f_a, \quad \hat{c}_b(f_b) = f_b^2 + 5f_b,$$

with marginals of the total cost functions given by:

$$\hat{c}'_a(f_a) = 2f_a + 5 \quad \hat{c}'_b(f_b) = 2f_b + 5,$$

which corresponds to the marginals of the total costs on the paths:

$$\hat{C}'_{p_1} = 2x_{p_1} + 5, \quad \hat{C}'_{p_2} = 2x_{p_2} + 5.$$

Assuming that both paths are used and using the expression for the travel demand being equal to the sum of the paths for the O/D pair, one obtains a single equation corresponding to the Kuhn-Tucker condition above:

$$2f_a + 5 + 0.1 = 2(10 - f_a) + 5 + 0.5,$$

the solution of which yields:

$$f_a^* = 5.1, \quad f_b^* = 4.9,$$

with

$$\hat{C}'_{p_1} + \sum_{a \in p_1} h_a = \hat{C}'_{p_2} + \sum_{a \in p_2} h_a = 15.3.$$

Note, however, that this flow pattern yields total emissions equal to 2.96, which exceed $\bar{Q} = 2$!

Therefore, the solution of the optimization problem satisfying optimality conditions (12) does not guarantee a sustainable transportation network.

The General Model

I now consider traffic networks which are sustainable and system-optimized, but on which the user link travel cost functions are no longer separable.

Hence, I assume the general situation in which the user link travel cost on link a is given by:

$$c_a = c_a(f), \quad \forall a \in L, \tag{13}$$

and, consequently, the total cost expression on a link a, denoted by $\hat{c}_a$, is given by:

$$\hat{c}_a = c_a(f) \times f_a, \quad \forall a \in L. \tag{14}$$

Note that, in the simple model, $\frac{\partial S(x)}{\partial x_p}$, where $S(x)$ denoted the total cost on the network, was precisely equal to $\hat{C}'_p = \sum_{a \in L} \hat{c}'_a$, since the user link cost travel functions, and, hence, the total link travel cost functions were separable.

In the general case, however, one can still define the "marginal of the total cost" on path p, denoted, again, by $\hat{C}'_p$ to be equal to $\frac{\partial S(x)}{\partial x_p}$, in which case the optimality conditions (10) and (11) coincide with the optimality conditions for the general problem, whose objective function remains that of (8.6), subject to constraints: (7)–(9).

Clearly, one can also derive a variational inequality formulation of the Kuhn-Tucker optimality conditions for this problem but this is not necessary since the problem, even in the general case, remains an *optimization problem* in the case of system-optimization, rather than an *equilibrium problem* as is the case in user-optimization.

Indeed, recall that in the case of nonseparable and asymmetric user link travel cost functions, one can no longer reformulate the traffic network equilibrium conditions as the optimality conditions of a convex optimization problem.

Consequently, in order to formulate, analyze, and solve such problems, one must appeal to variational inequality theory.

For completeness, however, as well as for the flexibility in applying, for example, such an algorithm as the modified projection method, which can be used to solve the sustainable S-O problem above, I also give the variational inequality formulation of conditions (10) and (11).

Of course, any general convex programming algorithm can be applied, at least in principle, to compute the solution to the system-optimization problem for sustainable transportation networks given by (6)–(9). Since the proof of the theorem is so similar to that of an earlier theorm, it is presented without proof.

Theorem 1 (Variational Inequality Formulation of a Sustainable System-Optimized Solution)

A traffic flow pattern and marginal cost of emission abatement $(x^*, \tau^*) \in \mathcal{K}^1$ *is a solution of the sustainable system-optimized model described above if and only if it is a solution to the variational inequality problem:*

Path Flow Formulation:

$$\sum_{w\in W}\sum_{p\in P_w}\left[\hat{C}'_p(x^*)+\tau^*\sum_{a\in L}h_a\delta_{ap}\right]\times\left[x_p-x^*_p\right]$$

$$+\left[\bar{Q}-\sum_{a\in L}h_a\sum_{p\in P}x^*_p\delta_{ap}\right]\times[\tau-\tau^*]\geq 0,\quad \forall(x,\tau)\in\mathcal{K}^1, \tag{15}$$

where $\mathcal{K}^1 \equiv \bar{K}_1 \times R^1_+$, *and* $\bar{K}_1 \equiv \{x | x \geq 0$, *and satisfies* (7)$\}$ *and, equivalently,* $(f^*, \tau^*) \in \mathcal{K}^2$ *is a solution of the sustainable S-O problem if and only if it satisfies the variational inequality problem:*

Link Load Formulation:

$$\sum_{a\in L}\left[\hat{c}'_a(f^*)+\tau^*h_a\right]\times[f_a-f^*_a]$$

$$+\left[\bar{Q}-\sum_{a\in L}h_af^*_a\right]\times[\tau-\tau^*]\geq 0,\quad\forall(f,\tau)\in\mathcal{K}^2, \tag{16}$$

where $\mathcal{K}^2 \equiv \bar{K}_2 \times R_+$ *and* $\bar{K}^2 \equiv \{f|$ *there exists an* $x \geq 0$, *satisfying* (7)$\}$.

Emission/Congestion Tolls for Sustainability

I now develop a framework for the construction of tolls, which serves the twofold purpose of alleviating congestion in that the total cost in the network as reflected by the total congestion is minimized, while, at the same time, guaranteeing that the environmental quality standard is satisfied, even though the travelers are now assumed to adopt, once again, user-optimizing behavior.

I assume that travelers seek to determine their paths from their origins to their destinations so as to minimize their travel "cost" where here cost is interpreted in a general manner to include not only, for example, travel time, but also the outlay of any necessary payments for use of the path.

The relationship between the tolls proposed in this section and sustainability of the transportation network is highlighted in the Table.

The Procedure

I now describe a procedure for the allocation of tolls which guarantees that the system-optimized flow pattern for a sustainable network, that is, one that satisfies optimality conditions (10) and the corresponding marginal cost of emission abatement, which satisfies, in turn, optimality conditions (11), is also a user-optimized pattern, after the imposition of the appropriate toll policy.

Note that, for the system-optimal solution, denoted by (x^*, τ^*), to also be user-optimized, it must satisfy the conditions: For each O/D pair $w \in W$, and each path $p \in P_w$:

$$C_p(x^*) + t_p \begin{cases} = & \lambda_w, \quad \text{if } x_p^* > 0 \\ \geq & \lambda_w, \quad \text{if } x_p^* = 0, \end{cases} \tag{17}$$

where t_p here denotes a path-toll policy.

Furthermore, the system-optimized flow pattern already satisfies conditions (10); that is: For each O/D pair $w \in W$, and each path $p \in P_w$:

$$\hat{C}'_p(x^*) + \tau^* \sum_{a \in L} h_a \delta_{ap} \begin{cases} = & \mu_w, \quad \text{if } x_p^* > 0 \\ \geq & \mu_w, \quad \text{if } x_p^* = 0, \end{cases} \tag{18}$$

For a solution to (18) to coincide with that of (17) implies that, for each path $p \in P$, one must have that:

$$t_p = \hat{C}'_p(x^*) - C_p(x^*) + \tau^* \sum_{a \in L} h_a \delta_{ap}. \tag{19}$$

Path-Toll Policy

Hence, equation (19) implies a procedure by which one can construct a path-toll policy. In particular, one first solves the problem (6)–(9) and then determines, for each path p in the network, the path-toll policy according to equation (19).

Link-Toll Policy

A link-toll policy, in turn, can also be determined according to:

$$t_a = \hat{c}'_a(f^*) - c_a(f^*) + \tau^* h_a, \quad \forall a \in L. \qquad (20)$$

Remark 2

It is worth highlighting the similarities between expressions (19) and (20) and the analogous expressions for path tolls and link tolls, respectively, arising in the classical toll policies, in which only congestion is considered and not pollution due to emissions.

Indeed, in the latter framework, there would be no such terms $\tau^* \sum_{a \in L} h_a \delta_{ap}$ and $\tau^* h_a$, and the flow pattern x^* and the link load pattern f^* would simply correspond to the patterns obtained by the solution of the system-optimization problem (6), subject only to (7) and (9) (when the link load/path flow expression is substituted directly into the formulation). These terms reflect precisely the cost associated with the emissions.

Numerical Example

I now present numerical examples of sustainable S-O transportation networks, to which the solutions are computed using the modified projection method. The code for this algorithm to solve this model was written in FORTRAN and the computer system used for the numerical work was the IBM SP2 located at the Computer Science Department at the University of Massachusetts at Amherst.

Example 3

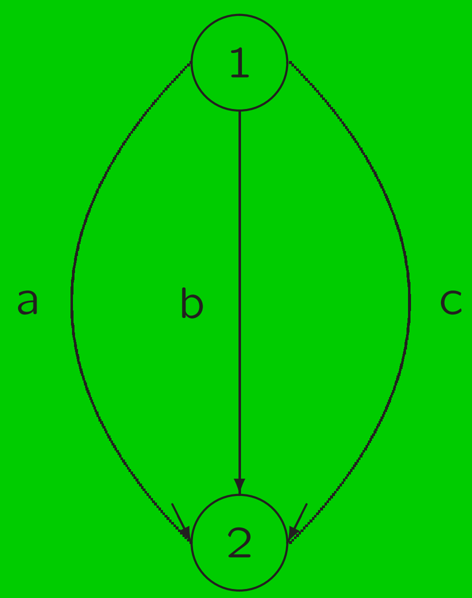
The network topology for this example is given in the Figure.

The network consists of two nodes, denoted by 1 and 2; three links, denoted by a, b, and c, and a single O/D pair $w_1 = (1, 2)$. I let $p_1 = a$, $p_2 = b$, and $p_3 = c$.

The travel demand $d_{w_1} = 10$. Recall that the user link travel cost functions are:

$$c_a(f_a) = 2f_a + 5, \quad c_b(f_b) = f_b + 8, \quad c_c(f_c) = 1.5f_c + 5.$$

The emissions are: $h_a = 0.1$, $h_b = 0.2$, and $h_c = 0.3$, with the environmental quality standard $\bar{Q} = 1.5$.



Network topology for Example 3

I set $\alpha = 0.4$ in the modified projection method with $\epsilon = .001$. An application of the modified projection method yielded the following sustainable S-O solution:

$$f_a^* = 5.40, \quad f_b^* = 4.20, \quad f_c^* = 0.40$$

This optimal link load pattern was induced by the optimal path flow pattern:

$$x_{p_1}^* = 5.40, \quad x_{p_2}^* = 4.20, \quad x_{p_3}^* = 0.40.$$

The optimal marginal cost of emission abatement was

$$\tau^* = 102.03.$$

The total cost in the network, as represented by the objective function $\sum_{a \in L} \hat{c}_a \times f_a$, was equal to 138.79.

Reference

The reference for this chapter is the textbook for this course, *Sustainable Transportation Networks*, Anna Nagurney (2000), Edward Elgar Publishers, Chelthenham, England.