Spatial Price Equilibrium

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Statement of the Problem

In the spatial price equilibrium problem, one seeks to compute the commodity supply prices, demand prices, and trade flows satisfying the equilibrium condition that the demand price is equal to the supply price plus the cost of transportation, if there is trade between the pair of supply and demand markets; if the demand price is less than the supply price plus the transportation cost, then there will be no trade.

• Enke (1951) established the connection between spatial price equilibrium problems and electronic circuit networks.

• Samuelson (1952) and Takayama and Judge (1964, 1971) showed that the prices and commodity flows satisfying the spatial price equilibrium conditions could be determined by solving an extremal problem, in other words, a mathematical programming problem.

Spatial price equilibrium models have been used to study problems in

• agriculture, • energy markets, • and mineral economics, as well as in finance.

We will study a variety of spatial price equilibrium models, along with the fundamentals of the qualitative theory and computational procedures.

Static Spatial Price Equilibrium Models

The distinguishing characteristic of spatial price equilibrium models lies in their recognition of the importance of space and transportation costs associated with shipping a commodity from a supply market to a demand market. These models are perfectly competitive partial equilibrium models, in that one assumes that there are many producers and consumers involved in the production and consumption, respectively, of one or more commodities.

As noted in Takayama and Judge (1971) distinct model formulations are needed, in particular, both quantity and price formulations, depending upon the availability and format of the data.

Quantity Formulation

In such models it is assumed that the supply price functions and demand price functions, which are a function of supplies and demands (that is, quantities), respectively, are given. First, a simple model is described and the variational inequality formulation of the equilibrium conditions derived. Then it is shown how this model can be generalized to multiple commodities.

Consider m supply markets and n demand markets involved in the production / consumption of a commodity.

Denote a typical supply market by i and a typical demand market by j.

Let s_i denote the supply of the commodity associated with supply market i and let π_i denote the supply price of the commodity associated with supply market i.

Let d_j denote the demand associated with demand market j and let ρ_j denote the demand price associated with demand market j.

Group the supplies and supply prices, respectively, into a column vector $s \in R^m$ and a row vector $\pi \in R^m$. Similarly, group the demands and the demand prices, respectively, into a column vector $d \in R^n$ and a row vector $\rho \in R^n$.

Let Q_{ij} denote the nonnegative commodity shipment between the supply and demand market pair (i,j) and let c_{ij} denote the nonnegative unit transaction cost associated with trading the commodity between (i,j). Assume that the transaction cost includes the cost of transportation; depending upon the application, one may also include a tax/tariff, fee, duty, or subsidy within this cost. Group then the commodity shipments into a column vector $Q \in R^{mn}$ and the transaction costs into a row vector $c \in R^{mn}$.

The Spatial Price Equilibrium Conditions

The market equilibrium conditions, assuming perfect competition, take the following form: For all pairs of supply and demand markets $(i,j) : i = 1,\ldots,m; j = 1,\ldots,n$:

$$\pi_i + c_{ij} \begin{cases} = \rho_j, & \text{if} \quad Q_{ij}^* > 0 \\ \geq \rho_j, & \text{if} \quad Q_{ij}^* = 0. \end{cases} \tag{1}$$

This condition states that if there is trade between a market pair (i,j), then the supply price at supply market i plus the transaction cost between the pair of markets must be equal to the demand price at demand market j in equilibrium; if the supply price plus the transaction cost exceeds the demand price, then there will be no shipment between the supply and demand market pair.

The following feasibility conditions must hold for every i and j:

$$s_i = \sum_{j=1}^{n} Q_{ij} \tag{2}$$

and

$$d_j = \sum_{i=1}^{m} Q_{ij}. \tag{3}$$

$K \equiv \{(s,Q,d) | (2) \text{ and } (3) \text{ hold}\}$.

The supply price, demand price, and transaction cost structure is now discussed. Assume that the supply price associated with any supply market may depend upon the supply of the commodity at every supply market, that is,

$$\pi = \pi(s) \tag{4}$$

where π is a known smooth function.

Similarly, the demand price associated with a demand market may depend upon, in general, the demand of the commodity at every demand market, that is,

$$\rho = \rho(d) \tag{5}$$

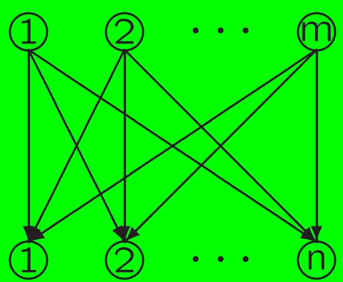
where ρ is a known smooth function.

The transaction cost between a pair of supply and demand markets may, in general, depend upon the shipments of the commodity between every pair of markets, that is,

$$c = c(Q) \tag{6}$$

where c is a known smooth function.

In the special case where the number of supply markets m is equal to the number of demand markets n, the transaction cost functions are assumed to be fixed, and the supply price functions and demand price functions are symmetric, i.e., $\frac{\partial \pi_i}{\partial s_k} = \frac{\partial \pi_k}{\partial s_i}$, for all $i = 1, \ldots, n$; $k = 1, \ldots, n$, and $\frac{\partial \rho_j}{\partial d_l} = \frac{\partial \rho_l}{\partial d_j}$, for all $j = 1, \ldots, n$; $l = 1, \ldots, n$, then the above model with supply price functions and demand price functions collapses to a class of single commodity models introduced in Takayama and Judge (1971) for which an equivalent optimization formulation exists.



Bipartite market network equilibrium model

We now present the variational inequality formulation of the equilibrium conditions.

Theorem 1 (Variational Inequality Formulation of the Quantity Model)

A commodity production, shipment, and consumption pattern $(s^*, Q^*, d^*) \in K$ *is in equilibrium if and only if it satisfies the variational inequality problem:*

$$\langle \pi(s^*), s - s^* \rangle + \langle c(Q^*), Q - Q^* \rangle - \langle \rho(d^*), d - d^* \rangle \geq 0, \quad \forall (s, Q, d) \in K. \tag{7}$$

Proof: First it is shown that if $(s^*, Q^*, d^*) \in K$ satisfies (1) then it also satisfies (7).

Note that for a fixed market pair (i, j), one must have that

$$(\pi_i(s^*) + c_{ij}(Q^*) - \rho_j(d^*)) \times (Q_{ij} - Q^*_{ij}) \geq 0 \tag{8}$$

for any nonnegative Q_{ij}. Indeed, if $Q^*_{ij} > 0$, then according to (1), $(\pi_i(s^*) + c_{ij}(Q^*) - \rho_j(d^*)) = 0$ and (8) must hold. On the other hand, if $Q^*_{ij} = 0$, then according to (1), $(\pi_i(s^*) + c_{ij}(Q^*) - \rho_j(d^*)) \geq 0$; and, consequently, (8) also holds. But it follows that (8) will hold for all (i, j); hence, summing over all market pairs, one has that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (\pi_i(s^*) + c_{ij}(Q^*) - \rho_j(d^*)) \times (Q_{ij} - Q^*_{ij}) \geq 0, \ \forall Q_{ij} \geq 0, \forall i, j. \tag{9}$$

Using now constraints (2) and (3), and some algebra, (9) yields

$$\sum_{i=1}^{m} \pi_i(s^*) \times (s_i - s^*_i) + \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}(Q^*) \times (Q_{ij} - Q^*_{ij})$$

$$- \sum_{j=1}^{n} \rho_j(d^*) \times (d_j - d^*_j) \geq 0, \quad \forall (s, Q, d) \in K, \tag{10}$$

which, in vector notation, gives us (7).

It is now shown that if $(s^*, Q^*, d^*) \in K$ satisfies (7) then it also satisfies equilibrium conditions (1).

For simplicity, utilize (7) expanded as (9). Let $Q_{ij} = Q^*_{ij}$, $\forall ij \neq kl$. Then (9) simplifies to:

$$(\pi_k(s^*) + c_{kl}(Q^*) - \rho_l(d^*)) \times (Q_{kl} - Q^*_{kl}) \geq 0 \qquad (11)$$

from which (1) follows for this kl and, consequently, for every market pair.

Variational inequality (7) may be put into standard form by defining the vector $x \equiv (s, Q, d) \in R^{m+mn+n}$ and the vector $F(x)^T \equiv (\pi(s), c(Q), \rho(d))$ which maps R^{m+mn+n} into R^{m+mn+n}.

Theorem 2

$F(x)$ as defined above is monotone, strictly monotone, or strongly monotone if and only if $\pi(s)$, $c(Q)$, and $\rho(d)$ are each monotone, strictly monotone, or strongly monotone in s, Q, d, respectively.

Since the feasible set K is not compact, existence of an equilibrium pattern (s^*, Q^*, d^*) does not immediately follow. Nevertheless, it follows from standard VI theory that if π, c, and ρ are strongly monotone, then existence and uniqueness of the equilibrium production, shipment, and consumption pattern are guaranteed.

The model is now illustrated with a simple example consisting of 2 supply markets and 2 demand markets.

Example 1

The supply price functions are:

$$\pi_1(s) = 5s_1 + s_2 + 2 \quad \pi_2(s) = 2s_2 + s_1 + 3.$$

The transaction cost functions are:

$$c_{11}(Q) = Q_{11} + .5Q_{12} + 1 \quad c_{12}(Q) = 2Q_{12} + Q_{22} + 1.5$$

$$c_{21}(Q) = 3Q_{21} + 2Q_{11} + 15 \quad c_{22}(Q) = 2Q_{22} + Q_{12} + 10.$$

The demand price functions are:

$$\rho_1(d) = -2d_1 - d_2 + 28.75 \quad \rho_2(d) = -4d_2 - d_1 + 41.$$

The equilibrium production, shipment, and consumption pattern is then given by:

$$s_1^* = 3 \quad s_2^* = 2$$

$$Q_{11}^* = 1.5 \quad Q_{12}^* = 1.5 \quad Q_{21}^* = 0 \quad Q_{22}^* = 2$$

$$d_1^* = 1.5 \quad d_2^* = 3.5,$$

with equilibrium supply prices, costs, and demand prices:

$$\pi_1 = 19 \quad \pi_2 = 10$$

$$c_{11} = 3.25 \quad c_{12} = 6.5 \quad c_{21} = 18 \quad c_{22} = 15.5$$

$$\rho_1 = 22.25 \quad \rho_2 = 25.5.$$

Note that supply market 2 does not ship to demand market 1. This is due, in part, to the high fixed cost associated with trading between this market pair.

A Spatial Price Model on a General Network

Consider a spatial price equilibrium problem that takes place on a general network. Markets at the nodes are denoted by i, j, etc., links are denoted by a, b, etc., paths connecting a pair of markets by p, q, etc. Flows in the network are generated by a commodity. Denote the set of nodes in the network by Z. Denote the set of H links by L and the set of paths by P. Let P_{ij} denote the set of paths joining markets i and j.

The supply price vectors, supplies, and demand price vectors and demands are defined as in the previous spatial price equilibrium model.

The transportation cost associated with shipping the commodity across link a is denoted by c_a. Group the costs into a row vector $c \in R^H$. Denote the load on a link a by f_a and group the link loads into a column vector $f \in R^H$.

Consider the general situation where the cost on a link may depend upon the entire link load pattern, that is,

$$c = c(f) \tag{12}$$

where c is a known smooth function.

Furthermore, the commodity being transported on path p incurs a transportation cost

$$C_p = \sum_{a \in L} c_a \delta_{ap}, \tag{13}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise, that is, the cost on a path is equal to the sum of the costs on the links comprising the path.

A flow pattern Q, where Q now, without any loss of generality, denotes the vector of path flows, induces a link load f through the equation

$$f_a = \sum_{p \in P} Q_p \delta_{ap}. \tag{14}$$

Conditions (2) and (3) become now, for each i and j:

$$s_i = \sum_{j \in Z, p \in P_{ij}} Q_p \tag{15}$$

and

$$d_j = \sum_{i \in Z, p \in P_{ij}} Q_p. \tag{16}$$

Any nonnegative flow pattern Q is termed feasible. Let K denote the closed convex set where

$$K \equiv \{(s, f, d) | \, \text{such that} \, (14) - -(16) \, \text{hold for} \, Q \geq 0\}.$$

Equilibrium conditions (1) now become in the framework of this model: For every market pair (i, j), and every path $p \in P_{ij}$:

$$\pi_i + C_p(f^*) \left\{ \begin{array}{lll} = \rho_j, & \text{if} & Q_p^* > 0 \\ \geq \rho_j, & \text{if} & Q_p^* = 0. \end{array} \right. \quad (17)$$

In other words, a spatial price equilibrium is obtained if the supply price at a supply market plus the cost of transportation is equal to the demand price at the demand market, in the case of trade between the pair of markets; if the supply price plus the cost of transportation exceeds the demand price, then the commodity will not be shipped between the pair of markets. In this model, a path represents a sequence of trade or transportation links; one may also append links to the network to reflect steps in the production process.

Now the variational inequality formulation of the equilibrium conditions is established. In particular, we have:

Theorem 3 (Variational Inequality Formulation of the Quantity Model on a General Network)

A commodity production, link load, and consumption pattern
$(s^*, f^*, d^*) \in K$*, induced by a feasible flow pattern* Q^**, is a spatial price equilibrium pattern if and only if it satisfies the variational inequality:*

$$\langle \pi(s^*), s-s^* \rangle + \langle c(f^*), f-f^* \rangle - \langle \rho(d^*), d-d^* \rangle \geq 0, \forall (s, f, d) \in K. \tag{18}$$

Proof: It is first established that a pattern $(s^*, f^*, d^*) \in K$ induced by a feasible Q^* and satisfying equilibrium conditions (17) also satisfies the variational inequality (18).

For a fixed market pair (i, j), and a path p connecting (i, j) one must have that

$$(\pi_i(s^*) + C_p(f^*) - \rho_j(d^*)) \times (Q_p - Q_p^*) \geq 0, \tag{19}$$

for any $Q_p \geq 0$.

Summing now over all market pairs (i, j) and all paths p connecting (i, j), one obtains

$$\sum_{ij} \sum_{p \in P_{ij}} (\pi_i(s^*) + C_p(f^*) - \rho_j(d^*)) \times (Q_p - Q_p^*) \geq 0. \tag{20}$$

Applying now (13)- (16) to (20), after some manipulations, yields

$$\sum_i \pi_i(s^*) \times (s_i - s_i^*) + \sum_a c_a(f^*) \times (f_a - f_a^*) - \sum_j \rho_j(d^*) \times (d_j - d_j^*)$$

$$\geq 0, \tag{21}$$

which, in vector notation, is variational inequality (18).

To prove the converse, utilize (21) expanded as (20). Specifically, set $Q_p = Q_p^*$ for all $p \neq q$, where $q \in P_{kl}$. Then (20) reduces to

$$(\pi_k(s^*) + C_q(f^*) - \rho_l(d^*)) \times (Q_q - Q_q^*) \geq 0, \qquad (22)$$

which implies equilibrium conditions (17) for any market pair k, l.

The proof is complete.

Note that if there is only a single path p joining a market pair (i, j) and no paths in the network share links then this model collapses to the spatial price model on a bipartite network depicted in Figure 1.

Both the above models can be generalized to multiple commodities. Let k denote a typical commodity and assume that there are J commodities in total. Then equilibrium conditions (1) would now take the form: For each commodity k; $k = 1, \ldots, J$, and for all pairs of markets (i, j); $i = 1, \ldots, m$; $j = 1, \ldots, n$:

$$\pi_i^k + c_{ij}^k \begin{cases} = \rho_j^k, & \text{if} \quad {Q_{ij}^k}^* > 0 \\ \geq \rho_j^k, & \text{if} \quad {Q_{ij}^k}^* = 0 \end{cases} \tag{23}$$

where π_i^k denotes the supply price of commodity k at supply market i, c_{ij}^k denotes the transaction cost associated with trading commodity k between (i, j), ρ_j^k denotes the demand price of commodity k at demand market j, and ${Q_{ij}^k}^*$ is the equilibrium flow of commodity k between i and j.

The conservation of flow equations (2) and (3) now become

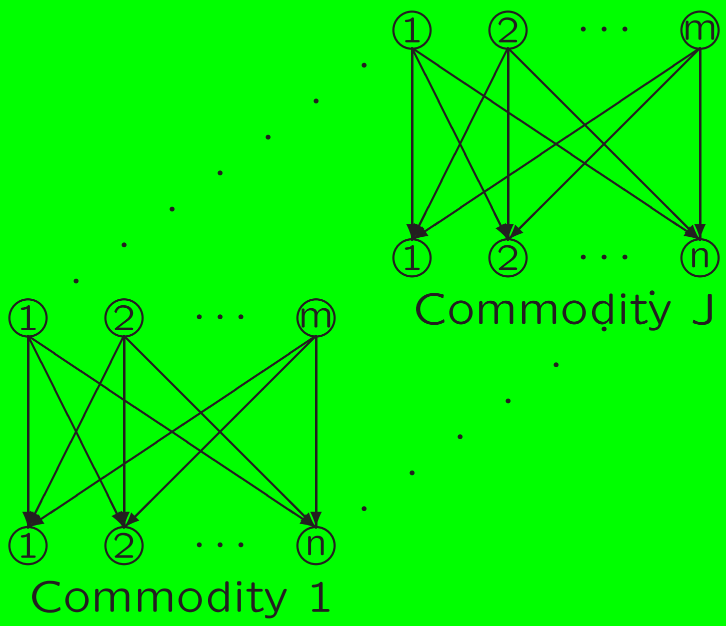
$$s_i^k = \sum_{j=1}^{n} Q_{ij}^k \tag{24}$$

and

$$d_j^k = \sum_{i=1}^{m} Q_{ij}^k \tag{25}$$

where s_i^k denotes the supply of commodity k at supply market i, d_j^k denotes the demand for commodity k at demand market j, and all Q_{ij}^k are nonnegative.

The variational inequality formulation of multicommodity spatial price equilibrium conditions (23) will have the same structure as the one governing the single commodity problem (cf. (7)), but now the vectors increase in dimension by a factor of J to accommodate all the commodities, that is, $\pi \in R^{Jm}$, $s \in R^{Jm}$, $\rho \in R^{Jn}$, $d \in R^{Jn}$, and $Q \in R^{Jmn}$. The feasible set K now contains (s, Q, d) such that (24) and (25) are satisfied. Note that the feasible set K can be expressed as a Cartesian product of subsets, where each subset corresponds to the constraints of the commodity.



Multicommodity model on a bipartite network

Optimization Reformulation in the Symmetric Case

If the supply price functions (4), demand price functions (5), and the transaction cost functions (6) have symmetric Jacobians, and the supply price and transaction cost functions are monotonically nondecreasing, and the demand price functions are monotonically nonincreasing, then the spatial price equilibrium supplies, flows, and demands could be obtained by solving the convex optimization problem:

$$\text{Minimize} \quad \sum_{i=1}^{m} \int_0^{s_i} \pi_i(x)dx + \sum_{i=1}^{m}\sum_{j=1}^{n} \int_0^{Q_{ij}} c_{ij}(y)dy$$

$$-\sum_{j=1}^{n} \int_0^{d_j} \rho_j(z)dz \tag{26}$$

subject to constraints (2) and (3) where $Q_{ij} \geq 0$, for all i and j. In particular, in the case of linear and separable supply price, demand price, and transaction cost functions, the demand market equilibration algorithm could then be used for the computation of the equilibrium pattern.

Price Formulation

Now we consider spatial price equilibrium models in which the supply and demand functions are available and are functions, respectively, of the supply and demand prices.

First consider the bipartite model. Assume, that there are m supply markets and n demand markets involved in the production/consumption of a commodity.

Consider the situation where the supply at a supply market may depend upon the supply prices at every supply market, that is,

$$s = s(\pi), \tag{27}$$

where s is a known smooth function.

The demand at a demand market, in turn, may depend upon the demand prices associated with the commodity at every demand market, i.e.,

$$d = d(\rho) \tag{28}$$

where d is a known smooth function.

The transaction costs are as in (6).

The equilibrium conditions (1) remain, but since the prices are now to be computed, because they are no longer functions as previously, but, rather, variables, one may write the conditions as: For all pairs of markets (i,j): $i = 1, \ldots, m$; $j = 1, \ldots, n$:

$$\pi_i^* + c_{ij} \begin{cases} = \rho_j^*, & \text{if} \quad Q_{ij}^* > 0 \\ \geq \rho_j^*, & \text{if} \quad Q_{ij}^* = 0, \end{cases} \tag{29}$$

to emphasize this point.

In view of the fact that one now has supply and demand functions, feasibility conditions (2) and (3) are now written as, in equilibrium:

$$s_i(\pi^*) \begin{cases} = \sum_{j=1}^n Q_{ij}^*, & \text{if} \quad \pi_i^* > 0 \\ \geq \sum_{j=1}^n Q_{ij}^*, & \text{if} \quad \pi_i^* = 0 \end{cases} \tag{30}$$

and

$$d_j(\rho^*) \begin{cases} = \sum_{i=1}^m Q_{ij}^*, & \text{if} \quad \rho_j^* > 0 \\ \leq \sum_{i=1}^m Q_{ij}^*, & \text{if} \quad \rho_j^* = 0. \end{cases} \tag{31}$$

The derivation of the variational inequality formulation of the equilibrium conditions (29) – (31) governing the price model is given in the subsequent theorem.

Theorem 4 (Variational Inequality Formulation of the Price Model)

The vector $x^* \equiv (\pi^*, Q^*, \rho^*) \in R^m_+ \times R^{mn}_+ \times R^n_+$ *is an equilibrium price and shipment vector if and only if it satisfies the variational inequality*

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in R^m_+ \times R^{mn}_+ \times R^n_+ \qquad (32)$$

where $F : R^{mn+m+n}_+ \mapsto R^{mn+m+n}$ *is the function defined by the row vector*

$$F(x) = (S(x), D(x), T(x)) \qquad (33)$$

where $S : R^{mn+m+n}_+ \mapsto R^m$, $T : R^{mn+m+n}_+ \mapsto R^{mn}$, *and* $D : R^{mn+m+n}_+ \mapsto R^n$ *are defined by:*

$$S_i = s_i(\pi) - \sum_{j=1}^{n} Q_{ij} T_{ij} = \pi_i + c_{ij}(Q) - \rho_j, D_j$$

$$= \sum_{i=1}^{m} Q_{ij} - d_j(\rho). \qquad (34)$$

Proof: Assume that $x^* = (\pi^*, Q^*, \rho^*)$ satisfies (29) – (31). We will show, first, that x^* must satisfy variational inequality (32). Note that (29) implies that

$$(\pi_i^* + c_{ij}(Q^*) - \rho_j^*) \times (Q_{ij} - Q_{ij}^*) \geq 0, \tag{35}$$

(30) implies that

$$(s_i(\pi^*) - \sum_{j=1}^{n} Q_{ij}^*) \times (\pi_i - \pi_i^*) \geq 0, \tag{36}$$

and (31) implies that

$$(\sum_{i=1}^{m} Q_{ij}^* - d_j(\rho^*)) \times (\rho_j - \rho_j^*) \geq 0. \tag{37}$$

Summing now (35) over all i, j, (36) over all i, and (37) over all j, one obtains

$$\sum_{i=1}^{m} \left[s_i(\pi^*) - \sum_{j=1}^{n} Q_{ij}^* \right] \times [\pi_i - \pi_i^*] + \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\pi_i^* + c_{ij}(Q^*) - \rho_j^* \right]$$

$$\times \left[Q_{ij} - Q_{ij}^* \right] + \sum_{j=1}^{n} \left[\sum_{i=1}^{m} Q_{ij}^* - d_j(\rho^*) \right] \times \left[\rho_j - \rho_j^* \right] \geq 0, \tag{38}$$

which is variational inequality (32).

Now the converse is established. Assume that $x^* = (\pi^*, Q^*, \rho^*)$ satisfies (32). We will show that it also satisfies conditions (29)-(31). Indeed, fix market pair kl, and set $\pi = \pi^*$, $\rho = \rho^*$, and $Q_{ij} = Q^*_{ij}$, for all $ij \neq kl$. Then variational inequality (32) reduces to:

$$(\pi^*_k + c_{kl}(Q^*) - \rho^*_l) \times (Q_{kl} - Q^*_{kl}) \geq 0, \tag{39}$$

which implies that (29) must hold.

Now construct another feasible x as follows. Let $Q_{ij} = Q^*_{ij}$, for all i, j, $\rho_j = \rho^*_j$, for all j, and let $\pi_i = \pi^*_i$ for all $i \neq k$. Then (32) reduces to

$$(s_k(\pi^*) - \sum_{j=1}^{n} Q^*_{kj}) \times (\pi_k - \pi^*_k) \geq 0, \tag{40}$$

from which (30) follows.

A similar construction on the demand price side yields

$$(\sum_{i=1}^{m} Q^*_{il} - d_l(\rho^*)) \times (\rho_l - \rho^*_l) \geq 0, \tag{41}$$

from which one can conclude (31).

The proof is complete.

We emphasize that, unlike the quantity model, the Jacobian matrix $\left[\frac{\partial F}{\partial x}\right]$ for the price model can never be symmetric, and, hence, (29) – (31) can never be cast into an equivalent convex minimization problem.

Recall that, strict monotonicity will guarantee uniqueness, provided that a solution exists. An existence condition is now presented that is weaker than coercivity or strong monotonicity.

Theorem 5

Assume that s, d, and c are continuous functions. Variational inequality (32) has a solution if and only if there exist positive constants r_1, r_2, and r_3, such that the variational inequality

$$\langle F(\bar{x}), x - \bar{x} \rangle \geq 0, \quad \forall x \in K_r \tag{42}$$

where

$$K_r = \{ \begin{bmatrix} \pi \\ Q \\ \rho \end{bmatrix} \in R^{mn+m+n} | \, \pi \leq r_1, Q \leq r_2, \rho \leq r_3 \} \tag{43}$$

has a solution $\bar{x} = \begin{bmatrix} \bar{\pi} \\ \bar{Q} \\ \bar{\rho} \end{bmatrix}$ *with the property:* $\bar{\pi} < r_1$, $\bar{Q} < r_2$, $\bar{\rho} < r_3$, *componentwise. Furthermore, such an* $\bar{x}$ *is a solution to variational inequality (32).*

Under the following conditions it is possible to construct r_1, r_2, and r_3 large enough so that the solution to the restricted variational inequality (43) will satisfy the boundedness condition with r_1, r_2, and r_3, and, thus, existence of an equilibrium will follow.

Theorem 6 (Existence)

If there exist μ, M, and $N > 0$, $\mu < N$, such that

$$s_i(\pi) > nM \quad \text{for any } \pi \text{ with } \pi_i \geq N, \forall i,$$

$$c_{ij}(Q) > \mu \quad \forall i, j, Q,$$

$$d_j(\rho) < M, \quad \text{for any } \rho \text{ with } \rho_j \geq \mu, \text{ and } \forall i,$$

then there exists an equilibrium point.

Sensitivity Analysis

Consider the network model governed by variational inequality (7) and subject to changes in the supply price functions, demand price functions, and transaction cost functions. In particular, change the supply price functions from $\pi(\cdot)$ to $\pi^*(\cdot)$, the demand price functions from $\rho(\cdot)$ to $\rho^*(\cdot)$, and the transaction cost functions from $c(\cdot)$ to $c^*(\cdot)$; what can be said about the corresponding equilibrium patterns (s, Q, d) and (s^*, Q^*, d^*)?

The following strong monotonicity condition is imposed on $\pi(\cdot)$, $c(\cdot)$, and $\rho(\cdot)$:

$$\langle \pi(s^1) - \pi(s^2), s^1 - s^2 \rangle + \langle c(Q^1) - c(Q^2), Q^1 - Q^2 \rangle$$

$$-\langle \rho(d^1) - \rho(d^2), d^1 - d^2 \rangle$$

$$\geq \alpha(\|s^1 - s^2\|^2 + \|Q^1 - Q^2\|^2 + \|d^1 - d^2\|^2), \quad (44)$$

for all $(s^1, Q^1, d^1), (s^2, Q^2, d^2) \in K$, where K was defined for this model earlier, and α is a positive constant.

A sufficient condition for (44) to hold is that for all $(s^1, Q^1, d^1) \in K$, $(s^2, Q^2, d^2) \in K$,

$$\langle \pi(s^1) - \pi(s^2), s^1 - s^2 \rangle \geq \beta \|s^1 - s^2\|^2$$

$$\langle c(Q^1) - c(Q^2), Q^1 - Q^2 \rangle \geq \gamma \|Q^1 - Q^2\|^2$$

$$-\langle \rho(d^1) - \rho(d^2), d^1 - d^2 \rangle \geq \delta \|d^1 - d^2\|^2, \qquad (45)$$

where $\beta > 0, \gamma > 0$, and $\delta > 0$.

The following theorem establishes that small changes in the supply price, demand price, and transaction cost functions induce small changes in the supplies, demands, and commodity shipment pattern.

Theorem 7

Let α be the positive constant in the definition of strong monotonicity. Then

$$\|((s^* - s), (Q^* - Q), (d^* - d))\|$$

$$\leq \frac{1}{\alpha} \|((\pi^*(s^*) - \pi(s^*)), (c^*(Q^*) - c(Q^*)), -(\rho^*(d^*) - \rho(d^*)))\|. \qquad (46)$$

Proof: The vectors (s,Q,d), (s^*,Q^*,d^*) must satisfy, respectively, the variational inequalities

$$\langle \pi(s), s'-s\rangle + \langle c(Q), Q'-Q\rangle - \langle \rho(d), d'-d\rangle \geq 0,$$

$$\forall (s',Q',d') \in K \tag{47}$$

and

$$\langle \pi^*(s^*), s'-s^*\rangle + \langle c^*(Q^*), Q'-Q^*\rangle - \langle \rho^*(d^*), d'-d^*\rangle \geq 0,$$

$$\forall (s',Q',d') \in K. \tag{48}$$

Writing (47) for $s'=s^*$, $Q'=Q^*$, $d'=d^*$, and (48) for $s'=s$, $Q'=Q$, $d'=d$, and adding the two resulting inequalities, one obtains

$$\langle \pi^*(s^*)-\pi(s), s-s^*\rangle + \langle c^*(Q^*)-c(Q), Q-Q^*\rangle$$

$$-\langle \rho^*(d^*)-\rho(d), d-d^*\rangle \geq 0 \tag{49}$$

or

$$\langle \pi^*(s^*)-\pi(s^*)+\pi(s^*)-\pi(s), s-s^*\rangle$$

$$+\langle c^*(Q^*)-c(Q^*)+c(Q^*)-c(Q), Q-Q^*\rangle$$

$$-\langle \rho^*(d^*)-\rho(d^*)+\rho(d^*)-\rho(d), d-d^*\rangle \geq 0. \tag{50}$$

Using now the monotonicity condition (44), (50) yields

$$\langle \pi^*(s^*) - \pi(s^*), s - s^* \rangle + \langle c^*(Q^*) - c(Q^*), Q - Q^* \rangle$$

$$- \langle \rho^*(d^*) - \rho(d^*), d - d^* \rangle$$

$$\geq \langle \pi(s^*) - \pi(s), s^* - s \rangle + \langle c(Q^*) - c(Q), Q^* - Q \rangle$$

$$- \langle \rho(d^*) - \rho(d), d^* - d \rangle$$

$$\geq \alpha(\|s^* - s\|^2 + \|Q^* - Q\|^2 + \|d^* - d\|^2). \tag{51}$$

Applying the Schwarz inequality to the left-hand side of (51) yields

$$\|((\pi^*(s^*) - \pi(s^*)), (c^*(Q^*) - c(Q^*)), -(\rho^*(d^*) - \rho(d^*)))\|$$

$$\|((s - s^*), (Q - Q^*), (d - d^*))\|$$

$$\geq \alpha \|((s - s^*), (Q - Q^*), (d - d^*))\|^2 \tag{52}$$

from which (46) follows, and the proof is complete.

The problem of how changes in the supply price, demand price, and transaction cost functions affect the direction of the change in the equilibrium supply, demand, and shipment pattern, and the incurred supply prices, demand prices, and transaction costs is now addressed.

Theorem 8

Consider the spatial price equilibrium problem with two supply price functions $\pi(\cdot)$, $\pi^*(\cdot)$, two demand price functions $\rho(\cdot)$, $\rho^*(\cdot)$, and two transaction cost functions $c(\cdot)$, $c^*(\cdot)$. Let (s, Q, d) and (s^*, Q^*, d^*) be the corresponding equilibrium supply, shipment, and demand patterns. Then

$$\sum_{i=1}^{m} [\pi_i^*(s^*) - \pi_i(s)] \times [s_i^* - s_i] + \sum_{i=1}^{m}\sum_{j=1}^{n} \left[c_{ij}^*(Q^*) - c_{ij}(Q)\right]$$

$$\times \left[Q_{ij}^* - Q_{ij}\right] - \sum_{j=1}^{n} \left[\rho_j^*(d^*) - \rho_j(d)\right] \times \left[d_j^* - d_j\right] \leq 0 \quad (53)$$

and

$$\sum_{i=1}^{m} [\pi_i^*(s^*) - \pi_i(s^*)] \times [s_i^* - s_i] + \sum_{i=1}^{m}\sum_{j=1}^{n} \left[c_{ij}^*(Q^*) - c_{ij}(Q^*)\right]$$

$$\times \left[Q_{ij}^* - Q_{ij}\right] - \sum_{j=1}^{n} \left[\rho_j^*(d^*) - \rho_j(d^*)\right] \times \left[d_j^* - d_j\right] \leq 0. \quad (54)$$

Proof: The above inequalities have been established in the course of proving the preceding theorem.

The following corollary establishes the direction of a change of the equilibrium supply at a particular supply market and the incurred supply price, subject to a specific change in the network.

Corollary 1

Assume that the supply price at supply market i is increased (decreased), while all other supply price functions remain fixed, that is, $\pi_i^*(s') \geq \pi_i(s')$, $(\pi_i^*(s') \leq \pi_i(s'))$ for some i, and $s' \in K$, and $\pi_j^*(s') = \pi_j(s')$ for all $j \neq i$, $s' \in K$. Assume also that $\frac{\partial \pi_j(s')}{\partial s_i} = 0$, for all $j \neq i$. If we fix the demand functions for all markets, that is, $\rho_j^*(d') = \rho_j(d')$, for all j, and $d' \in K$, and the transaction cost functions, that is, $c_{ij}^*(Q') = c_{ij}(Q')$, for all i, j, and $Q' \in K$, then the supply at supply market i cannot increase (decrease) and the incurred supply price cannot decrease (increase), i.e., $s_i^* \leq s_i$ $(s_i^* \geq s_i)$, and $\pi_i^*(s^*) \geq \pi_i(s)$ $(\pi_i(s^*) \leq \pi_i(s))$.

One can also obtain similar corollaries for changes in the demand price functions at a fixed demand market, and changes in the transaction cost functions, respectively, under analogous conditions.

Policy Interventions

Now policy interventions are incorporated directly into both quantity and price formulations of spatial price equilibrium models within the variational inequality framework. First, a quantity model with price controls is presented, and then a price model with both price controls and trade restrictions.

Quantity Formulation

The notation for the bipartite network model is retained, but now, introduce u_i to denote the nonnegative possible excess supply at supply market i and v_j the nonnegative possible excess demand at demand market j. Group then the excess supplies into a column vector u in R^m and the excess demands into a column vector v in R^n.

The following equations must now hold:

$$s_i = \sum_{j=1}^{n} Q_{ij} + u_i, \quad i = 1, \ldots, m \tag{55}$$

and

$$d_j = \sum_{i=1}^{m} Q_{ij} + v_j, \quad j = 1, \ldots, n. \tag{56}$$

Let $K^1 = \{(s, d, Q, u, v) | \text{(55), (56) hold}\}$.

Assume that there is a fixed minimum supply price $\underline{\pi}_i$ for each supply market i and a fixed maximum demand price $\bar{\rho}_j$ at each demand market j. Thus $\underline{\pi}_i$ represents the price floor imposed upon the producers at supply market i, whereas $\bar{\rho}_j$ represents the price ceiling imposed at the demand market j. Group the supply price floors into a row vector $\underline{\pi}$ in R^m and the demand price ceilings into a row vector $\bar{\rho}$ in R^n. Also, define the vector $\tilde{\pi}$ in R^{mn} consisting of m vectors, where the i-th vector, $\{\tilde{\pi}_i\}$, consists of n components $\{\pi_i\}$. Similarly, define the vector $\tilde{\rho}$ in R^{mn} consisting of m vectors $\{\tilde{\rho}_j\}$ in R^n with components $\{\rho_1, \rho_2, \ldots, \rho_n\}$.

The economic market conditions for the above model, assuming perfect competition, take the following form: For all pairs of supply and demand markets $(i, j); i = 1, \ldots, m; j = 1, \ldots, n:$

$$\pi_i + c_{ij} \begin{cases} = \rho_j, & \text{if } Q^*_{ij} > 0 \\ \geq \rho_j, & \text{if } Q^*_{ij} = 0 \end{cases} \tag{57}$$

$$\pi_i \begin{cases} = \underline{\pi}_i, & \text{if } u^*_i > 0 \\ \geq \underline{\pi}_i, & \text{if } u^*_i = 0 \end{cases} \tag{58}$$

$$\rho_j \begin{cases} = \bar{\rho}_j, & \text{if } v^*_j > 0 \\ \leq \bar{\rho}_j, & \text{if } v^*_j = 0. \end{cases} \tag{59}$$

Assume that the level of generality of the governing functions is as in the spatial price equilibrium models without policy interventions at the beginning of these lectures.

These conditions are now illustrated with an example consisting of two supply markets and a single demand market.

Example 2

The supply price functions are:

$$\pi_1(s) = 2s_1 + s_2 + 5 \quad \pi_2(s) = s_2 + 10.$$

The transaction cost functions are:

$$c_{11}(Q) = 5Q_{11} + Q_{21} + 9 \quad c_{21}(Q) = 3Q_{21} + 2Q_{11} + 19.$$

The demand price function is:

$$p_1(d) = -d_1 + 80.$$

The supply price floors are:

$$\underline{\pi}_1 = 21 \quad \underline{\pi}_2 = 16.$$

The demand price ceiling is:

$$\bar{\rho}_1 = 60.$$

The production, shipment, consumption, and excess supply and demand pattern satisfying conditions (57)–(59) is:

$$s_1^* = 5 \quad s_2^* = 6, \quad Q_{11}^* = 5 \quad Q_{21}^* = 5, \quad d_1^* = 20,$$

$$u_1^* = 0 \quad u_2^* = 1, \quad v_1^* = 10,$$

with induced supply prices, transaction costs, and demand prices:

$$\pi_1 = 21 \quad \pi_2 = 16, \quad c_{11} = 39 \quad c_{21} = 44, \quad \rho_1 = 60.$$

Define now the vectors $\widehat{\pi} = \pi \in R^m$, and $\widehat{\rho} = \rho \in R^n$. In view of conditions (55) and (56), one can express $\widehat{\pi}$ and $\widehat{\rho}$ in the following manner:

$$\widehat{\pi} = \widehat{\pi}(Q, u) \quad \text{and} \quad \widehat{\rho} = \widehat{\rho}(Q, v). \tag{60}$$

Also define the vector $\tilde{\widehat{\pi}} \in R^{mn}$ consisting of m vectors, where the i-th vector, $\{\tilde{\widehat{\pi}}_i\}$, consists of n components $\{\widehat{\pi}_i\}$ and the vector $\tilde{\widehat{\rho}} \in R^{mn}$ consisting of m vectors $\{\tilde{\widehat{\rho}}_j\} \in R^n$ with components $\{\widehat{\rho}_1, \widehat{\rho}_2, \ldots, \widehat{\rho}_n\}$.

The above system (57), (58), and (59) can be formulated as a variational inequality problem, as follows.

Theorem 11 (Variational Inequality Formulation of the Quantity Model with Price Floors and Ceilings)

A pattern of total supplies, total demands, and commodity shipments, and excess supplies and excess demands $(s^*, d^*, Q^*, u^*, v^*) \in K^1$ *satisfies inequalities (57), (58), and (59) governing the disequilibrium market problem if and only if it satisfies the variational inequality*

$$\langle \pi(s^*), s - s^* \rangle - \langle \underline{\pi}, u - u^* \rangle + \langle c(Q^*), Q - Q^* \rangle$$

$$-\langle \rho(d^*), d - d^* \rangle + \langle \bar{\rho}, v - v^* \rangle \geq 0, \quad \forall (s, d, Q, u, v) \in K^1 \tag{61}$$

or, equivalently, the variational inequality

$$\langle \tilde{\tilde{\pi}}(Q^*, u^*) + c(Q^*) - \tilde{\tilde{\rho}}(Q^*, v^*), Q - Q^* \rangle$$

$$+\langle \hat{\pi}(Q^*, u^*) - \underline{\pi}, u - u^* \rangle + \langle \bar{\rho} - \hat{\rho}(Q^*, v^*), v - v^* \rangle \geq 0,$$

$$\forall (Q, u, v) \in K^2 \equiv R_+^{mn} \times R_+^m \times R_+^n. \tag{62}$$

Proof: Assume that a vector $(s^*, d^*, Q^*, u^*, v^*) \in K^1$ satisfies (57), (58), and (59). Then for each pair (i, j), and any $Q_{ij} \geq 0$:

$$(\pi_i(s^*) + c_{ij}(Q^*) - \rho_j(d^*)) \times (Q_{ij} - Q^*_{ij}) \geq 0. \qquad (63)$$

Summing over all pairs (i, j), one has that

$$\langle \tilde{\pi}(s^*) + c(Q^*) - \tilde{\rho}(d^*), Q - Q^* \rangle \geq 0. \qquad (64)$$

Using similar arguments yields

$$\langle (\pi(s^*) - \underline{\pi}), u - u^* \rangle \geq 0 \quad \text{and} \quad \langle (\bar{\rho} - \rho(d^*)), v - v^* \rangle \geq 0. \qquad (65)$$

Summing then the inequalities (64) and (65), one obtains

$$\langle \tilde{\pi}(s^*) + c(Q^*) - \tilde{\rho}(d^*), Q - Q^* \rangle + \langle \pi(s^*) - \underline{\pi}, u - u^* \rangle$$
$$+ \langle \bar{\rho} - \rho(d^*), v - v^* \rangle \geq 0, \qquad (66)$$

which, after the incorporation of the feasibility constraints (55) and (56), yields (61).

Also, by definition of $\hat{\pi}$ and $\hat{\rho}$, one concludes that if $(Q^*, u^*, v^*) \in K^2$ satisfies (57), (58), and (59), then

$$\langle \tilde{\tilde{\pi}}(Q^*, u^*) + c(Q^*) - \tilde{\tilde{\rho}}(Q^*, v^*), Q - Q^* \rangle$$

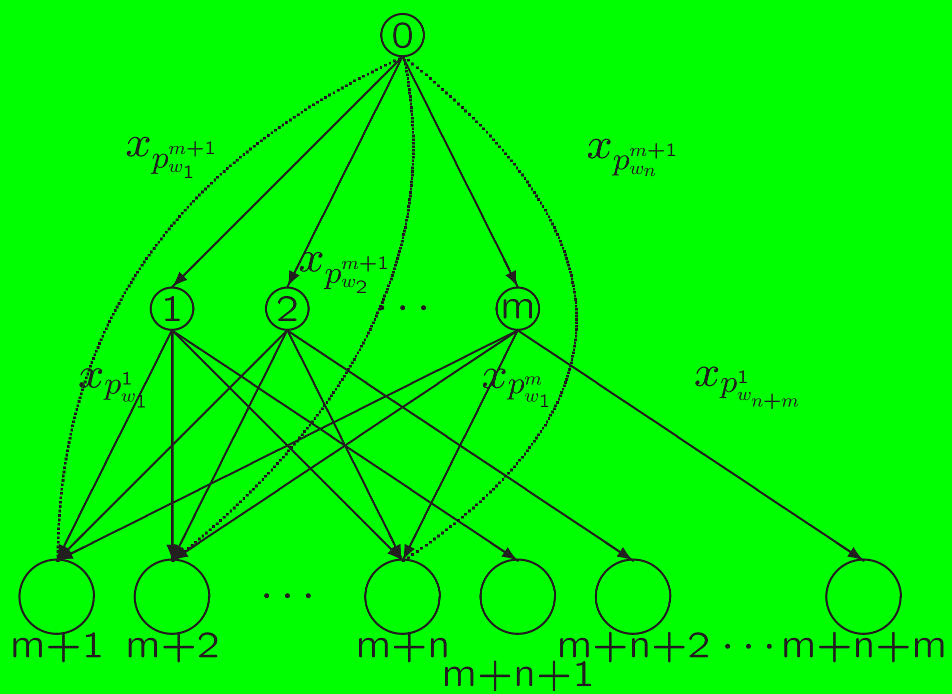
$$+ \langle \hat{\pi}(Q^*, u^*) - \underline{\pi}, u - u^* \rangle + \langle (\bar{\rho} - \hat{\rho}(Q^*, v^*)), v - v^* \rangle \geq 0. \quad (67)$$

Assume now that variational inequality (61) holds. Let $u = u^*$ and $v = v^*$. Then

$$\langle \tilde{\pi}(s^*) + c(Q^*) - \tilde{\rho}(d^*), Q - Q^* \rangle \geq 0, \quad (68)$$

which, in turn, implies that (57) holds. Similar arguments demonstrate that (58) and (59) also then hold.

By definition, the same inequalities can be established when utilizing the functions $\hat{\pi}(Q, u)$ and $\hat{\rho}(Q, v)$.



Network equilibrium representation of market disequilibrium

First, the existence conditions are given.

Denote the row vector $F(Q,u,v)$ by

$$F(Q,u,v) \equiv (\tilde{\tilde{\pi}}(Q,u)+c(Q)-\tilde{\bar{\rho}}(Q,v), \hat{\pi}(Q,u)-\underline{\pi}, \bar{\rho}-\hat{\rho}(Q,v)). \tag{69}$$

Variational inequality (62) will admit at least one solution provided that the function $F(Q,u,v)$ is coercive. More precisely, one has the following:

Theorem 12 (Existence Under Coercivity)

Assume that the function $F(Q,u,v)$ is coercive, that is, there exists a point $(Q^0,u^0,v^0) \in K^2$, such that

$$\lim_{\|(Q,u,v)\|\to\infty} \frac{\langle F(Q,u,v)-F(Q^0,u^0,v^0), \begin{bmatrix} Q-Q^0 \\ u-u^0 \\ v-v^0 \end{bmatrix} \rangle}{\|(Q-Q^0,u-u^0,v-v^0)\|} = \infty, \tag{70}$$

$$\forall (Q,u,v) \in K^2.$$

Then variational inequality (62) admits at least one solution or, equivalently, a disequilibrium solution exists.

One of the sufficient conditions ensuring (70) in Theorem 12 is that the function $F(Q, u, v)$ is strongly monotone, that is, the following inequality holds:

$$\langle F(Q^1, u^1, v^1) - F(Q^2, u^2, v^2), \begin{bmatrix} Q^1 \\ u^1 \\ v^1 \end{bmatrix} - \begin{bmatrix} Q^2 \\ u^2 \\ v^2 \end{bmatrix} \rangle$$

$$\geq \alpha \| \begin{bmatrix} Q^1 - Q^2 \\ u^1 - u^2 \\ v^1 - v^2 \end{bmatrix} \|^2, \tag{71}$$

$$\forall (Q^1, u^1, v^1), (Q^2, u^2, v^2) \in K^2,$$

where α is a positive constant.

Under condition (71) uniqueness of the solution pattern (Q, u, v) is guaranteed.

Through the subsequent lemmas, it is shown that strong monotonicity of $F(Q,u,v)$ is equivalent to the strong monotonicity of the transaction cost $c(Q)$, the supply price $\pi(s)$, and the demand price $\rho(d)$ functions, which is a commonly imposed condition in the study of the spatial price equilibrium problem.

Lemma 1

Let (Q,s,d) be a vector associated with $(Q,u,v) \in K^2$ via (55) and (56). There exist positive constants m_1 and m_2 such that:

$$\|(Q,u,v)^T\|^2_{R^{mn+m+n}} \leq m_1 \|(Q,s,d)^T\|^2_{R^{mn+m+n}} \tag{72}$$

and

$$\|(Q,s,d)^T\|^2_{R^{mn+m+n}} \leq m_2 \|(Q,u,v)^T\|^2_{R^{mn+m+n}} \tag{73}$$

where $\|.\|_{R^k}$ denotes the norm in the space R^k.

Lemma 2

$F(Q,u,v)$ is a strongly monotone function of (Q,u,v) if and only if $\pi(s), c(Q)$, and $-\rho(d)$ are strongly monotone functions of s, Q, and d, respectively.

At this point, we state the following:

Proposition 1 (Existence and Uniqueness Under Strong Monotonicity)

Assume that $\pi(s), c(Q)$, and $-\rho(d)$ are strongly monotone functions of s, Q, and d, respectively. Then there exists precisely one disequilibrium point $(Q^*, u^*, v^*) \in K^2$.

Lemma 3

$F(Q, u, v)$ is strictly monotone if and only if $\pi(s), c(Q)$, and $-\rho(d)$ are strictly monotone functions of s, Q, and d, respectively.

It is now clear that the following statement is true:

Theorem 13 (Uniqueness Under Strict Monotonicity)

Assume that $\pi(s), c(Q)$, and $-\rho(d)$ are strictly monotone in s, Q, and d, respectively. Then the disequilibrium solution $(Q^*, u^*, v^*) \in K^2$ is unique, if one exists.

By further observation, one can see that if $\pi(s)$ and $-\rho(d)$ are monotone, then the disequilibrium commodity shipment Q^* is unique, provided that $c(Q)$ is a strictly monotone function of Q.

Existence and uniqueness of a disequilibrium solution (Q^*, u^*, v^*), therefore, crucially depend on the strong (strict) monotonicity of the functions $c(Q)$, $\pi(s)$, and $-\rho(d)$. If the Jacobian matrix of the transaction cost function $c(Q)$ is positive definite (strongly positive definite), that is,

$$x^T \nabla c(Q) x > 0 \quad \forall x \in R^{mn}, Q \in K_1, x \neq 0 \tag{74}$$

$$x^T \nabla c(Q) x \geq \alpha \|x\|^2, \alpha > 0, \quad \forall x \in R^{mn}, Q \in K_1, \tag{75}$$

then the function $c(Q)$ is strictly (strongly) monotone. Monotonicity of $c(Q)$ is not economically unreasonable, since the transaction cost c_{ij} from supply market i to demand market j can be expected to depend mainly upon the shipment Q_{ij} which implies that the Jacobian matrix $\nabla c(Q)$ is diagonally dominant; hence, $\nabla c(Q)$ is positive definite.

Next, the economic meaning of monotonicity of the supply price function $\pi(s)$ and the demand price function $\rho(d)$ is explored.

Lemma 4

Suppose that $f : D \mapsto V$ is continuously differentiable on set D. Let $f^{-1} : V \mapsto D$ be the inverse function of f, where D and V are subsets of R^k. $\nabla f(x)$ is positive definite for all $x \in D$ if and only if $\nabla(f^{-1}(y))$ is positive definite for all $y \in V$.

Proof: Since $\nabla f(x)$ is positive definite, we have that

$$w^T \nabla f(x) w > 0 \quad \forall w \in R^k, x \in D, w \neq 0. \tag{76}$$

It is well-known that

$$\nabla(f^{-1}) = (\nabla f)^{-1}. \tag{77}$$

(76) can be written as:

$$w^T (\nabla f)^T (\nabla f)^{-1} (\nabla f) w > 0, \quad \forall w \in R^k, x \in D, w \neq 0. \tag{78}$$

Letting $z = \nabla f \cdot w$ in (78) and using (77) yields

$$z^T \nabla(f^{-1}(y)) z > 0, \quad \forall z \in R^k, z \neq 0, y \in V. \tag{79}$$

Thus, $\nabla(f^{-1}(y))$ is positive definite. Observing that each step of the proof is convertible, one can easily prove the converse part of the lemma.

Denote the inverse of the supply price function $\pi(s)$ by π^{-1} and the inverse of the demand price function $\rho(d)$ by ρ^{-1}. Then

$$s = \pi^{-1}(\pi) \qquad d = \rho^{-1}(\rho). \tag{80}$$

By virtue of Lemma 4, $\pi(s)$ is a strictly (strongly) monotone function of s, provided that $\nabla_\pi s(\pi)$ is positive definite (strongly positive definite) for all $\pi \in R^m_+$. Similarly, $-\rho(d)$ is a strictly (strongly) monotone function of d provided that $-\nabla_\rho d(\rho)$ is positive definite (strongly positive definite) for all $\rho \in R^n_+$. In reality, the supply s_i is mainly affected by the supply price π_i, for each supply market $i; i = 1, \ldots, m$, and the demand d_j is mainly affected by the demand price ρ_j for each demand market $j; j = 1, \ldots, n$. Thus, in most cases, one can expect the matrices $\nabla_\pi s(\pi)$ and $-\nabla_\rho d(\rho)$ to be positive definite (strongly positive definite).

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