Permits for User-Optimized, Fixed Demand Networks

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Introduction

In this lecture, I develop a framework for sustainable transportation systems through the perspective of pollution permits.

I consider fixed demand traffic networks in which the travel behavior on the transportation network is that of user-optimization.

For such networks, a permit system on links is constructed which guarantees that the environmental quality standard is satisfied.

I assume throughout this lecture that the transportation network in question is viable.

In particular, a variational inequality framework is utilized for the modeling, qualitative analysis, and computation of equilibrium patterns in the congested urban transportation systems in the presence of emission pollution permits.

Qualitative analysis of the model is conducted and existence and uniqueness results of the solution obtained.

I also propose an algorithm, with convergence results, to compute the equilibrium link load, marginal cost of emission abatement, license, and license price pattern.

Numerical examples are included to illustrate this approach.

In this lecture, the topic of sustainable transportation systems is explored through the prism of pollution permits. Recall that marketable pollution permits were provided by Montgomery (1972), who showed that perfectly competitive firms could trade permits in an effort to comply with preexisting regional environmental standards with the quantity of pollution fixed by the total number of permits. The system of marketable pollution permits could also address the spatial dimension of the pollution problem by incorporating ambient concentrations within the model.

In this lecture, in contrast, I consider users of a transportation network who are mobile, rather than fixed in location, as are firms. Nevertheless, the pollution dispersion is still spatial in nature as it was in the preceding chapters.

The Model with Fixed Travel Demands and Pollution Permits

Here, a single pollutant and the case of a single receptor point for emissions are considered.

This permit system model readily generalizes to multiple pollutants and multiple receptor points but with an associated increase in notation.

The necessary notation is presented first. As in the preceding lectures, consider a transportation network $G = [N, L]$ consisting of the set of nodes N and a set of directed links L.

Let a, b, etc., denote the links and let p, q, etc., denote the paths, which are assumed to be acyclic.

Assume that there are J origin/destination (O/D) pairs in the network, with a typical O/D pair denoted by w, and the set of O/D pairs denoted by W.

Let P_w denote the set of paths connecting O/D pair w and let P denote the set of paths in the network.

The flow on a link a is denoted by f_a, and the user cost associated with traveling on link a by c_a. Group the link loads into a column vector $f \in R^n$, and the link user travel costs into a row vector $c \in R^n$, where n is the number of links in the network.

The user travel cost on a link will, in general, depend upon the entire link load pattern, that is,

$$c = c(f), \tag{1}$$

where c is a known smooth function.

A user traveling on path p incurs a user travel cost C_p, where

$$C_p(f) = \sum_{a \in L} c_a(f)\delta_{ap}, \tag{2}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise.

Here I consider the case of fixed travel demands where the demand for O/D pair is denoted by d_w. The non-negative flow on path p is denoted by x_p, with the path flows grouped into a column vector $x \in R^{n_P}_+$, where n_P denotes the number of paths in the network.

The following conservation of flow equations must be satisfied by the flows in the network:

$$d_w = \sum_{p \in P_w} x_p, \quad \forall w \in W, \tag{3}$$

and

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in L. \tag{4}$$

Recall that the conservation of flow equation (3) states that the sum of the path flows on paths connecting an O/D pair must be equal to the travel demand for that O/D pair. Equation (4), in turn, states that the flow on a link equals the sum of the path flows on paths that use that link.

Let K denote the feasible set defined as follows:

$$K \equiv \{f, \exists \text{an } \; x \geq 0, \text{satisfying (3) and (4)}\}.$$

Let l_a denote the number of license permits on link a that allows travelers to emit pollutants at a certain rate.

Let l_a^0 denote the initial allocation of licenses on link a, which is assumed to be nonnegative.

Group the licenses into the column vector $l \in R_+^n$.

Let h_a denote the emission factor associated with link $a \in L$.

Price and Cost Structure

I now discuss the price and cost structure associated with the pollution permits.

Let ρ denote the price of a license in the transportation network and let τ_a denote the marginal cost of emission abatement on link a.

Note that, one now has not a single marginal cost of emission abatement for the entire network but one for each link of the network.

Without any loss of generality, one can group the marginal costs of abatement into the column vector $\tau \in R^n_+$.

Equilibrium Conditions

Next, I construct the equilibrium conditions. They consist of systems of equalities and inequalities which must hold, respectively, for the path flows, the marginal costs of emission abatement, the licenses, and the license price.

Subsequently, the variational inequality formulation of the governing equilibrium conditions is derived.

Equilibrium Conditions

The traveler on a path p is now subject not only to the user travel cost, given by (2), but also is subject to the payment of the price or cost of emissions. In particular, one now has that the cost on a link a is given by $c_a(f) + h_a\tau_a$, and, hence, the *generalized* cost on a path p, denoted by $\bar{C}_p(f,\tau)$ is given by $\bar{C}_p(f,\tau) = C_p(f) + \sum_{a\in L} h_a\tau_a\delta_{ap}$.

Consequently, the traffic network equilibrium conditions, in the presence of the pollution permit system, take on the form: For all O/D pairs $w \in W$ and each path $p \in P_w$:

$$\bar{C}_p(f^*,\tau^*) = C_p(f^*) + \sum_{a\in L} h_a\tau_a^*\delta_{ap} \begin{cases} = & \lambda_w, \quad \text{if } x_p^* > 0 \\ \geq & \lambda_w, \quad \text{if } x_p^* = 0, \end{cases} \tag{5}$$

where τ_a^* denotes the equilibrium marginal cost of emission abatement associated with link a and discussed further below.

Equilibrium conditions (5) state that a traveler on a path of the network is now subjected to payment of the true cost of his emissions while traveling on the path p.

The emission payment for traveling on path p is equal to the sum over all links that comprise the path p of the marginal cost of emission abatement times the emission factor on the links.

Note that, in this framework, the transportation authority is responsible for informing the travelers of the license prices and the corresponding payments required, as well as the availability of the licenses or permits on the links.

One also has the following conditions, which must hold in equilibrium, akin to those that must hold in the case of firms and marketable pollution permits (cf. Nagurney and Dhanda 1996).

For each link $a \in L$, the equilibrium marginal cost of emission abatement τ_a^* must satisfy the following system: For each link $a \in L$:

$$h_a f_a^* \begin{cases} = & l_a^*, \quad \text{if } \tau_a^* > 0 \\ \leq & l_a^*, \quad \text{if } \tau_a^* = 0. \end{cases} \tag{6}$$

In other words, if the marginal cost of emission abatement, τ_a^*, is positive in equilibrium for a link a, then the emissions on that link are precisely equal to the pollution license holdings for that link; if the number of licenses exceeds the emissions on a link, then, in equilibrium, the marginal cost of abatement is zero.

The following condition must also be met at equilibrium: For each link $a \in L$:

$$\tau_a^* \begin{cases} = & \rho^*, \quad \text{if } l_a^* > 0 \\ \leq & \rho^*, \quad \text{if } l_a^* = 0. \end{cases} \tag{7}$$

Hence, in equilibrium, a positive holding of licenses for a link implies that the marginal cost of emission abatement must be equal to the price of the license on a link. However, if the price of the license exceeds the marginal cost of emission abatement, then the number of licenses for that link will be zero.

Finally, the equilibrium price, ρ^*, of a license must satisfy the following equilibrium condition:

$$\sum_{a \in L} (l_a^0 - l_a^*) \begin{cases} = & 0, \quad \text{if } \rho^* > 0 \\ \geq & 0, \quad \text{if } \rho^* = 0. \end{cases} \tag{8}$$

Expression (8) corresponds to the well-known economic equilibrium conditions that state that, in equilibrium, if a price of a good (which in this case is the license) is positive, then the market for that good must clear; that is, the supply of the licenses, which is equal to $\sum_{a \in L} l_a^0$, must be equal to the demand for the licenses in equilibrium, which is given by $\sum_{a \in L} l_a^*$. On the other hand, if the price of a license in the transportation network is zero, then we may have an excess supply of the licenses in this network.

Let $\mathcal{K}$ denote the feasible set such that $\mathcal{K} \equiv K \times R_+^{2n+1}$, since both vectors l and τ are n-dimensional vectors and ρ is single dimensional.

Definition 1 (Pollution Permit System Traffic Network Equilibrium)

A vector $(f^*, \tau^*, l^*, \rho^*) \in \mathcal{K}$ *is an equilibrium of the traffic network equilibrium emission permits market model if and only if it satisfies the systems of equalities and inequalities (5), (6), (7), and (8).*

The variational inequality formulation of the equilibrium conditions for the model is now derived. Subsequently, a special case is considered.

Theorem 1 (Variational Inequality Formulation of Pollution Permit System Traffic Network Equilibrium)

A vector of link loads, marginal costs of emission abatement, licenses, and license price, $(f^*, \tau^*, l^*, \rho^*) \in \mathcal{K}$, is an equilibrium of the traffic network equilibrium problem with emission pollution permits if and only if it is a solution to the variational inequality problem:

$$\sum_{a \in L} (c_a(f^*) + h_a \tau_a^*) \times (f_a - f_a^*) + \sum_{a \in L} (l_a^* - h_a f_a^*) \times (\tau_a - \tau_a^*)$$

$$+ \sum_{a \in L} (\rho^* - \tau_a^*) \times (l_a - l_a^*) + \sum_{a \in L} (l_a^0 - l_a^*) \times (\rho - \rho^*) \geq 0,$$

$$\forall (f, \tau, l, \rho) \in \mathcal{K}. \quad (9)$$

Proof:

I first establish that a solution to the equilibrium conditions (5), (6), (7), and (8) satisfies the variational inequality problem (9).

Equilibrium conditions (5) imply that, for a fixed O/D pair $w \in W$ and a fixed path $p \in P_w$:

$$(C_p(f^*) + \sum_{a \in L} h_a \tau_a^* \delta_{ap}) \times (x_p - x_p^*) \geq 0, \quad \forall x_p \in R_+. \quad (10)$$

Summing (10) over all paths p and O/D pairs w, and using (2), (3), and (4) yields:

$$\sum_{a\in L}(c_a(f^*)+h_a\tau_a^*)\times(f_a-f_a^*)\geq 0, \quad \forall f\in K. \tag{11}$$

From (6) and (7) one has, in turn, that, for a fixed link a:

$$(l_a^*-h_af_a^*)\times(\tau_a-\tau_a^*)\geq 0, \quad \forall \tau_a\in R_+, \tag{12}$$

and

$$(\rho^*-\tau_a^*)\times(l_a-l_a^*)\geq 0, \quad \forall l_a\in R_+. \tag{13}$$

Summing now (12) and (13) over all links a, one obtains:

$$\sum_{a\in L}(l_a^*-h_af_a^*)\times(\tau_a-\tau_a^*)+\sum_{a\in L}(\rho-\tau_a^*)\times(l_a-l_a^*)\geq 0,$$

$$\forall(\tau,l)\in R_+^{2n}. \tag{14}$$

From equilibrium conditions (8), one can conclude that:

$$\sum_{a\in L}(l_a^0-l_a^*)\times(\rho-\rho^*)\geq 0, \quad \forall \rho\in R_+. \tag{15}$$

Summing now (11), (14), and (15) yields variational inequality (9).

I now prove that a solution to the variational inequality problem (8) also satisfies the equilibrium conditions (5), (6), (7), and (8).

Let $(f^*, \tau^*, l^*, \rho^*) \in \mathcal{K}$ be a solution to (12). Let $f_a = f_a^*$, for all $a \in L$, $l_a = l_a^*$, for all $a \in L$, and $\rho = \rho^*$, and substitute these values in (9). One then obtains:

$$\sum_{a \in L} (l_a^* - h_a f_a^*) \times (\tau_a - \tau_a^*) \geq 0, \tag{16}$$

which implies the equilibrium conditions (6).

Similarly, if one lets $f_a = f_a^*$, for all $a \in L$, $\tau_a = \tau_a^*$, for all $a \in L$, and $\rho = \rho^*$, and substitutes these values in (9), one gets:

$$\sum_{a \in L} (\rho^* - \tau_a^*) \times (l_a - l_a^*) \geq 0, \tag{17}$$

which implies the equilibrium conditions (7).

Also, if one lets $l_a = l_a^*$, $\tau_a = \tau_a^*$, for all $a \in L$, and $\rho = \rho^*$, and substitutes these values into (9), one obtains:

$$\sum_{a \in L} (c_a(f^*) + h_a \tau_a^*) \times (f_a - f_a^*) \geq 0, \tag{18}$$

which implies the equilibrium conditions (5).

Finally, if one lets $f_a = f^*_a$, for all $a \in L$, $\tau_a = \tau^*_a$, and $l_a = l^*_a$, for all $a \in L$, then upon substitution into (9), one obtains:

$$\sum_{a \in L} (l^0_a - l^*_a) \times (\rho - \rho^*) \geq 0, \tag{19}$$

which implies the economic equilibrium conditions (8). This completes the proof of the theorem.

The variational inequality (9) is now put into standard form. Define column vector $X \equiv (f, \tau, l, \rho) \in \mathcal{K}$ and the column vector $F(X)$, where

$$F(X) \equiv (C(X), T(X), L(X), P(X)).$$

$C(X)$, $L(X)$, $T(X)$ are each n-dimensional vectors with component a given, respectively, as follows:

$$C_a(X) : c_a(f) + h_a \tau_a,$$

$$T_a(X) : l_a - h_a f_a,$$

$$L_a(X) : \rho - \tau_a,$$

whereas $P(X)$ is the one-dimensional vector with the single component:

$$P(X) : \sum_{a \in L} (l^0_a - l_a).$$

Thus, variational inequality (9) can be expressed as

$$\langle F(X^*)^T, X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{20}$$

I now turn to studying whether the equilibrium pattern is independent of the initial allocation of the licenses on the links and how to guarantee that the environmental emission standards imposed by the governing body are met in equilibrium.

The question as to whether the initial allocation of licenses affects the equilibrium pattern is answered in the following corollary.

Corollary 1 (Equilibrium Pattern Independence from Initial License Allocation)

If $l_a^0 \geq 0$ *for all* $a \in L$, *and* $\sum_{a \in L} l_a^0 = \bar{Q}$, *with* $\bar{Q}$ *fixed and positive, then the equilibrium pattern* $(f^*, \tau^*, l^*, \rho^*)$ *is independent of the initial allocation.*

Proof:

The terms in the variational inequality (9) are either independent of l_a^0 or depend only on the sum, $\sum_{a \in L} l_a^0$. The conclusion follows.

In the following proposition, it is shown that the environmental standards are met by the equilibrium pattern, provided that the sum of the initial allocation of licenses is equal to the imposed environmental standard given by $\bar{Q}$.

Proposition 1 (Achievement of Environmental Quality Standard)

If $\sum_{a\in L} l_a^0 = \bar{Q}$, *the equilibrium vector achieves the environmental quality standard,* $\bar{Q}$, *provided that the transportation network is viable.*

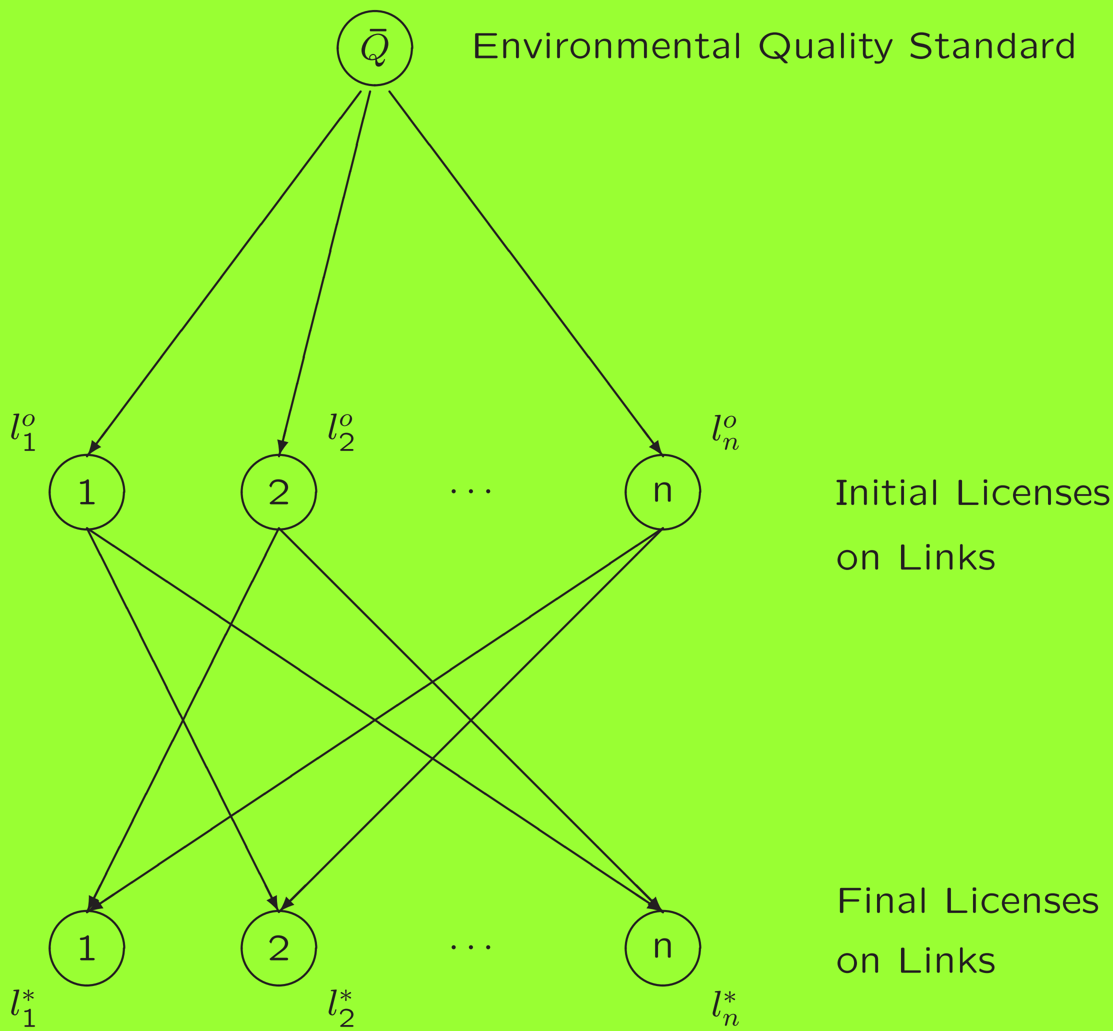
Proof:

One has from equilibrium conditions (6) and (8) that

$$\sum_{a\in L} h_a f_a^* \leq \sum_{a\in L} l_a^* \leq \sum_{a\in L} l_a^0 = \bar{Q}.$$

Hence, the environmental standards are met by the equilibrium pattern.

In the Figure, a graphical depiction is provided of the initial license allocation summing up to the environmental quality standard, and the final license holdings on the links. In the Table, further reinforcement is provided that sustainability is achieved under the pollution permit system.



Initial license allocation and final license holdings

Relationship between a viable user-optimized network, pollution permits, and sustainability

User-Optimized Viable Network $+$ **License Allocation:** $\sum_{i \in I} \sum_{a \in L_i} l_a^0 = \bar{Q}$	$\Rightarrow$	**Sustainability**

Qualitative Properties

I now present qualitative properties of the equilibrium. In particular, I provide existence and uniqueness results and also establish properties of the function $F(X)$ under which convergence of the algorithm discussed next is guaranteed.

Theorem 2 (Existence)

If $(f^*, \tau^*, l^*, \rho^*) \in \mathcal{K}$ satisfies the variational inequality (9), then the equilibrium link load vector f^* is a solution to the variational inequality problem:

$$\sum_{a \in L} c_a(f^*) \times (f_a - f_a^*) \geq 0, \quad \forall f \in K^1, \tag{21}$$

where

$$K^1 \equiv \{f, \textit{ such that there exist vectors } x \geq 0 \textit{ and }$$
$$l \geq 0 \textit{ so that } (3), (4); h_a f_a \leq l_a, \forall a \in L,$$
$$\textit{and } \sum_{a \in L} l_a^0 \geq \sum_{a \in L} l_a \textit{ are satisfied}\}.$$

A solution to (21) is guaranteed to exist since K^1 is compact and c is assumed to be continuous. Furthermore, there exist vectors: $l^* \in R^n_+$, $\tau^* \in R^n_+$, and $\rho^* \in R_+$ with $(f^*, \tau^*, l^*, \rho^*)$ being a solution to the variational inequality problem (9) and, hence, in equilibrium.

Proof:

I will prove this theorem by contradiction. Assume that (21) is not true, that is:

$$\sum_{a\in L} c_a(f^*) \times (f_a - f_a^*) < 0, \quad \text{for some } f \in K^1. \tag{22}$$

According to (9), one then has that

$$\sum_{a\in L} h_a\tau_a^* \times (f_a - f_a^*) + \sum_{a\in L}(l_a^0 - l_a^*) \times (\rho - \rho^*)$$

$$+\sum_{a\in L}(\rho^* - \tau_a^*) \times (l_a - l_a^*) + \sum_{a\in L}(l_a^* - h_a f_a^*) \times (\tau_a - \tau_a^*) \geq$$

$$-\sum_{a\in L} c_a(f^*) \times (f_a - f_a^*) > 0. \tag{23}$$

Letting τ_a and ρ equal zero, substitution into (23), after some algebraic manipulations, yields:

$$\sum_{a\in L}[l_a^0 - l_a][-\rho^*] + \sum_{a\in L}[l_a - h_a f_a][-\tau_a^*] > 0. \tag{24}$$

However, the inequality on the left-hand side of (24) is clearly nonpositive and, hence, a contradiction to the initial assumption. Thus, the variational inequality (21) must be satisfied.

Moreover, K^1 is compact, since $\sum_{a \in L} l_a \leq \sum_{a \in L} l_a^0$ and the travel demands are bounded. Thus, f and l lie in a compact set. Hence, the existence of f^* and l^* is guaranteed from the standard theory of variational inequalities (cf. Kinderlehrer and Stampacchia 1980).

Also, according to the Lagrange Multiplier Theorem, there exist multipliers $t^* \in R_+^n$ and $\rho^* \in R_+$ associated with the constraints of K^1 and these, together, with the (f^*, l^*) above must satisfy the variational inequality (9). This completes the proof of the theorem.

A uniqueness result is presented in the subsequent theorem.

Theorem 3 (Uniqueness)

Assume that the user link cost functions c are strictly monotone in f, that is:

$$\langle (c(f^1) - c(f^2))^T, f^1 - f^2 \rangle \geq 0, \quad \forall f^1, f^2 \in K, \quad f^1 \neq f^2. \tag{25}$$

Then the equilibrium link load pattern f^* is unique.

Proof:

Assume, on the contrary, that there are two distinct equilibrium patterns, denoted by X^1 and X^2. Then, both must satisfy the variational inequality (9). Hence, one must have that

$$\langle F(X^1)^T, X - X^1 \rangle \geq 0, \quad \forall X \in \mathcal{K} \tag{26}$$

and

$$\langle F(X^2)^T, X - X^2 \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{27}$$

Substituting X^2 for X in (26) and X^1 for X in (27) and adding the two resultant inequalities, after algebraic simplifications, yields:

$$\sum_{a \in L} (c_a(f^1) - c_a(f^2)) \times (f_a^2 - f_a^1)) \geq 0, \tag{28}$$

but due to the assumption of strict monotonicity this implies that this inequality must hold as an equality and, hence, one must have that $f^1 = f^2$.

Certain qualitative properties of the function $F(\cdot)$ are now investigated.

Lemma 1

Assume that the user link cost functions c are monotone increasing in the link loads, that is, for every $f^1, f^2 \in K$, we have that:

$$\langle (c(f^1) - c(f^2))^T, f^1 - f^2 \rangle \geq 0, \quad \forall f^1, f^2 \in K. \tag{29}$$

Then $F(X)$ is monotone, that is:

$$\langle (F(X^1) - F(X^2))^T, X^1 - X^2 \rangle \geq 0, \quad \forall X^1, X^2 \in \mathcal{K}. \tag{30}$$

Proof:

From the definition of $F(X)$ for this model, the left-hand side term of inequality (30) is given by

$$\sum_{a\in L}(c_a(f^1) - c_a(f^2)) \times (f_a^1 - f_a^2)$$

$$+\sum_{a\in L}(\tau_a^1 h_a - \tau_a^2 h_a) \times (f_a^1 - f_a^2) + \sum_{a\in L}((l_a^1 - h_a f^1) - (l_a^2 - h_a f_a^2))$$

$$\times(\tau_a^1 - \tau_a^2)$$

$$+\sum_{a\in L}((l_a^0 - l_a^1) - (l_a^0 - l_a^2)) \times (\rho^1 - \rho^2)$$

$$+\sum_{a\in L}((\rho^1 - \tau_a^1) - (\rho^2 - \tau_a^2)) \times (l_a^1 - l_a^2). \qquad (31)$$

After rearranging the terms and simplifying, (31) reduces to

$$\sum_{a\in L}(c_a(f^1) - c_a(f^2)) \times (f_a^1 - f_a^2), \qquad (32)$$

which by the assumption of monotonicity, is greater than or equal to zero. The proof is complete.

Theorem 4 (Lipschitz Continuity)

If the user link travel cost functions and the disutility functions have bounded first-order derivatives, then the function $F(X)$ is Lipschitz continuous, that is, there exists a positive constant $\bar{L}$, such that

$$\| F(X^1) - F(X^2) \| \leq \bar{L} \, \| X^1 - X^2 \|, \quad \forall X^1, X^2 \in \mathcal{K}. \quad (33)$$

Proof:

Follows along the same arguments as the proof of Lemma 3 in Nagurney (1994).

The Algorithm

The modified projection method of Korpelevich (1977) is proposed for the solution of variational inequality (9) governing the traffic network equilibrium model for pollution permits.

The algorithm resolves the problems into simpler variational inequality subproblems. Here, I show what form the modified projection method takes in the specific application.

Modified Projection Method for the Policy Model with Pollution Permits Satisfying Variational Inequality (9)

Step 0: Initialization

Set $(f^0, \tau^0, l^0 \rho^0) \in \mathcal{K}$. Let $\mathcal{T} = 1$ and set α such that $0 < \alpha \leq \frac{1}{\bar{L}}$, where $\bar{L}$ is the Lipschitz constant for the problem.

Step 1: Computation

Compute $(\bar{f}^{\mathcal{T}}, \bar{\tau}^{\mathcal{T}}, \bar{l}^{\mathcal{T}}, \bar{\rho}^{\mathcal{T}}) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\sum_{a \in L}(\bar{f}_a^{\mathcal{T}} + \alpha(c_a(f^{\mathcal{T}-1}) + h_a \tau_a^{\mathcal{T}-1}) - f_a^{\mathcal{T}-1}) \times (f_a - \bar{f}_a^{\mathcal{T}})$$

$$+ \sum_{a \in L}(\bar{\tau}_a^{\mathcal{T}} + \alpha(l_a^{\mathcal{T}-1} - h_a f_a^{\mathcal{T}-1}) - \tau_a^{\mathcal{T}-1}) \times (\tau_a - \bar{\tau}_a^{\mathcal{T}})$$

$$+ \sum_{a \in L}(\bar{l}_a^{\mathcal{T}} + \alpha(\rho^{\mathcal{T}-1} - \tau_a^{\mathcal{T}-1}) - l_a^{\mathcal{T}-1}) \times (l_a - \bar{l}_a^{\mathcal{T}})$$

$$+ (\bar{\rho}^{\mathcal{T}} + \alpha(\sum_{a \in L} l_a^0 - \sum_{a \in L} l_a^{\mathcal{T}-1}) - \rho^{\mathcal{T}-1}) \times (\rho - \bar{\rho}^{\mathcal{T}}) \geq 0,$$

$$\forall (f, \tau, l, \rho) \in \mathcal{K}. \tag{34}$$

Step 2: Adaptation

Compute $(f^{\mathcal{T}}, \tau^{\mathcal{T}}, l^{\mathcal{T}}, \rho^{\mathcal{T}}) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\sum_{a\in L}(f_a^{\mathcal{T}} + \alpha(c_a(\bar{f}^{\mathcal{T}}) + h_a\bar{\tau}_a^{\mathcal{T}}) - f_a^{\mathcal{T}-1}) \times (f_a - f_a^{\mathcal{T}})$$

$$+\sum_{a\in L}(\tau_a^{\mathcal{T}} + \alpha(\bar{l}_a^{\mathcal{T}} - h_a\bar{f}_a^{\mathcal{T}}) - \tau_a^{\mathcal{T}-1}) \times (\tau_a - \tau_a^{\mathcal{T}})$$

$$+\sum_{a\in L}(l_a^{\mathcal{T}} + \alpha(\bar{\rho}^{\mathcal{T}} - \bar{\tau}_a^{\mathcal{T}}) - l_a^{\mathcal{T}-1}) \times (l_a - l_a^{\mathcal{T}})$$

$$+(\rho^{\mathcal{T}} + \alpha(\sum_{a\in L} l_a^0 - \sum_{a\in L}\bar{l}_a^{\mathcal{T}}) - \rho^{\mathcal{T}-1}) \times (\rho - \rho^{\mathcal{T}}) \geq 0,$$

$$\forall (f, d, l, \tau, \rho) \in \mathcal{K}. \qquad (35)$$

Step 3: Convergence Verification

If $|f_a^{\mathcal{T}} - f_a^{\mathcal{T}-1}| \leq \epsilon$, $|l_a^{\mathcal{T}} - l_a^{\mathcal{T}-1}| \leq \epsilon$, $|\tau_a^{\mathcal{T}} - \tau_a^{\mathcal{T}-1}| \leq \epsilon$, for all $a \in L$, and $|\rho^{\mathcal{T}} - \rho^{\mathcal{T}-1}| \leq \epsilon$, with $\epsilon > 0$, a pre-specified tolerance, then stop; otherwise, set $\mathcal{T} := \mathcal{T} + 1$, and go to Step 1.

The decomposed subproblems in (34) and (35) can be computed very efficiently. Observe that since the feasible set $\mathcal{K}$ here is a Cartesian product, where $\mathcal{K} \equiv K \times R_+^{2n+1}$, the above variational inequality subproblems can be decomposed across K, which has the network structure of the problem, and across the coordinates of the nonnegative orthant.

Hence, (34), for example, yields the subproblem in K in link variables given by:

$$\text{Minimize}_{f \in K} \langle \bar{f}^{\mathcal{T}T}, \bar{f}^{\mathcal{T}} \rangle + \langle g^T, f^{\mathcal{T}} \rangle, \tag{36}$$

where $\bar{f}^{\mathcal{T}}$ is the column vector with component $a = \bar{f}_a^{\mathcal{T}}$, g is the column vector with component

$$g_a = \alpha(c_a(f^{\mathcal{T}-1}) + h_a \tau_a^{\mathcal{T}-1}) - f_a^{\mathcal{T}-1}).$$

The preceding subproblem is a quadratic programming problem or, equivalently, the optimization reformulation of the traffic network equilibrium conditions in the case of linear and separable user link travel cost functions.

Hence, this problem can be solved in many different ways.

A possible approach is given later in this lecture.

In addition, (34) yields subproblems in the marginal cost of emission abatement, the license, and the price variables, which are defined, in turn, on the nonnegative orthant and can be solved explicitly and exactly in closed form as follows.

For each link a, $a \in L$, compute the marginal cost of emission abatement according to:

$$\bar{\tau}_a^{\mathcal{T}} = \max\{0, -\alpha(l_a^{\mathcal{T}-1} - h_a f_a^{\mathcal{T}-1}) + \tau_a^{\mathcal{T}-1}\} \tag{37}$$

and for each link a, $a \in L$, compute the license thus:

$$\bar{l}_a^{\mathcal{T}} = \max\{0, -\alpha(\rho^{\mathcal{T}-1} - \tau_a^{\mathcal{T}-1}) + l_a^{\mathcal{T}-1}\}. \tag{38}$$

Finally, one can compute the license price as follows:

$$\bar{\rho}^{\mathcal{T}} = \max\{0, -\alpha(\sum_{a\in L} l_a^0 - \sum_{a\in L} l_a^{\mathcal{T}-1})\}. \tag{39}$$

The problem (35) also yields analogous subproblems to those above, where the first one is simply an optimization reformulation of the classical traffic network equilibrium problem with fixed travel demands, and the other subproblems can be solved exactly and in closed form in a similar manner as described above.

Convergence for the algorithm is given in the following theorem.

Theorem 5 (Convergence)

If the user link travel cost functions c are assumed to be monotone and have bounded first-order derivatives, then the modified projection method described above converges to the solution of the variational inequality (9).

Proof:

From Lemma 1, if c is monotone, one has that $F(X)$ is monotone. By Theorem 4, $F(X)$ is Lipschitz continuous. Hence, according to Korpelevich (1977), the modified projection method is guaranteed to converge.

Numerical Examples

Now, numerical examples are presented in order to illustrate both the model and the algorithm.

In particular, the modified projection method was implemented in FORTRAN and the system utilized for the numerical work was the IBM SP2 located at the Computer Science Department at the University of Massachusetts at Amherst.

For the solution of the standard traffic network equilibrium problem encountered in both the computation and adaptation steps (cf. (34) and (35)) I utilized the equilibration method (cf. Dafermos and Sparrow 1969).

The convergence criterion was given by $|x_p^{\mathcal{T}} - x_p^{\mathcal{T}-1}| \leq \epsilon$, for all $p \in P$, $|\tau_a^{\mathcal{T}} - \tau_a^{\mathcal{T}-1}| \leq \epsilon$, for all $a \in L$, $|l_a^{\mathcal{T}} - l_a^{\mathcal{T}-1}| \leq \epsilon$, for all $a \in L$, and $|\rho^{\mathcal{T}} - \rho^{\mathcal{T}-1}| \leq \epsilon$.

The modified projection method was initialized by setting the flow on a path equal to the travel demand for the O/D pair that the path belongs to divided by the number of paths. All other variables were initialized to zero.

Example 1

The first numerical example had the network topology depicted in the Figure.

The emission factors were given by: $h_a = 0.1$, $h_b = 0.2$, and $h_c = 0.3$, and the environmental quality standard $\bar{Q} = 1$ with a travel demand $d_{w_1} = 10$.

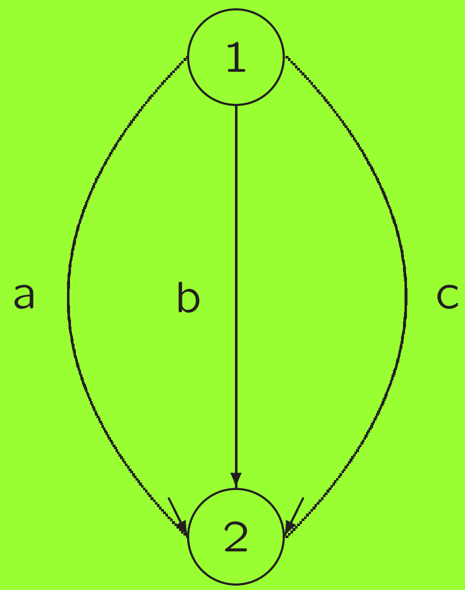
The user link travel cost functions were:

$$c_a(f_a) = 2f_a + 5, \quad c_b(f_b) = f_b + 8, \quad c_c(f_c) = 1.5f_c + 5.$$

I set the initial license allocations to $l_a^0 = l_b^0 = l_c^0 = \frac{1}{3}$ with $l_a^0 + l_b^0 + l_c^0 = 1 = \bar{Q}$. The convergence tolerance $\epsilon = .001$.

An application of the modified projection method with $\alpha = 0.7$ yielded the following solution:

$$f_a^* = 10.00, \quad f_b^* = 0.00, \quad f_c^* = 0.00.$$



Network topology for Examples 1 and 2

This equilibrium link load pattern was induced by the equilibrium path flow pattern:

$$x_{p_1}^* = 10.00, \quad x_{p_2}^* = 0.00, \quad x_{p_3}^* = 0.00.$$

The generalized user travel costs on the paths were: $\bar{C}_{p_1} = \bar{C}_{p_2} = \bar{C}_{p_3} = 41.99$.

The equilibrium licenses were:

$$l_a^* = 1.00, \quad l_b^* = 0.00, \quad l_c^* = 0.00.$$

The equilibrium marginal costs of emission abatement were:

$$\tau_a^* = 169.97, \quad \tau_b^* = 169.96, \quad \tau_c^* = 123.31.$$

The price of a license in equilibrium was:

$$\rho^* = 1.00.$$

In this example, the market for licenses cleared, that is, the excess supply of licenses was zero.

It is easy to verify that, indeed, equilibrium conditions (5)–(8) are satisfied by this flow, marginal cost, licenses, and price pattern. Moreover, the environmental standard was met since the total emissions $\sum_{a \in L} h_a f_a^* = 1$.

Example 2: Variant of Example 1

I then made the following change to Example 1. The environmental quality standard was relaxed from $\bar{Q} = 1$ to $\bar{Q} = 1.5$.

Recall that $\bar{Q} = 1.5$ is still lower than the total number of emissions emitted, which was equal to 2.1, if there is no permit system in place. The initial license allocation was as follows: $l_a^0 = 0.25$, $l_b^0 = 0.25$, and $l_c^0 = 1.00$. We set $\alpha = 0.4$.

An application of the modified projection method yielded the following solution:

$$f_a^* = 5.75, \quad f_b^* = 3.39, \quad f_c^* = 0.85.$$

This equilibrium link load pattern was induced by the equilibrium path flow pattern:

$$x_{p_1}^* = 5.75, \quad x_{p_2}^* = 3.39, \quad x_{p_3}^* = 0.85.$$

The generalized user travel costs on the paths were: $\bar{C}_{p_1} = \bar{C}_{p_2} = \bar{C}_{p_3} = 21.62$.

The equilibrium licenses were:

$$l_a^* = 0.57, \quad l_b^* = 0.68, \quad l_c^* = 0.25.$$

The equilibrium marginal costs were:

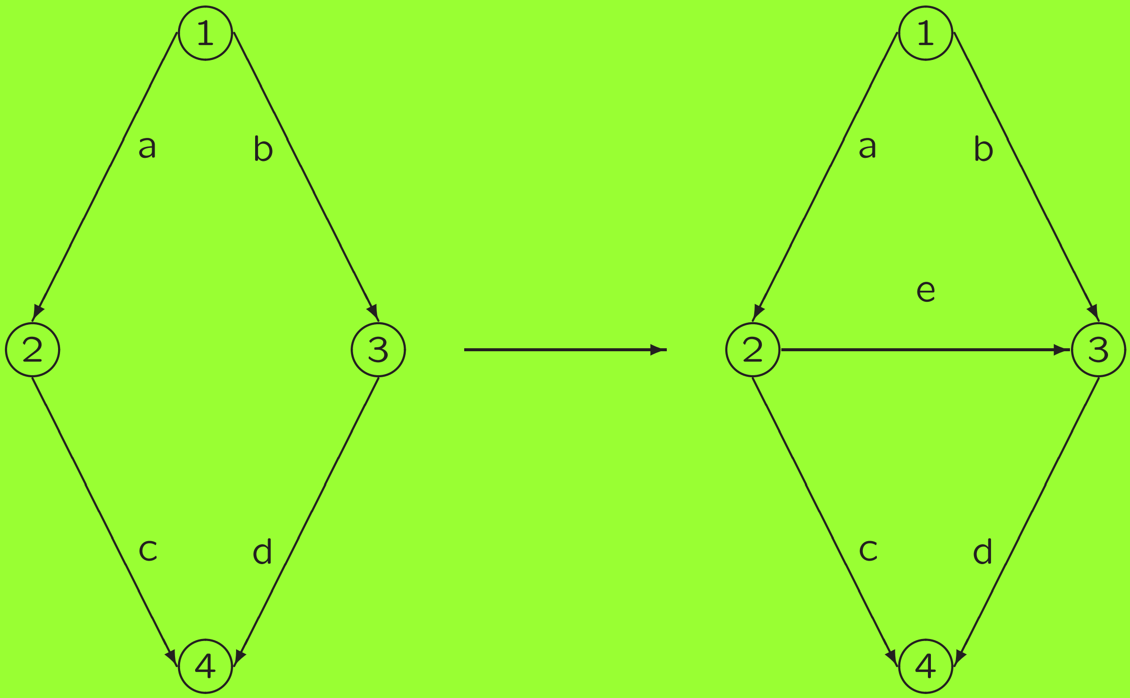
$$\tau_a^* = 51.15, \quad \tau_b^* = 51.15, \quad \tau_c^* = 51.15.$$

The price of a license in equilibrium was:

$$\rho^* = 51.15.$$

In this example, the market for licenses cleared, that is, the excess supply of licenses was zero.

It is easy to verify that, indeed, equilibrium conditions (5)–(8) are satisfied by this flow, marginal cost, licenses, and price pattern. Moreover, the environmental standard was met since the total emissions $\sum_{a \in L} h_a f_a^* = 1.5$.



Network topology for Example 3: Braess network

Example 3

Assume a network as depicted in the Figure in which there are five links: a, b, c, d, e; four nodes: 1, 2, 3, 4, and a single O/D pair $w_1 = (1, 4)$. There are, hence, three paths available to travelers between this O/D pair: $p_1 = (a, c)$, $p_2 = (b, d)$, and $p_3 = (a, e, c)$.

The link travel cost functions are:

$$c_a(f_a) = 10f_a, \quad c_b(f_b) = f_b + 50,$$

$$c_c(f_c) = f_c + 50, \quad c_d(f_d) = 10f_d, \quad c_e(f_e) = f_e + 10.$$

Assume a fixed travel demand $d_{w_1} = 6$.

The emission factors were set to $h_a = 0.1$, for all $a \in L$.

I utilized $\alpha = 0.1$ in the modified projection method for this example and allocated the initial licenses as follows: $l_a^0 = l_b^0 = l_c^0 = l_d^0 = l_e^0 = 0.24$ with the total environmental quality standard being equal to 1.2 which was the level in the original network before the addition of the new road e.

The convergence tolerance ϵ was set to .00001.

The modified projection method yielded the following equilibrium pattern:

$$f_a^* = 3.00, \quad f_b^* = 3.00, \quad f_c^* = 3.00,$$

$$f_d^* = 3.00, \quad f_e^* = 0.00.$$

This equilibrium link load pattern was induced by the equilibrium path flow pattern:

$$x_{p_1}^* = 3.00, \quad x_{p_2}^* = 3.00, \quad x_{p_3}^* = 0.00.$$

The generalized user travel costs on the used paths were:

$$\bar{C}_{p_1} = \bar{C}_{p_2} = \bar{C}_{p_3} = 108.98.$$

The equilibrium licenses were:

$$l_a^* = 0.30, \quad l_b^* = 0.30, \quad l_c^* = 0.30,$$

$$l_d^* = 0.30, \quad l_e^* = 0.00.$$

The equilibrium marginal costs of emission abatement were:

$$\tau_a^* = 129.88, \quad \tau_b^* = 126.07, \quad \tau_c^* = 129.88,$$

$$\tau_d^* = 129.88, \quad \tau_e^* = 129.88.$$

The price of a license in equilibrium was:

$$\rho^* = 129.88.$$

In this example, the market for licenses cleared, that is, the excess supply of licenses was zero. The total calculated emissions were 1.2, which was precisely the imposed environmental quality standard.

Note that the use of pollution permits precludes the utilization of the new path p_3 since to do so would increase the emissions over and above the desired environmental quality standard.

Example 4

I then considered the network in the following figure which consists of ten nodes, thirteen links and two O/D pairs: $w_1 = (1, 8)$ and $w_2 = (2, 10)$ with travel demands $d_{w_1} = 5$ and $d_{w_2} = 5$.

The user link travel cost functions were:

$$c_1(f) = .00005f_1^4 + 5f_1 + 2f_2 + 5,$$

$$c_2(f) = .00003f_2^4 + 4f_2 + f_1 + 2,$$

$$c_3(f) = .00005f_3^4 + 3f_3 + f_4 + 3,$$

$$c_4(f) = .00003f_4^4 + 6f_4 + 3f_5 + 4,$$

$$c_5(f) = 4f_5 + f_{12} + 8,$$

$$c_6(f) = .00007f_6^4 + 7f_6 + 4f_{12} + 6,$$

$$c_7(f) = 8f_7 + 2f_{13} + 7,$$

$$c_8(f) = .00001f_8^4 + 7f_8 + 3f_{12} + 6,$$

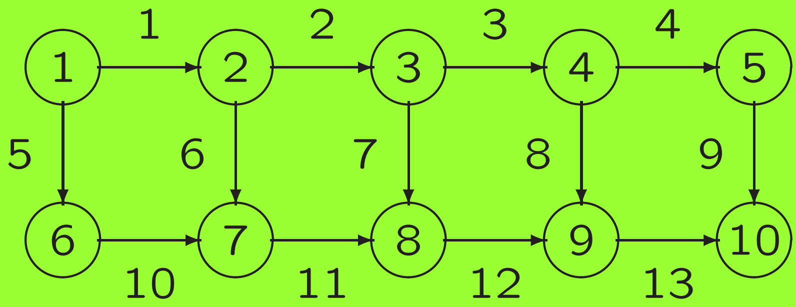
$$c_9(f) = 8f_9 + 3f_{11} + 5,$$

$$c_{10}(f) = .00003f_{10}^4 + 6f_{10} + f_1 + 3,$$

$$c_{11}(f) = .00004f_{11}^4 + 4f_{11} + f_2 + 4,$$

$$c_{12}(f) = .00002f_{12}^4 + 6f_{12} + f_1 + 5,$$

$$c_{13}(f) = .00003f_{12}^4 + 9f_{13} + 2f_4 + 3.$$



Network topology for Example 4

The paths were:

For O/D pair w_1:

$$p_1 = (1, 2, 7), \quad p_2 = (1, 6, 11), \quad p_3 = (5, 10, 11)$$

and for O/D pair w_2:

$$p_4 = (2, 3, 4, 9), \quad p_5 = (2, 3, 8, 13), \quad p_6 = (2, 7, 12),$$

$$p_7 = (6, 11, 12, 13).$$

The parameter α in the modified projection method was set to 0.1 and ϵ was set to .0001.

I set the emission parameters $h_a = 0.5 \times a$, for all $a \in L$.

The initial licenses were set as: $l_a^0 = a$, for all links $a \in L$.

The modified projection method yielded the equilibrium link load pattern given by:

$$f_1^* = 3.88, \quad f_2^* = 7.36, \quad f_3^* = 5.00, \quad f_4^* = 3.51,$$

$$f_5^* = 1.12, \quad f_6^* = 1.52, \quad f_7^* = 2.36, \quad f_8^* = 1.49,$$

$$f_9^* = 3.51, \quad f_{10}^* = 1.12, \quad f_{11}^* = 2.64, \quad f_{12}^* = 0.00,$$

$$f_{13}^* = 1.49,$$

with an equilibrium path flow pattern:

For O/D pair w_1:

$$x_{p_1}^* = 2.36, \quad x_{p_2}^* = 1.52, \quad x_{p_3}^* = 1.12,$$

and for O/D pair w_2:

$$x_{p_4}^* = 3.51, \quad x_{p_5}^* = 1.49, \quad x_{p_6}^* = 0.00, \quad x_{p_7}^* = 0.00,$$

and with generalized user travel costs:

For O/D pair w_1:

$$\bar{C}_{p_1} = \bar{C}_{p_2} = \bar{C}_{p_3} = 140.43,$$

and for O/D pair w_2:

$$\bar{C}_{p_4} = \bar{C}_{p_5} = 193.14, \quad \bar{C}_{p_6} = 203.16, \quad \bar{C}_{p_7} = 203.16.$$

The computed licenses were:

$$l_1^* = 1.93, \quad l_2^* = 7.36, \quad l_3^* = 7.50, \quad l_4^* = 7.02,$$

$$l_5^* = 2.80, \quad l_6^* = 4.56, \quad l_7^* = 8.26, \quad l_8^* = 5.95,$$

$$l_9^* = 15.81, \quad l_{10}^* = 5.61, \quad l_{11}^* = 14.52, \quad l_{12}^* = 0.00,$$

$$l_{13}^* = 9.67.$$

The equilibrium price was $\rho^* = 7.41$, which was also the value of all the marginal costs on the links with positive license holdings. The equilibrium marginal cost on the link with zero license holdings was:

$$\tau_{12}^* = 7.41.$$

The market for licenses also cleared for this example. The total emissions generated were precisely equal to the environmental quality standard $\bar{Q} = 91$.

The results in this lecture are from Nagurney (1999c). Nagurney (1999d) presents alternative permit schemes by O/D pairs and by paths. Additional relevant citations, in addition to the text, are given below.

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