Emission Pricing for Sustainability – Permits for User-Optimized, Fixed Demand Networks

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Introduction

In this lecture, the focus is on policy instruments for sustainable transportation networks when the travelers behave in a user-optimized fashion.

Here it is assumed that the networks are viable in that they satisfy the appropriate definition of viability for the given situation.

I first consider traffic networks with given origin/destination pairs and fixed travel demands, and provide an emission pricing scheme which guarantees that the network will be sustainable, that is, that the environmental quality standard will be met and that the traffic flow pattern will be in equilibrium.

I then extend the pricing concept for the case of Situations 1 through 3, as previously discussed.

Subsequently, I turn to elastic demand traffic network problems and provide a mechanism for emission pricing for such networks which guarantees sustainability, that is, that the environmental quality standard is met and the traffic flow pattern is in equilibrium.

I also discuss a computational procedure which can be applied to compute the equilibrium patterns in both the fixed demand and the elastic demand traffic network policy models presented in this lecture.

Summary of policy instruments for sustainability of U-O networks developed in Chapter 5

	Fixed demand networks	
Emission pricing for:	Baseline:	• Known O/D pairs and demands
	Situation 1:	• Known origin nodes and trip productions
	Situation 2:	• Known destination nodes and trip attractions
	Situation 3:	• Known origin nodes, destination nodes, and total number of trips
	Elastic demand networks	
Emission pricing for:	Baseline:	• Known O/D pairs

Pricing for Sustainable Fixed Demand Networks

I now develop a traffic network policy model which guarantees sustainability of the network in question, under the assumption that the travelers behave in a user-optimized manner.

The policy that is presented is that of emission pricing but, unlike some other tolls, here it is guaranteed that the tolls are equitable, in that the travelers pay according to their emissions.

In order to fix ideas, a simplified model is first presented, which is classical in the sense that the user link travel cost functions are assumed to be separable.

From the optimality conditions for this problem, I then argue that they are also appropriate as equilibrium conditions for the more general case in which the user travel cost function on each link may depend not only on the flow on a particular link but also on the flows on other links in the network.

Notation

The notation utilized in this lecture is the same as in previous lectures of this course, except where noted.

A Simple Model

Consider a traffic network consisting of the graph $G = [N, L]$, where N denotes the set of nodes and L the set of links.

It is assumed, as given, a vector of travel demands d associated with the origin/destination pairs. Also, I consider the "classical" form of the user link travel cost functions, due to Beckmann, McGuire, and Winsten (1956), where the user travel cost on link a, denoted by c_a, is separable, that is:

$$c_a = c_a(f_a), \quad \forall a \in L. \tag{1}$$

Moreover, I assume that this function is increasing in the flow for each link in the network.

The travel cost on a path p, hence, is equal to the sum of the travel costs on links that comprise that path, that is:

$$C_p = \sum_{a \in L} c_a(f_a)\delta_{ap}, \quad \forall a \in L. \tag{2}$$

Recall that, in the case of such functions, the traffic network equilibrium conditions, which state that, in equilibrium, all used paths for each O/D pair have equal and minimal travel costs, can be reformulated as a solution to an optimization problem, where the objective function is given by:

$$\text{Minimize} \quad \sum_{a \in L} \int_0^{f_a} c_a(x)dx. \qquad (3)$$

In the simple pricing model for sustainability one retains, hence, the objective function of the classical traffic network equilibrium model, as well as the constraints, but now one adds the environmental quality constraint.

After making the substitution for the link load $f_a = \sum_{p \in P} x_p \delta_{ap}$ in the objective function (in order to simplify the derivation), one obtains the following problem:

$$\text{Minimize} \quad \sum_{a \in L} \int_0^{\sum_{p \in P} x_p \delta_{ap}} c_a(x) dx \tag{4}$$

subject to:

$$\sum_{p \in P_w} x_p = d_w, \quad \forall w \in W, \tag{5}$$

$$\sum_{a \in L} h_a \sum_{p \in P} x_p \delta_{ap} \leq \bar{Q}, \tag{6}$$

$$x_p \geq 0, \quad \forall p \in P. \tag{7}$$

Observe that conditions (5)–(7) correspond precisely to Linear System 1, the existence of a solution to which guarantees viability of a transportation network with given O/D pairs and travel demands.

Optimality Conditions

I now derive the optimality conditions for the optimization problem given by (4)–(7).

Since the user travel cost functions are increasing functions of the flow, the objective function is convex and the constraints, which are linear, are also convex. Hence, the Kuhn-Tucker optimality conditions can be stated as follows: $x^* \in R_+^{n_P}$ is an optimal solution if it satisfies the travel demands and satisfies the following system of equalities and inequalities: For each O/D pair $w \in W$ and each path $p \in P_w$:

$$\bar{C}_p(x^*, \tau^*) = C_p(x^*) + \tau^* \sum_{a \in L} h_a \delta_{ap} \begin{cases} = & \lambda_w, \quad \text{if } x_p^* > 0 \\ \geq & \lambda_w, \quad \text{if } x_p^* = 0, \end{cases} \tag{8}$$

where τ^* is the Lagrange multiplier associated with the environmental quality constraint (5) with τ^* having the interpretation here as being the marginal cost of emission abatement.

In addition, one must have that:

$$\bar{Q} - \sum_{a \in L} h_a \sum_{p \in P} x_p^* \delta_{ap} \begin{cases} = & 0, \quad \text{if } \tau^* > 0 \\ \geq & 0, \quad \text{if } \tau^* = 0. \end{cases} \tag{9}$$

Note that $\bar{C}_p$ in (8) denotes the *generalized* cost associated with traveling now on path p after the imposition of the tolls, where the toll on a path p is equal to $\tau^* \sum_{a \in L} h_a \delta_{ap}$.

Hence, the higher the emission factors on a utilized path, the higher the toll payment for travelers on that path.

This optimization problem can be solved by any general purpose convex programming algorithm. However, later, I will present an algorithm which can be applied to solve all the models presented in this lecture.

Emission Pricing Policies

According to (8) and (9), the following emission toll policies satisfy the equilibrium conditions:

Link Pricing Policy

The link pricing policy given by $t_a = \tau^* h_a$, $\forall a \in L$, where τ^* is the equilibrium marginal cost of emission abatement and t_a denotes the toll on link a, guarantees that the transportation network is sustainable.

Path Pricing Policy

The path pricing policy given by $t_p = \tau^* \sum_{a \in L} h_a \delta_{ap}$ for all $p \in P$, where t_p denotes the toll on path p guarantees that the network will be sustainable.

An example is now presented, in which the tolls can be solved for explicitly, as well as the equilibrium flow pattern.

Example 1

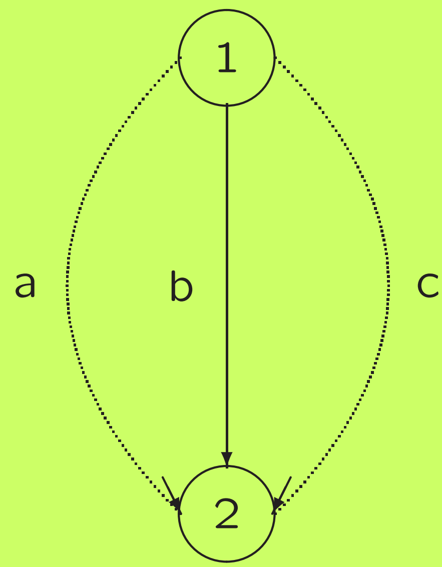
The network topology is depicted in the Figure. The network consist of two nodes, denoted by 1 and 2; three links, denoted by a, b, and c, and a single O/D pair $w_1 = (1, 2)$. Let $p_1 = a$, $p_2 = b$, and $p_3 = c$. The travel demand $d_{w_1} = 10$. The user link travel cost functions are:

$$c_a(f_a) = 2f_a + 5,\ c_b(f_b) = f_b + 8,\ c_c(f_c) = 1.5f_c + 5.$$

The emissions are: $h_a = 0.1$, $h_b = 0.2$, and $h_c = 0.3$, with the environmental quality standard $\bar{Q} = 1$.

One is interested in determining a pricing policy, as described above.

First, note that the viable solution $x_{p_1} = 10$, and $x_{p_2} = x_{p_3} = 0$ is, in fact, the only feasible solution with conditions (9) holding as an equality.



Network topology for Example 1

Hence, one has, from conditions (8), that for path p_1, which is the only path that will be utilized, that

$$\hat{C}_{p_1} = c_a + \tau^*.1 = 2f_a^* + 5 + \tau^*.1 = 25 + \tau^*.1.$$

Moreover, since paths p_2 and p_3 will not be utilized, one has also that:

$$\bar{C}_{p_2} = c_b + \tau^*.2 = f_b + 8 + \tau^*.2 = 8 + \tau^*.2 \geq \bar{C}_{p_1} = 25 + \tau^*.1.$$

In addition, one has that:

$$\bar{C}_{p_3} = c_c + .3\tau^* = 1.5f_c + 5 + .3\tau^* = 5 + .3\tau^* \geq \bar{C}_{p_1} = 25 + \tau^*.1.$$

But these two inequalities imply that:

$$17 \leq .1\tau^* \quad \text{and} \quad 20 \leq .2\tau^*,$$

or that $\tau^* \geq 170$.

Letting then $\tau^* = 170$, one obtains the following link emission toll policy: $t_a = h_a \tau^* = 17$, $t_b = h_b \tau^* = 34$, and $t_c = h_c \tau^* = 51$, yielding generalized path travel costs of:

$$\bar{C}_{p_1} = 42, \quad \bar{C}_{p_2} = 42, \quad \bar{C}_{p_3} = 56.$$

Note that one has equity here since the travelers who travel on path p_1 do now pay a positive toll according to their emissions.

The General Model

Observe that conditions (8) and (9), although derived from an optimization problem, may be interpreted as equilibrium conditions.

I now consider these conditions as equilibrium conditions, and present a generalized version of the preceding model, which considers user link travel cost functions which are no longer separable.

Moreover, the equilibrium conditions will be shown to satisfy a variational inequality problem.

Indeed, I now consider user link travel cost functions, which may, in general, depend upon the entire vector of link loads; that is, I assume that now

$$c_a = c_a(f), \quad \forall a \in L. \tag{10}$$

Definition 1 (Traffic Network Equilibrium in the Presence of Emission Tolls)

Given O/D pairs and fixed travel demands, in the presence of emission tolls and user-optimized behavior, a path flow pattern and the marginal cost of emission abatement (x^*, τ^*) *is said to be in equilibrium if it satisfies equilibrium conditions (8) and (9). Moreover, such a traffic network is sustainable.*

I now establish that the solution to the systems of equations and inequalities (8) and (9) satisfies a variational inequality problem.

Theorem 1 (Variational Inequality Formulation of Traffic Network Equilibrium in the Presence of Emission Tolls)

A traffic flow pattern and marginal cost of emission abatement $(x^*, \tau^*) \in \mathcal{K}^1$ *is an equilibrium of the traffic network toll policy model described above if and only if it is a solution to the variational inequality problem:*

Path Flow Formulation:

$$\sum_{w \in W} \sum_{p \in P_w} \left[C_p(x^*) + \tau^* \sum_{a \in L} h_a \delta_{ap} \right] \times \left[x_p - x_p^* \right]$$

$$+ \left[\bar{Q} - \sum_{a \in L} h_a \sum_{p \in P} x_p^* \delta_{ap} \right] \times [\tau - \tau^*] \geq 0, \quad \forall (x, \tau) \in \mathcal{K}^1, \tag{11}$$

where $\mathcal{K}^1 \equiv \bar{K}^1 \times R_+^1$ *and* $\bar{K}^1 \equiv \{x | x \geq 0$ *and satisfies* $(5)\}$, *or, equivalently,* $(f^*, \tau^*) \in \mathcal{K}^2$ *is an equilibrium link load and marginal cost of emission abatement pattern if and only if it satisfies the variational inequality problem:*

Link Load Formulation:

$$\sum_{a \in L} [c_a(f^*) + \tau^* h_a] \times [f_a - f_a^*]$$

$$+ \left[\bar{Q} - \sum_{a \in L} h_a f_a^* \right] \times [\tau - \tau^*] \geq 0, \quad \forall (f, \tau) \in \mathcal{K}^2, \tag{12}$$

where $\mathcal{K}^2 \equiv \bar{K}^2 \times R_+^1$, *and* $\bar{K}^2 \equiv \{f|$ *there exists an* $x \geq 0$ *satisfying* $(5)\}$.

Proof:

I first establish that if a path flow pattern and marginal cost of emission abatement are in equilibrium, that is, they satisfy equilibrium conditions (8) and (9), then this pattern also satisfies variational inequality (11), equivalently, variational inequality (12).

Note that from (8) one has that, for a fixed O/D pair w and a fixed path $p \in P_w$:

$$\left[C_p(x^*) + \tau^* \sum_{a \in L} h_a \delta_{ap} - \lambda_w\right] \times \left[x_p - x_p^*\right] \geq 0. \qquad (13)$$

Indeed, since if $x_p^* > 0$, then the left-hand side of inequality (13) must be precisely equal to zero, in which case (13) must hold. On the other hand, if $x_p^* = 0$, then $\left[x_p - x_p^*\right] \geq 0$, since the path flows must be nonnegative, and the left-hand side of (13) is also nonnegative due to (8) and, hence, since the product of two nonnegative terms is also nonnegative, the inequality in (13) is also satisfied.

Furthermore, since (13) holds for all paths connecting O/D pair w, we can sum the inequality in (13) over all such paths, that is:

$$\sum_{p \in P_w} \left[C_p(x^*) + \tau^* \sum_{a \in L} h_a \delta_{ap} - \lambda_w\right] \times \left[x_p - x_p^*\right] \geq 0, \qquad (14)$$

but since the path flows must sum up to the travel demands (cf. (5)), (14) reduces to:

$$\sum_{p \in P_w} \left[C_p(x^*) + \tau^* \sum_{a \in L} h_a \delta_{ap} \right] \times \left[x_p - x_p^* \right] \geq 0. \tag{15}$$

But (15) also holds for any O/D pair and, hence, summing over all O/D pairs, yields the inequality:

$$\sum_{w \in W} \sum_{p \in P_w} \left[C_p(x^*) + \tau^* \sum_{a \in L} h_a \delta_{ap} \right] \times \left[x_p - x_p^* \right] \geq 0,$$

$$\forall x \in \bar{K}^1. \tag{16}$$

Using now the relationships between the path travel cost and the link travel costs (see (2)), as well as the path flow pattern and the link load pattern, inequality (16) is equivalent, in link loads, to the inequality:

$$\sum_{a \in L} [c_a(f^*) + \tau^* h_a] \times [f_a - f_a^*] \geq 0, \quad \forall f \in \bar{K}^2. \tag{17}$$

I now turn to equilibrium conditions (9) and, arguing as above, but for τ^*, note that these equilibrium conditions imply the inequality:

$$\left[\bar{Q} - \tau^* \sum_{a \in L} h_a \sum_{p \in P} x_p^* \delta_{ap}\right] \times [\tau - \tau^*] \geq 0, \quad \forall \tau \in R_+^1. \quad (18)$$

Summing now inequality (16) and (18), one obtains variational inequality (11) (which is in path flows since path flows are the variables). Similarly, summing inequalities (17) and (18), and using the relationship: $f_a = \sum_{p \in P} x_p \delta_{ap}$, $\forall a \in L$, one obtains variational inequality (12).

I now show that a solution to variational inequality (11) and, equivalently, a solution to variational inequality (12), satisfies the equilibrium conditions (8) and (9).

Indeed, consider first variational inequality (11), and make the following substitutions: Let $\tau = \tau^*$, and let $x_p = x_p^*$, for all paths $p \in P_\omega$, for all $\omega \neq w$. Further, for paths $p \in P_w$, let r denote some path such that $x_r > 0$ (we know that such a path must exist since the travel demand for each O/D pair is positive), and select some path $q \in P_w$; for all paths $p \neq q \subset r$, let $x_p = x_p^*$. Let $x_r = x_r^* - \delta$, for some small $\delta > 0$, and let $x_q = x_q^* + \delta$.

Clearly, such a constructed path flow pattern is feasible. Substitution into variational inequality (11), after some algebraic simplifications, yields then:

$$C_q(x^*) + \tau^* \sum_{a \in L} h_a \delta_{aq} \leq C_r(x^*) + \tau^* \sum_{a \in L} h_a \delta_{ar}. \tag{19}$$

Now, if $x_q^* > 0$, one can construct an analogous path flow (and marginal cost of emission abatement pattern) but with a δ reallocation from path q to path r yielding:

$$C_q(x^*) + \tau^* \sum_{a \in L} h_a \delta_{aq} \geq C_r(x^*) + \tau^* \sum_{a \in L} h_a \delta_{ar}. \tag{20}$$

However, in order for (19) and (20) to hold simultaneously, one must have that they hold with an equality and, thus, I have shown the first part of equilibrium conditions, that is, if the flows on paths are positive then their generalized path travel costs must be equal.

On the other hand, inequality (19) holds even if the path flow on q is zero (in this case one cannot make a construction such as (11) since it would mean that the resulting flow on q would be negative and, hence, infeasible), which implies the second condition of (8), that is, that unused paths or those with zero flow have generalized path travel costs that cannot be less than those on used paths.

In order to establish that a solution to variational inequality (11) also satisfies equilibrium conditions (9), I make the following construction: set $x_p = x_p^*$, for all paths $p \in P$, and substitute into variational inequality (11), which yields:

$$\left[\bar{Q} - \tau^* \sum_{a \in L} h_a \sum_{p \in P} x_p^* \delta_{ap}\right] \times [\tau - \tau^*] \geq 0, \ \forall \tau \in R_+^1. \quad (21)$$

It is easy to see that (21) implies equilibrium conditions (9), since if $\tau^* = 0$, one knows that $\tau - \tau^* = \tau \geq 0$, and for the entire expression to be nonnegative, it must be that $\bar{Q} - \tau^* \sum_{a \in L} h_a \delta_{ap} \geq 0$. On the other hand, if $\tau^* > 0$, then since the inequality must hold for any $\tau \geq 0$, one can let $\tau = \tau^* + \delta$ and substitute the resultant into (21), yielding:

$$\left[\bar{Q} - \tau^* \sum_{a \in L} h_a \sum_{p \in P} x_p^* \delta_{ap}\right] \times \delta \geq 0. \quad (22)$$

Similarly, one can let $\tau = \tau^* - \delta$ for the same small $\delta > 0$, in which case substitution into (21) yields:

$$\left[\bar{Q} - \tau^* \sum_{a \in L} h_a \sum_{p \in P} x_p^* \delta_{ap}\right] \delta \leq 0. \quad (23)$$

However, for both (22) and (23) to hold, one must conclude that for $\tau^* > 0$:

$$\bar{Q} - \tau^* \sum_{a \in L} h_a \sum_{p \in P} x_p^* \delta_{ap} = 0. \tag{24}$$

I have, thus, also verified the second part of equilibrium conditions (8).

Hence, the proof of equivalence in the case of the path flow variational inequality (11) is complete. What remains to be shown only is that a solution to the link load formulation (12) also satisfies equilibrium conditions (8), since in the course of establishing the proof for (11), I have already established, since $f_a = \sum_{p \in P} x_p \delta_{ap}$, $\forall a \in L$, that (9) must hold. But the first term of variational inequality (12) is derived from the first term of variational inequality (11) and, thus, the conclusion must follow.

I now put both variational inequality problem (11) and (12) into standard form. I first consider variational inequality (11), which is in path flow variables, and define the column vector $X \equiv (x, \tau) \in \mathcal{K}^1$ and the column vector $F(X)$, where

$$F(X) \equiv (C(X), T(X)).$$

$C(X)$ is the n_P-dimensional vector with component p given as follows:

$$C_p(X) : C_p(x) + \tau \sum_{a \in L} h_a \delta_{ap},$$

whereas $T(X)$ is the one-dimensional vector with component:

$$T(X) : \bar{Q} - \sum_{a \in L} h_a \sum_{p \in P} x_p \delta_{ap}.$$

Clearly, the variational inequality (11) can now be put into standard form where $\mathcal{K} \equiv \mathcal{K}^1$.

Next, consider variational inequality (12), which is in link load variables and define the column vector $X \equiv (f, \tau)$ and the column vector $F(X)$, where

$$F(X) \equiv (c(X), T(X)).$$

$c(X)$ is the n-dimensional vector with component a given by:

$$c_a(X) = c_a(f) + h_a \tau,$$

whereas $T(X)$ is as defined for the variational inequality in path flow variables above.

Pricing in Alternative Situations

Here I develop pricing policies and address transportation networks in the alternative situations.

Furthermore, I introduce, for the first time, the concept of an *expanded* or supernetwork, which is based on the original transportation network in the given situation but which allows one to determine not only the equilibrium path flow akin to those in the networks with fixed travel demands, but also the travel demands themselves.

It is assumed that the given networks are viable according to the respective definitions. The notion of an expanded network was also utilized in Nagurney (1999a) to transform multimodal networks under distinct situations into a multimodal traffic network equilibrium problem with known O/D pairs and travel demands.

That work was an extension of the idea developed for single modal networks by Dafermos (1976) who called such network problems *integrated* traffic network equilibrium problems.

The models therein, however, in contrast to those described here, were not concerned with sustainability issues in an environmental setting. See also the recent book on supernetworks by Nagurney and Dong (2002).

Emission Pricing for Networks with Known Origins and Trip Productions

Consider now traffic networks in Situation 1 in which the origins are known as well as the trip productions.

Assume user-optimized behavior on such networks and a general user link travel cost function on each link $a \in L$ given by (10).

I first state the equilibrium conditions for a network in such a situation (in the absence of any policies such as tolls).

I then provide a construction of the network in expanded form which allows one to reduce either the equilibrium problem without or with tolls to the corresponding problem with known O/D pairs and travel demands.

Definition 2 (Equilibrium Conditions for User-Optimization in Situation 1)

A traffic flow pattern on a transportation network in Situation 1 is said to be in equilibrium if all used paths emanating from each origin node are equal and minimal, that is, one has that for each $y \in Y$, and $p \in P_y$:

$$C_p(x^*) \begin{cases} = & \lambda_y, \quad \text{if } x_p^* > 0 \\ \geq & \lambda_y, \quad \text{if } x_p^* = 0, \end{cases} \tag{25}$$

where $\sum_{p \in P_{O_y}} x_p^* = O_y$, $\forall y \in Y$.

I now need to introduce the expanded or supernetwork concept in this situation, that is, when one is given known origins and trip productions.

Construction of Expanded Network for Situation 1

I do the network construction as follows: Construct a "super" demand or sink node, denoted by ψ, and from each destination node construct then a single link terminating in the destination node ψ.

Associate with each link a in the original network a user travel cost c_a, which is assumed to take on the general form given by (10). Associate with each of the artificial links terminating in node ψ a user travel cost equal to zero.

I now denote by $\hat{p}$ the path, which consists of the links in path p plus the artificial link terminating in ψ.

In addition, define then the O/D pairs on the expanded network consisting, respectively, of each of the original origin nodes y and with destination node ψ. Associate with each such O/D pair $\hat{w}$ the corresponding "travel demand," given by O_y.

Note that, in the model with known origins and trip productions, it is path travel costs on paths emanating from each origin to all the destinations that are equalized for all used paths.

Equilibrium conditions (8) and (9) are then applied to the expanded network. Equivalently, in order to obtain the equilibrium path flows and marginal cost of emission abatement, one can solve variational inequality (11) or (12). Note that, once, the solution is obtained to the pricing problem on the expanded network, one can then recover the travel demands to the original network in Situation 1, by simply summing the resulting path flows on the paths between each origin y and the respective destination nodes.

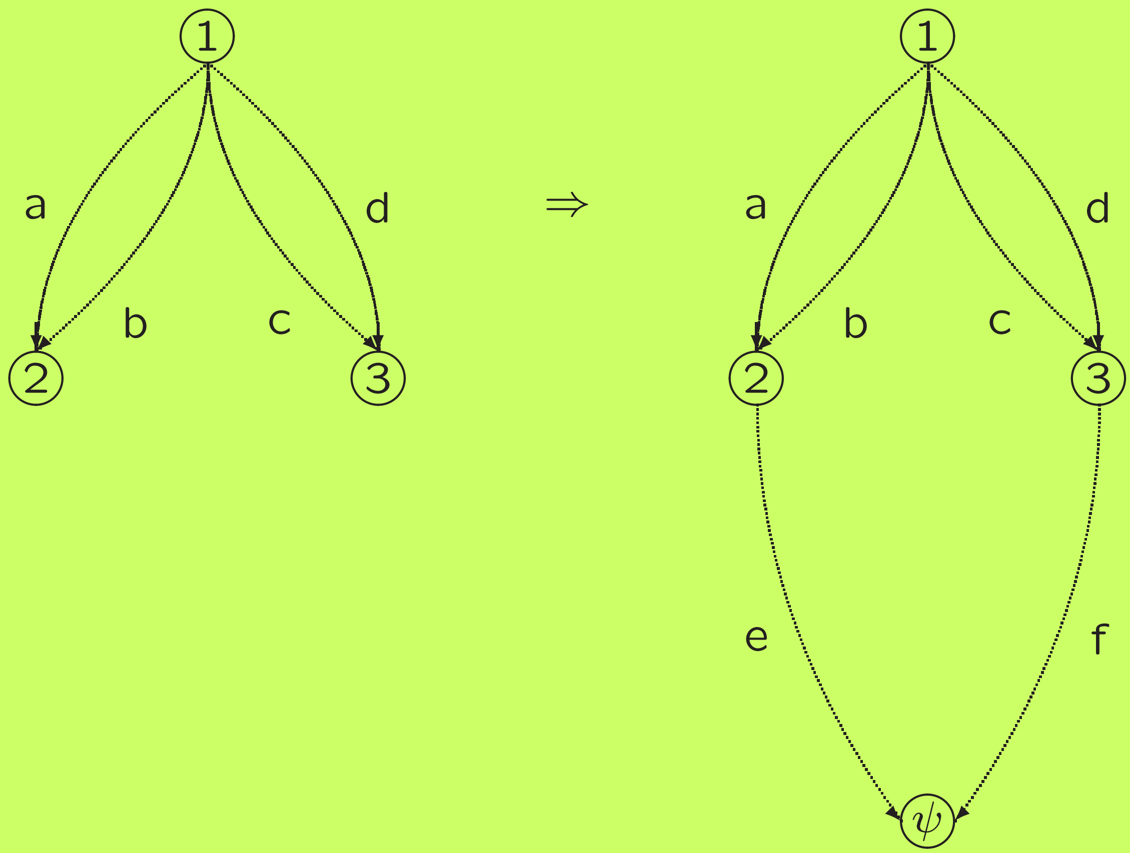
An example is now given.

Example 2

Consider the network with topology as given in the Figure, along with the expanded or supernetwork construction.

In this network there are three nodes: 1, 2, 3, and four links: a, b, c, d, in the original network and four nodes and six links in the expanded network. There are two destinations: 2 and 3 and, hence, according to the network expansion construction, one has a single "O/D" pair denoted by $\hat{w}_1 = (1, \psi)$ and four paths connecting the O/D pair: $\hat{p}_1 = (a, e)$, $\hat{p}_2 = (b, e)$, $\hat{p}_3 = (c, f)$, and $\hat{p}_4 = (d, f)$.

The emission factors are: $h_a = 0.5$, $h_b = 0.5$, $h_c = 0.3$, and $h_d = 0.3$, and that the desired environmental quality standard $\bar{Q} = 4$.



Original Network　　　Expanded Network

Expanded network for Example 2

I now impose a travel cost structure as follows:

$$c_a(f_a) = f_a + 5,\; c_b(f_b) = f_b + 5,$$

$$c_c(f_c) = 2f_c + 1,\; c_d(f_d) = 2f_d + 1,$$

with user travel costs on the "new" links given by:

$$c_e(f_e) = c_f(f_f) = 0.$$

Furthermore, from $O_1 = 10$ one constructs the travel demand on the expanded network given by $d_{\hat{w}_1} = 10$.

Utilizing the equilibrium conditions (8) and by the "symmetry" in the network and the user link travel cost functions, one can conclude that $f_a^* = f_b^*$ and $f_c^* = f_d^*$.

Further, I posit that, since the emissions factors are low on the links relative to the travel demand and the environmental quality standard, $\tau^* = 0$.

Hence, the solution of the emission pricing problem collapses to the solution of a variational inequality problem governing the well-known traffic network equilibrium conditions without any policies, or, equivalently, since the user travel cost functions on the links are separable, to the solution of the optimization reformulation.

Due, however, to the simplicity of the example, solutions of the resulting algebraic equations yields:

$$x^*_{\hat{p}_1} = 2, \quad x^*_{\hat{p}_2} = 2, \quad x^*_{\hat{p}_3} = 3, \quad x^*_{\hat{p}_4} = 3,$$

with induced link loads:

$$f^*_a = f^*_b = 2, \quad f^*_c = f^*_d = 3,$$

and

$$f^*_e = 4, \quad f^*_f = 6.$$

The incurred generalized path travel costs are:

$$\bar{C}_{\hat{p}_1} = \bar{C}_{\hat{p}_2} = \bar{C}_{\hat{p}_3} = \bar{C}_{\hat{p}_4} = 7,$$

which are also the path travel costs, respectively, on the paths p_1, p_2, p_3, and p_4.

Note that, hence, in the original network, the equilibrium solution must be:

$$x^*_{p_1} = 2, \quad x^*_{p_2} = 2, \quad x^*_{p_3} = 3, \quad x^*_{p_4} = 3,$$

and

$$d_{w_1} = 4, \quad d_{w_2} = 6.$$

The total emissions generated are: $h_a f^*_a + h_b f^*_b + h_c f^*_c + h_d f^*_d = 1 + 1 + 0.6 + 0.6 = 3.2$, which is less than $\bar{Q} = 4$.

Thus, both equilibrium conditions (8) and (9) hold, respectively, for the flow pattern and the marginal cost of emission abatement.

Emission Pricing for Networks with Known Destinations and Trip Attractions

I here consider transportation networks in Situation 2, in which now one is given the known destinations and the trip attractions associated with the destinations. I assume user-optimized behavior in that the travelers will select their origins as well as their paths to the given destinations so that the path travel costs on used paths to each destination are equal and minimal.

The statement of the equilibrium conditions is:

Definition 3 (Equilibrium Conditions for User-Optimization in Situation 2)

A traffic network is said to be in equilibrium in Situation 2 if the following conditions hold: For each destination node $z \in Z$, and each path $p \in P_z$:

$$C_p(x^*) \begin{cases} = & \lambda_z, \quad \text{if } x_p^* > 0 \\ \geq & \lambda_z, \quad \text{if } x_p^* = 0, \end{cases} \tag{26}$$

where $\sum_{p \in P_{D_z}} x_p^* = D_z, \forall z \in Z$.

Analogously to Situation 1, one can construct an expanded network in Situation 2, which can be used to: (1) determine the user-optimized traffic flow pattern without any policies and (2) determine the user-optimized traffic flow pattern in the presence of toll policies.

Construction of Expanded Network for Situation 2

The construction of the expanded or supernetwork is as follows: Construct a super origin or source node denoted by ξ and from node ξ construct as many links as there are origin nodes, with each such link originating at the supersource node and terminating at the origin node. These links are artificial links.

Let the new paths in the network, which originate at the supersource node and terminate in one of the destination or attraction nodes, be denoted by $\widehat{p}$, where $\widehat{p}$ consists of the appropriate artificial link in addition to the links on path p.

The user link costs on the artificial links are set equal to zero.

The O/D pairs on the expanded network become $\hat{w} = (\xi, z)$ for each destination node z with associated demand $d_{\hat{w}} = D_z$.

The equilibrium conditions in the case of tolls according to (8) and (9) are now applicable to the expanded network where we make the substitutions of $\hat{w}$ for w and $\hat{p}$ for p.

Moreover, as in Situation 1, one can recover the path flows and travel demand for the original network in this case.

Pricing for Networks with Known Number of Trips

I now turn to networks in Situation 3, in which the known number of trips for the entire network is given and denoted by T.

Assume that the network is viable in this situation. I first state the equilibrium conditions without the imposition of any toll policy. In this situation, travelers select their origins, their destinations, as well as their travel paths.

Hence, in this situation, it is the travel costs on used paths from each origin to each destination which are equalized and minimal in equilibrium. Formally stated, one has the following:

Definition 4 (Equilibrium Conditions for User-Optimization in Situation 3)

A traffic flow pattern is said to be in equilibrium in Situation 3 if for all $p \in P$:

$$C_p(x^*) \begin{cases} = & \lambda, \quad \text{if } x_p^* > 0 \\ \geq & \lambda, \quad \text{if } x_p^* = 0, \end{cases} \tag{27}$$

where $\sum_{p \in P} x_p^* = T$.

Construction of Expanded Network in Situation 3

In order to construct an expanded network or supernetwork, which enables one to reduce this equilibrium problem to that governing the case of known O/D pairs and travel demands, one proceeds as follows: Construct a supersource node ξ and a super sink node ψ.

Then construct an artificial link from the supersource node to each origin node.

In addition, construct a link from each destination node to the supersink node.

Denote by $\hat{p}$ for this situation a path which consists of p but has the appropriate artificial link from the supersource node attached to it as well as the artificial link at its terminus ending at the supersink node. There is then defined to be a single artificial O/D pair on the expanded network given by $\hat{w} = (\xi, \psi)$.

The equilibrium conditions (27) are then subsumed by the classical traffic network equilibrium conditions for the expanded network (with only a single O/D pair).

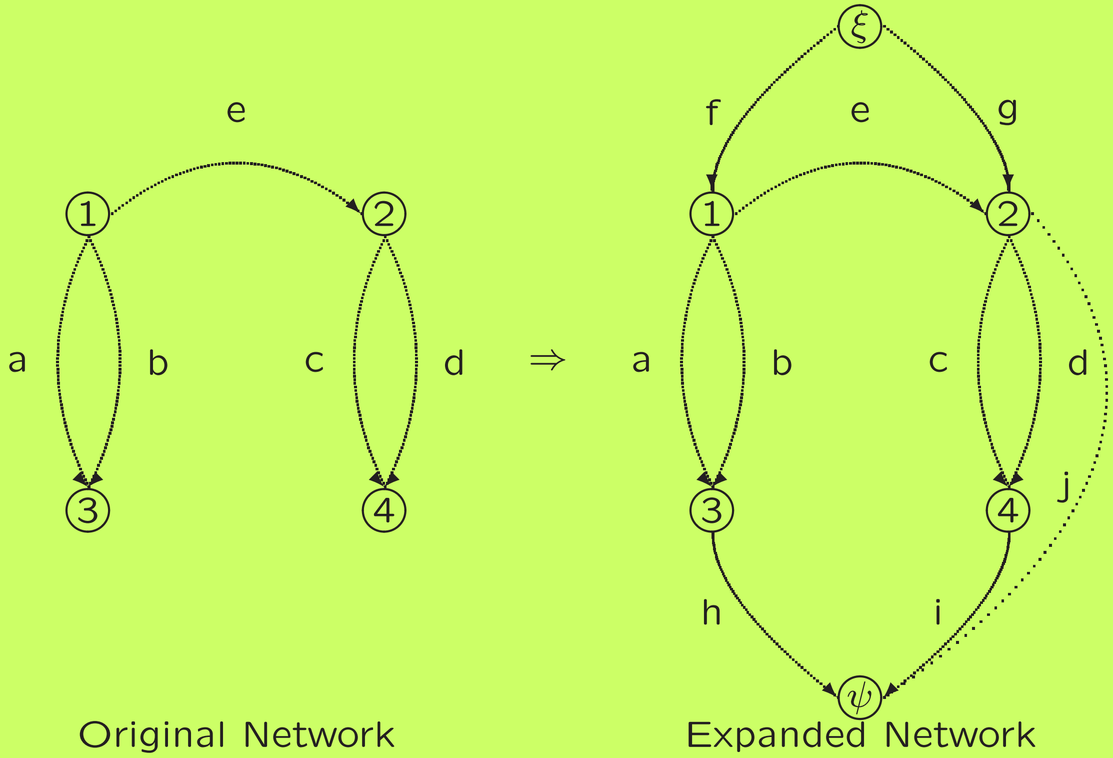
Furthermore, the equilibrium conditions (8) and (9) can be now applied directly to the expanded network with the appropriate substitutions for paths and O/D pair to obtain the equilibrium conditions for the emission pricing problem in the case of Situation 3.

Example 3

Consider the network in the Figure consisting of four nodes: 1, 2, 3, 4, and five links: a, b, c, d, and e. Construct the expanded network, which is also depicted in the Figure, as follows: Add a single supersource node ξ and a supersink node ψ and links f and g originating in node ψ and terminating, respectively, in the origin nodes 1 and 2.

Also, construct links h, i, and j originating in the destination nodes 2, 3, and 4 and terminating in the super sink node ψ.

There is then a single (artificial) O/D pair on the expanded network $\hat{w} = (\xi, \psi)$ with an associated (artificial) travel demand $d_{\hat{w}} = T = 6$.



Expanded network for Example 3

The emission factors are: $h_a = 0.1$, $h_b = 0.1$, $h_c = 0.2$, $h_d = 0.2$, and $h_e = 0.7$. Assume that the desired environmental quality standard $\bar{Q} = 0.6$.

The cost structure is now described. The user link cost functions are:

$$c_a(f_a) = f_a + 1,\ c_b(f_b) = f_b + 1,\ c_c(f_c) = f_c + 1,$$

$$c_d(f_d) = f_d + 1,\ c_e(f_e) = 2f_e + 5,$$

with the travel costs on the artificial links f, g, h, and i being set equal to zero, by construction.

The paths, hence, on the expanded network are: $\hat{p}_1 = (f, a, h)$, $\hat{p}_2 = (f, b, h)$, $\hat{p}_3 = (f, e, j)$, $\hat{p}_4 = (g, c, i)$, and $\hat{p}_5 = (g, d, i)$.

It is straightforward to verify that the path flow pattern: $x^*_{\hat{p}_1} = x^*_{\hat{p}_2} = 3$, with all other path flows equal to zero, and $\tau^* = 0$ is an equilibrium solution for the problem.

Summary of expanded network constructions

Situation 1	
Known origins and trip productions:	Construct a supersink node ξ and add a zero cost link from each destination node to ξ.
Situation 2	
Known destinations:	Construct a supersource node ψ add a zero cost link from ψ to each origin node.
Situation 3	
Known total number:	Construct a supersource node ψ and a supersink node ξ and add a zero cost link from ψ to each origin node and a zero cost link from each destination node to ψ.

Pricing for Sustainable Elastic Demand Traffic Networks

The topic now is that of pricing of elastic demand traffic networks for sustainability where the behavior is that of user-optimization.

As before, I assume that the travel cost on a link may depend upon, in general, the entire link load pattern, that is, we assume user link travel cost functions of the form given by (10).

In addition, since the demand is no longer fixed, as in the elastic demand traffic network models, I assume now that one has associated with each O/D pair w a travel disutility λ_w, such that

$$\lambda_w = \lambda_w(d), \quad \forall w \in W, \tag{28}$$

that is, I assume that, in general, the travel disutility associated with traveling between O/D pair w may depend upon the vector of travel demands.

In this case, one may adapt equilibrium conditions (8) directly and equilibrium conditions (9) are still applicable, yielding the following definition:

Definition 5 (Traffic Network Equilibrium with Elastic Demands in the Presence of Emission Pricing)

For each O/D pair $w \in W$, and each path $p \in P_w$:

$$\bar{C}_p(x^*, \tau^*) = C_p(x^*) + \tau^* \sum_{a \in L} h_a \delta_{ap} \begin{cases} = & \lambda_w(d^*), \quad \text{if } x_p^* > 0 \\ \geq & \lambda_w(d^*), \quad \text{if } x_p^* = 0, \end{cases} \tag{29}$$

In addition, one must have that:

$$\bar{Q} - \sum_{a \in L} h_a \sum_{p \in P} x_p^* \delta_{ap} \begin{cases} = & 0, \quad \text{if } \tau^* > 0 \\ \geq & 0, \quad \text{if } \tau^* = 0, \end{cases} \tag{30}$$

where the path flow pattern must satisfy the demand conservation of flow equations.

Hence, in this problem, one also needs to determine the travel demands, in addition to the path flows and the marginal cost of emission abatement.

I now provide the variational inequality formulations in path flows and in link loads of the equilibrium conditions (akin to (11) and (12) for the model with fixed travel demands). They are presented without proof since the proofs are similar to that of Theorem 1.

Theorem 2 (Variational Inequality Formulations of Elastic Demand Traffic Network Emission Pricing Policy Model)

A traffic flow and demand pattern and marginal cost of emission abatement $(x^*, d^*, \tau^*) \in \mathcal{K}^3$ *is an equilibrium of the traffic network emission pricing policy model with elastic demands if and only if it is a solution to the variational inequality problem:*

Path Flow Formulation:

$$\sum_{w \in W} \sum_{p \in P_w} \left[C_p(x^*) + \tau^* \sum_{a \in L} h_a \delta_{ap} \right] \times \left[x_p - x_p^* \right]$$

$$- \sum_{w \in W} \lambda_w(d^*) \times (d_w - d_w^*)$$

$$+ \left[\bar{Q} - \sum_{a \in L} h_a \sum_{p \in P} x_p \delta_{ap} \right] \times [\tau - \tau^*] \geq 0, \quad \forall (x, \tau, d) \in \mathcal{K}^3, \tag{31}$$

where $\mathcal{K}^3 \equiv \bar{K}^3 \times R^1_+$ *and* $\bar{K}^3 \equiv \{(x,d)|x \geq 0$, *and the demand equations hold*$\}$, *or, equivalently,* $(f^*, d^*, \tau^*) \in \mathcal{K}^4$ *is an equilibrium link load, travel demand, and marginal cost of emission abatement pattern if and only if it satisfies the variational inequality problem:*

Link Load Formulation:

$$\sum_{a \in L} [c_a(f^*) + h_a \tau^*] \times [f_a - f^*_a] - \sum_{w \in W} \lambda_w(d^*) \times (d_w - d^*_w)$$

$$\left[\bar{Q} - \sum_{a \in L} h_a f^*_a\right] \times [\tau - \tau^*] \geq 0, \quad \forall (f, d, \tau) \in \mathcal{K}^4, \qquad (32)$$

where $\mathcal{K}^4 \equiv \bar{K}^4 \times R^1_+$ *and* $\bar{K}^4 \equiv \{f|$ *there exists an* $x \geq 0$, *so that the demand equations hold*$\}$.

I now show that variational inequalities (31) and (32) can be put into standard form. I first establish this for variational inequality (31) which is in path flow variables.

Define the column vector $X \equiv (x, d, \tau) \in \mathcal{K}^3$ and the column vector $F(X) \equiv (C(X), L(X), T(X)$, where $C(X)$ and $T(X)$ are as defined for the fixed demand model and $L(X)$ is the J-dimensional vector with component w given by:

$$L_w(X) : \lambda_w(d).$$

For variational inequality (32) in link load variables, define the column vector $X \equiv (f, d, \tau) \in \mathcal{K}^4$ and the column vector

$$F(X) \equiv (c(X), L(X), T(X)),$$

where $c(X)$ is as defined for the fixed demand emission pricing model, and $L(X)$ and $T(X)$ are as defined above.

In the special case in which the user link travel cost functions are separable and given by (1), and the travel disutility functions are also separable, that is,

$$\lambda_w = \lambda_w(d_w), \quad \forall w \in W, \tag{33}$$

one may obtain a solution to the simple case of the traffic network policy model with elastic travel demands. Indeed, one has:

Corollary 1 (Optimization Reformulation in a Special Case)

The solution to the traffic network equilibrium policy model in the case of separable user link travel cost functions and travel disutility functions can be obtained by solving the optimization problem:

$$\textit{Minimize} \quad \sum_{a\in L} \int_0^{\sum_{p\in P} x_p\delta_{ap}} c_a(x)dx - \sum_{w\in W} \int_0^{d_w} \lambda_w(y)dy \tag{34}$$

subject to:

$$\sum_{p\in P_w} x_p = d_w, \quad \forall w \in W, \tag{35}$$

$$\sum_{a\in L} h_a \sum_{p\in P} x_p\delta_{ap} \leq \bar{Q}, \tag{36}$$

$$x_p \geq 0, \quad \forall p \in P. \tag{37}$$

A Computational Procedure

An algorithm is now presented which can be applied to solve any variational inequality problem in which the function F that enters the variational inequality is monotone and Lipschitz continuous, provided that a solution exists. The algorithm is the modified projection method of Korpelevich (1977).

Recall that *monotonicity* of a function F requires that F satisfies the following condition:

$$\langle (F(X^1) - F(X^2))^T, X^1 - X^2 \rangle \geq 0, \quad \forall X^1, X^2 \in \mathcal{K}, \tag{38}$$

whereas *Lipschitz continuity* of a function F requires that there exists a positive constant $\bar{L}$, such that

$$\|F(X^1) - F(X^2)\| \leq \bar{L}\|X^1 - X^2\|, \quad \forall X^1, X^2 \in \mathcal{K}. \tag{39}$$

The statement of the modified projection method is as follows, where $\mathcal{T}$ denotes an iteration counter:

Modified Projection Method

Step 0: Initialization

Set $X^0 \in K$. Let $\mathcal{T} = 1$ and let α be a scalar such that $0 < \alpha < \frac{1}{L}$, where L is the Lipschitz continuity constant (cf. (39)).

Step 1: Computation

Compute $\bar{X}^{\mathcal{T}}$ by solving the variational inequality subproblem:

$$\langle (\bar{X}^{\mathcal{T}} + \alpha F(X^{\mathcal{T}-1})^T - X^{\mathcal{T}-1})^T, X - \bar{X}^{\mathcal{T}} \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{40}$$

Step 2: Adaptation

Compute $X^{\mathcal{T}}$ by solving the variational inequality subproblem:

$$\langle (X^{\mathcal{T}} + \alpha F(\bar{X}^{\mathcal{T}})^T - X^{\mathcal{T}-1})^T, X - X^{\mathcal{T}} \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{41}$$

Step 3: Convergence Verification

If $\max |X_l^{\mathcal{T}} - X_l^{\mathcal{T}-1}| \leq \epsilon$, for all l, with $\epsilon > 0$, a prespecified tolerance, then stop; else, set $\mathcal{T} =: \mathcal{T} + 1$, and go to Step 1.

I now discuss the modified projection method more fully. I first recall the definition of the projection of X on the closed convex set $\mathcal{K}$, with respect to the Euclidean norm, and denoted by $P_{\mathcal{K}}X$, as

$$y = P_{\mathcal{K}}X = \text{arg min}_{z\in\mathcal{K}}\|X - z\|. \tag{42}$$

In particular, note that $\bar{X}^{\mathcal{T}}$ generated by the modified projection method as the solution to the variational inequality subproblem (40) is actually the projection of $(X^{\mathcal{T}-1} - \alpha F(X^{\mathcal{T}-1})^T)$ on the closed convex set $\mathcal{K}$. In other words,

$$\bar{X}^{\mathcal{T}} = P_{\mathcal{K}}\left[X^{\mathcal{T}-1} - \alpha F(X^{\mathcal{T}-1})^T\right]. \tag{43}$$

Similarly, $X^{\mathcal{T}}$ generated by the solution to variational inequality subproblem (41) is the projection of $X^{\mathcal{T}-1} - \alpha F(\bar{X}^{\mathcal{T}})^T$ on $\mathcal{K}$, that is,

$$X^{\mathcal{T}} = P_{\mathcal{K}}\left[X^{\mathcal{T}-1} - \alpha F(\bar{X}^{\mathcal{T}})^T\right]. \qquad (44)$$

I now give an explicit statement of the modified projection method for the solution of variational inequality problem (12) for the fixed demand traffic network equilibrium model with emission pricing.

Modified Projection Method for the Solution of Variational Inequality (12)

Step 0: Initialization

Set $(f^0, \tau^0) \in \mathcal{K}^2$. Let $\mathcal{T} = 1$ and set α such that $0 < \alpha \leq \frac{1}{\bar{L}}$, where $\bar{L}$ is the Lipschitz constant for the problem.

Step 1: Computation

Compute $(\bar{f}^{\mathcal{T}}, \bar{\tau}^{\mathcal{T}}) \in \mathcal{K}^2$ by solving the variational inequality subproblem:

$$\sum_{a \in L} (\bar{f}_a^{\mathcal{T}} + \alpha(c_a(f^{\mathcal{T}-1}) + h_a \tau^{\mathcal{T}-1}) - f_a^{\mathcal{T}-1}) \times (f_a - \bar{f}_a^{\mathcal{T}})$$

$$+(\bar{\tau}^{\mathcal{T}} + \alpha(\bar{Q} - \sum_{a \in L} h_a f_a^{\mathcal{T}-1}) - \tau^{\mathcal{T}-1}) \times (\tau - \bar{\tau}^{\mathcal{T}}) \geq 0,$$

$$\forall (f, \tau) \in \mathcal{K}^2. \quad (45)$$

Step 2: Adaptation

Compute $(f^{\mathcal{T}}, \tau^{\mathcal{T}}) \in \mathcal{K}^2$ by solving the variational inequality subproblem:

$$\sum_{a \in L} (f_a^{\mathcal{T}} + \alpha(c_a(\bar{f}^{\mathcal{T}}) + h_a \bar{\tau}^{\mathcal{T}}) - f_a^{\mathcal{T}-1}) \times (f_a - f_a^{\mathcal{T}})$$

$$+(\tau^{\mathcal{T}} + \alpha(\bar{Q} - \sum_{a \in L} h_a \bar{f}_a^{\mathcal{T}}) - \tau^{\mathcal{T}-1}) \times (\tau - \tau^{\mathcal{T}}) \geq 0,$$

$$\forall (f, d) \in \mathcal{K}^2. \quad (46)$$

Step 3: Convergence Verification

If $| f_a^{\mathcal{T}} - f_a^{\mathcal{T}-1} | \leq \epsilon$, for all $a \in L$ and $| \tau^{\mathcal{T}} - \tau^{\mathcal{T}-1} | \leq \epsilon$, with $\epsilon > 0$, a pre-specified tolerance, then stop; otherwise, set $\mathcal{T} := \mathcal{T} + 1$, and go to Step 1.

The decomposed subproblems take on a simple form which can be computed very efficiently. Note that since the feasible set $\mathcal{K}^2$ is a Cartesian product, where $\mathcal{K}^2 \equiv \bar{K}^2 \times R^1_+$, the above variational inequality subproblems can be decomposed across $\bar{K}^2$, which has the network structure of the problem, and across the nonnegative orthant.

In fact, (45) yields the subproblem in $\bar{K}^2$ in link variables given by:

$$\text{Minimize}_{f\in\bar{K}^2}\langle\bar{f}^{\mathcal{T}T},\bar{f}^{\mathcal{T}}\rangle+\langle g^T,f^{\mathcal{T}}\rangle, \qquad (47)$$

where $\bar{f}^{\mathcal{T}}$ is the column vector with component $a=\bar{f}_a^{\mathcal{T}}$, g is the column vector with component $g_a=\alpha(c_a(f^{\mathcal{T}-1})+h_a\tau^{\mathcal{T}-1})-f_a^{\mathcal{T}-1})$.

Observe that this subproblem is a quadratic programming problem or, equivalently, the optimization reformulation of the traffic network equilibrium conditions in the case of linear and separable user link travel cost functions.

Hence, this problem can be solved in numerous ways.

In addition, (45) yields a subproblem in the marginal cost of emission abatement variable, which can be solved explicitly and exactly in closed form as follows: Set

$$\bar{\tau}^{\mathcal{T}} = \max\{0, -\alpha(\bar{Q} - \sum_{a \in L} h_a f_a^{\mathcal{T}-1}) + \tau^{\mathcal{T}-1}\}. \quad (48)$$

The solution of the induced subproblems in (46) can be solved in an analogous fashion.

I now state the convergence result for the algorithm for the fixed demand emission pricing model.

It is presented without proof since the proof is similar to the convergence proofs of the modified projection method given in other lectures.

Theorem 3 (Convergence)

If the user link travel cost functions c are assumed to be monotone and have bounded first-order derivatives, then the modified projection method described above converges to the solution of the variational inequality (12).

I now show the realization of the modified projection method for the solution of the variational inequality in link loads for the elastic demand emission pricing policy model.

Modified Projection Method for the Solution of Variational Inequality (32)

Step 0: Initialization

Set $(f^0, d^0, \tau^0) \in \mathcal{K}^4$. Let $\mathcal{T} = 1$ and set α such that $0 < \alpha \leq \frac{1}{\bar{L}}$, where $\bar{L}$ is the Lipschitz constant for the problem.

Step 1: Computation

Compute $(\bar{f}^{\mathcal{T}}, \bar{d}^{\mathcal{T}}, \bar{\tau}^{\mathcal{T}}) \in \mathcal{K}^4$ by solving the variational inequality subproblem:

$$\sum_{a \in L} (\bar{f}_a^{\mathcal{T}} + \alpha(c_a(f^{\mathcal{T}-1}) + h_a \tau^{\mathcal{T}-1}) - f_a^{\mathcal{T}-1}) \times (f_a - \bar{f}_a^{\mathcal{T}})$$

$$+ \sum_{w \in W} (\bar{d}_w^{\mathcal{T}} - \alpha \lambda_w (d^{\mathcal{T}-1}) - d_w^{\mathcal{T}-1}) \times (d_w - \bar{d}_w^{\mathcal{T}})$$

$$+ (\bar{\tau}^{\mathcal{T}} + \alpha(\sum_{a \in L} \bar{Q} - \sum_{a \in L} h_a f_a^{\mathcal{T}-1}) - \tau^{\mathcal{T}-1}) \times (\tau - \bar{\tau}^{\mathcal{T}}) \geq 0,$$

$$\forall (f, d, \tau) \in \mathcal{K}^4. \qquad (49)$$

Step 2: Adaptation

Compute $(f^{\mathcal{T}}, d^{\mathcal{T}}, \tau^{\mathcal{T}}) \in \mathcal{K}^4$ by solving the variational inequality subproblem:

$$\sum_{a \in L} (f_a^{\mathcal{T}} + \alpha(c_a(\bar{f}^{\mathcal{T}}) + h_a \bar{\tau}^{\mathcal{T}}) - f_a^{\mathcal{T}-1}) \times (f_a - f_a^{\mathcal{T}})$$

$$+ \sum_{w \in W} (d_w^{\mathcal{T}} - \alpha \lambda_w(\bar{d}^{\mathcal{T}}) - d_w^{\mathcal{T}-1}) \times (d_w - d_w^{\mathcal{T}})$$

$$+(\tau^{\mathcal{T}} + \alpha(\sum_{a \in L} \bar{Q} - \sum_{a \in L} h_a \bar{f}_a^{\mathcal{T}}) - \tau^{\mathcal{T}-1}) \times (\tau - \tau^{\mathcal{T}}) \geq 0,$$

$$\forall (f, d, \tau) \in \mathcal{K}^4. \tag{50}$$

Step 3: Convergence Verification

If $|f_a^{\mathcal{T}} - f_a^{\mathcal{T}-1}| \leq \epsilon$, for all $a \in L$, $|d_w^{\mathcal{T}} - d_w^{\mathcal{T}-1}| \leq \epsilon$, for all $w \in W$, and $|\tau^{\mathcal{T}} - \tau^{\mathcal{T}-1}| \leq \epsilon$, with $\epsilon > 0$, a pre-specified tolerance, then stop; otherwise, set $\mathcal{T} := \mathcal{T} + 1$, and go to Step 1.

The decomposed subproblems (48) and (49) can be computed very efficiently.

Note that since the feasible set $\mathcal{K}^4$ is again a Cartesian product, where $\mathcal{K}^4 \equiv \bar{K}^4 \times R^1_+$, the above projections can be decomposed across $\bar{K}^4$, which has the network structure of the problem, and across the nonnegative orthant.

Hence, (49), for example, yields the subproblem in K^4 in link and demand variables given by:

$$\text{Minimize}_{(f,d)\in\bar{K}^2} \langle \bar{f}^{\mathcal{T}T}, \bar{f}^{\mathcal{T}} \rangle + \langle g^T, f^{\mathcal{T}} \rangle + \langle \bar{d}^{\mathcal{T}T}, \bar{d}^{\mathcal{T}} \rangle - r^T, \quad (51)$$

where $\bar{f}^{\mathcal{T}}$ is the column vector with component $a = \bar{f}^{\mathcal{T}}_a$, g is the column vector with component

$$g_a = \alpha(c_a(f^{\mathcal{T}-1}) + h_a \tau^{\mathcal{T}-1}) - f_a^{\mathcal{T}-1}$$

,

$$\bar{d}^{\mathcal{T}}$$

is the column vector with component $w = \bar{d}^{\mathcal{T}}_w$, and r is the column vector with component $w = \alpha\lambda_w(d^{\mathcal{T}-1}) + d_w^{\mathcal{T}-1}$.

This subproblem is a quadratic programming problem or, equivalently, the optimization reformulation of the traffic network equilibrium conditions in the case of linear and separable user link travel cost and travel disutility functions. Hence, this problem can be also solved in numerous ways (see, e.g., Dhanda, Nagurney, and Ramanujam 1999).

In addition, one can reformulate the elastic demand problem as a fixed demand problem over an appropriately constructed abstract network (see Gartner 1980) and, therefore, one can apply then the equilibration algorithm of Dafermos and Sparrow (1969).

Furthermore, (49) yields a subproblem in the marginal cost of emission abatement, which can be solved using the expression (5.48) above.

Convergence for the algorithm is given in the following theorem, which is also presented without proof since the proof is similar to other proofs of convergence given in this lecture series.

Theorem 4 (Convergence)

If the user link travel cost functions c and the travel disutility functions $-\lambda$ are assumed to be monotone and have bounded first-order derivatives, then the modified projection method described above converges to the solution of the variational inequality (32).

Numerical Examples

Several numerical examples are now presented.

I consider the solution of the fixed demand emission pricing policy model and apply the modified projection method for solving the governing variational inequality given by (12). The modified projection method was implemented in FORTRAN and the system utilized was the IBM SP2 located at the Computer Science Department at the University of Massachusetts at Amherst for the numerical work.

For the solution of the standard traffic network equilibrium problem encountered in both the computation and adaptation steps (cf. (45) and (46)) I utilized the equilibration method (cf. Dafermos and Sparrow 1969. The convergence criterion used was: $|x_p^{\mathcal{T}} - x_p^{\mathcal{T}-1}| \leq \epsilon$, for all $p \in P$; $|\tau^{\mathcal{T}} - \tau^{\mathcal{T}-1}| \leq \epsilon$.

The modified projection method was initialized by setting the flow on a path equal to the travel demand for the O/D pair that the path belongs to divided by the number of paths. All other variables were initialized to zero.

Example 4: Variant of Example 1

The first numerical example was identical to Example 1 except for the following change: I relaxed the environmental quality standard from $\bar{Q} = 1$ to $\bar{Q} = 1.5$. Note that $\bar{Q} = 1.5$ is still lower than the total number of emissions emitted, which was equal to 2.1, if there is no emission pricing policy system in place. I set $\alpha = 0.4$ in the modified projection method.

An application of the modified projection method yielded the following solution:

$$f_a^* = 5.79, \quad f_b^* = 3.40, \quad f_c^* = 0.81.$$

This equilibrium link load pattern was induced by the equilibrium path flow pattern:

$$x_{p_1}^* = 5.79, \quad x_{p_2}^* = 3.40, \quad x_{p_3}^* = 0.81.$$

The generalized user travel costs on the paths were: $\bar{C}_{p_1} = \bar{C}_{p_2} = \bar{C}_{p_3} = 21.62$.

The equilibrium marginal cost of emission abatement was:

$$\tau^* = 51.79.$$

The environmental standard was met since the total emissions:
$\sum_{a \in L} h_a f_a^* = \bar{Q} = 1.5$.

Clearly, both equilibrium conditions (8) and (9) are satisfied by the computed flow and marginal cost of emission abatement pattern.

Example 5

I then considered the transportation network depicted in the Figure consisting of ten nodes, thirteen links and two O/D pairs: $w_1 = (1, 8)$ and $w_2 = (2, 10)$ with travel demands $d_{w_1} = 5$ and $d_{w_2} = 5$.

The user link travel cost functions were:

$$c_1(f) = .00005f_1^4 + 5f_1 + 2f_2 + 5,\ c_2(f) = .00003f_2^4 + 4f_2 + f_1 + 2,$$

$$c_3(f) = .00005f_3^4 + 3f_3 + f_4 + 3,\ c_4(f) = .00003f_4^4 + 6f_4 + 3f_5 + 4,$$

$$c_5(f) = 4f_5 + f_{12} + 8,\ c_6(f) = .00007f_6^4 + 7f_6 + 4f_{12} + 6,$$

$$c_7(f) = 8f_7 + 2f_{13} + 7,\ c_8(f) = .00001f_8^4 + 7f_8 + 3f_{12} + 6,$$

$$c_9(f) = 8f_9 + 3f_{11} + 5,\ c_{10}(f) = .00003f_{10}^4 + 6f_{10} + f_1 + 3,$$

$$c_{11}(f) = .00004f_{11}^4 + 4f_{11} + f_2 + 4,\ c_{12}(f) = .00002f_{12}^4 + 6f_{12} + f_1 + 5,$$

$$c_{13}(f) = .00003f_{12}^4 + 9f_{13} + 2f_4 + 3.$$

The paths were:

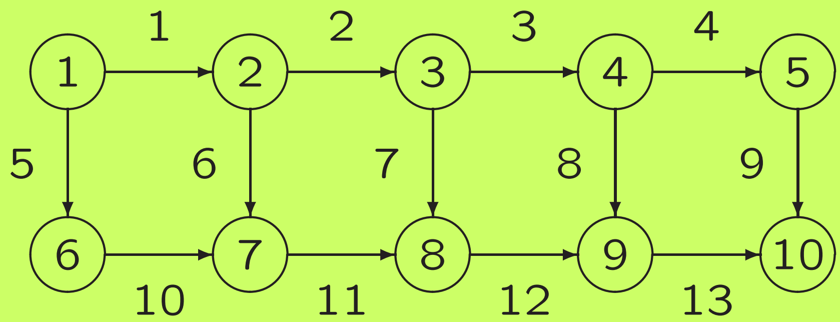
For O/D pair w_1:

$$p_1 = (1, 2, 7), \quad p_2 = (1, 6, 11), \quad p_3 = (5, 10, 11)$$

and for O/D pair w_2:

$$p_4 = (2, 3, 4, 9), \quad p_5 = (2, 3, 8, 13), \quad p_6 = (2, 7, 12),$$

$$p_7 = (6, 11, 12, 13).$$



Network topology for Example 5

The parameter α in the modified projection method was set to 0.1, whereas ϵ was set to .0001.

The emission parameters were: $h_a = 0.5 \times a$, $\forall a \in L$, and the environmental quality standard $\bar{Q} = 91$.

The modified projection method yielded the equilibrium link load pattern given by:

$$f_1^* = 3.88, \quad f_2^* = 7.36, \quad f_3^* = 5.00, \quad f_4^* = 3.51,$$

$$f_5^* = 1.12, \quad f_6^* = 1.52, \quad f_7^* = 2.36, \quad f_8^* = 1.49,$$

$$f_9^* = 3.51, \quad f_{10}^* = 1.12, \quad f_{11}^* = 2.64, \quad f_{12}^* = 0.00,$$

$${}^*_{13} = 1.49,$$

with an equilibrium path flow pattern:

For O/D pair w_1:

$$x_{p_1}^* = 2.36, \quad x_{p_2}^* = 1.52, \quad x_{p_3}^* = 1.12,$$

and for O/D pair w_2:

$$x_{p_4}^* = 3.51, \quad x_{p_5}^* = 1.49, \quad x_{p_6}^* = 0.00, \quad x_{p_7}^* = 0.00,$$

and with generalized user travel costs:

For O/D pair w_1:

$$\bar{C}_{p_1} = \bar{C}_{p_2} = \bar{C}_{p_3} = 140.43,$$

For O/D pair w_2:

$$\bar{C}_{p_4} = \bar{C}_{p_5} = 193.14, \quad \bar{C}_{p_6} = 203.16, \quad \bar{C}_{p_7} = 203.16.$$

The equilibrium marginal cost of emission abatement was $\tau^* = 7.41$.

The total emissions generated were precisely equal to the environmental quality standard $\bar{Q} = 91$. The equilibrium conditions (8) and (9) were also met by the computed flow and marginal cost of emission abatement pattern.

References

In addition to the text, **Sustainable Transportation Network**, Anna Nagurney (2000), Edward Elgar Publishers, the following references are also relevant to this lecture.

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