Oligopolies and Nash Equilibrium

Anna Nagurney
Isenberg School of Management
University of Massachusetts
Amherst, MA 01003

Oligopoly theory dates to Cournot (1838), who investigated competition between two producers, the so-called duopoly problem, and is credited with being the first to study noncooperative behavior.

In an oligopoly, it is assumed that there are several firms, which produce a product and the price of the product depends on the quantity produced.

Examples of oligopolies include large firms in computer, automobile, chemical or mineral extraction industries, among others.

Nash (1950, 1951) subsequently generalized Cournot's concept of an equilibrium for a behavioral model consisting of n agents or players, each acting in his/her own self-interest, which has come to be called a noncooperative game. Specifically, consider m players, each player i having at his/her disposal a strategy vector $x_i = \{x_{i1}, \ldots, x_{in}\}$ selected from a closed, convex set $K_i \subset R^n$, with a utility function $u_i : K \mapsto R^1$, where $K = K_1 \times K_2 \times \ldots \times K_m \subset R^{mn}$. The rationality postulate is that each player i selects a strategy vector $x_i \in K_i$ that maximizes his/her utility level $u_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m)$ given the decisions $(x_j)_{j \neq i}$ of the other players.

In this framework one then has:

Definition 1 (Nash Equilibrium)

A Nash equilibrium is a strategy vector

$$x^* = (x_1^*, \ldots, x_m^*) \in K,$$

such that

$$u_i(x_i^*, \hat{x}_i^*) \geq u_i(x_i, \hat{x}_i^*), \quad \forall x_i \in K_i, \forall i, \tag{1}$$

where $\hat{x}_i^* = (x_1^*, \ldots, x_{i-1}^*, x_{i+1}^*, \ldots, x_m^*)$.

It has been shown (cf. Hartman and Stampacchia (1966) and Gabay and Moulin (1980)) that Nash equilibria satisfy variational inequalities. In the present context, under the assumption that each u_i is continuously differentiable on K and concave with respect to x_i, one has

Theorem 1 (Variational Inequality Formulation of Nash Equilibrium)

Under the previous assumptions, x^* is a Nash equilibrium if and only if $x^* \in K$ is a solution of the variational inequality

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K, \tag{2}$$

where $F(x) \equiv (-\nabla_{x_1} u_1(x), \ldots, -\nabla_{x_m} u_m(x))$ is a row vector and where $\nabla_{x_i} u_i(x) = (\frac{\partial u_i(x)}{\partial x_{i1}}, \ldots, \frac{\partial u_i(x)}{\partial x_{in}})$.

Proof: Since u_i is a continuously differentiable function and concave with respect to x_i, the equilibrium condition (1), for a fixed i, is equivalent to the variational inequality problem

$$-\langle \nabla_{x_i} u_i(x^*), x_i - x_i^* \rangle \geq 0, \quad \forall x_i \in K_i, \tag{3}$$

which, by summing over all players i, yields (2).

If the feasible set K is compact, then existence is guaranteed under the assumption that each u_i is continuously differentiable. Rosen (1965) proved existence under similar conditions. Karamardian (1969), on the other hand, relaxed the assumption of compactness of K and provided a proof of existence and uniqueness of Nash equilibria under the strong monotonicity condition. As shown by Gabay and Moulin (1980), the imposition of a coercivity condition on $F(x)$ will guarantee existence of a Nash equilibrium x^* even if the feasible set is no longer compact. Moreover, if $F(x)$ satisfies the strict monotonicity condition, uniqueness of x^* is guaranteed, provided that the equilibrium exists.

Classical Oligopoly Problems

We now consider the classical oligopoly problem in which there are m producers involved in the production of a homogeneous commodity. The quantity produced by firm i is denoted by q_i, with the production quantities grouped into a column vector $q \in R^m$. Let f_i denote the cost of producing the commodity by firm i, and let p denote the demand price associated with the good. Assume that

$$f_i = f_i(q_i) \tag{4}$$

and

$$p = p(\sum_{i=1}^{m} q_i). \tag{5}$$

The profit for firm i, u_i, can then be expressed as

$$u_i(q) = p(\sum_{i=1}^{m} q_i)q_i - f_i(q_i). \tag{6}$$

Assuming that the competitive mechanism is one of noncooperative behavior, in view of Theorem 1, one can write down immediately:

Theorem 2 (Variational Inequality Formulation of Classical Cournot-Nash Oligopolistic Market Equilibrium)

Assume that the profit function $u_i(q)$ is concave with respect to q_i, and that $u_i(q)$ is continuously differentiable. Then $q^* \in R^m_+$ is a Nash equilibrium if and only if it satisfies the variational inequality

$$\sum_{i=1}^{m} \left[\frac{\partial f_i(q_i^*)}{\partial q_i} - \frac{\partial p(\sum_{i=1}^{m} q_i^*)}{\partial q_i} q_i^* - p(\sum_{i=1}^{m} q_i^*) \right] \times [q_i - q_i^*] \geq 0,$$

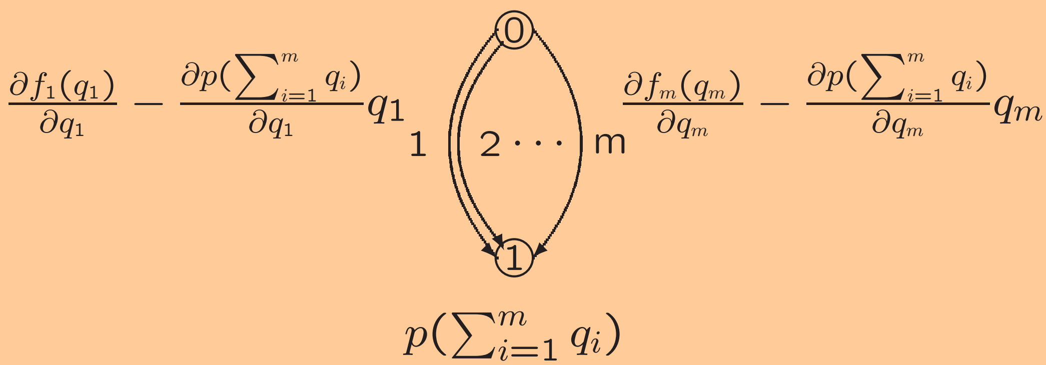
$$\forall q \in R^m_+. \quad (7)$$

We now establish the equivalence between the classical oligopoly model and a network equilibrium model. For a graphic depiction, see Figure 1.

Let 0 be the origin node and 1 the destination node. Construct m links connecting 0 to 1. The cost on a link i is then given by:

$$\left[\frac{\partial f_i(q_i)}{\partial q_i} - \frac{\partial p(\sum_{i=1}^m q_i)}{\partial q_i} q_i\right],$$

and the inverse demand associated with the origin/destination (O/D) pair $(0,1)$ is given by $p(\sum_{i=1}^m q_i)$. The flow on link i corresponds to q_i and the demand associated with the O/D pair to $\sum_{i=1}^m q_i$. Hence, the classical oligopoly model is isomorphic to a network equilibrium model with a single O/D pair, m paths corresponding to the m links, and with elastic demand.



Network equilibrium representation of an oligopoly model

Computation of Classical Oligopoly Problems

First consider a special case of the oligopoly model described above, characterized by quadratic cost functions, and a linear inverse demand function. The former model has received attention in the literature (cf. Gabay and Moulin (1980), and the references therein), principally because of its stability properties. It is now demonstrated that a demand market equilibration algorithm can be applied for the explicit computation of the Cournot-Nash equilibrium pattern. The algorithm is called the oligopoly equilibration algorithm, OEA. After its statement, it is applied to compute the solution to a three-firm example.

Assume a quadratic production cost function for each firm, that is,

$$f_i(q_i) = a_i q_i^2 + b_i q_i + e_i, \tag{8}$$

and a linear inverse demand function, that is,

$$p(\sum_{i=1}^{m} q_i) = -o \sum_{i=1}^{m} q_i + r, \tag{9}$$

where $a_i, b_i, e_i, o, r > 0$, for all i. Then the expression for the cost on link i is given by: $(2a_i + o)q_i + b_i$, for all $i = 1, \ldots, m$.

The oligopoly equilibration algorithm is now stated.

OEA

Step 0: Sort

Sort the b_i's; $i = 1, \ldots, m$, in nondescending order and relabel them accordingly. Assume, henceforth, that they are relabeled. Also, define $b_{m+1} \equiv \infty$ and set $I := 1$. If $b_1 \geq r$, stop; set $q_i = 0$; $i = 1, \ldots, m$, and exit; otherwise, go to Step 1.

Step 1: Computation

Compute

$$\lambda^I = \frac{\frac{r}{o} + \sum_{i=1}^{I} \frac{b_i}{(2a_i+o)}}{\frac{1}{o} + \sum_{i=1}^{I} \frac{1}{(2a_i+o)}}. \tag{10}$$

Step 2: Evaluation

If $b_I < \lambda^I \leq b_{I+1}$, set $j := I$; $\lambda := \lambda^I$, and go to Step 3; otherwise, set $I := I + 1$, and go to Step 1.

Step 3: Update

Set

$$q_i = \frac{\lambda - b_i}{(2a_i + o)}, \quad i = 1, \ldots, j$$

$$q_i = 0, \quad i = j + 1, \ldots, m.$$

An example is now presented.

Example 1

In this oligopoly example there are three firms. The data are as follows:

producer cost functions:

$$f_1(q_1) = q_1^2 + q_1 + 1$$

$$f_2(q_2) = .5q_2^2 + 4q_2 + 2$$

$$f_3(q_3) = q_3^2 + .5q_3 + 5,$$

inverse demand function:

$$p(\sum_{i=1}^{3} q_i) = -\sum_{i=1}^{3} q_i + 5.$$

Step 0 of OEA consists of sorting the b_i terms, which yields: $.5 \leq 1 \leq 4$, with the reordering of the links being: link 3, link 1, link 2. Set $I := 1$ and compute:

$$\lambda^1 = \frac{5 + \frac{.5}{(2+1)}}{1 + \frac{1}{(2+1)}} = 3\frac{7}{8}.$$

Since $.5 < 3\frac{7}{8} \not\leq 1$, increment I to 2 and compute:

$$\lambda^2 = \frac{5 + \frac{1}{6} + \frac{1}{3}}{1 + \frac{1}{3} + \frac{1}{3}} = \frac{5\frac{1}{2}}{1\frac{2}{3}} = 3\frac{3}{10}.$$

Since $1 < 3\frac{3}{10} \leq 4$, stop; $j = 2$, $\lambda = 3\frac{3}{10}$. Update the production outputs as follows:

$$q_1 = \frac{23}{30}, \quad q_2 = 0, \quad q_3 = \frac{14}{15}; \quad \sum_{i=1}^{3} q_i = 1\frac{7}{10}.$$

We now turn to the computation of Cournot-Nash equilibria in the case where the production cost functions (4) are not limited to being quadratic, and the inverse demand function (cf. (5)) is not limited to being linear. In particular, an oligopoly iterative scheme is presented for the solution of variational inequality (7) governing the Cournot-Nash model. It is then shown that the scheme induces the projection method and the relaxation method; each of these methods, in turn, decomposes the problem into very simple subproblems.

The Iterative Scheme

Construct a smooth function $g(q, y) : R^m_+ \times R^m_+ \mapsto R^m$ with the following properties:

(i). $g(q, q) = -\nabla^T u(q), \quad \forall q \in R^m_+.$

(ii). For every $q \in R^m_+, y \in R^m_+$, the $m \times m$ matrix $\nabla_q g(q, y)$ is positive definite.

Any smooth function $g(q, y)$ with the above properties generates the following algorithm:

Step 0: Initialization

Start with $q^0 \in R^m_+$. Set $k := 1$.

Step 1: Construction and Computation

Compute q^k by solving the variational inequality subproblem:

$$\langle g(q^k, q^{k-1})^T, q - q^k \rangle \geq 0, \quad \forall q \in R^m_+. \tag{11}$$

Step 2: Convergence Verification

If $|q^k - q^{k-1}| \leq \epsilon$, with $\epsilon > 0$, a prespecified tolerance, then stop; otherwise, set $k := k + 1$, and go to Step 1.

The above algorithm generates a well-defined sequence $\{q^k\}$, such that if $\{q^k\}$ converges, say $q^k \to q^*$, as $k \to \infty$, then q^* is an equilibrium quantity vector, that is, a solution of variational inequality (7).

Projection Method

The projection method then corresponds to the choice

$$g(q,y) = -\nabla^T u(q) + \frac{1}{\rho} G(q-y), \tag{12}$$

where ρ is a positive scalar and G is a fixed, symmetric positive definite matrix. It is easy to verify that conditions (i) and (ii) are satisfied. Note that in the application of the projection method to the Cournot oligopoly model, each subproblem (11) can be solved exactly at iteration k as follows:

$$q_i^k = \max\{0, \frac{\rho \frac{\partial u_i(q^{k-1})}{\partial q_i} + G_{ii} q_i^{k-1}}{G_{ii}}\}, \quad \text{for} \quad i = 1, \ldots, m, \tag{13}$$

where G_{ii} is the i-th diagonal element of G. In particular, if one selects $G = I$, then (11) simplifies even further to:

$$q_i^k = \max\{0, \rho \frac{\partial u_i(q^{k-1})}{\partial q_i} + q_i^{k-1}\}, \quad \text{for} \quad i = 1, \ldots, m. \tag{14}$$

Relaxation Method

The relaxation/diagonalization method, on the other hand, corresponds to the selection

$$g_i(q,y) = -\frac{\partial u_i}{\partial q_i}(y_1,\ldots,y_{i-1},q_i,y_{i+1},\ldots,y_m), \text{ for } i=1,\ldots,m. \tag{15}$$

In this case, properties (i) and (ii) are also satisfied.

Note that in the realization of the relaxation method at each step k one must solve

$$\max_{q_i \geq 0} u_i(q_i, \hat{q}_i^{k-1}) \tag{16}$$

for each i, where $\hat{q}_i^{k-1} \equiv \{q_1^{k-1},\ldots,q_{i-1}^{k-1},q_{i+1}^{k-1},\ldots,q_m^{k-1}\}$.

Specifically, this subproblem can be solved by the following rule:

$$q_i^k = \max\{0, \bar{q}_i\}, \tag{17}$$

where $\bar{q}_i$ is the solution of the one-variable nonlinear equation

$$f_i'(q_i) - p'(q_i + \sum_{j=1, j\neq i}^{m} q_j^{k-1}) q_i - p(q_i + \sum_{j=1, j\neq i}^{m} q_j^{k-1}) = 0. \tag{18}$$

Note that the solution of (18) which is needed for (17) would usually be solved iteratively, unlike (13) which is an analytical expression for the determination of each q_i^k.

First, some theoretical results are presented and then a numerical example is given.

We now state the convergence conditions for the general iterative scheme over an unbounded feasible set.

Theorem 3 (Convergence of General Iterative Scheme)

Assume that there exists a constant $\theta > 0$*, such that*

$$\|g_q^{-\frac{1}{2}}(q^1, y^1)\nabla_y g(q^2, y^2) g_q^{-\frac{1}{2}}(q^3, y^3)\| \le \theta < 1 \qquad (19)$$

for all (q^1, q^2, q^3),$(y^1, y^2, y^3)$$\in R^m_+$*, and that the infimum over* $K \times K$ *of the minimum eigenvalue of* $\nabla_x g(x, y)$ *is positive. Then the sequence* $\{q^k\}$ *obtained by solving variational inequality (7) converges.*

The following example is taken from Murphy, Sherali, and Soyster (1982) and solved now using both the projection method and the relaxation method.

Example 2

The oligopoly consists of five firms, each with a production cost function of the form

$$f_i(q_i) = c_i q_i + \frac{\beta_i}{(\beta_i + 1)} K_i^{-\frac{1}{\beta_i}} q_i^{\frac{(\beta_i+1)}{\beta_i}}, \tag{20}$$

with the parameters given in the Table. The demand price function is given by

$$p(\sum_{i=1}^{5} q_i) = 5000^{\frac{1}{1.1}} (\sum_{i=1}^{5} q_i)^{-\frac{1}{1.1}}. \tag{21}$$

Parameters for the five-firm oligopoly example

Firm i	c_i	K_i	β_i
1	10	5	1.2
2	8	5	1.1
3	6	5	1.0
4	4	5	.9
5	2	5	.8

Both the projection method and the relaxation method were implemented in FORTRAN, compiled using the FORTVS compiler, optimization level 3. The convergence criterion was $|q_i^k - q_i^{k-1}| \leq .001$, for all i, for both methods. A bisecting search method was used to solve the single variable problem (17) for each firm i, in the relaxation method. The matrix G was set to the identitiy matrix I for the projection method with $\rho = .9$. The system used was an IBM 3090/600J at the Cornell Theory Center. Both algorithms were initialized at $q^0 = (10, 10, 10, 10, 10)$.

The projection method required 33 iterations but only .0013 CPU seconds for convergence, whereas the relaxation method required only 23 iterations, but .0142 CPU seconds for convergence. Both methods converged to q^*=(36.93,41.81,43.70,42.65,39.17), reported to the same number of decimal places.

A Spatial Oligopoly Model

Now a generalized version of the classical oligopoly model is presented. Assume that there are m firms and n demand markets that are generally spatially separated. Assume that the homogeneous commodity is produced by the m firms and is consumed at the n markets. As before, let q_i denote the nonnegative commodity output produced by firm i and now let d_j denote the demand for the commodity at demand market j. Let T_{ij} denote the nonnegative commodity shipment from supply market i to demand market j. Group the production outputs into a column vector $q \in R^m_+$, the demands into a column vector $d \in R^n_+$, and the commodity shipments into a column vector $T \in R^{mn}_+$.

The following conservation of flow equations must hold:

$$q_i = \sum_{j=1}^{n} T_{ij}, \quad \forall i \tag{22}$$

$$d_j = \sum_{i=1}^{m} T_{ij}, \quad \forall j \tag{23}$$

where $T_{ij} \geq 0$, $\forall i, j$.

As previously, associate with each firm i a production cost f_i, but allow now for the more general situation where the production cost of a firm i may depend upon the entire production pattern, that is,

$$f_i = f_i(q). \tag{24}$$

Similarly, allow the demand price for the commodity at a demand market to depend, in general, upon the entire consumption pattern, that is,

$$p_j = p_j(d). \tag{25}$$

Let t_{ij} denote the transaction cost, which includes the transportation cost, associated with trading (shipping) the commodity between firm i and demand market j. Here we permit the transaction cost to depend, in general, upon the entire shipment pattern, that is,

$$t_{ij} = t_{ij}(T). \tag{26}$$

The profit u_i of firm i is then

$$u_i = \sum_{j=1}^{n} p_j T_{ij} - f_i - \sum_{j=1}^{n} t_{ij} T_{ij}. \tag{27}$$

In view of (22) and (23), one may write

$$u = u(T). \tag{28}$$

Now consider the usual oligopolistic market mechanism, in which the m firms supply the commodity in a noncooperative fashion, each one trying to maximize its own profit. We seek to determine a nonnegative commodity distribution pattern T for which the m firms will be in a state of equilibrium as defined below.

Definition 2 (Spatial Cournot-Nash Equilibrium)

A commodity shipment distribution $T^* \in R^{mn}_+$ *is said to constitute a Cournot-Nash equilibrium if for each firm* $i; i = 1, \ldots, m$,

$$u_i(T_i^*, \hat{T}_i^*) \geq u_i(T_i, \hat{T}_i^*), \quad \forall T_i \in R^n_+, \tag{29}$$

where

$$T_i \equiv \{T_{i1}, \ldots, T_{in}\} \quad \textit{and} \quad \hat{T}_i^* \equiv (T_1^*, \ldots, T_{i-1}^*, T_{i+1}^*, \ldots, T_m^*).$$

The variational inequality formulation of the Cournot-Nash equilibrium is given in the following theorem.

Theorem 4 (Variational Inequality Formulation of Cournot-Nash Equilibrium)

Assume that for each firm i the profit function $u_i(T)$ is concave with respect to the variables $\{T_{i1}, \ldots, T_{in}\}$, and continuously differentiable. Then $T^* \in R_+^{mn}$ is a Cournot-Nash equilibrium if and only if it satisfies the variational inequality

$$-\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\partial u_i(T^*)}{\partial T_{ij}} \times (T_{ij} - T_{ij}^*) \geq 0, \quad \forall T \in R_+^{mn}. \tag{30}$$

Upon using (22) and (23), (30) takes the form:

$$\sum_{i=1}^{m} \frac{\partial f_i(q^*)}{\partial q_i} \times (q_i - q_i^*) + \sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij}(T^*) \times (T_{ij} - T_{ij}^*)$$

$$-\sum_{j=1}^{n} p_j(d^*) \times (d_j - d_j^*)$$

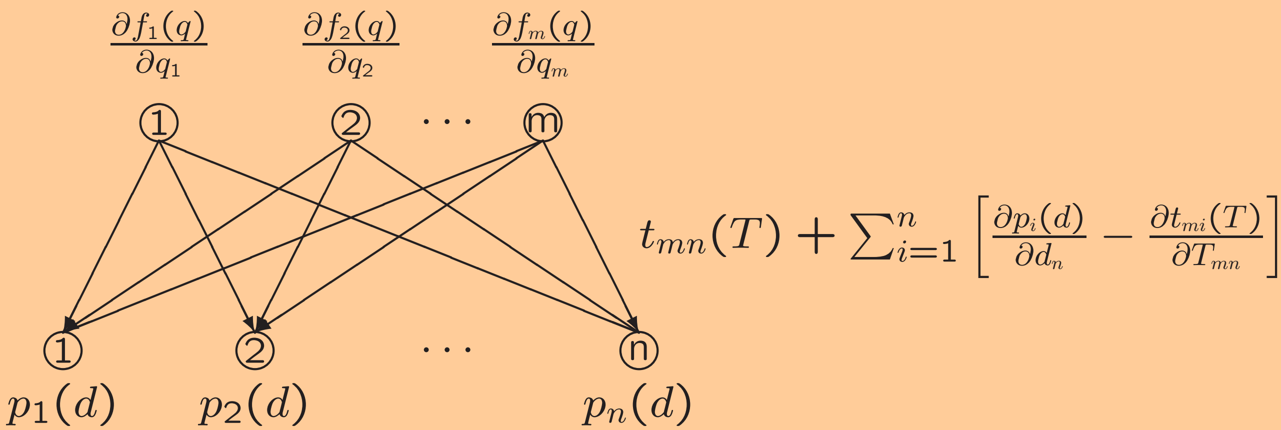
$$-\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{n} \left[\frac{\partial p_l(d^*)}{\partial d_j} - \frac{\partial t_{il}(T^*)}{\partial T_{ij}}\right] T_{il}^*(T_{ij} - T_{ij}^*) \geq 0,$$

$$\forall (q, T, d) \in K, \tag{31}$$

where $K \equiv \{(q,T,d) | T \geq 0, \text{and} \, (22) \, \text{and} \, (23) \text{ hold}\}$.

Note that, in the special case, where there is only a single demand market and the transaction costs are identically equal to zero, variational inequality (31) collapses to variational inequality (7).

The underlying network structure of the model is depicted in Figure 2 with the cost on link (i,j) given by $t_{ij}(T) + \sum_{l=1}^{n} \left[\frac{\partial p_l(d)}{\partial d_j} - \frac{\partial t_{il}(T)}{\partial T_{ij}} \right] T_{il}$.



Network structure of the spatial oligopoly problem

Relationship Between Spatial Oligopolies and Spatial Price Equilibrium Problems

Consider now a spatial oligopoly model of the type just discussed but endowed with the following structure.

The m firms are grouped into J groups: $S_1, \ldots, S_J$, called supply markets with m_a firms in supply market S_a, that is, $\sum_{a=1}^{J} m_a = m$ and $\cup_{a=1}^{J} S_a = \{1, 2, \ldots, m\}$. The firms in supply market S_a ship to demand market j a shipment Q_{aj} of the commodity given by

$$Q_{aj} = \sum_{i \in S_a} T_{ij}, \quad a = 1, \ldots, J; j = 1, \ldots, n. \tag{32}$$

The total production s_a of all firms in S_a is

$$s_a = \sum_{j=1}^{n} Q_{aj} = \sum_{i \in S_a} q_i = \sum_{j=1}^{n} \sum_{i \in S_a} T_{ij}. \tag{33}$$

Assume that the supply markets represent geographic locations and thus all firms belonging to the same supply market face identical production and transaction costs. This is expressed through the following assumptions:

(a). All firms in supply market S_a have the same production cost g_a, that is,

$$f_i = g_a, \quad \text{if} \quad i \in S_a. \tag{34}$$

(b). All firms in supply market S_a face the same transaction cost c_{aj} to the demand market j, that is,

$$t_{ij} = c_{aj}, \quad \text{if} \quad i \in S_a. \tag{35}$$

(c). The production cost of any firm in supply market S_a is determined solely by the production pattern, that is,

$$g = g(s) \tag{36}$$

where g and s are vectors in R^m with components g_a and s_a and g is a known smooth function.

(d). The transaction cost of any firm in a supply market S_a to the demand market j is determined solely by the shipment distribution

$$c = c(Q), \tag{37}$$

where c and Q are $J \times n$ matrices with components c_{aj} and Q_{aj} and c is a known smooth function.

Finally, the demand price at any demand market may depend, as in the general model, upon the commodity demand pattern, namely,

$$p = p(d), \tag{38}$$

where p and d are vectors in R^n with components p_j and d_j and p is a known smooth function.

In the present case if $i \in S_a$, we have by virtue of (34), (36), and (33),

$$\frac{\partial f_i}{\partial q_i} = \sum_b \frac{\partial g_a}{\partial s_b} \frac{\partial s_b}{\partial q_i} = \frac{\partial g_a}{\partial s_a}, \qquad .39)$$

and, due to (35), (37), and (32),

$$\frac{\partial t_{ij}}{\partial T_{il}} = \sum_{b,\gamma} \frac{\partial c_{aj}}{\partial Q_{b\gamma}} \frac{\partial Q_{b\gamma}}{\partial T_{il}} = \frac{\partial c_{aj}}{\partial Q_{al}}. \tag{40}$$

Using (39), (33), (35), (32), and (40), we may now write variational inequality (31) in the form:

$$\sum_a \frac{\partial g_a(s^*)}{\partial s_a}(s_a - s_a^*) + \sum_{aj} c_{aj}(Q^*) \times (Q_{aj} - Q_{aj}^*)$$

$$- \sum_j p_j(d^*) \times (d_j - d_j^*)$$

$$- \sum_{a,j,l} \left[\frac{\partial p_j(d^*)}{\partial d_l} - \frac{\partial c_{aj}(Q^*)}{\partial Q_{al}} \right] \sum_{i \in S_a} T_{ij}^*(T_{il} - T_{il}^*) \geq 0. \tag{41}$$

Let T^* be any solution of variational inequality (30). Construct any $\tilde{T} \in R_+^{mn}$ such that for any $j = 1, \ldots, n$, $a = 1, \ldots, J$ the set $\tilde{T}_{ij}$ with $i \in S_a$ is any permutation of the set T^*_{ij} with $i \in S_a$. Then it follows from (41) that $\tilde{T}$ is also a solution of variational inequality (30). Hence, (30) admits a unique solution, so that $T^* = \tilde{T}$, T^* must be symmetric, that is,

$$T^*_{ij} = \frac{1}{m_a} Q^*_{aj}, \quad a = 1, \ldots, J; j = 1, \ldots, n; i \in S_a. \tag{42}$$

The connection between oligopolistic equilibrium and spatial price equilibrium is now established.

Fix the number of supply and demand markets at J and n, respectively, as well as the function g in (36), the function c in (37), and the function p in (38), and construct a sequence of oligopolistic models of the type described in this section with $m_a \to \infty$, for $a = 1, \ldots, J$. Construct the corresponding sequence of symmetric oligopolistic equilibria $T^*_{(k)}$ which induces sequences $(s^*_{(k)}, Q^*_{(k)}, d^*_{(k)})$ of supply, shipment, and demand patterns.

Theorem 5

Any convergent subsequence of the sequence $(s^*_{(k)}, d^*_{(k)}, Q^*_{(k)})$ converges to (s^*, d^*, Q^*) which satisfies the spatial price variational inequality with $\pi_a = \frac{\partial g_a}{\partial s_a}$ (and $\rho_j \equiv p_j$, for all j). Thus, (s^*, d^*, Q^*) is a spatial price equilibrium with demand price functions p, transaction cost functions c, and supply price functions $\pi(s)$ with $\pi_a = \frac{\partial g_a}{\partial s_a}$, the marginal cost.

Sensitivity Analysis

We now discuss sensitivity analysis in the framework of Nash equilibria. The results are readily adaptable to the oligopoly models. First, consider the comparison of two equilibria. We begin with the following lemma.

Lemma 1

Let u and u^* denote two utility functions, and let x and x^* denote, respectively, their associated Nash equilibrium strategy vectors. Assume that u_i and u_i^* are concave with respect to $x_i \in K_i$ and $x_i^* \in K_i$, for each i, and continuously differentiable. Then

$$\langle \nabla u^*(x^*) - \nabla u(x), x^* - x \rangle \geq 0. \tag{45}$$

Moreover, when $-\nabla u$ is strictly monotone, then

$$\langle \nabla u^*(x^*) - \nabla u(x^*), x^* - x \rangle \geq 0, \tag{46}$$

with equality holding only when $x = x^*$.

Proof: Since x and x^* are both Nash equilibrium vectors, by Theorem 1 they must satisfy, respectively, the variational inequalities:

$$\langle \nabla u(x), y - x \rangle \leq 0, \quad \forall y \in K, \tag{47}$$

$$\langle \nabla u^*(x^*), y - x^* \rangle \leq 0, \quad \forall y \in K. \tag{48}$$

Letting $y = x^*$ in (47), and $y = x$ in (48), and summing the resulting inequalities, yields (45).

From (45) one has that

$$\langle \nabla u^*(x^*) - \nabla u(x) + \nabla u(x^*) - \nabla u(x^*), x^* - x \rangle \geq 0. \tag{49}$$

When $-\nabla u(x)$ is strictly monotone, (49) yields

$$\langle \nabla u^*(x^*) - \nabla u(x^*), x^* - x \rangle \geq -\langle \nabla u(x^*) - \nabla u(x), x^* - x \rangle \geq 0, \tag{50}$$

and, consequently, (46) follows with equality holding only when $x = x^*$.

We now present another result.

Theorem 6

Let u and u^* denote two utility functions, and x and x^* the corresponding Nash equilibrium strategy vectors. Assume that ∇u satisfies the strong monotonicity assumption

$$\langle \nabla u(x) - \nabla u(y), x - y \rangle \leq -\alpha \|x - y\|^2, \quad \forall x, y \in K, \quad (51)$$

where $\alpha > 0$. Then

$$\|x^* - x\| \leq \frac{1}{\alpha} \|\nabla u^*(x^*) - \nabla u(x^*)\|. \quad (52)$$

Proof: From Lemma 1 one has that (45) holds and from (45) one has that

$$\langle \nabla u^*(x^*) - \nabla u(x) + \nabla u(x^*) - \nabla u(x^*), x^* - x \rangle \geq 0. \quad (53)$$

But from the strong monotonicity condition (51), (53) yields

$$\langle \nabla u^*(x^*) - \nabla u(x^*), x^* - x \rangle \geq -\langle \nabla u(x^*) - \nabla u(x), x^* - x \rangle$$

$$\geq \alpha \|x^* - x\|^2. \quad (54)$$

By virtue of the Schwartz inequality, (54) yields

$$\alpha \|x^* - x\|^2 \leq \|\nabla u^*(x^*) - \nabla u(x^*)\| \|x^* - x\| \quad (55)$$

from which (52) follows and the proof is complete.

Below are references cited in the lecture as well as additional supplementary ones.

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