Migration Equilibrium

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Human migration is a topic that has received attention from economists, demographers, sociologists, and geographers. In this lecture, we focus on the development of a network framework using variational inequality theory in an attempt to formalize this challenging problem domain. In particular, we explore the utilization of variational inequality theory in conceptualizing complex problems in migration networks.

A series of migration models is presented of increasing complexity and generality. We assume that each class of migrant has a utility associated with locations, where the utilities are functions of the population distribution pattern. The framework is similar in spirit to the one developed by Beckmann (1957), who also focused on migratory flows and assumed that the attractiveness of a location was a function of the population distribution pattern.

Costless Migration

We first describe a model of human migration, which is shown to have a simple, abstract network structure in which the links correspond to locations and the flows on the links to populations of a particular class at the particular location.

Assume a closed economy in which there are n locations, typically denoted by i, and J classes, typically denoted by k. Assume further that the attractiveness of any location i as perceived by class k is represented by a utility u_i^k. Let $\bar{p}^k$ denote the fixed and known population of class k in the economy, and let p_i^k denote the population of class k at location i. Group the utilities into a row vector $u \in R^{Jn}$ and the populations into a column vector $p \in R^{Jn}$. Assume no births and no deaths in the economy.

The conservation of flow equation for each class k is given by

$$\bar{p}^k = \sum_{i=1}^{n} p_i^k \tag{1}$$

where $p_i^k \geq 0$, $\forall k=1,\ldots,J$; $i=1,\ldots,n$. Equation (1) states that the population of each class k must be conserved in the economy.

Let $K \equiv \{p | p \geq 0$, and satisfy (1)$\}$.

Assume that the migrants are rational and that migration will continue until no individual of any class has any incentive to move since a unilateral decision will no longer yield an increase in the utility. Mathematically, hence, a multiclass population vector $p^* \in K$ is said to be in equilibrium if for each class k; $k=1,\ldots,J$:

$$u_i^k \begin{cases} = \lambda^k, & \text{if} \quad p_i^{k^*} > 0 \\ \leq \lambda^k, & \text{if} \quad p_i^{k^*} = 0. \end{cases} \tag{2}$$

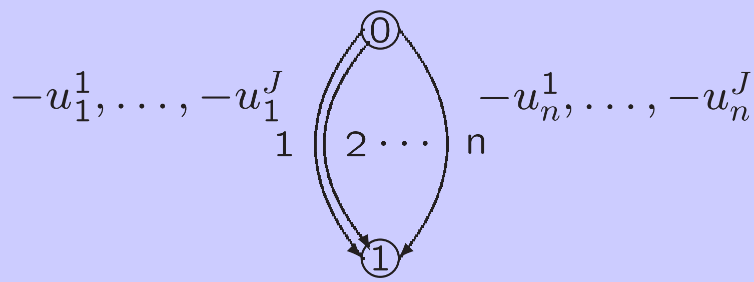
Equilibrium conditions (2) state that for a given class k only those locations i with maximal utility equal to an indicator λ^k will have a positive volume of the class. Moreover, the utilities for a given class are equilibrated across the locations.

The function structure is now addressed. Assume that, in general, the utility associated with a particular location as perceived by a particular class, may depend upon the population associated with every class and every location, that is, assume that

$$u = u(p). \tag{3}$$

Note that in allowing the utility to depend upon the populations of the classes, we are, in essence, using populations as a proxy for amenities associated with a particular location; at the same time, such a utility function can handle the negative externalities associated with overpopulation, such as congestion, increased crime, competition for scarce resources, etc.

The above migration model is equivalent to a network equilibrium model with a single origin/destination pair and fixed demands. Indeed, make the identification as follows. Construct a network consisting of two nodes, an origin node 0 and a destination node 1, and n links connecting the origin node to the destination node (cf. Figure 1).



$$\bar{p}^1 = \sum_{i=1}^{n} p_i^1, \ldots, \bar{p}^J = \sum_{i=1}^{n} p_i^J$$

Network equilibrium formulation of costless migration

If we associate then with each link i, J costs: $-u_i^1, \ldots, -u_i^J$, and link flows represented by $p_i^1, \ldots, p_i^J$. This model is, hence, equivalent to a multimodal traffic network equilibrium model with fixed demand for each mode, a single origin/destination pair, and J paths connecting the O/D pair. Of course, one can make J copies of the network, in which case each k-th network will correspond to class k with the cost functions on the links defined accordingly. This identification enables us to immediately write down the following:

Theorem 1 (Variational Inequality Formulation of Costless Migration Equilibrium)

A population pattern $p^* \in K$ is in equilibrium if and only if it satisfies the variational inequality problem:

$$\langle -u(p^*), p - p^* \rangle \geq 0, \quad \forall p \in K. \tag{4}$$

Qualitative Properties and Computation of Solutions

Existence of an equilibrium then follows from the standard theory, since the feasible set K is compact, assuming that the utility functions are continuous. Uniqueness of the equilibrium population pattern also follows from standard variational inequality theory, provided that the $-u$ function is strictly monotone. In the context of applications, this monotonicity condition implies that the utility associated with a given class and location is expected to be a decreasing function of the population of that class at that location; hence, for uniqueness to be guaranteed, "congestion" of the system is critical.

This model is amenable to solution by a variety of algorithms, including, the projection method. The projection method will resolve the solution of variational inequality (4) into separable quadratic programming problems, if the matrix G is chosen to be diagonal, which can then, in turn, be solved exactly, and in closed form, using the fixed demand market exact equilibration algorithm.

Note that the network equilibrium equivalent of the above model is constructed over an abstract network in that the nodes do not correspond to locations in space; in contrast, the links are identified with locations in space.

Migration with Migration Costs

Now a network model of human migration equilibrium is developed, which allows not only for multiple classes but for migration costs between locations. In this framework the cost of migration reflects both the cost of transportation (a proxy for distance) and the "psychic" costs associated with dislocation.

The importance of translocation costs in migration decision-making is well-documented in the literature from both theoretical and empirical perspectives.

Economic research, however, has emphasized the development of equilibrium models in which the population is assumed to be perfectly mobile and the costs of migration insignificant. In such models, as in the model just described, individuals and/or households are assumed to select a location until the utilities are equalized across the economy.

Assume, as before, a closed economy in which there are n locations, typically denoted by i, and J classes, typically denoted by k. Further, assume that the attractiveness of any location i as perceived by class k is represented by a utility u_i^k. Let $\bar{p}_i^k$ denote the initial fixed population of class k in location i, and let p_i^k denote the population of class k in location i. Group the utilities into a row vector $u \in R^{Jn}$ and the populations into a column vector $p \in R^{Jn}$. Again, assume the situation in which there are no births and no deaths in the economy.

Associate with each class k and each pair of locations i, j a nonnegative cost of migration c_{ij}^k and let the migration flow of class k from origin i to destination j be denoted by f_{ij}^k.

The migration costs are grouped into a row vector $c \in R^{Jn(n-1)}$ and the flows into a column vector $f \in R^{Jn(n-1)}$. Assume that the migration costs reflect not only the cost of physical movement but also the personal and psychic cost as perceived by a class in moving between locations.

The conservation of flow equations, given for each class k and each location i, assuming no repeat or chain migration, are

$$p_i^k = \bar{p}_i^k + \sum_{l \neq i} f_{li}^k - \sum_{l \neq i} f_{il}^k \tag{5}$$

and

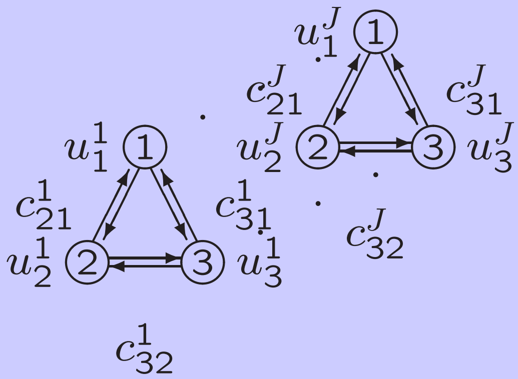
$$\sum_{l \neq i} f_{il}^k \leq \bar{p}_i^k, \tag{6}$$

$f_{il}^k \geq 0$, $\forall k = 1, \ldots, J$; $l \neq i$.

$$K \equiv \{(p, f) | f \geq 0, (p, f) \text{ satisfy } (5), (6)\}.$$

Equation (5) states that the population at location i of class k is given by the initial population of class k at location i plus the migration flow into i of that class minus the migration flow out of i for that class. Equation (6) states that the flow out of i by class k cannot exceed the initial population of class k at i, since no chain migration is allowed.

The multiclass network model with migration costs is now constructed. In particular, construct n nodes, $i = 1, \ldots, n$, to represent the locations and a link (i, j) connecting each pair of nodes. There are, hence, n nodes in the network and $n(n-1)$ links. With each link (i, j) associate k costs c_{ij}^k and corresponding flows f_{ij}^k. With each node i associate k utilities u_i^k and the initial positive populations $\bar{p}_i^k$. A graphic depiction of a three-location migration network is given in Figure 2, where the classes are layered. Of course, rather than a multiclass network, one can construct J copies of the network topology given in Figure 2 to represent the classes where the costs on the links and the utilities are defined accordingly.



The multiclass migration network with three locations

We are now ready to state the equilibrium conditions. As before, assume that migrants are rational and that migration will continue until no individual has any incentive to move since a unilateral decision will no longer yield a positive net gain (gain in utility minus migration cost). Mathematically, the multiclass equilibrium conditions are stated as follows. A multiclass population and flow pattern $(p^*, f^*) \in K$ is in equilibrium, if for each class $k; k = 1, \ldots, J$, and each pair of locations i, j; $i = 1, \ldots, n$; $j \neq i$:

$$u_i^k + c_{ij}^k \begin{cases} = u_j^k - \lambda_i^k, & \text{if } f_{ij}^{k^*} > 0 \\ \geq u_j^k - \lambda_i^k, & \text{if } f_{ij}^{k^*} = 0 \end{cases} \tag{7}$$

and

$$\lambda_i^k \begin{cases} \geq 0, & \text{if } \sum_{l \neq i} f_{il}^{k^*} = \bar{p}_i^k \\ = 0, & \text{if } \sum_{l \neq i} f_{il}^{k^*} < \bar{p}_i^k. \end{cases} \tag{8}$$

Equilibrium conditions (7) and (8), although similar in structure to the equilibrium conditions governing the multicommodity spatial price equilibrium problem, differ significantly in that the indicator λ_i^k is present. The necessity of λ_i^k, and, in particular, condition (8), are now interpreted.

Observe that, unlike spatial price equilibrium problems (or the related traffic network equilibrium problem with elastic demand), the level of the population $\bar{p}_i^k$ may not be large enough so that the gain in utility $u_j^k - u_i^k$ is exactly equal to the cost of migration c_{ij}^k. Nevertheless, the utility gain minus the migration cost will be maximal and nonnegative. Moreover, the net gain will be equalized for all locations and classes which have a positive flow out of a location. In fact, λ_i^k is exactly the equalized net gain for all individuals of class k of location i.

First, the function structure is discussed and then the variational inequality formulation of the equilibrium conditions (7) and (8) is derived.

Assume, as before, that the utility associated with a particular location and class can depend upon the population associated with every class and every location, that is,

$$u = u(p). \tag{9}$$

Assume also that the cost associated with migrating between two locations as perceived by a particular class can depend, in general, upon the flows of every class between every pair of locations, that is,

$$c = c(f). \tag{10}$$

The variational inequality formulation of the migration equilibrium conditions is given by:

Theorem 2 (Variational Inequality Formulation of Migration Equilibrium with Migration Costs)

A population and migration flow pattern $(p^*, f^*) \in K$ *satisfies equilibrium conditions (7) and (8) if and only if it satisfies the variational inequality problem*

$$\langle -u(p^*), p - p^* \rangle + \langle c(f^*), f - f^* \rangle \geq 0, \quad \forall (p, f) \in K. \quad (11)$$

Proof: We first show that if a pattern (p^*, f^*) satisfies equilibrium conditions (7) and (8), subject to constraints (5) and (6), then it also satisfies the variational inequality in (11).

Suppose that (p^*, f^*) satisfies the equilibrium conditions. Then

$$f_{ij}^{k^*} \geq 0 \quad \text{and} \quad \sum_{l \neq i} f_{il}^{k^*} \leq \bar{p}_i^k, \quad \forall i, j, k.$$

For fixed class k we define $\Gamma_1^k = \{l | f_{il}^{k^*} > 0\}$ and $\Gamma_2^k = \{l | f_{il}^{k^*} = 0\}$. Then

$$\sum_{l \neq i} \left[u_i^k(p^*) + c_{il}^k(f^*) - u_l^k(p^*) \right] \times \left[f_{il}^k - f_{il}^{k^*} \right]$$

$$= \sum_{l \in \Gamma_1^k} \left[u_i^k(p^*) + c_{il}^k(f^*) - u_l^k(p^*) \right] \times \left[f_{il}^k - f_{il}^{k^*} \right]$$

$$+ \sum_{l \in \Gamma_2^k} \left[u_i^k(p^*) + c_{il}^k(f^*) - u_l^k(p^*) \right] \times \left[f_{il}^k - f_{il}^{k^*} \right]$$

$$\geq -\lambda_i^k \sum_{l \in \Gamma_1^k} (f_{il}^k - f_{il}^{k^*}) + \sum_{l \in \Gamma_2^k} (-\lambda_i^k)(f_{il}^k)$$

$$= -\lambda_i^k (\sum_{l \neq i} f_{il}^k - \sum_{l \neq i} f_{il}^{k^*}) \begin{cases} = 0, & \text{if } \sum_{l \neq i} f_{il}^{k^*} < \bar{p}_i^k \\ \geq 0, & \text{if } \sum_{l \neq i} f_{il}^{k^*} = \bar{p}_i^k \end{cases}$$

holds for all such locations i.

Therefore, for this class k and all locations i, $f_{il}^{k^*} \geq 0$, $\sum_{l \neq i} f_{il}^{k^*} \leq \bar{p}_i^k$, and

$$\sum_{i=1}^{n} \sum_{l \neq i} \left[u_i^k(p^*) + c_{il}^k(f^*) - u_l^k(p^*) \right] \times \left[f_{il}^k - f_{il}^{k^*} \right] \geq 0. \quad (12)$$

But inequality (12) holds for each k; hence,

$$\sum_{k=1}^{J}\sum_{i=1}^{n}\sum_{l\neq i}\left[u_i^k(p^*)+c_{il}^k(f^*)-u_l^k(p^*)\right]\times\left[f_{il}^k-f_{il}^{k^*}\right]\geq 0. \tag{13}$$

Observe now that inequality (13) can be rewritten as:

$$\sum_{k=1}^{J}\sum_{l=1}^{n}u_l^k(p^*)\times((\sum_{j\neq l}f_{lj}^k-\sum_{j\neq l}f_{jl}^k)-(\sum_{j\neq l}f_{lj}^{k^*}-\sum_{j\neq l}f_{jl}^{k^*}))$$

$$+\sum_{k=1}^{J}\sum_{i=1}^{N}\sum_{l\neq i}c_{il}^k(f^*)\times(f_{il}^k-f_{il}^{k^*})\geq 0. \tag{14}$$

Using constraint (5), and substituting it into (14), one concludes that

$$-\sum_{k=1}^{J}\sum_{l=1}^{n}u_l^k(p^*)\times(p_l^k-p_l^{k^*})+\sum_{k=1}^{J}\sum_{i=1}^{n}\sum_{l\neq i}c_{il}^k(f^*)\times(f_{il}^k-f_{il}^{k^*})\geq 0, \tag{15}$$

or, equivalently, in vector notation,

$$\langle -u(p^*),p-p^*\rangle+\langle c(f^*),f-f^*\rangle\geq 0, \qquad \forall (p,f)\in K. \tag{16}$$

We now show that if a pattern $(p^*, f^*) \in K$ satisfies variational inequality (11), then it also satisfies equilibrium conditions (7) and (8). Suppose that (p^*, f^*) satisfies variational inequality (5.11). Then

$$\langle -u(p^*), p\rangle + \langle c(f^*), f\rangle \geq \langle -u(p^*), p^*\rangle + \langle c(f^*), f^*\rangle, \forall (p, f) \in K.$$

Hence, (p^*, f^*) solves the minimization problem

$$\text{Min}_{(p,f)\in K}\langle -u(p^*), p\rangle + \langle c(f^*), f\rangle, \tag{17}$$

or, equivalently, (17) may be expressed solely in terms of f, that is,

$$\text{Min}_{f'\in K_1}\langle -\widehat{u}(Af^*), Af\rangle + \langle c(f^*), f\rangle \tag{18}$$

where $K_1 \equiv \{f | f \geq 0, \text{satisfying}\,(5.6)\}$, A is the arc-node incidence matrix in (5), and $\widehat{u}(Af^*) \equiv u(p^*)$.

Since the constraints in K are linear, one has the following Kuhn-Tucker conditions: There exist

$$\lambda = (\lambda_i^k) \geq 0, \tag{19}$$

such that

$$\lambda_i^k (\sum_{l\neq i} f_{il}^{k^*} - \bar{p}_i^k) = 0 \tag{20}$$

and

$$u_i^k - u_j^k + c_{ij}^k + \lambda_i^k \geq 0 \tag{21}$$

$$(u_i^k - u_j^k + c_{ij}^k + \lambda_i^k) f_{ij}^{k^*} = 0. \tag{22}$$

Clearly, equilibrium conditions (7) and (8) follow from (19) – (22). The proof is complete.

Qualitative Properties

Existence of at least one solution to variational inequality (11) follows from the standard theory of variational inequalities, under the sole assumption of continuity of the utility and migration cost functions u and c, since the feasible convex set K is compact. Uniqueness of the equilibrium population and migration flow pattern (p^*, f^*) follows under the assumption that the utility and movement cost functions are strictly monotone, that is,

$$-\langle u(p^1) - u(p^2), p^1 - p^2 \rangle + \langle c(f^1) - c(f^2), f^1 - f^2 \rangle > 0,$$

$$\forall (p^1, f^1), (p^2, f^2) \in K, \text{ such that } (p^1, f^1) \neq (p^2, f^2). \quad (23)$$

We now interpret monotonicity condition (23) in terms of the applications. Under reasonable economic situations, the monotonicity condition (23) can be verified.

Essentially, it is assumed that the system is subject to congestion; hence, the utilities are decreasing with larger populations, and the movement costs are increasing with larger migration flows.

Furthermore, each utility function $u_i^k(p)$ depends mainly on the population p_i^k, and each movement cost $c_{ij}^k(f)$ depends mainly on the flow f_{ij}^k. Mathematically, the strict monotonicity condition will hold, for example, when $-\nabla u$ and ∇c are diagonally dominant.

Migration with Class Transformations

A network model of human migration equilibrium is now developed, which allows not only for multiple classes and migration costs between locations but also for class transformations.

In this model users select the class/location combination that will yield the greatest net gain, where the net gain is defined as the gain in utility minus the migration cost.

The cost here reflects both the cost associated with translocation and the cost associated with training, education, and the like, if there is migration across classes either within a location or across locations. This model may also be viewed as a framework for labor movements.

As in the preceding two migration models, assume a closed economy in which there are n locations, typically denoted by i, and J classes, typically denoted by k. The utility functions and the population vectors are as defined for the preceding model. However, now associate with each pair of class/location combinations k, i and l, j a nonnegative cost of migration c_{ij}^{kl} and let the migration flow of class k from origin i to class l at destination j be denoted by f_{ij}^{kl}.

Note that in the case where the destination class l is identical to the origin class k, then the migration cost c_{ij}^{kk} represents the cost of translocation, which includes not only the cost of physical movement but also the psychic cost as perceived by this class in moving between the pair of locations.

On the other hand, when the destination location j is equal to the origin location i, the cost c_{ii}^{kl} represents the cost of transforming from class k to class l while staying in location i. Hence, the migration cost here is interpreted in a general setting as including the cost of migrating from class to class. The migration costs are grouped into a row vector $c \in R^{Jn(Jn-1)}$, and the flows into a column vector $f \in R^{Jn(Jn-1)}$.

The conservation of flow equations are given for each class k and each region i, assuming no repeat or chain migration, by

$$p_i^k = \bar{p}_i^k + \sum_{(l,h)\neq(k,i)} f_{hi}^{lk} - \sum_{(l,h)\neq(k,i)} f_{ih}^{kl} \tag{24}$$

and

$$\sum_{(l,h)\neq(k,i)} f_{ih}^{kl} \leq \bar{p}_i^k, \tag{25}$$

where $f_{ih}^{kl} \geq 0$, for all (k,l); k=1, ..., J; l=1, ..., J, (h,i); $h = 1, \ldots, n$; i=1, ..., n. Let $K \equiv \{(p,f)|f \geq 0$, and satisfy (24), (25).

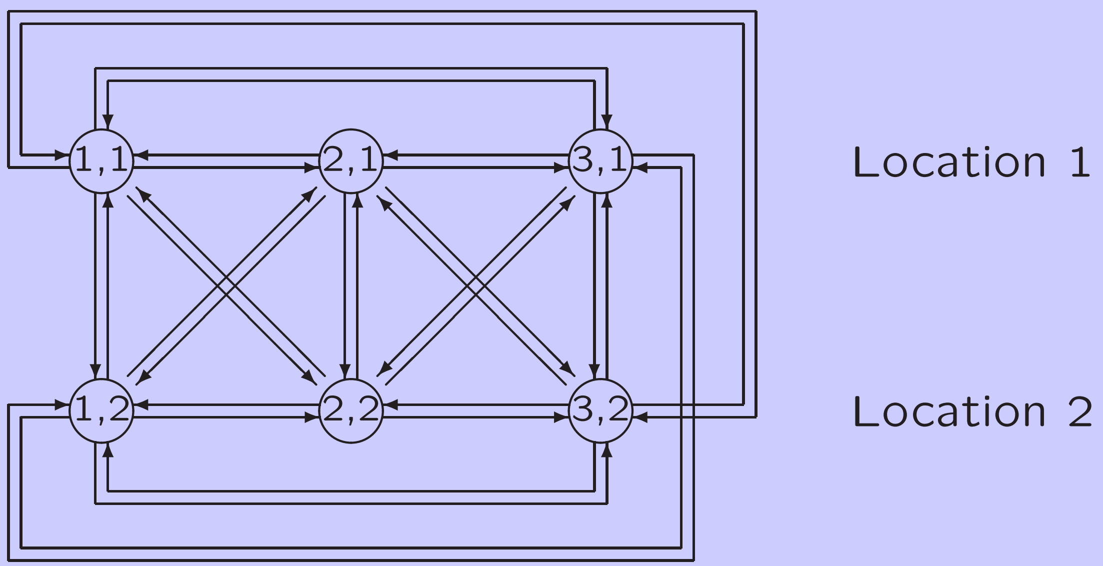
Equation (24) states that the population in location i of class k is given by the initial population of class k in location i plus the migration flow into i of that class and transformations of other classes into that class from this and other locations minus the migration flow out of i for that class and transformations of that class to other classes at this and other locations. Equation (25) states that the flow out of i by class k cannot exceed the initial population of class k at i, since no chain migration is allowed.

The general network model with class transformations is now presented. For each class k, construct n nodes, (k, i); $i = 1, \ldots, n$, to represent the locations and a link (ki, kj) connecting each such pair of nodes.

These links, hence, represent migration links within a class. From each node (k, i) construct $Jn-1$ links joining each node (k, i) to node (l, h) where $l \neq k$; $l = 1, \ldots, J$; $h = 1, \ldots, n$.

These links represent migration links which are class transformation links. There are, hence, a total of Jn nodes in the network and $Jn(Jn-1)$ links. Note that each node may be interpreted as a state in class/location space.

With each link (ki, lj) associate the cost c_{ij}^{kl} and the corresponding flow f_{ij}^{kl}. With each node (k, i) associate the utility u_i^k and the initial positive population $\bar{p}_i^k$. A graphical depiction of a two-region, three-class migration network is given in Figure 3.



The transformation network for two locations and three classes

We are now ready to state the equilibrium conditions, following those presented for the model with migration costs. Assume that migrants are rational and that migration will continue until no individual has any incentive to move since a unilateral decision will no longer yield a positive net gain (gain in utility minus migration cost).

Mathematically, the multiclass equilibrium conditions are stated as follows. A multiclass population and flow pattern $(p^*, f^*) \in K$ is said to be in equilibrium if for each pair (k,i) and (l,j); $(k,l), k = 1,\ldots,J$; $l = 1,\ldots,J$, (i,j), $i = 1,\ldots,n$; $j = 1,\ldots,n$:

$$u_i^k + c_{ij}^{kl} \begin{cases} = u_j^l - \lambda_i^k, & \text{if } f_{ij}^{kl^*} > 0 \\ \geq u_j^l - \lambda_i^k, & \text{if } f_{ij}^{kl^*} = 0 \end{cases} \tag{26}$$

and

$$\lambda_i^k \begin{cases} \geq 0, & \text{if } \sum_{(l,h)\neq(k,i)} f_{ih}^{kl^*} = \bar{p}_i^k \\ = 0, & \text{if } \sum_{(l,h)\neq(k,i)} f_{ih}^{kl^*} < \bar{p}_i^k. \end{cases} \tag{27}$$

Observe that the population $\bar{p}_i^k$ may not be large enough so that the gain in utility $u_j^l - u_i^k$ is exactly equal to the cost of migration c_{ij}^{kl}. Nevertheless, the utility gain minus the migration cost will be maximal and nonnegative. Moreover, the net gain will be equalized for all classes/locations which have a positive flow out of a location of that class. In fact, λ_i^k is exactly the equalized net gain for all individuals of class k in location i. In the case where no class transformations are allowed, in other words, $l = k$, then the above equilibrium conditions collapse to those given for the model with migration costs.

Assume that, in general, the utility associated with a particular location and class can depend upon the population associated with every class and every location, as similarly assumed in the preceding migration models.

Also assume that, in general, the cost associated with migrating between two distinct pairs of classes/locations as perceived by a particular class can depend, in general, upon the flows of every class between every pair of locations, as well as the flows between every pair of classes.

The equilibrium conditions are illustrated through the following example.

Example 1

Consider the migration problem with two classes and two locations where the utility functions are:

$$u_1^1(p) = -p_1^1 + 5 \quad u_1^2(p) = -p_1^2 - .5p_1^1 + 20$$

$$u_2^1(p) = -p_2^1 + 15 \quad u_2^2(p) = -p_2^2 + .5p_1^1 + 10$$

and assume that the migration cost functions are:

$$c_{11}^{12}(f) = f_{11}^{12} + .5f_{12}^{12} + 1 \quad c_{11}^{21}(f) = f_{11}^{21} + 1$$

$$c_{12}^{11}(f) = f_{12}^{11} + .2f_{12}^{12} + 10 \quad c_{21}^{11}(f) = f_{21}^{11} + 10$$

$$c_{12}^{12}(f) = f_{12}^{12} + .1f_{12}^{11} + 5 \quad c_{21}^{21}(f) = f_{21}^{21} + 20$$

$$c_{12}^{22}(f) = f_{12}^{22} + .3f_{21}^{12} + 2 \quad c_{21}^{22}(f) = f_{21}^{22} + 3$$

$$c_{12}^{21}(f) = f_{12}^{21} + 15 \quad c_{21}^{12}(f) = f_{21}^{12} + .2f_{21}^{11} + 15$$

$$c_{22}^{12}(f) = f_{22}^{12} + 10 \quad c_{22}^{21}(f) = 3f_{22}^{21} + 2f_{11}^{21} + 1.$$

The fixed populations are:

$$\bar{p}_1^1 = 1 \quad \bar{p}_1^2 = 5 \quad \bar{p}_2^1 = 1 \quad \bar{p}_2^2 = 3,$$

with associated initial utilities

$$u_1^1 = 4 \quad u_1^2 = 15 \quad u_2^1 = 14 \quad u_2^2 = 7.$$

The equilibrium populations and the flow pattern are:

$$p_1^{1^*} = 0 \quad p_1^{2^*} = 7 \quad p_2^{1^*} = 2 \quad p_2^{2^*} = 1$$

$$f_{11}^{12^*} = f_{21}^{22^*} = f_{22}^{21^*} = 1, \quad \text{all other} \quad f_{ij}^{kl^*} = 0,$$

and with associated equilibrium utilities

$$u_1^1 = 5 \quad u_1^2 = 13 \quad u_2^1 = 13 \quad u_2^2 = 9.$$

We now verify that this population and flow pattern satisfies equilibrium conditions (26) and (27).

Class 1, Location 1

Observe that in this case the final population is $p_1^{1^*} = 0$, and, hence, the original population was exhausted. Note that

$u_1^1 + c_{11}^{12} = 5 + 2 = u_1^2 = 13 - \lambda_1^1$, where $\lambda_1^1 = 6$ and $f_{11}^{12^*} > 0$

$$u_1^1 + c_{12}^{12} = 5 + 5 \geq u_2^2 = 9, \quad \text{and} \quad f_{12}^{12^*} = 0$$

$$u_1^1 + c_{12}^{11} = 5 + 10 \geq u_2^1 = 13, \quad \text{and} \quad f_{12}^{11^*} = 0.$$

Class 2, Location 2

Note that here the final population is $p_2^{2^*} = 1$, and, hence, this population is not exhausted. Note also that

$$u_2^2 + c_{21}^{22} = 9 + 4 = u_1^2 = 13, \quad \text{and} \quad f_{21}^{22^*} > 0$$

$$u_2^2 + c_{22}^{21} = 9 + 4 = u_2^1 = 13, \quad \text{and} \quad f_{22}^{21^*} > 0$$

$$u_2^2 + c_{21}^{21} = 9 + 20 \geq u_1^1 = 5, \quad \text{and} \quad f_{21}^{21^*} = 0.$$

Both class 1, location 2 and class 2, location 1 have zero migration flow out with the equilibrium conditions $u_i^k + c_{ij}^{kl} \geq u_j^l$ holding, as is easy to verify. Thus, the above population and flow distribution patterns satisfy the migration equilibrium conditions (26) and (27), and the conservation of flow equations (24) and (25) also hold.

The variational inequality formulation of the above migration equilibrium conditions is given below. The proof follows from similar arguments as given in the proof of the preceding variational inequality.

Theorem 3 (Variational Inequality Formulation of Migration Equilibrium with Class Transformations)

A population and migration flow pattern $(p^*, f^*) \in K$ *satisfies equilibrium conditions (26) and (27) if and only if it satisfies the variational inequality problem*

$$\langle -u(p^*), p - p^* \rangle + \langle c(f^*), f - f^* \rangle \geq 0, \quad \forall (p, f) \in K, \tag{28}$$

where $K \equiv \{(p, f) | f \geq 0,$ *and* (p, f) *satisfy* $(24), (25)\}$.

Existence of at least one solution to variational inequality (28) is again guaranteed by the standard theory under the sole assumption of continuity of the utility and migration cost functions u and c, since the feasible set K is compact. Uniqueness of the equilibrium population and migration flow pattern follows from the assumption that the utility and migration cost functions are strictly monotone.

The above model can be further interpreted in the context of the migration network model described before as follows. If one makes the identification that each node in the network model (cf. Figure 3) is, indeed, a "location," albeit a location in class/location space, then the model developed here with J classes and n regions is structurally isomorphic to the human migration model of Section 2 in the case of a single class and Jn locations, in which asymmetric utility functions and migration cost functions are, of course, permitted. The model just described, nevertheless, is the richer model conceptually and more general from an application point of view.

Furthermore, the development here illustrates and yet another network equilibrium model in which the network representation is fundamental to the formulation, understanding, and, as shall be demonstrated in the subsequent section, the ultimate solution of the problem at hand.

Computation of Migration Equilibria

The variational inequality decomposition algorithm for the solution of the multiclass human migration equilibrium problem is now presented. Note that, as discussed above, the network model with class transformations can be reformulated as the model with migration costs with the appropriate identification between nodes corresponding to locations and nodes corresponding to class/location combinations. Hence, the algorithm described below is applicable to both models. The decomposition algorithm is based crucially on the special structure of the underlying network (cf. Figure 2).

In particular, note that the feasible set K for variational inequality (11) can be expressed as the Cartesian product

$$K = \prod_{k=1}^{J} K^k, \tag{29}$$

where $K^k \equiv \{(p^k, f^k) | p^k = \{p_i^k; i = 1, \ldots, n\}\}$; $f^k = \{f_{ij}^k, i = 1, \ldots, n; j = 1, \ldots, n; j \neq i\}$, and satisfying (5) and (6).

One can, hence, decompose the variational inequality governing the multiclass migration network equilibrium problem into J simpler variational inequalities in lower dimensions. Each variational inequality in the decomposition corresponds to a particular class which, after linearizing, is equivalent to a quadratic programming problem and can be solved by the migration equilibration algorithm developed in Nagurney (1989).

That algorithm is a relaxation scheme and proceeds from location (node) to location (node), at each step computing the migratory flow out of the location exactly and in closed form. This can be accomplished because the special network structure of the problem lies in that each of the paths from an origin location to the $n-1$ potential destination locations are disjoint.

The statement of the decomposition algorithm by classes is as follows

The Linearization Decomposition Algorithm by Classes

Step 0: Initialization

Given an initial feasible solution (p^0, f^0), set $t := 0$ and $k := 1$.

Step 1: Linearization and Computation

Solve for $(p^k)^{t+1}, (f^k)^{t+1}$ in the following separable variational inequality:

$$\sum_{i=1}^{n}(q_i^k - (p_i^k)^{t+1})$$

$$\times(-u_i^k((p^1)^{t+1}, \ldots, (p^{k-1})^{t+1}, (p^k)^t, \ldots, (p^J)^t)$$

$$-\frac{\partial u_i^k}{\partial p_i^k}((p^1)^{t+1}, \ldots, (p^{k-1})^{t+1}, (p^k)^t, \ldots, (p^J)^t)$$

$$\times((p_i^k)^{t+1} - (p_i^k)^t))$$

$$+\sum_i \sum_{j \neq i}(g_{ij}^k - (f_{ij}^k)^{t+1})$$

$$\times(c_{ij}^k((f^1)^{t+1}, \ldots, (f^{k-1})^{t+1}, (f^k)^t, \ldots, (f^J)^t)$$

$$+\frac{\partial c_{ij}^k}{\partial f_{ij}^k}((f^1)^{t+1},\ldots,(f^{k-1})^{t+1},(f^k)^t,\ldots,(f^J)^t)$$

$$\times(((f_{ij}^k)^{k+1}-(f_{ij}^k)^t))\geq 0 \quad (30)$$

$$\forall q_i^k \geq 0,\ g_{ij}^k \geq 0,$$

such that $\sum_{j\neq i} g_{ij}^k \leq \bar{p}_i^k$ and $q_i^k = \bar{p}_i^k - \sum_{j\neq i}(g_{ij}^k - g_{ji}^k)$.

If $k < J$, then let $k := k+1$, and go to Step 1; otherwise, go to Step 2.

Step 2: Convergence Verification

If equilibrium conditions (7) and (8) hold for a given prespecified tolerance $\epsilon > 0$, then stop; otherwise, let $t := t + 1$, and go to Step 1.

The global convergence proof for the above linearized decomposition algorithm is now stated. In addition, sufficient conditions that guarantee the convergence are also given.

Let

$$A(p,f) = \begin{bmatrix} A_1(p,f) & & \\ & \ddots & \\ & & A_J(p,f) \end{bmatrix} \tag{31}$$

where

$$A_k(p,f) = \begin{bmatrix} -\frac{\partial u_1^k}{\partial p_1^k} & & & & & \\ & \ddots & & & & \\ & & -\frac{\partial u_n^k}{\partial p_n^k} & & & \\ & & & \ddots & & \\ & & & & \frac{\partial c_{ij}^k}{\partial f_{ij}^k} & \\ & & & & & \ddots \end{bmatrix}_{n^2 \times n^2} \tag{32}$$

and (p,f) is feasible.

Theorem 4 (Convergence of the Linearized Decomposition Algorithm)

Suppose that there exist symmetric positive definite matrices G_k such that $A_k(p,f) - G_k$ is positive semidefinite for all feasible (p,f) and that there exists a $\beta \in (0,1]$ such that

$$\|G_k^{-1}(-u_1^k(p)+u_1^k(q)+\frac{\partial u_1^k}{\partial p_1^k}(q)\times(p_1^k-q_1^k),\ldots,-u_n^k(p)+u_n^k(q)$$

$$+\frac{\partial u_n^k}{\partial p_n^k}(q)\times(p_n^k-q_n^k),\ldots,c_{ij}^k(f)-c_{ij}^k(g)$$

$$-\frac{\partial c_{ij}^k}{\partial f_{ij}^k}(g)\times(f_{ij}^k-g_{ij}^k)\ldots)_{n^2}^T\|_k$$

$$\leq \beta \max_{\beta} \|(p_1^k-q_1^k,\ldots,p_n^k-q_n^k,\ldots,f_{ij}^k-g_{ij}^k,\ldots)_{n^2}\|_k \quad (33)$$

where $\|\cdot\|_k = (\cdot^T G_k \cdot)^{\frac{1}{2}}$. Then the linearized decomposition algorithm by classes converges to the unique solution of the variational inequality geometrically.

In the case when $-u$, c are separable, that is,

$$u_i^k(p) = u_i^k(p_i^k), \quad c_{ij}^k(f) = c_{ij}^k(f_{ij}^k), \tag{34}$$

the positive semidefiniteness of $A_k(p, f) - G_k$ is equivalent to the strong monotonicity of $(-u^k, c^k)$ for each block k.

In fact, if $A_k(p, f) - G_k$ is positive semidefinite, then

$$\sum_{i=1}^{n} (-u_i^k(p) + u_i^k(q)) \times (p_i^k - q_i^k)$$

$$+ \sum_{i=1}^{n} \sum_{j \neq i} (c_{ij}^k(f) - c_{ij}^k(g)) \times (f_{ij}^k - g_{ij}^k)$$

$$= \sum_{i=1}^{n} (-\frac{\partial u_i^k}{\partial p_i^k}(\eta)) \times (p_i^k - q_i^k)^2 + \sum_{i,j \neq i} \frac{\partial c_{ij}^k}{\partial f_{ij}^k}(\zeta) \times (f_{ij}^k - g_{ij}^k)^2$$

$$\geq \alpha (\sum_{i=1}^{n} (p_i^k - q_i^k)^2 + \sum_{i=1}^{n} \sum_{j \neq i} (f_{ij}^k - g_{ij}^k)^2), \tag{35}$$

that is, $(-u^k, c^k)$ is strongly monotone. The converse is clear from the above inequality.

The norm inequality condition is actually a measure of linearity of $-u$ and c. In particular, when $-u$, c are linear and separable, the inequality is automatically satisfied, since the lefthand side is zero. Of course, the variational inequality can be solved for each class by the migration equilibration algorithm in this extremal case. A not-too-large perturbation from this case means not-too-strong interactions among classes and locations.

Numerical Results

The numerical results for the decomposition algorithm are presented in this section.

The algorithm was implemented in FORTRAN and compiled using the FORTVS compiler, optimization level 3. The special-purpose migration equilibration algorithm outlined in Nagurney (1989) was used for the embedded quadratic programming problems. The system used was an IBM 3090/600J at the Cornell National Supercomputer Facility. All of the CPU times reported are exclusive of input/output times, but include initialization times. The initial pattern for all the runs was set to $(p^0, f^0) = 0$. The convergence tolerance used was $\epsilon = .01$, with the equilibrium conditions serving as the criteria.

We first considered migration examples without class transformations with asymmetric and nonlinear utility and migration cost functions. The utility functions were of the form

$$u_i^k(p) = -\alpha_{ii}^{kk}(p_i^k)^2 - \sum_{l,j} a_{ij}^{kl} p_j^l + b_i^k, \tag{36}$$

and the migration cost functions were of the form

$$c_{ij}^k(f) = \gamma_{ijij}^k (f_{ij}^k)^2 + \sum_{l,rs} g_{ijrs}^{kl} f_{rs}^l + h_{ij}^k. \tag{37}$$

The data were generated randomly and uniformly in the ranges as follows: $\alpha_{ii}^{kk} \in [1, 10] \times 10^{-6}$,$\gamma_{ijij}^{kk} \in [.1, .5,] \times 10^{-6}$,$-a_{ij}^{kk} \in [1, 10]$, $b_i^k \in [10, 100]$, $g_{ijij}^{kk} \in [.1, .5]$, and $h_{ij}^k \in [1, 5]$, for all i, j, k, with the diagonal terms generated so that strict diagonal dominance of the respective Jacobians of the utility and movement cost functions held, thus guaranteeing uniqueness of the equilibrium pattern (p^*, f^*).

The number of cross-terms for the functions (5.36) and (37) was set at five. The initial population $\bar{p}_i^k$ was generated randomly and uniformly in the range $[10, 30]$, for all i, k.

Numerical results for nonlinear multiclass migration networks

Number of Locations	Number of Classes 5	10
	CPU Time in sec. (# of Iterations)	
10	.24(4)	.41(3)
20	1.18(4)	2.38(4)
30	3.87(4)	9.73(4)
40	8.73(4)	17.01(5)
50	16.22(5)	33.07(4)

In Table 1 we varied the number of locations from 10 through 50, in increments of 10, and fixed the number of classes at 5 and 10.

As can be seen from the Table 1, the decomposition algorithm by classes required only several iterations for convergence. As expected, the problems with 10 classes required, typically, at least twice the CPU time for computation as did the problems with 5 classes. Finally, note that, although the decomposition algorithm by classes implemented here was a serial algorithm, the parallel version converges under the same conditions as given in Theorem 4. Hence, the parallel analogue allows for implementation on parallel computers. We now turn to the computation of large-scale migration network equilibrium problems with class transformations and present numerical results for the linearization decomposition algorithm.

We now report the numerical results for multiclass migration problems with class transformations in Table 2. As in the previous examples, we considered examples with asymmetric and nonlinear utility and migration cost functions, that is, the utility functions were of the form

$$u_i^k(p) = -\alpha_i^k (p_i^k)^2 - \sum_{l,j} a_{ij}^{kl} p_j^l + b_i^k, \tag{38}$$

and the migration cost functions were of the form

$$c_{ij}^{kl}(f) = \gamma_{ij}^{kl} (f_{ij}^{kl})^2 + \sum_{uv,rs} g_{ijrs}^{kluv} f_{rs}^{uv} + h_{ij}^{kl}. \tag{39}$$

The data were generated in a similar fashion to the preceding examples, i.e., randomly and uniformly in the ranges as follows: $\alpha_i^k \in [1, 10] \times 10^{-6}$, $\gamma_{ij}^{kl} \in [.1, .5] \times 10^{-6}$, $a_{ii}^{kk} \in [1, 10]$, $b_i^k \in [10, 100]$, $g_{ijij}^{klkl} \in [.1, .5]$, and $h_{ij}^{kl} \in [1, 5]$, for all i, j, k, l, with the off-diagonal terms generated so that strict diagonal dominance of the respective Jacobians of the utility and migration cost functions held, thus guaranteeing uniqueness of the equilibrium pattern (p^*, f^*). However, the Jacobians were asymmetric.

The number of cross-terms for the functions (38) and (39) was set at 5. The initial population $\bar{p}_i^k$ was generated randomly and uniformly in the range $[10, 30]$, for all i, k.

Numerical results for nonlinear multiclass migration networks with class transformations

Number of Locations	Number of Classes	Number of (Nodes; Links)	CPU Time in sec. (# of Iterations)
10	5	(50; 2,450)	2.70(4)
20	5	(100; 9,900)	16.89(4)
30	5	(150; 22,350)	80.40(5)
40	5	(200; 39,800)	171.95(6)
50	5	(250; 72,250)	321.06(5)
10	10	(100; 9,900)	23.71(4)
20	10	(200; 39,800)	131.63(5)
30	10	(300; 89,700)	512.03(4)

In Table 2 the problems ranged in size from 10 regions, 5 classes through 50 regions, 5 classes, to 30 regions, 10 classes. The problems, hence, ranged in size from 50 nodes and 2,450 links to 300 nodes and 89,700 links. The number of nodes and the number of links for each problem are also reported in the tables.

As can be seen from the two tables, the linearization decomposition algorithm required only several iterations for convergence. The problems solved here represent large-scale problems from both numerical as well as application-oriented perspectives. Although the class transformation problems solved here cannot directly be compared to those solved without class transformations, some inferences can, nevertheless, be made.

The problems in Table 2 are more time-consuming to solve for a fixed number of locations and classes. This is due, in part, to the fact that a problem with J classes and n locations, in the absence of class transformations, has only $Jn(n-1)$ links, whereas a problem with the same number of classes and regions in the presence of class transformations has the number of links now equal to $Jn(Jn-1)$.

Hence, the dimensionality of a given problem now increases in terms of the number of links by a factor on the order of the number of classes J.

The largest problem solved in Table 1 had 50 regions and 10 classes and consisted of 24,500 links, whereas the largest problem solved in Table 2 consisted of 30 regions and 10 classes and had 89,700 links.

The literature on human migration is extensive and spans disciplines ranging from economics through geography to sociology. Some precursors to a network formalism are the contributions of Beckmann (1957), Tobler (1981), and Dorigo and Tobler (1983). Tobler (1981) and Dorigo and Tobler (1983) establish connections between migration problems and transportation problems. The importance of migration cost in migration decision-making has been documented in the literature from both theoretical and empirical perspectives (cf. Tobler (1981) and Sjaastad (1962), and the references therein), and such costs are explicitly included in our more general migration models. Some surveys of the migration literature are Greenwood (1975, 1985).

A related problem has been studied by Faxen and Thore (1990) who utilize a network analysis for studying labor markets and discuss the relationship between their model and classical spatial price equilibrium models. Here our emphasis has been on developing the fundamentals of a unifying network framework for the study of human population movements. Of course, our model of class transformations captures labor movements as well.

Below we include the references to the above material as well as additional ones for the interested reader.

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