Financial Equilibrium

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Financial applications have provided in the past several decades a stimulus for the development of both modeling and methodological advances. Financial theory, in particular, dating to the seminal work of Markowitz (1959) and Sharpe (1970), has built a strong platform for both scholarly investigations and, ultimately, empirical practice.

The introduction of new technologies and financial instruments, coupled with the complexity of the economic interactions and the scale and scope of financial problems, identify this problem domain as one in which computational research will continue to play a pivotal role.

In this lecture, a theoretical framework is developed for the formulation, analysis, and computation of financial equilibria using variational inequality theory. Here, as in more classical models, portfolio optimization remains the behavioral assumption underlying a given sector, but, in contrast, the focus is on multiple sectors, where each sector seeks to determine its optimal composition of both assets and liabilities.

The models, although theoretical, are developed with empirical application in mind. Specifically, the framework fits well with flow-of-funds accounts (cf. Cohen (1987)).

Flow-of-funds accounts trace their history to the work of Quesnay (1785) in which the modeling of the circular flow of funds as a network problem also has its roots.

More recently, Thore (1969, 1970) and Thore and Kydland (1972) have introduced network models of financial credit activity. In this lecture, the underlying network structure of competitive financial equilibrium problems will also be explored.

In the general competitive financial equilibrium models considered here, the equilibrium yields both asset and liability volumes, as well as the instrument prices.

General financial equilibrium problems can be expected to be large-scale in practice, since one may wish to disaggregate sectors and instruments as finely as required. Hence, decomposition algorithms that resolve such large-scale problems into simpler subproblems are especially appealing.

Towards this end, we propose a variational inequality decomposition algorithm, based on the modified projection method, which in many applications yields network subproblems which not only can be solved using equilibration algorithms but can also be implemented on parallel architectures.

Quadratic Utility Functions

A general equilibrium model of financial flows and prices is developed here that assumes quadratic utility functions. The equilibrium conditions are first derived and then the governing variational inequality formulation is presented. Subsequently, the qualitative analysis of the model is conducted.

Consider an economy consisting of m sectors, with a typical sector denoted by i, and with n instruments, with a typical instrument denoted by j. Denote the volume of instrument j held in sector i's portfolio as an asset, by x_{ij}, and the volume of instrument j held in sector i's portfolio as a liability, by y_{ij}. The assets in sector i's portfolio are grouped into a column vector $x_i \in R^n$, and the liabilities are grouped into the column vector $y_i \in R^n$. Further group the sector asset vectors into the column vector $x \in R^{mn}$, and the sector liability vectors into the column vector $y \in R^{mn}$.

Explicit recognition of both sides of the sectoral balance sheet is included here in order to maintain the strategic distinction between acquisitions net of sales (denoted as asset holdings) and issues net of paybacks (denoted as holdings of liability) that may be important in empirical applications.

Each sector's utility can be defined as a function of the expected future portfolio value. The expected value of the future portfolio may be described by two characteristics: the expected mean value and the uncertainty surrounding the expected mean. In this model, the expected mean portfolio value of the next period is assumed to be equal to the market value of the current period portfolio.

Each sector's uncertainty, or assessment of risk, with respect to the future value of the portfolio is based on a variance-covariance matrix denoting the sector's assessment of the standard deviation of prices for each instrument. The $2n \times 2n$ variance-covariance matrix associated with sector i's assets and liabilities is denoted by Q^i.

In this model it is assumed that the total volume of each balance sheet side is exogenous. Moreover, under the assumption of perfect competition, each sector will behave as if it has no influence on instrument prices or on the behavior of the other sectors.

Let r_j denote the price of instrument j, and group the instruments into the column vector $r \in R^n$.

Since each sector's expectations are formed by reference to current market activity, sector utility maximization can be written in terms of optimizing the current portfolio. Sectors may trade, issue, or liquidate holdings in order to optimize their portfolio compositions.

Each sector i's portfolio optimization problem is as follows. Sector i seeks to determine its optimal composition of instruments held as assets and as liabilities, so as to minimize the risk while at the same time maximizing the value of its asset holdings and minimizing the value of its liabilities. The portfolio optimization problem for sector i is, hence, given by:

$$\text{Minimize} \quad \begin{bmatrix} x_i \\ y_i \end{bmatrix}^T Q^i \begin{bmatrix} x_i \\ y_i \end{bmatrix} - \sum_{j=1}^{n} r_j (x_{ij} - y_{ij})$$

subject to:

$$\sum_{j=1}^{n} x_{ij} = s_i, \quad \sum_{j=1}^{n} y_{ij} = s_i \tag{1}$$

$$x_{ij} \geq 0, \quad y_{ij} \geq 0; \quad j = 1, \ldots, n, \tag{2}$$

where the instrument price vector r is exogenous to the individual sector optimization problem.

Constraints (1) represent the accounting identity reflecting that the accounts for sector i must balance, where s_i is the total financial volume held by sector i. Constraints (10.2) are the nonnegativity assumption. Let P_i denote the closed convex set of (x_i, y_i) satisfying constraints (1) and (12).

Since Q^i is a variance-covariance matrix, it will be assumed here that this matrix is positive definite and, therefore, the objective function for each sector i's portfolio optimization problem is strictly convex.

In Figure 1 we depict the network structure of the individual sectors' portfolio optimization problems.

Necessary and sufficient conditions for a portfolio $(x_i^*, y_i^*) \in P_i$ to be optimal is that it satisfy the following system of inequalities and equalities.

For each instrument j; $j = 1, \ldots, n$:

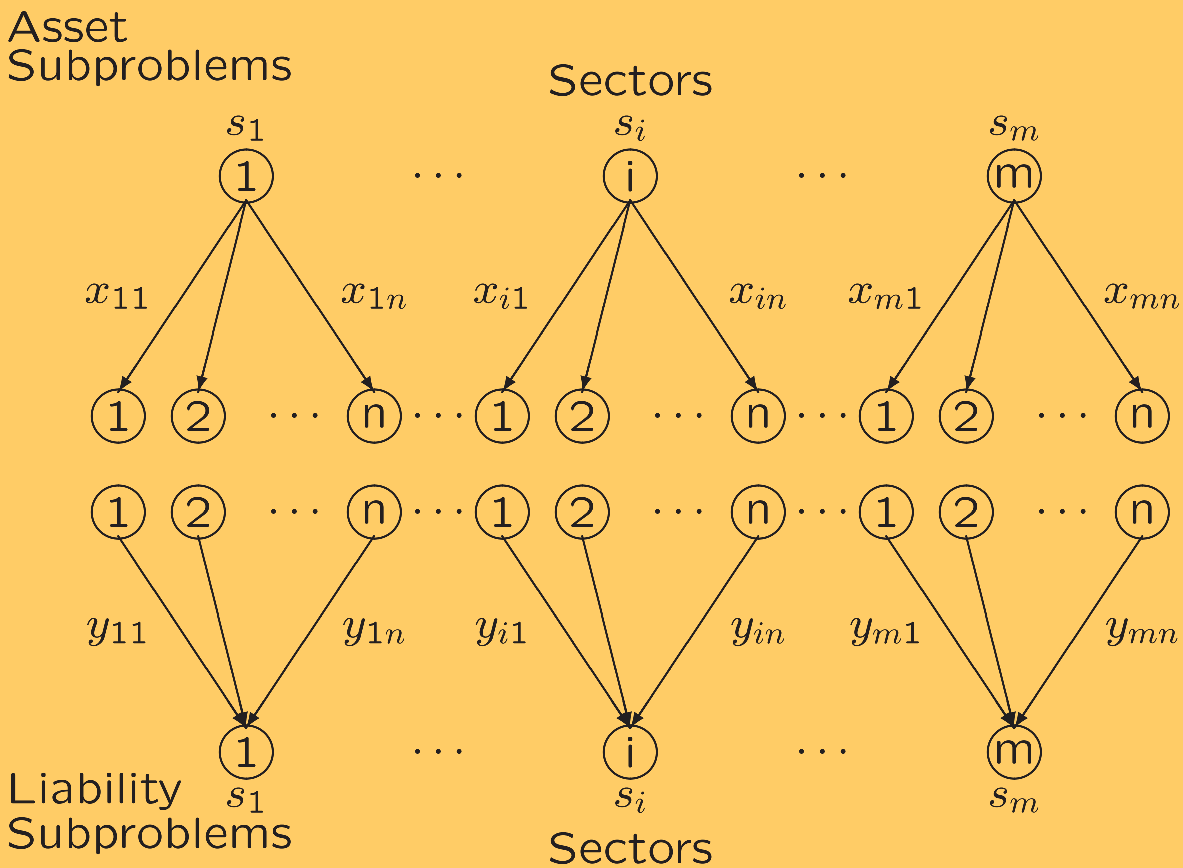
$$2{Q^i_{(11)j}}^T \cdot x_i^* + 2{Q^i_{(21)j}}^T \cdot y_i^* - r_j^* - \mu_i^1 \geq 0$$

$$2{Q^i_{(22)j}}^T \cdot y_i^* + 2{Q^i_{(12)j}}^T \cdot x_i^* + r_j^* - \mu_i^2 \geq 0$$

$$x_{ij}^* \cdot (2{Q^i_{(11)j}}^T \cdot x_i^* + 2{Q^i_{(21)j}}^T \cdot y_i^* - r_j^* - \mu_i^1) = 0 \qquad (3)$$

$$y_{ij}^* \cdot (2{Q^i_{(22)j}}^T \cdot y_i^* + 2{Q^i_{(12)j}}^T \cdot x_i^* + r_j^* - \mu_i^2) = 0,$$

where r_j^* denotes the price for instrument j, which is assumed to be fixed from the perspective of the sectors. Note that Q^i has been partitioned as $Q^i = \begin{bmatrix} Q^i_{11} & Q^i_{12} \\ Q^i_{21} & Q^i_{22} \end{bmatrix}$, and is symmetric. Further, $Q^i_{(\alpha\beta)j}$ denotes the j-th column of $Q^i_{(\alpha\beta)}$, with $\alpha = 1, 2; \beta = 1, 2$. The terms μ_i^1 and μ_i^2 are the Lagrange multipliers of constraints (1).



Network structure of the sectors' optimization problems

A similar set of inequalities and equalities will hold for each of the m sectors.

The inequalities governing the instrument prices in the economy are now described. These prices provide feedback from the economic system to the sectors in regard to the equilibration of the total assets and total liabilities of each instrument. Here it is assumed that there is free disposal and, hence, the instrument prices will be nonnegative.

The economic system conditions insuring market clearance then take on the following form.

For each instrument j; $j = 1, \ldots, n$:

$$\sum_{i=1}^{m} (x^*_{ij} - y^*_{ij}) \begin{cases} = 0, & \text{if } r^*_j > 0 \\ \geq 0, & \text{if } r^*_j = 0. \end{cases} \tag{4}$$

In other words, if the price is positive, then the market must clear for that instrument; if there is an excess supply of an instrument in the economy, then its price must be zero. Combining the above sector and market inequalities and equalities yields the following.

Definition 1 (Financial Equilibrium)

A vector $(x^*, y^*, r^*) \in \prod_{i=1}^{m} P_i \times R^n_+$ *is an equilibrium of the financial model if and only if it satisfies the system of equalities and inequalities (3) and (4), for all sectors* i*;* $i = 1, \ldots, m$*, and for all instruments* j*;* $j = 1, \ldots, n$*, simultaneously.*

Now we are ready to establish the variational inequality governing the equilibrium conditions of our financial model.

Theorem 1 (Variational Inequality Formulation of Financial Equilibrium)

A vector of sector assets, liabilities, and instrument prices (x^*, y^*, r^*) *is a financial equilibrium if and only if it satisfies the following variational inequality problem.*

Determine $(x^*, y^*, r^*) \in \prod_{i=1}^{m} P_i \times R^n_+$*, satisfying:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[2({Q^i_{(11)_j}}^T \cdot x^*_i + {Q^i_{(21)_j}}^T \cdot y^*_i) - r^*_j\right] \times \left[x_{ij} - x^*_{ij}\right]$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\left[2({Q^i_{(22)_j}}^T \cdot y^*_i + {Q^i_{(12)_j}}^T \cdot x_i) + r^*_j\right] \times \left[y_{ij} - y^*_{ij}\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}(x^*_{ij} - y^*_{ij})\right]\times\left[r_j - r^*_j\right] \geq 0,\ \forall (x, y, r) \in \prod_{i=1}^{m} P_i \times R^n_+. \quad (5)$$

Proof: Assume that $(x^*, y^*, r^*) \in \prod_{i=1}^{m} P_i \times R_+^n$ is an equilibrium point. Then inequalities (3) and (4) hold for all i and j. Hence, one has that

$$\sum_{j=1}^{n} \left[2({Q^i_{(11)_j}}^T \cdot x_i^* + {Q^i_{(21)_j}}^T \cdot y_i^*) - r_j^* - \mu_i^1 \right] \times \left[x_{ij} - x_{ij}^* \right] \geq 0,$$

from which it follows, after applying constraint (1), that

$$\sum_{j=1}^{n} \left[2({Q^i_{(11)_j}}^T \cdot x_i^* + {Q^i_{(21)_j}}^T \cdot y_i^*) - r_j^* \right] \times \left[x_{ij} - x_{ij}^* \right] \geq 0. \tag{6}$$

Similarly, one can obtain

$$\sum_{j=1}^{n} \left[2({Q^i_{(22)_j}}^T \cdot y_i^* + {Q^i_{(12)_j}}^T \cdot x_i^*) + r_j^* \right] \times \left[y_{ij} - y_{ij}^* \right] \geq 0. \tag{7}$$

Summing now inequalities (6) and (7) for all i, one concludes that for $(x^*, y^*) \in \prod_{i=1}^{m} P_i$,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left[2({Q^i_{(11)_j}}^T \cdot x_i^* + {Q^i_{(21)_j}}^T \cdot y_i^*) - r_j^* \right] \times \left[x_{ij} - x_{ij}^* \right]$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{n} \left[2({Q^i_{(22)_j}}^T \cdot y_i^* + {Q^i_{(12)_j}}^T \cdot x_i^*) + r_j^* \right] \times \left[y_{ij} - y_{ij}^* \right] \geq 0,$$

$$\forall (x, y) \in \prod_{i=1}^{m} P_i. \tag{8}$$

From inequalities (4) one can further conclude that $r_j^* \geq 0$ must satisfy

$$\sum_{i=1}^{m}(x_{ij}^* - y_{ij}^*) \times (r_j - r_j^*) \geq 0, \quad \forall r_j \geq 0, \tag{9}$$

and, therefore, $r^* \in R_+^n$ must satisfy

$$\sum_{j=1}^{n}\sum_{i=1}^{m}(x_{ij}^* - y_{ij}^*) \times (r_j - r_j^*) \geq 0, \quad \forall r \in R_+^n. \tag{10}$$

Summing now inequalities (8) and (10), one obtains the variational inequality (5).

We now establish that a solution to variational inequality (5) will also satisfy equilibrium conditions (3) and (4).

If $(x^*, y^*, r^*) \in \prod_{i=1}^m P_i \times R_+^n$ is a solution of variational inequality (5), let $x_i = x_i^*; y_i = y_i^*$; for all i. Then one has that

$$\sum_{j=1}^{n}\left[\sum_{i=1}^{m}(x_{ij}^* - y_{ij}^*)\right] \times \left[r_j - r_j^*\right] \geq 0, \quad \forall r \in R_+^n,$$

which implies condition (4).

Finally, let $r_j = r_j^*$, for all j, in which case substitution into (5) yields:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[2({Q^i_{(11)j}}^T \cdot x_i^* + {Q^i_{(21)j}}^T \cdot y_i^*) - r_j^*\right] \times \left[x_{ij} - x_{ij}^*\right]$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\left[2({Q^i_{(22)j}}^T \cdot x_i^* + {Q^i_{(12)j}}^T \cdot y_i^*) + r_j^*\right] \times \left[y_{ij} - y_{ij}^*\right] \geq 0, \tag{11}$$

which implies (3).The proof is complete.

General Utility Functions

The quadratic financial model is now generalized and the variational inequality formulation of the equilibrium conditions presented.

Assume that each sector seeks to maximize its utility, where the utility function, $U_i(x_i, y_i, r)$, is given by:

$$U_i(x_i, y_i, r) = u_i(x_i, y_i) + \langle r^T, x_i - y_i \rangle. \qquad (12)$$

The optimization problem for sector i can then be expressed as:

$$\text{Maximize}_{(x_i, y_i) \in P_i} \quad U_i(x_i, y_i, r) \qquad (13)$$

where P_i is a closed, convex, non-empty, and bounded subset of R^{2n}, denoting the feasible set of asset and liability choices.

Note that in this model we no longer require the constraint set P_i to be of the form given by equations (1) and inequalities (2).

Nevertheless, the model introduced in this section captures the general financial equilibrium modeldescribed earlier as a special case, where $u_i(x_i, y_i) = -\begin{bmatrix} x_i \\ y_i \end{bmatrix}^T Q^i \begin{bmatrix} x_i \\ y_i \end{bmatrix}$.

Assuming that each sector's utility function is concave, necessary and sufficient conditions for an optimal portfolio (x_i^*, y_i^*), given a fixed vector of instrument prices r^*, are that $(x_i^*, y_i^*) \in P_i$, and satisfy the inequality:

$$-\langle \nabla_{x_i} U_i(x_i^*, y_i^*, r^*)^T, x_i - x_i^* \rangle - \langle \nabla_{y_i} U_i(x_i^*, y_i^*, r^*)^T, y_i - y_i^* \rangle \geq 0,$$

$$\forall (x_i, y_i) \in P_i, \qquad (14)$$

where $\nabla_{x_i} U_i(\cdot)$ denotes the gradient of $U_i(\cdot)$ with respect to x_i, or, equivalently, in view of (12),

$$-\langle (\nabla_{x_i} u_i(x_i^*, y_i^*) + r^*)^T, x_i - x_i^* \rangle$$

$$-\langle (\nabla_{y_i} u_i(x_i^*, y_i^*) - r^*)^T, y_i - y_i^* \rangle \geq 0, \quad \forall (x_i, y_i) \in P_i. \quad (15)$$

A similar inequality will hold for each of the m sectors.

The system of equalities and inequalities governing the instrument prices in the economy as in (4) is still valid. Hence, one can immediately write down the following economic system conditions.

For each instrument $j; j = 1, \ldots, n$:

$$\sum_{i=1}^{m} x_{ij}^{*} - \sum_{i=1}^{m} y_{ij}^{*} \begin{cases} = 0, & \text{if } r_j^{*} > 0 \\ \geq 0, & \text{if } r_j^{*} = 0. \end{cases} \tag{16}$$

In other words, as before, if there is an excess supply of an instrument in the economy, then its price must be zero; if the price of an instrument is positive, then the market for that instrument must clear.

Combining the above sector and market inequalities and equalities, yields the following.

Definition 2 (Financial Equilibrium with General Utility Functions)

A vector $(x^*, y^*, r^*) \in \prod_{i=1}^m P_i \times R_+^n$ *is an equilibrium of the financial model developed above if and only if it satisfies inequalities (15) and (16), for all sectors* i; $i = 1, \ldots, m$, *and for all instruments* j; $j = 1, \ldots, n$, *simultaneously.*

The variational inequality formulation of the equilibrium conditions of the model is now presented. The proof of this theorem is similar to that of Theorem 1.

Theorem 2 (Variational Inequality Formulation of Financial Equilibrium with General Utility Functions)

A vector of assets and liabilities of the sectors, and instrument prices $(x^*, y^*, r^*) \in \prod_{i=1}^m P_i \times R_+^n$ *is a financial equilibrium if and only if it satisfies the variational inequality problem:*

$$-\sum_{i=1}^{m} \langle (\nabla_{x_i} u_i(x_i^*, y_i^*) + r^*)^T, x_i - x_i^* \rangle$$

$$-\sum_{i=1}^{m} \langle (\nabla_{y_i} u_i(x_i^*, y_i^*) - r^*)^T, y_i - y_i^* \rangle$$

$$+\sum_{j=1}^{n} \left[\sum_{i=1}^{m} x_{ij}^* - \sum_{i=1}^{m} y_{ij}^* \right] \times \left[r_j - r_j^* \right] \geq 0, \ \forall (x, y, r) \in \prod_{i=1}^{m} P_i \times R_+^n. \tag{17}$$

Qualitative Properties

Certain qualitative properties of the equilibrium of the financial model just outlined are investigated. First, the existence result is given. In particular, we establish in the following theorem that the asset and liability vector that satisfies variational inequality (17) also satisfies a variational inequality defined on a compact set. Moreover, the Lagrange multipliers corresponding to the constraints of the new variational inequality problem are equilibrium prices of the original variational inequality problem.

Theorem 3 (Existence)

If $(x^*, y^*, r^*) \in \prod_{i=1}^m P_i \times R^n_+$ *is an equilibrium, that is, satisfies variational inequality (17), then the equilibrium asset and liability vector* (x^*, y^*) *is a solution of the variational inequality:*

$$-\sum_{i=1}^{m} \langle \nabla_{x_i} u_i(x_i^*, y_i^*)^T, x_i - x_i^* \rangle - \sum_{i=1}^{m} \langle \nabla_{y_i} u_i(x_i^*, y_i^*)^T, y_i - y_i^* \rangle \geq 0,$$

$$\forall (x, y) \in S, \tag{18}$$

where $S \equiv \{(x,y) | (x,y) \in \prod_{i=1}^m P_i; \sum_{i=1}^m x_{ij} - y_{ij} \geq 0; j = 1, \ldots, n\}$*, and is non-empty.*

Conversely, if (x^*, y^*) *is a solution of (18), there exists an* $r^* \in R^n_+$*,* (x^*, y^*, r^*) *is a solution of (17), and, thus, an equilibrium.*

Proof: Assume that (x^*, y^*, r^*) is an equilibrium. Then (x^*, y^*, r^*) satisfies (17). Let $x_i = x_i^*; y_i = y_i^*$; for all i, and $r = 0$; substitution of these vectors into (17) yields:

$$-\sum_{j=1}^{n}\left[\sum_{i=1}^{m} x_{ij}^* - y_{ij}^*\right] r_j^* \geq 0. \tag{19}$$

Letting now $r = r^*$, substitution into (17) yields:

$$-\sum_{i=1}^{m}\langle(\nabla_{x_i} u_i(x_i^*, y_i^*) + r^*)^T, x_i - x_i^*\rangle$$

$$-\sum_{i=1}^{m}\langle(\nabla_{y_i} u_i(x_i^*, y_i^*) - r^*)^T, y_i - y_i^*\rangle \geq 0$$

or $-\sum_{i=1}^{m}\langle\nabla_{x_i} u_i(x_i^*, y_i^*)^T, x_i - x_i^*\rangle - \sum_{i=1}^{m}\langle\nabla_{y_i} u_i(x_i^*, y_i^*)^T, y_i - y_i^*\rangle$

$$\geq \sum_{j=1}^{n} r_j^* \left[\sum_{i=1}^{m}(x_{ij} - y_{ij}) - \sum_{i=1}^{m}(x_{ij}^* - y_{ij}^*)\right]. \tag{20}$$

But, the right-hand side of inequality (20) is ≥ 0, because of (19) and the constraint set S. Thus, we have established that (x^*, y^*) satisfying (17), also satisfies (18).

Observe that there always exists an asset and liability pattern (x^*, y^*) satisfying (18), since the feasible set S is compact. Further, by the Lagrange Multiplier Theorem, one is guaranteed the existence of multipliers $r^* \in R^n_+$, corresponding to the constraints defining S, and for such an (x^*, y^*, r^*) one has that

$$-\sum_{i=1}^{m} \langle (\nabla_{x_i} u_i(x_i^*, y_i^*) + r^*)^T, x_i - x_i^* \rangle$$

$$-\sum_{i=1}^{m} \langle (\nabla_{y_i} u_i(x_i^*, y_i^*) - r^*)^T, y_i - y_i^* \rangle$$

$$+\sum_{j=1}^{n} \left[\sum_{i=1}^{m} x_{ij}^* - \sum_{i=1}^{m} y_{ij}^* \right] \times \left[r_j - r_j^* \right] \geq 0.$$

The proof is complete.

We now show that if the utility functions U_i are strictly concave for all i, then the equilibrium asset and liability pattern (x^*, y^*) is also unique.

If the U_i are strictly concave, then

$$-\sum_{i=1}^{m}\langle(\nabla_{x_i}u_i(x_i^1,y_i^1)-\nabla_{x_i}u_i(x_i^2,y_i^2))^T, x_i^1-x_i^2\rangle$$

$$-\sum_{i=1}^{m}\langle(\nabla_{y_i}u_i(x_i^1,y_i^1)-\nabla_{y_i}u_i(x_i^2,y_i^2))^T, y_i^1-y_i^2\rangle > 0,$$

$$\forall(x^1,y^1)\neq(x^2,y^2)\in\prod_{i=1}^{m}P_i. \qquad (21)$$

Assume now that there are two distinct equilibrium solutions (x^1, y^1, r^1) and (x^2, y^2, r^2). Then

$$-\sum_{i=1}^{m}\langle(\nabla_{x_i}u_i(x_i^1,y_i^1)+r^1)^T, x_i-x_i^1\rangle$$

$$-\sum_{i=1}^{m}\langle(\nabla_{y_i}u_i(x_i^1,y_i^1)-r^1)^T, y_i-y_i^1\rangle$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}x_{ij}^1-\sum_{i=1}^{m}y_{ij}^1\right]\times\left[r_j-r_j^1\right]\geq 0,$$

$$\forall(x',y',r')\in\prod_{i=1}^{n}P_i\times R_+^n \qquad (22)$$

and

$$-\sum_{i=1}^{m}\langle(\nabla_{x_i}u_i(x_i^2,y_i^2)+r^2)^T, x_i-x_i^2\rangle$$

$$-\sum_{i=1}^{m}\langle(\nabla_{y_i}u_i(x_i^2,y_i^2)-r^2)^T, y_i-y_i^2\rangle$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}x_{ij}^2-\sum_{i=1}^{m}y_{ij}^2\right]\times\left[r_j-r_j^2\right]\geq 0,$$

$$\forall(x,y,r)\in\prod_{i=1}^{m}P_i\times R_+^n. \quad (23)$$

Let $(x,y,r)=(x^2,y^2,r^2)$, and substitute into (22). Also, let $(x,y,r)=(x^1,y^1,r^1)$ and substitute into inequality (23). Adding the resulting inequalities, yields

$$-\sum_{i=1}^{m}\langle(\nabla_{x_i}u_i(x_i^1,y_i^1)-\nabla_{x_i}u_i(x_i^2,y_i^2))^T, x_i^2-x_i^1\rangle$$

$$-\sum_{i=1}^{m}\langle(\nabla_{y_i}u_i(x_i^1,y_i^1)-\nabla_{y_i}u_i(x_i^2,y_i^2))^T, y_i^2-y_i^1\rangle\geq 0. \quad (24)$$

But (24) is a contradiction to (21). Hence, we have thus established what follows.

Theorem 4 (Uniqueness of Asset and Liability Pattern)

If the utility functions U_i *are strictly concave for all sectors* i*, then the equilibrium asset and liability pattern* (x^*, y^*) *is unique.*

Observe that in the above analysis, if the utility functions had been assumed to be concave, rather than strictly concave, then existence would still have been guaranteed, but one would no longer be able to guarantee uniqueness of the equilibrium asset and liability pattern.

Policy Interventions

Here the general model of competitive financial equilibrium described earlier is considered and generalized to allow for the incorporation of policy interventions in the form of taxes and price controls.

From the policy intervention aspect, denote the price ceiling associated with instrument j by $\hat{r}_j$, and group the ceilings into a vector $\hat{r} \in R^n$. Note that, ceilings have been imposed on commodity prices in spatial price equilibrium problems.

Denote the given tax rate levied on sector i's net yield on financial instrument j, as τ_{ij} and group the tax rates into the vector $\tau \in R^{mn}$. Assume that the tax rates lie in the interval $[0, 1)$. Therefore, the government in this model has the flexibility of levying a distinct tax rate across both sectors and instruments.

Assume that each sector seeks to maximize its utility, where the utility function, $U_i(x_i, y_i, r)$, is now given by

$$U_i(x_i, y_i, r) = u_i(x_i, y_i) + \sum_{j=1}^{n} (1 - \tau_{ij}) r_j (x_{ij} - y_{ij}). \quad (25)$$

The optimization problem for sector i can, thus, be expressed as:

$$\text{Maximize}_{(x_i, y_i) \in P_i} U_i(x_i, y_i, r) \quad (26)$$

where P_i is a closed, convex, non-empty, and bounded subset of R^{2n}, denoting the feasible set of asset and liability choices.

Observe that the objective function (25) differs from the objective function (12) in that the second term now incorporates the tax rate through the presence of the $(1 - \tau_{ij})$ term premultiplying the $r_j(x_{ij} - y_{ij})$ term, with the former term acting, in effect, as a discount rate.

Assume that, as previously, each sector is risk–averse so that his/her utility function is a strictly concave function. Also, assume that the utility function has bounded second order derivatives for all its entries in the feasible set. This assumption is imposed from the point of view of establishing convergence of the algorithm subsequently. One should note that this condition is satisfied by the quadratic utility functions.

Given a fixed instrument price vector r^*, the necessary and sufficient conditions for an optimal portfolio (x_i^*, y_i^*) of sector i is that $(x_i^*, y_i^*) \in P_i$, and satisfies the inequality:

$$-\langle \nabla_{x_i} U_i(x_i^*, y_i^*, r^*)^T, x_i - x_i^* \rangle - \langle \nabla_{y_i} U_i(x_i^*, y_i^*, r^*)^T, y_i - y_i^* \rangle \geq 0,$$

$$\forall (x_i, y_i) \in P_i, \tag{27}$$

or, equivalently, in view of (25),

$$-\langle \nabla_{x_i} u_i(x_i^*, y_i^*)^T + r^{*T}(I - \tau_i), x_i - x_i^* \rangle$$

$$-\langle \nabla_{y_i} u_i(x_i^*, y_i^*)^T - r^{*T}(I - \tau_i), y_i - y_i^* \rangle \geq 0, \tag{28}$$

for all $(x_i, y_i) \in P_i$, where

$$\tau_i = \begin{bmatrix} \tau_{i1} & & \\ & \ddots & \\ & & \tau_{in} \end{bmatrix}.$$

Similar inequalities will hold for each of the m sectors.

We now describe the inequalities governing the instrument prices in the economy in the presence of price ceilings.

For each instrument $j; j = 1, \ldots, n$:

$$\sum_{i=1}^{m}(1-\tau_{ij})(x_{ij}^* - y_{ij}^*) \begin{cases} \leq 0, & \text{if } r_j^* = \hat{r}_j \\ = 0, & \text{if } 0 < r_j^* < \hat{r}_j \\ \geq 0, & \text{if } r_j^* = 0. \end{cases} \tag{29}$$

In other words, if there is an effective excess supply of that instrument in the economy, then its price must be zero; if the price of an instrument is positive, but not at the ceiling, then the market for that instrument must clear; finally, if there is an effective excess demand for an instrument in the economy, then the price must be at the ceiling. Let $\hat{S} \equiv \{r | 0 \leq r \leq \hat{r}\}$, and $K \equiv \prod_{i=1}^{m} P_i \times \hat{S}$.

Combining the above sector and market inequalities and equalities, yields the following.

Definition 3 (Financial Equilibrium with Policy Interventions)

A vector $(x^*, y^*, r^*) \in K$ is an equilibrium point of the financial model with policy interventions developed above if and only if it satisfies the system of equalities and inequalities (27) (or (28)), and (29), for all sectors i; $i = 1, \ldots, m$, and for all instruments j; $j = 1, \ldots, n$, simultaneously.

We now derive the variational inequality formulation of the equilibrium conditions of the above model.

Theorem 5 (Variational Inequality Formulation of Financial Equilibrium with Policy Interventions)

A vector of assets and liabilities of the sectors, and instrument prices, (x^*, y^*, r^*), is a financial equilibrium with policy interventions if and only if it satisfies the variational inequality problem:

Determine $(x^*, y^*, r^*) \in K$, satisfying:

$$-\sum_{i=1}^{m} \langle \nabla_{x_i} U_i(x_i^*, y_i^*, r^*)^T, x_i - x_i^* \rangle$$

$$-\sum_{i=1}^{m} \langle \nabla_{y_i} U_i(x_i^*, y_i^*, r^*)^T, y_i - y_i^* \rangle$$

$$+\sum_{j=1}^{n} [\sum_{i=1}^{m} (1 - \tau_{ij})(x_{ij}^* - y_{ij}^*)] \times [r_j - r_j^*] \geq 0,$$

$$\forall (x, y, r) \in K. \quad (30)$$

Proof: Assume that $(x^*, y^*, r^*) \in K$ is an equilibrium point. Then inequalities (27) or (28) and (29) hold for all i and j. Hence, from (28), after summing over all sectors, one obtains:

$$-\sum_{i=1}^{m} \langle \nabla_{x_i} U_i(x_i^*, y_i^*, r^*)^T, x_i - x_i^* \rangle$$

$$-\sum_{i=1}^{m} \langle \nabla_{y_i} U_i(x_i^*, y_i^*, r^*)^T, y_i - y_i^* \rangle \geq 0,$$

$$\forall (x, y) \in \prod_{i=1}^{m} P_i. \tag{31}$$

Also, from inequality (29) one can conclude that $0 \leq r_j^* \leq \hat{r}_j$ must satisfy

$$\sum_{i=1}^{m} (1 - \tau_{ij})(x_{ij}^* - y_{ij}^*) \times (r_j - r_j^*) \geq 0, \quad 0 \leq r_j \leq \hat{r}_j, \tag{32}$$

and, therefore, $r^* \in \hat{S}$ must satisfy

$$\sum_{j=1}^{n} \sum_{i=1}^{m} (1 - \tau_{ij})(x_{ij}^* - y_{ij}^*) \times (r_j - r_j^*) \geq 0, \quad \forall r \in \hat{S}. \tag{33}$$

Summing inequalities (31) and (33), one obtains the variational inequality (30).

We now establish that a solution to (30) will also satisfy equilbrium conditions (27) (or (28)), and (29).

If $(x^*, y^*, r^*) \in K$ is a solution of (30), let $x_i = x_i^*, y_i = y_i^*$, for all i, and substitute the resultants into (30). Then it follows that

$$\sum_{j=1}^{n} \sum_{i=1}^{m} (1 - \tau_{ij})(x_{ij}^* - y_{ij}^*)(r_j - r_j^*) \geq 0, \quad \forall r \in \widehat{S}, \tag{34}$$

which implies condition (29).

Similarly, let $r_j = r_j^*$, for all j, in which case substitution into (30) yields

$$-\sum_{i=1}^{m} \langle \nabla_{x_i} U_i(x_i^*, y_i^*, r^*)^T, x_i - x_i^* \rangle$$

$$-\sum_{i=1}^{m} \langle \nabla_{y_i} U_i(x_i^*, y_i^*, r^*)^T, y_i - y_i^* \rangle \geq 0, \tag{35}$$

which implies that (27) must hold. The proof is complete.

Qualitative Properties

We now address the qualitative properties of the equilibrium pattern through the study of variational inequality (30).

Since the feasible set K is compact, and the function that enters variational inequality (30) is assumed to be continuous, it thus follows from the standard theory of variational inequalities that the solution (x^*, y^*, r^*) to (30) is guaranteed to exist.

Note that the utility functions $U_i(x_i, y_i, r); i = 1, \ldots, m$, are strictly concave and the terms related with r are linear with respect to (x_i, y_i); therefore, each $u_i(x_i, y_i); i = 1, \ldots, m$, is strictly concave. By the theorem of convex functions, one has that

$$-\sum_{i=1}^{m} \langle \left(\nabla_{x_i} u_i(x_i^1, y_i^1) - \nabla_{x_i} u_i(x_i^2, y_i^2)\right)^T, x_i^1 - x_i^2 \rangle$$

$$-\sum_{i=1}^{m} \langle \left(\nabla_{y_i} u_i(x_i^1, y_i^1) - \nabla_{y_i} u_i(x_i^2, y_i^2)\right)^T, y_i^1 - y_i^2 \rangle > 0, \quad (36)$$

for any distinct $(x^1, y^1), (x^2, y^2) \in \prod_{i=1}^{m} P_i$.

Observe further that

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}(1-\tau_{ij})(r_j^1-r_j^2)(x_{ij}^1-x_{ij}^2)$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}(1-\tau_{ij})(r_j^1-r_j^2)(y_{ij}^1-y_{ij}^2)$$

$$+\sum_{j=1}^{n}\sum_{i=1}^{m}(1-\tau_{ij})(x_{ij}^1-x_{ij}^2)(r_j^1-r_j^2)$$

$$-\sum_{j=1}^{n}\sum_{i=1}^{m}(1-\tau_{ij})(y_{ij}^1-y_{ij}^2)(r_j^1-r_j^2)=0 \qquad (37)$$

for all $(x^1, y^1, r^1), (x^2, y^2, r^2) \in K$.

Hence, by summing (36) and (37), one obtains

$$-\sum_{i=1}^{m}\langle(\nabla_{x_i}U_i(x_i^1,y_i^1,r^1)-\nabla_{x_i}U_i(x_i^2,y_i^2,r^2))^T,x_i^1-x_i^2\rangle$$

$$-\sum_{i=1}^{m}\langle\nabla_{y_i}(U_i(x_i^1,y_i^1,r^1)-\nabla_{y_i}U_i(x_i^2,y_i^2,r^2))^T,y_i^1-y_i^2\rangle$$

$$+\sum_{j=1}^{n}[\sum_{i=1}^{m}(1-\tau_{ij})(x_{ij}^1-x_{ij}^2)$$

$$-\sum_{i=1}^{m}(1-\tau_{ij})(y_{ij}^1-y_{ij}^2)] \times [r_j^1-r_j^2] > 0, \qquad (38)$$

for any distinct $(x^1, y^1), (x^2, y^2) \in \prod_{i=1}^{m} P_i$, and for any $r^1, r^2 \in \widehat{S}$.

The above inequality yields the following.

Theorem 6

The function that enters the variational inequality (30) is strictly monotone for $(x, y) \in \prod_{i=1}^{m} P_i$*, and monotone for* (x, y, r) *in its feasible set* K*.*

Following Theorem 5 and the previous discussion one obtains the following.

Theorem 7 (Existence and Uniqueness of the Asset and Liability Pattern)

The equilibrium asset and liability pattern (x^*, y^*) *exists and is unique.*

Computation of Financial Equilibria

Here the modified projection method of is proposed for the computation of the general financial equilibrium problems. We begin with its realization in the solution of variational inequality (30), governing the general competitive financial equilibrium model with taxes and price controls developed earlier, and then specialize it to models with a network structure. The algorithm resolves the large–scale problems into simpler variational inequality subproblems.

The Financial Modified Projection Method

Step 0: Initialization

Set $(x^0, y^0, r^0) \in K$. Let $k := 1$. Let ρ be a positive scalar.

Step 1: Construction and Computation

Compute $(\bar{x}^{k-1}, \bar{y}^{k-1}, \bar{r}^{k-1}) \in K$ by solving the variational inequality subproblem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}[\bar{x}_{ij}^{k-1} + \rho(-\frac{\partial u_i(x_i^{k-1}, y_i^{k-1})}{\partial x_{ij}} - (1-\tau_{ij})r_j^{k-1}) - x_{ij}^{k-1}]$$

$$\times[x_{ij} - \bar{x}_{ij}^{k-1}]$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}[\bar{y}_{ij}^{k-1} + \rho(-\frac{\partial u_i(x_i^{k-1}, y_i^{k-1})}{\partial y_{ij}} + (1-\tau_{ij})r_j^{k-1}) - y_{ij}^{k-1})]$$

$$\times[y_{ij} - \bar{y}_{ij}^{k-1}]$$

$$+\sum_{j=1}^{n}[\bar{r}_j^{k-1} + \rho\sum_{i=1}^{m}(1-\tau_{ij})(x_{ij}^{k-1} - y_{ij}^{k-1}) - r_j^{k-1}] \times [r_j - \bar{r}_j^{k-1}]$$

$$\geq 0, \qquad \forall (x, y, r) \in K. \tag{39}$$

Step 2: Adaptation

Compute $(x^k, y^k, r^k) \in K$ by solving the variational inequality subproblem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}[x_{ij}^k + \rho(-\frac{\partial u_i(\bar{x}_i^{k-1}, \bar{y}_i^{k-1})}{\partial x_{ij}} - (1-\tau_{ij})\bar{r}_j^{k-1}) - x_{ij}^{k-1}]$$

$$\times [x_{ij} - x_{ij}^k]$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}[y_{ij}^k + \rho(-\frac{\partial u_i(\bar{x}_i^{k-1}, \bar{y}_i^{k-1})}{\partial y_{ij}} + (1-\tau_{ij})\bar{r}_j^{k-1}) - y_{ij}^{k-1}]$$

$$\times [y_{ij} - y_{ij}^k]$$

$$+\sum_{j=1}^{n}[r_j^k + \rho\sum_{i=1}^{m}(1-\tau_{ij})(\bar{x}_{ij}^{k-1} - \bar{y}_{ij}^{k-1}) - r_j^{k-1}] \times [r_j - r_j^k] \geq 0,$$

$$\forall (x, y, r) \in K. \tag{40}$$

Step 3: Convergence Verification

If $|x_{ij}^k - x_{ij}^{k-1}| \leq \epsilon$, $|y_{ij}^k - y_{ij}^{k-1}| \leq \epsilon$, $|r_j^k - r_j^{k-1}| \leq \epsilon$, with $\epsilon > 0$, a prespecified tolerance, then stop; otherwise, set $k := k + 1$, and go to Step 1.

Observe that both (39) and (40) are equivalent to optimization problems, in particular, to quadratic programming problems, of the form:

$$\text{Minimize}_{X\in K}\langle X^T, X\rangle + \langle h^T, X\rangle$$

where $X \equiv (x, y, r) \in R^{2mn+n}$, and $h \in R^{2mn+n}$ consists of the fixed linear terms in the equality subproblems (39) and (40).

Convergence of the algorithm follows under the assumption that the function F that enters the variational inequality is monotone and Lipschitz continuous, where $0 < \rho < 1/L$, and L is the Lipschitz constant. We now state the following.

Lemma 1 (Lipschitz Continuity)

The function $F(x, y, r)$ that enters the variational inequality (30) is Lipschitz continuous, that is, for all $(x^1, y^1, r^1), (x^2, y^2, r^2) \in K$,

$$\|F(x^1, y^1, r^1) - F(x^2, y^2, r^2)\| \leq L\|(x^1, y^1, r^1) - (x^2, y^2, r^2)\| \tag{41}$$

with Lipschitz constant $L > 0$, under the assumption that the utility function has bounded second order derivatives for all its entries in the feasible set.

Proof: $F(x, y, r)$ can be represented as

$$F(x, y, r)$$

$$= \begin{bmatrix} F_1(x, y, r) \\ \vdots \\ F_l(x, y, r) \\ \vdots \\ F_{2mn+n}(x, y, r) \end{bmatrix} = \begin{bmatrix} -\nabla_x U(x, y, r) \\ -\nabla_y U(x, y, r) \\ \sum_{i=1}^{m} (1 - \tau_{ij})(x_{ij} - y_{ij}) \\ \vdots \\ \sum_{i=1}^{m} (1 - \tau_{ij})(x_{in} - y_{in}) \end{bmatrix}. \tag{42}$$

Several relationships are now presented, the proofs of which are given immediately following.

$$\|F(x^1, y^1, r^1) - F(x^2, y^2, r^2)\|^2$$

$$= \sum_{l=1}^{2mn+n} [F_l(x^1, y^1, r^1) - F_l(x^2, y^2, r^2)]^2 \qquad (43)$$

$$= \sum_{l=1}^{2mn+n} [\nabla^T F_l(\tilde{x}^l, \tilde{y}^l, \tilde{r}^l)(x^1 - x^2, y^1 - y^2, r^1 - r^2)]^2 \quad (44)$$

$$\leq \sum_{l=1}^{2mn+n} \|\nabla F_l(\tilde{x}^l, \tilde{y}^l, \tilde{r}^l)\|^2 \|(x^1 - x^2, y^1 - y^2, r^1 - r^2)\|^2 \quad (45)$$

$$\leq \sum_{l=1}^{2mn+n} L_l^2 \|(x^1 - x^2, y^1 - y^2, r^1 - r^2)\|^2 \qquad (46)$$

$$\leq L^2 \|(x^1 - x^2, y^1 - y^2, r^1 - r^2)\|^2. \qquad (47)$$

Since $F(x, y, r)$ is differentiable, applying the Mean Value Theorem to each component $F_l(x, y, r)$ of $F(x, y, r)$, one knows that there exist ($\tilde{x}^l$,$\tilde{y}^l$,$\tilde{r}^l$) such that

$$(\tilde{x}^l, \tilde{y}^l, \tilde{r}^l) = \theta_l(x^1, y^1, r^1) + (1 - \theta_l)(x^2, y^2, r^2),$$

$0 < \theta_l < 1$, and hence, $(\tilde{x}^l, \tilde{y}^l, \tilde{r}^l) \in K$; $l = 1, \cdots, 2mn + n$, so that (43) equals (44).

Applying the Schwartz inequality, one then obtains (45) from (44).

Further, since each U_i has bounded second order derivatives for all the variables, that is equivalent to saying that ∇F_l; $l = 1, \ldots, 2mn$, are all bounded over the feasible set. Since F_l; $l = 2mn + 1, \ldots, 2mn + n$, are linear functions of (x, y, r), the ∇F_l; $l > 2mn$, are also bounded. Therefore, there exist $L_l > 0$; $l = 1, \cdots, 2mn + n$, such that $\|\nabla F_l(x, y, r)\| \leq L_l$;
$l = 1, \cdots, 2mn + n$; $\forall (x, y, r) \in K$. Hence, inequality (46) is obtained from (45).

Now, let

$$L = \max_{1 \leq l \leq 2mn+n} \{L_l\},$$

from which inequality (47) follows. The proof is complete.

Combining Theorem 6 and Lemma 1, one obtains the following.

Theorem 8 (Convergence)

The decomposition algorithm converges to the equilibrium asset, liability, and price pattern (x^*, y^*, r^*) *satisfying the variational inequality problem (30), with* ρ *such that* $0 < \rho < \frac{1}{L}$.

We now describe the application of the modified projection method to the computation of the quadratic financial equilibrium model developed earlier in the lecture, governed by variational inequality (5). This model has the notable feature that the feasible set underlying it has the structure of a network problem. Hence, the decomposition algorithm will resolve the problem into quadratic programming problems which can then be solved in closed form using the exact equilibration algorithms. Note that the network approach is not limited to financial problems with quadratic utility functions. Rather, it is the structure of the feasible set that predicates whether or not a network approach can be used.

The Modified Projection Method for the Quadratic Model

Step 0: Initialization

Set $(x^0, y^0, r^0) \in \prod_{i=1}^m P_i \times R_+^n$. Let $k := 1$. Let ρ be such that $0 < \rho < \frac{1}{L}$.

Step 1: Construction and Computation

Compute $(\bar{x}^{k-1}, \bar{y}^{k-1}, \bar{r}^{k-1}) \in \prod_{i=1}^m P_i \times R_+^n$ by solving the variational inequality subproblem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\bar{x}_{ij}^{k-1} + \rho(2({Q^i_{(11)j}}^T \cdot x_i^{k-1} + {Q^i_{(21)j}}^T \cdot y_i^{k-1}) - r_j^{k-1})\right.$$

$$\left.-x_{ij}^{k-1}\right] \times \left[x_{ij} - \bar{x}_{ij}^{k-1}\right]$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\bar{y}_{ij}^{k-1} + \rho(2({Q^i_{(22)j}}^T \cdot y_i^{k-1} + {Q^i_{(12)j}}^T \cdot x_i^{k-1}) + r_j^{k-1})\right.$$

$$\left.-y_{ij}^{k-1}\right] \times \left[y_{ij} - \bar{y}_{ij}^{k-1}\right]$$

$$+\sum_{j=1}^{n}\left[\bar{r}_j^{k-1} + \rho\sum_{i=1}^{m}(x_{ij}^{k-1} - y_{ij}^{k-1}) - r_j^{k-1}\right] \times \left[r_j - \bar{r}_j^{k-1}\right] \geq 0, \tag{48}$$

$$\forall (x, y, r) \in \prod_{i=1}^{m} P_i \times R_+^n.$$

Step 2: Adaptation

Compute $(x^k, y^k, r^k) \in \prod_{i=1}^m P_i \times R^n_+$ by solving the variational inequality subproblem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[x_{ij}^{k} + \rho(2({Q_{(11)_j}^{i}}^{T} \cdot \bar{x}_i^{k-1} + {Q_{(21)_j}^{i}}^{T} \cdot \bar{y}_i^{k-1}) - \bar{r}_j^{k-1})\right.$$

$$\left. - x_{ij}^{k-1}\right] \times \left[x_{ij} - x_{ij}^{k}\right]$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{n}\left[y_{ij}^{k} + \rho(2({Q_{(22)_j}^{i}}^{T} \cdot \bar{y}_i^{k-1} + {Q_{(12)_j}^{i}}^{T} \cdot \bar{x}_i^{k-1})\right.$$

$$\left. + \bar{r}_j^{k}) - y_{ij}^{k-1}\right] \times \left[y_{ij} - y_{ij}^{k}\right]$$

$$+ \sum_{j=1}^{n}\left[r_j^{k} + \rho\sum_{i=1}^{m}(\bar{x}_{ij}^{k-1} - \bar{y}_{ij}^{k-1}) - r_j^{k-1}\right] \times \left[r_j - r_j^{k}\right] \geq 0, \tag{49}$$

$$\forall (x, y, r) \in \prod_{i=1}^{m} P_i \times R^n_+.$$

Step 3: Convergence Verification

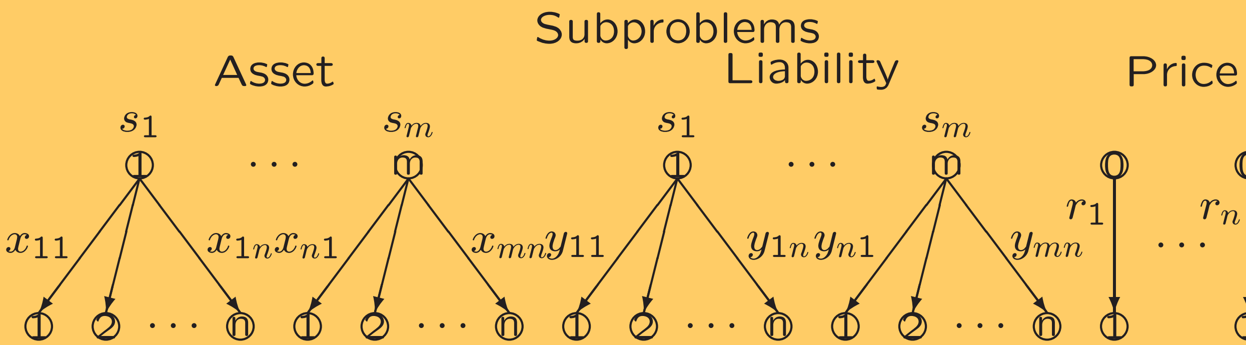
If $|x_{ij}^{k} - x_{ij}^{k-1}| \leq \epsilon$, $|y_{ij}^{k} - y_{ij}^{k-1}| \leq \epsilon$, $|r_j^{k} - r_j^{k-1}| \leq \epsilon$, for all i, j with $\epsilon > 0$, a prespecified tolerance, then stop; otherwise, set $k := k + 1$, and go to Step 1.

We now give an interpretation of the algorithm as an adjustment process. In (48) each sector i at each time period k receives instrument price signals r^{k-1}, and determines its optimal asset and liability pattern $(\bar{x}_i^{k-1}, \bar{y}_i^{k-1})$; at the same time, the system determines the prices $\bar{r}^{k-1}$ in reponse to the difference of the total volume of each instrument held as an asset minus the total volume held as a liability at time period $k-1$. The agents and the system then improve upon their approximations through the solution of (49). The process continues until stability is reached, that is, the asset and liability volumes, and the instrument prices change negligibly between time periods.

Observe now that both (48) and (49) are equivalent to optimization problems, in particular, to quadratic programming problems, of the form:

$$\text{Minimize}_{X \in \prod_{i=1}^{m} P_i \times R_+^n} \quad \langle X^T, X \rangle + \langle h^T, X \rangle$$

where $X \equiv \{(x, y, r) \in R^{2mn+n}\}$, and $h \in R^{2mn+n}$ consists of the fixed linear terms in inequality subproblems (48) and (49). Moreover, the above optimization problem is separable in x, y, and r, and, in view of the feasible set, has the network structure depicted in Figure 2. Each of the $2mn + n$ network subproblems can, thus, be allocated to a distinct processor for one type of parallel decomposition.



Parallel structure of financial network subproblems

Numerical Results

The numerical solution of financial equilibrium models is now addressed through several examples. In particular, the quadratic model with policy interventions is considered.

Assume an economy with two sectors and with three financial instruments. Assume that the "size" of each sector, denoted by s_i, is given by $s_1 = 1$ and $s_2 = 2$. Each sector realizes that the future values of its portfolio are random variables that can be described by mean values and dispersions around the means. Each sector believes that the mean of the expected value is equal to the current value. The variance-covariance matrices of the two sectors are:

$$Q^1 = \begin{bmatrix} 1 & .15 & .3 & -.2 & -.1 & 0 \\ .15 & 1 & .1 & -.1 & -.2 & 0 \\ .3 & .1 & 1 & -.3 & 0 & -.1 \\ -.2 & -.1 & -.3 & 1 & 0 & .3 \\ -.1 & -.2 & 0 & 0 & 1 & .2 \\ 0 & 0 & -.1 & .3 & .2 & 1 \end{bmatrix}$$

and

$$Q^2 = \begin{bmatrix} 1 & .4 & .3 & -.1 & -.1 & 0 \\ .4 & 1 & .5 & 0 & -.05 & 0 \\ .3 & .5 & 1 & 0 & 0 & -.1 \\ -.1 & 0 & 0 & 1 & .5 & 0 \\ -.1 & -.05 & 0 & .5 & 1 & .2 \\ 0 & 0 & -.1 & 0 & .2 & 1 \end{bmatrix}.$$

Note that the terms in the blocks: Q^1_{12}, Q^1_{21}, Q^2_{12}, Q^2_{21}, are not positive, since the returns flowing in from an asset item must covary negatively with the interest expenses flowing out into the portfolio's liabilities. (For details see Francis and Archer (1979).)

The above data were used to construct examples governed by variational inequality (30). The algorithm was coded in FORTRAN, compiled using the FORTVS compiler, optimization level 3, and the numerical runs were done on an IBM 3090/600J. For each of the subsequent examples, the variables were initialized as follows: $r^0_j = 1$, for all j, $x_{ij} = \frac{s_i}{n}$, for all j, $y_{ij} = \frac{s_i}{n}$, for all j. The ρ parameter was set to .35. The convergence tolerance ϵ was set to 10^{-3}.

Example 1

In the first example, the taxes were set to 0 for all sectors and instruments, and the price control ceilings $\widehat{r}$ to 2 for all instruments.

The numerical results for this example follow.

Equilibrium Prices:

$$r_1^* = .91404 \quad r_2^* = .94535 \quad r_3^* = 1.14058$$

Equilibrium Asset Holdings:

$$x_{11}^* = .28736 \quad x_{12}^* = .40063 \quad x_{13}^* = .31200$$

$$x_{21}^* = .75644 \quad x_{22}^* = .56740 \quad x_{23}^* = .67616$$

Equilibrium Liability Holdings:

$$y_{11}^* = .32035 \quad y_{12}^* = .51047 \quad y_{13}^* = .16917$$

$$y_{21}^* = .72447 \quad y_{22}^* = .45723 \quad y_{23}^* = .81830.$$

The algorithm converged in 17 iterations and required 3.62 miliseconds of CPU time for convergence, not including input/output time. Note that in this example, the solution is one in which the policies, in essence, have no effect. Hence, this algorithm may also be used to compute solutions to financial models in the absence of taxes and price controls, provided that the taxes are set to zero and the price ceilings are set at a high enough level. The resulting model is then a special case of our more general one.

Example 2

In the second example, the taxes were kept at zero, but now the price ceilings were tightened to .5 for each instrument. The numerical results for this example follow.

Equilibrium Prices:

$$r_1^* = .27083 \quad r_2^* = .30192 \quad r_3^* = .49716$$

Equilibrium Asset Holdings:

$$x_{11}^* = .28730 \quad x_{12}^* = .40043 \quad x_{13}^* = .31227$$

$$x_{21}^* = .75653 \quad x_{22}^* = .56752 \quad x_{23}^* = .67595$$

Equilibrium Liability Holdings:

$$y_{11}^* = .32005 \quad y_{12}^* = .51074 \quad y_{13}^* = .16920$$

$$y_{21}^* = .72464 \quad y_{22}^* = .45708 \quad y_{23}^* = .81828.$$

The algorithm converged in 18 iterations and required 3.82 miliseconds of CPU time for convergence. Note that in this example, the equilibrium prices all lie within the tighter bounds. In particular, the price of instrument 3 is approximately at its upper bound of .5.

Example 3

In the third example, the tax rate was raised from zero to .15 for all sectors and instruments, and the instrument price ceilings were retained at .5. The numerical results for this example follow.

Equilibrium Prices:

$$r_1^* = .23256 \quad r_2^* = .26871 \quad r_3^* = .49995$$

Equilibrium Asset Holdings:

$$x_{11}^* = .28726 \quad x_{12}^* = .40035 \quad x_{13}^* = .31239$$

$$x_{21}^* = .75663 \quad x_{22}^* = .56777 \quad x_{23}^* = .67560$$

Equilibrium Liability Holdings:

$$y_{11}^* = .31965 \quad y_{12}^* = .51098 \quad y_{13}^* = .16938$$

$$y_{21}^* = .72460 \quad y_{22}^* = .45680 \quad y_{23}^* = .81860.$$

The algorithm converged in 19 iterations and required 4.04 miliseconds of CPU time for convergence.

Example 4

In the fourth example, the price ceilings were kept at .5, but now the tax rate was increased from .15 to .30. The numerical results for this example follow.

Equilibrium Prices:

$$r_1^* = .17990 \quad r_2^* = .22313 \quad r_3^* = .5000$$

Equilibrium Asset Holdings:

$$x_{11}^* = .28782 \quad x_{12}^* = .40104 \quad x_{13}^* = .31114$$

$$x_{21}^* = .75776 \quad x_{22}^* = .56804 \quad x_{23}^* = .67420$$

Equilibrium Liability Holdings:

$$y_{11}^* = .31846 \quad y_{12}^* = .51107 \quad y_{13}^* = .17046$$

$$y_{21}^* = .72386 \quad y_{22}^* = .45497 \quad y_{23}^* = .82117.$$

The algorithm converged in 24 iterations and required 5.09 miliseconds for convergence.

Example 5

In the final example, the tax rate was at $\tau = .3$, but the price ceilings were raised to $\hat{r} = 2$. The numerical results are as follows.

Equilibrium Prices:

$$r_1^* = .87731 \quad r_2^* = .92179 \quad r_3^* = 1.20088$$

Equilibrium Asset Holdings:

$$x_{11}^* = .28710 \quad x_{12}^* = .40066 \quad x_{13}^* = .31224$$

$$x_{21}^* = .75613 \quad x_{22}^* = .56744 \quad x_{23}^* = .67643$$

Equilibrium Liability Holdings:

$$y_{11}^* = .32066 \quad y_{12}^* = .51040 \quad y_{13}^* = .16894$$

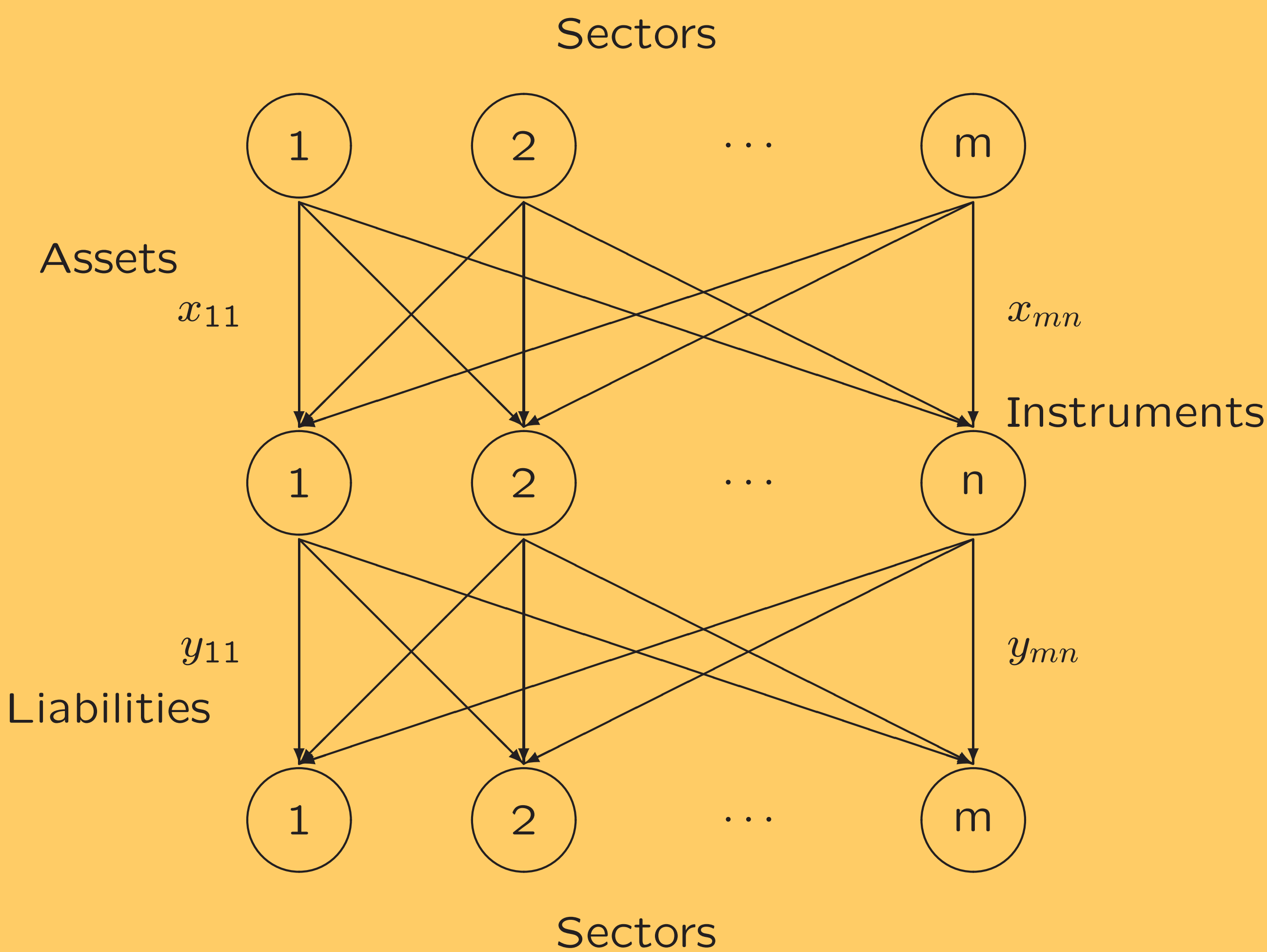
$$y_{21}^* = .72478 \quad y_{22}^* = .45746 \quad y_{23}^* = .81776.$$

The algorithm converged in 17 iterations for this example and required 3.59 miliseconds of CPU time for convergence.

For each of the above five examples, the algorithm yielded asset and liability patterns such that the difference between the total effective volume of an instrument held as an asset is approximately equal to the total volume of the instrument held as a liability, which the instrument price is not at one of the bounds. Hence, the market clears for each such instrument, and the price of each instrument is positive in equilibrium.

This lecture considers general financial equilibrium problems in a macroeconomic framework. In particular, multi-sector, multi-instrument models are developed which allow for the inclusion of policy interventions in the form of price ceilings and taxes. The behavioral assumption is that of utility/portfolio optimization for each sector. This assumption is in concert with classical single-agent, portfolio optimization models. The network structure of the models reveals itself through the decomposition algorithm that is proposed.

Note that this framework may also readily incorporate transaction costs directly into the utility functions, that is, into the objective function of each sector. The framework developed in this chapter may, hence, be used in a variety of policy settings. In addition, it can serve as a platform for the development of other models.



The network structure at equilibrium

In the references, additional citations are included that may be of interest to the reader.

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