A Supply Chain Network Equilibrium Model with Random Demands

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Abstract:

In this paper, we develop a supply chain network model consisting of manufacturers and retailers in which the demands associated with the retail outlets are random. We model the optimizing behavior of the various decision-makers, derive the equilibrium conditions, and establish the finite-dimensional variational inequality formulation. We provide qualitative properties of the equilibrium pattern in terms of existence and uniqueness results and also establish conditions under which the proposed computational procedure is guaranteed to converge. Finally, we illustrate the model through several numerical examples for which the equilibrium prices and product shipments are computed. This is the first supply chain network equilibrium model with random demands for which modeling, qualitative analysis, and computational results have been obtained.

Key Words: supply chain management, variational inequalities, network equilibrium, random demands

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1. Introduction

The topic of supply chain modeling and analysis has been of great interest, both from practical and research perspectives, due to its import in the efficient and cost-effective production and flow of goods and services in the network economy. Approaches that have been utilized to study supply chains have often been multidisciplinary in nature since such multitiered networks of suppliers, manufacturers, retailers, and consumers involve aspects of manufacturing, transportation and logistics, as well as retailing/marketing.

The body of literature on supply chains is vast (cf. Stadtler and Kilger (2000) and the references therein) with the associated research being both conceptual in nature (see, e.g., Poirier (1996, 1999), Mentzer (2000), Bovet (2000)), due to the complexity of the problem and the numerous decision-makers in the transactions, as well as analytical (cf. Federgruen and Zipkin (1986), Federgruen (1993), Slats et al. (1995), Bramel and Simchi-Levi (1997), Ganeshan et al. (1998), Miller (2001), Hensher, Button, and Brewer (2001) and the references therein).

Recently, there has been a notable effort expended on the development of decentralized supply chain network models in which the complexity of the interactions among the various decision-makers is captured and studied. For example, Lee and Billington (1993) emphasized the need for the development of decentralized models that allow for a generalized network structure and simplicity in computation in regards to the study of supply chains. Anupindi and Bassok (1996), on the other hand, focused on the challenges of systems consisting of decentralized retailers with information sharing. Lederer and Li (1997), in turn, modeled the competition among firms that produce goods or services for customers who are sensitive to delay time.

Nagurney, Dong, and Zhang (2002a) developed a supply chain network equilibrium model consisting of three tiers of decision-makers on the network and established that the governing equilibrium conditions which reflected the optimality conditions of the decision-makers consisting of manufacturers, retailers, and consumers along with the market equilibrium conditions could be formulated and studied in a unified manner as a finite-dimensional variational inequality problem. Such a modeling approach was subsequently extended by Nagurney, Loo, Dong, and Zhang (2002) to include electronic commerce in the form of business to business (B2B) and business to consumer (B2C) transactions and by Nagurney et al. (2002) to address the disequilibrium dynamics. More recently, Dong, Zhang, and Nagurney (2002) introduced multicriteria decision-making into supply chain network equilibrium modeling and computations. Additional background on related models as well as complementary ones in finance and in transportation can be found in the book by Nagurney and Dong (2002). For background on variational inequalities with a special emphasis on network economics, see the book by Nagurney (1999).

The afore-mentioned variational inequality models of supply chain networks, however, assumed that the underlying functions, be they, cost, revenue, or profit, were known with certainty. In this paper, in contrast, we relax this assumption for the demand functions at the level of the retailers. This result is significant since, in practice, retailers may not know the demands for a product with certainty but may, nevertheless, possess some information such as the density function based on historical data and/or forecasted data. Moreover, even with this extension, we are able to not only derive the optimality conditions for both manufacturers and the retailers, but also to establish that the governing equilibrium conditions in the random demand case satisfy a finite-dimensional variational inequality. Moreover, we provide reasonable conditions on the underlying functions in order to establish qualitative properties of the equilibrium price and product shipment pattern. Furthermore, we give conditions that, if satisfied, guarantee convergence of the proposed algorithmic scheme.

We note that Mahajan and Ryzin (2001) considered retailers under uncertain demand and focused on inventory competition. However, they assumed that the price of the product is exogenous. In this paper, in contrast, we assume competition, uncertain demand, and provide a means to determine the equilibrium prices both at the retailers and at the manufacturers. Lippman and McCardle (1997), in turn, developed a model of inventory competition for firms but assumed an aggregated random demand. In this paper, we allow each retailer to handle his own uncertain demand and to engage in competition, which seems closer to actual practice. More recently, Iida (2002) presented a production-inventory model with uncertain production capacity and uncertain demand.

The paper is organized as follows. In Section 2, we construct the supply chain network model with random demands at the retailer tier of nodes. We model the behavior of both the manufacturers and the retailers who are faced with random demands. The manufacturers are assumed to be profit-maximizers, to produce a homogeneous product, and to compete with other manufacturers in a noncooperative fashion. They seek to determine their profit-maximizing outputs and shipments of the product to the retailers. The retailers, who are faced with random demand for the product at their respective outlets, are also assumed to be profit-maximizers with a penalty associated with shortage of the product as well as with excess supply. The retailers also compete with one another in a noncooperative manner. In Section 2, we derive the optimality conditions and establish the governing equilibrium concept. We then give the variational inequality formulation, which is utilized in Section 3 to obtain qualitative properties of the equilibrium state as well as properties of the function that enters the variational inequality required for convergence of the algorithmic scheme.

In Section 4, we outline the algorithm and give convergence results. The algorithm is then applied in Section 5 to compute the equilibrium price and product shipment pattern in several supply chain examples. In Section 6 we summarize the results in this paper and present suggestions for future research.

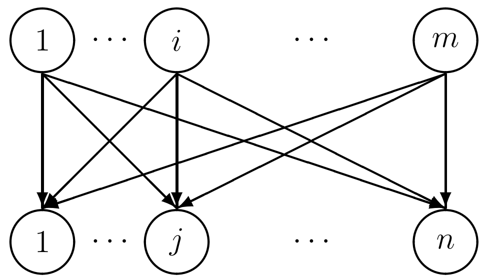

Figure 1: The Network Structure of the Supply Chain

2. The Supply Chain Network Equilibrium Model with Random Demands

In this Section, we develop the supply chain network equilibrium model with random demands at the retailer level. The representative decision-makers in our framework are the manufacturers and the retailers, with the consumers being represented through the random demands for the product at the different retail outlets. The supply chain network structure is as depicted in Figure 1.

In particular, we consider m manufacturers involved in the production of a homogeneous product, which can then be purchased by n retailers, who, in turn, respond to the consumers via the random demand functions. We denote a typical manufacturer by i and a typical retailer by j. The supply chain network, as depicted in Figure 1, consists of two tiers of nodes, with the manufacturers associated with the nodes in the top tier of the supply chain network and the retailers with the bottom tier of nodes. The links in the supply chain network denote transportation/transaction links.

We now turn to the discussion of the behavior of the various decision-makers in the supply chain. We first focus on the manufacturers. We then turn to the retailers.

The Manufacturers and their Optimality Conditions

Let q_i denote the nonnegative production output of the product by manufacturer i and group the production outputs of all manufacturers into the column vector $q \in R^m_+$. We assume that each manufacturer i is faced with a production cost function f_i, which can depend, in general, on the entire vector of production outputs, that is,

$$f_i = f_i(q), \quad \forall i. \tag{1}$$

A manufacturer may ship the product to the retailers, with the amount of the product shipped (or transacted) between manufacturer i and retailer j denoted by q_{ij}. We group the product shipments between all manufacturers and all retailers into the mn-dimensional column vector Q.

We associate with each manufacturer and retailer pair (i, j) a transaction cost, denoted by c_{ij}. The transaction cost includes the cost of shipping the product. We assume that the transaction cost between a manufacturer and retailer pair depends upon the volume of flow of the product between that pair, that is:

$$c_{ij} = c_{ij}(q_{ij}), \quad \forall i, j. \tag{2}$$

The quantity produced by manufacturer i must satisfy the following conservation of flow equation:

$$q_i = \sum_{j=1}^{n} q_{ij}, \tag{3}$$

which states that the quantity produced by manufacturer i is equal to the sum of the quantities shipped from the manufacturer to all retailers.

The total costs incurred by a manufacturer i are equal to the sum of his production cost plus the total transaction costs. His revenue, in turn, is equal to the price that the manufacturer charges for the product (and the retailers are willing to pay) times the total quantity purchased of the product from the manufacturer by all the retail outlets. We let ρ_{1ij} denote the price charged for the product by manufacturer i to retailer j and later in the paper we discuss how this price, in equilibrium, which is denoted by ρ^*_{1ij}, is arrived at. We group the prices of the manufacturers into the mn-dimensional column vector ρ_1.

Noting the conservation of flow equations (3), we can express the criterion of profit maximization for manufacturer i as:

$$\text{Maximize} \quad \sum_{j=1}^{n} \rho_{1ij} q_{ij} - f_i(Q) - \sum_{j=1}^{n} c_{ij}(q_{ij}), \tag{4}$$

subject to $q_{ij} \geq 0$, for all j.

We assume that the manufacturers compete in a noncooperative fashion. Also, we assume that the production cost functions and the transaction cost functions for each manufacturer are continuous and convex. Given that the governing optimization/equilibrium concept underlying noncooperative behavior is that of Nash (1950, 1951), which states that each manufacturer will determine his optimal production quantity and shipments, given the optimal ones of the competitors, the optimality conditions for all manufacturers *simultaneously* can be expressed as the following variational inequality (cf. Bazaraa, Sherali, and Shetty (1993), Gabay and Moulin (1980); see also Dafermos and Nagurney (1987) and Nagurney (1999)): determine $Q^* \in R_+^{mn}$ satisfying:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial f_i(Q^*)}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} - \rho_{1ij} \right] \times \left[q_{ij} - q_{ij}^* \right] \geq 0, \quad \forall Q \in R_+^{mn}. \tag{5}$$

The optimality conditions as expressed by (5) have a nice economic interpretation, which is that a manufacturer will ship a positive amount of the product to a retailer (and the flow on the corresponding link will be positive) if the price that the retailer is willing to pay for the product is precisely equal to the manufacturer's marginal production and transaction cost associated with that retailer. If the sum of the manufacturer's marginal production and transaction cost exceeds what the retailer is willing to pay for the product, then there will be zero shipment of the product between the pair.

The Retailers and their Optimality Conditions

The retailers, in turn, must decide how much to order from the manufacturers in order to cope with the random demand while still seeking to maximize their profits. A retailer j is faced with what we term a *handling* cost, which may include, for example, the display and storage cost associated with the product. We denote this cost by c_j and, in the simplest case, we would have that c_j is a function of $\sum_{i=1}^{m} q_{ij}$, that is, the holding cost of a retailer is a function of how much of the product he has obtained from the various manufacturers. However, for the sake of generality, and to enhance the modeling of competition, we allow the function to, in general, depend also on the amounts of the product held by other retailers and, therefore, we may write:

$$c_j = c_j(Q), \quad \forall j. \tag{6}$$

Let ρ_{2j} denote the demand price of the product associated with retailer j. We assume that $\hat{d}_j(\rho_{2j})$ is the demand for the product at the demand price of ρ_{2j} at retail outlet j, where $\hat{d}_j(\rho_{2j})$ is a random variable with a density function of $\mathcal{F}_j(x, \rho_{2j})$, with ρ_{2j} serving as a parameter. Hence, we assume that the density function may vary with the demand price. Let P_j be the probability distribution function of $\hat{d}_j(\rho_{2j})$, that is, $P_j(x, \rho_{2j}) = P(\hat{d}_j \leq x) = \int_0^x \mathcal{F}_j(x, \rho_{2j})dx$.

Let $s_j = \sum_{i=1}^{m} q_{ij}$, in turn, denote the total supply at retailer j that he obtains from all the manufacturers. Then, retailer j can sell to the consumers no more than the minimum of his supply or his demand, that is, the actual sale of j cannot exceed $\min\{s_j, \hat{d}_j\}$. Let

$$\Delta_j^+ \equiv \max\{0, s_j - \hat{d}_j\} \tag{7}$$

and

$$\Delta_j^- \equiv \max\{0, \hat{d}_j - s_j\}, \tag{8}$$

where Δ_j^+ is a random variable representing the excess supply (inventory), whereas Δ_j^- is a random variable representing the excess demand (shortage).

Note that the expected values of excess supply and excess demand of retailer j are scalar functions of s_j and ρ_{2j}. In particular, let e_j^+ and e_j^- denote, respectively, the expected values: $E(\Delta_j^+)$ and $E(\Delta_j^-)$, that is,

$$e_j^+(s_j, \rho_{2j}) \equiv E(\Delta_j^+) = \int_0^{s_j} (s_j - x)\mathcal{F}_j(x, \rho_{2j})dx, \tag{9}$$

$$e_j^-(s_j, \rho_{2j}) \equiv E(\Delta_j^-) = \int_{s_j}^{\infty} (x - s_j)\mathcal{F}_j(x, \rho_{2j})dx. \tag{10}$$

Assume that the unit penalty of having excess supply at retail outlet j is λ_j^+ and that the unit penalty of having excess demand is λ_j^-, where the λ_j^+ and the λ_j^- are assumed to be nonnegative. Then, the expected total penalty of retailer j is given by

$$E(\lambda_j^+\Delta_j^+ + \lambda_j^-\Delta_j^-) = \lambda_j^+ e_j^+(s_j, \rho_{2j}) + \lambda_j^- e_j^-(s_j, \rho_{2j}).$$

Assuming, as already mentioned, that the retailers are also profit-maximizers, the expected revenue of retailer j is $E(\rho_{2j}\min\{s_j, \hat{d}_j\})$. Hence, the optimization problem of a retailer j can be expressed as:

$$\text{Maximize} \quad E(\rho_{2j}\min\{s_j, \hat{d}_j\}) - E(\lambda_j^+\Delta_j^+ + \lambda_j^-\Delta_j^-) - c_j(Q) - \sum_{i=1}^{m} \rho_{1ij} q_{ij} \tag{11}$$

subject to:

$$q_{ij} \geq 0, \text{ for all } i. \tag{12}$$

Objective function (11) expresses that the expected profit of retailer j, which is the difference between the expected revenues and the sum of the expected penalty, the handling cost, and the payout to the manufacturers, should be maximized.

Applying now the definitions of Δ_j^+, and Δ_j^-, we know that $\min\{s_j, \hat{d}_j\} = \hat{d}_j - \Delta_j^-$. Therefore, the objective function (11) can be expressed as

$$\text{Maximize} \quad \rho_{2j} d_j(\rho_{2j}) - \rho_{2j} e_j^-(s_j, \rho_{2j}) - \lambda_j^+ e_j^+(s_j, \rho_{2j}) - \lambda_j^- e_j^-(s_j, \rho_{2j}) - c_j(Q) - \sum_{i=1}^{m} \rho_{1ij} q_{ij} \tag{13}$$

where $d_j(\rho_{2j}) \equiv E(\hat{d}_j)$ is a scalar function of ρ_{2j}.

We now consider the optimality conditions of the retailers assuming that each retailer is faced with the optimization problem (11), subject to (12), which represents the nonnegativity assumption on the variables. Here, we also assume that the retailers compete in a noncooperative manner so that each maximizes his profits, given the actions of the other retailers. Note that, at this point, we consider that retailers seek to determine the amount that they wish to obtain from the manufacturers. First, however, we make the following derivation and introduce the necessary notation:

$$\frac{\partial e_j^+(s_j, \rho_{2j})}{\partial q_{ij}} = P_j(s_j, \rho_{2j}) = P_j(\sum_{i=1}^{m} q_{ij}, \rho_{2j}) \tag{14}$$

$$\frac{\partial e_j^-(s_j)}{\partial q_{ij}} = P_j(s_j, \rho_{2j}) - 1 = P_j(\sum_{i=1}^m q_{ij}, \rho_{2j}) - 1. \quad (15)$$

Assuming that the handling cost for each retailer is continuous and convex, then the optimality conditions for all the retailers satisfy the variational inequality: determine $Q^* \in R_+^{mn}$, satisfying:

$$\sum_{i=1}^m \sum_{j=1}^n \left[\lambda_j^+ P_j(\sum_{i=1}^m q_{ij}^*, \rho_{2j}) - (\lambda_j^- + \rho_{2j})(1 - P_j(\sum_{i=1}^m q_{ij}^*, \rho_{2j})) + \frac{\partial c_j(Q^*)}{\partial q_{ij}} + \rho_{1ij}\right]$$

$$\times \left[q_{ij} - q_{ij}^*\right] \geq 0, \quad \forall Q \in R_+^{mn}. \quad (16)$$

In this derivation, as in the derivation of inequality (5), we have not had the prices charged be variables. They become endogenous variables in the complete supply chain network equilibrium model.

We now highlight the economic interpretation of the retailers' optimality conditions. In inequality (16), we can infer that, if a manufacturer i transacts with a retailer j resulting in a positive flow of the product between the two, then the selling price at retail outlet j, ρ_{2j}, with the probability of $(1 - P_j(\sum_{i=1}^m q_{ij}^*, \rho_{2j}))$, that is, when the demand is not less then the total order quantity, is precisely equal to the retailer j's payment to the manufacturer, ρ_{1ij}, plus his marginal cost of handling the product and the penalty of having excess demand with probability of $P_j(\sum_{i=1}^m q_{ij}^*, \rho_{2j})$, (which is the probability when actual demand is less than the order quantity), subtracted by the penalty of having shortage with probability of $(1 - P_j(\sum_{i=1}^m q_{ij}^*, \rho_{2j}))$ (when the actual demand is greater than the order quantity).

The Equilibrium Conditions

We now turn to a discussion of the market equilibrium conditions. Subsequently, we construct the equilibrium conditions for the entire supply chain.

The equilibrium conditions associated with the transactions that take place between the retailers and the consumers are the *stochastic economic equilibrium conditions*, which, mathematically, take on the following form: For any retailer j; $j = 1, \ldots, n$:

$$\hat{d}_j(\rho_{2j}^*) \begin{cases} \leq \sum_{i=1}^m q_{ij}^* & \textbf{a.e.}, \text{ if } \rho_{2j}^* = 0 \\ = \sum_{i=1}^m q_{ij}^* & \textbf{a.e.}, \text{ if } \rho_{2j}^* > 0, \end{cases} \quad (17)$$

where **a.e.** means that the corresponding equality or inequality holds almost everywhere.

Conditions (17) state that, if the equilibrium demand price at outlet j is positive, that is, $\rho_{2j}^* > 0$, then the quantities purchased by the retailer from the manufacturers in the aggregate, that is, $\sum_{i=1}^m q_{ij}^*$, is equal to the demand, with exceptions of zero probability. These conditions correspond to the well-known economic equilibrium conditions (cf. Nagurney (1999) and the references therein). Related equilibrium conditions, but in a deterministic version, were proposed in Nagurney, Dong, and Zhang (2002a).

Equilibrium conditions (17) are equivalent to the following variational inequality problem, after taking the expected value and summing over all retailers j: determine $\rho_2^* \in R_+^n$ satisfying

$$\sum_{j=1}^n (\sum_{i=1}^m q_{ij}^* - d_j(\rho_{2j}^*)) \times \left[\rho_{2j} - \rho_{2j}^*\right] \geq 0, \qquad \forall \rho_2 \in R_+^n, \tag{18}$$

where ρ_2 is the n-dimensional column vector with components: $\{\rho_{21}, \ldots, \rho_{2n}\}$.

The Equilibrium Conditions of the Supply Chain

In equilibrium, we must have that the sum of the optimality conditions for all manufacturers, as expressed by inequality (5), the optimality conditions for all retailers, as expressed by inequality (16), and the market equilibrium conditions, as expressed by inequality (18) must be satisfied. Hence, the shipments that the manufacturers ship to the retailers must be equal to the shipments that the retailers accept from the manufacturers. We state this explicitly in the following definition:

Definition 1: Supply Chain Network Equilibrium with Random Demands

The equilibrium state of the supply chain with random demands is one where the product flows between the two tiers of the decision-makers coincide and the product shipments and prices satisfy the sum of the optimality conditions (5) and (16) and the conditions (18).

The summation of inequalities (5), (16), and (18) (with the prices at the manufacturers and the retailers denoted, respectively, by their values at the equilibrium and denoted by ρ_1^* and ρ_2^*), after algebraic simplification, yields the following result:

Theorem 1: Variational Inequality Formulation

The equilibrium conditions governing the supply chain network model with random demands are equivalent to the solution of the variational inequality problem given by: determine $(Q^*, \rho_2^*) \in R_+^{mn+n}$ *satisfying:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(Q^*)}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} + \frac{\partial c_j(Q^*)}{\partial q_{ij}} + \lambda_j^+ P_j(\sum_{i=1}^{m} q_{ij}^*, \rho_{2j}^*) - (\lambda_j^- + \rho_{2j}^*)(1 - P_j(\sum_{i=1}^{m} q_{ij}^*, \rho_{2j}^*))\right]$$

$$\times \left[q_{ij} - q_{ij}^*\right] + \sum_{j=1}^{n}\left[\sum_{i=1}^{m} q_{ij}^* - d_j(\rho_{2j}^*)\right] \times \left[\rho_{2j} - \rho_{2j}^*\right] \geq 0, \quad \forall (Q, \rho_2) \in R_+^{mn+n}. \qquad (19)$$

For easy reference in the subsequent sections, variational inequality problem (19) can be rewritten in standard variational inequality form (cf. Nagurney (1999)) as follows:

$$\langle F(X^*)^T, X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K} \equiv R_+^{mn+n}, \qquad (20)$$

where $X \equiv (Q, \rho_2)$, and $F(X) \equiv (F_{ij}, F_j)_{i=1,\ldots,m; j=1,\ldots,n}$. The expression $\langle \cdot, \cdot \rangle$ denotes the inner product in n-dimensional Euclidean space where here $N = mn + n$.

The variables in the variational inequality problem are: the equilibrium product shipments from the manufacturers to the retailers, Q^* (from which one can then recover the production outputs through (3)), and the equilibrium demand prices of the product at the retailers, ρ_2^*.

We now discuss how to recover the prices ρ_{1ij}^*, for all i and j, from the solution of variational inequality (19). (In Section 4 we describe an algorithm for computing the solution.) The prices ρ_{1ij}^* (cf. (5)) can be obtained as follows: if $q_{ij}^* > 0$, then set $\rho_{1ij}^* = \left[\frac{\partial f(Q^*)}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}}\right]$; equivalently, (cf. (16)) set $\rho_{1ij}^* = -\lambda_j^+ P_j(\sum_{i=1}^m q_{ij}^*, \rho_{2j}^*) + (\lambda_j^- + \rho_{2j}^*)(1 - P_j(\sum_{i=1}^m q_{ij}^*, \rho_{2j}^*)) - \frac{\partial c_j(Q^*)}{\partial q_{ij}}$.

Note that in this model, the equilibrium prices associated with the manufacturers as well as those associated with the retailers are endogenous to the model with the manufacturers' and the retailers' product shipments at equilibrium being determined at the equilibrium price vectors.

3. Qualitative Properties

In this Section, we provide some qualitative properties of the solution to variational inequality (19). In particular, we derive existence and uniqueness results. We also investigate properties of the function F (cf. (20)) that enters the variational inequality of interest here.

Our previous assumptions about the production cost functions, transaction cost functions, and retailers' handling cost functions imply that the vector function that enters into the variational inequality (20) is continuous. However, the feasible set is not compact. Therefore, we cannot derive the existence of a solution simply from the assumption of continuity of the functions. Nevertheless, we can impose a rather weak condition to guarantee existence of a solution pattern.

Let

$$\mathcal{K}_b = \{(Q, \rho_2) | 0 \leq Q \leq b_1; \ 0 \leq \rho_2 \leq b_2\}, \tag{21}$$

where $b = (b_1, b_2) \geq 0$ and $Q \leq b_1; \rho_{2j} \leq b_2$ means that $q_{ij} \leq b_1$ and $\rho_{2j} \leq b_2$ for all i, j. Then $\mathcal{K}_b$ is a bounded closed convex subset of R^{mn+n}. Thus, the following variational inequality

$$\langle F(X^b)^T, X - X^b \rangle \geq 0, \quad \forall X^b \in \mathcal{K}_b, \tag{22}$$

admits at least one solution $X^b \in \mathcal{K}_b$, from the standard theory of variational inequalities, since $\mathcal{K}_b$ is compact and F is continuous. Following Kinderlehrer and Stampacchia (1980) (see also Theorem 1.5 in Nagurney (1999)), we then have:

Theorem 2

Variational inequality (20) admits a solution if there exists a $b > 0$, such that variational inequality (22) admits a solution in $\mathcal{K}_b$ with

$$Q^{1b} < b_1, \quad \rho_2^b < b_2. \tag{23}$$

Theorem 3: Existence

Suppose that there exist positive constants M, N, R *such that:* $\forall Q$ *with* $q_{ij} \geq N$, $\forall i, j$:

$$\frac{\partial f_i(Q)}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij})}{\partial q_{ij}} + \frac{\partial c_j(Q)}{\partial q_{ij}} + \lambda_j^+ P_j(s_j, \rho_{2j}) - (\lambda_j^- + \rho_{2j})(1 - P_j(s_j, \rho_{2j})) \geq M, \tag{24}$$

and

$$d_j(\rho_{2j}) \leq N, \quad \forall \rho_2 \quad \text{with} \quad \rho_{2j} \geq R, \quad \forall j. \tag{25}$$

Then, variational inequality (20) admits at least one solution.

Proof: Follows using analogous arguments as the proof of existence for Proposition 1 in Nagurney and Zhao (1993) (see also existence proof in Nagurney, Dong, and Zhang (2002b)). □

Assumptions (24) and (25) can be economically justified as follows. In particular, when the product shipment, q_{ij}, between manufacturer i and retailer j is large, one can expect the corresponding sum of the marginal costs associated with the production, transaction, and holding to exceed a positive lower bound, say M. At same time, the large q_{ij} causes a greater s_j, which in turn causes the probability distribution $P_j(s_j, \rho_{2j})$ to be close to 1. Consequently, the sum of the last two terms on the left-hand side of (24), $\lambda_j^+ P_j(s_j, \rho_{2j}) - (\lambda_j^- + \rho_{2j})(1 - P(s_j, \rho_{2j}))$ is seen to be positive. Therefore, the left-hand side of (24) is greater than or equal to the lower bound M. On the other hand, a high price ρ_{2j} at retailer j will drive the demand at that retailer down, in line with the decreasing nature of any demand function, which ensures (25).

We now recall the concept of an additive production cost, which was introduced by Zhang and Nagurney (1996) in the stability analysis of dynamic spatial oligopolies, and has also been employed in the qualitative analysis by Nagurney, Zhang, and Dong (2002b) for the study of spatial economic networks with multicriteria producers and consumers.

Definition 2: Additive Production Cost

Suppose that for each manufacturer i, *the production cost* f_i *is additive, that is*

$$f_i(q) = f_i^1(q_i) + f_i^2(\bar{q}_i), \tag{26}$$

where $f_i^1(q_i)$ is the internal production cost that depends solely on the manufacturer's own output level q_i, which may include the production operation and the facility maintenance, etc., and $f_i^2(\bar{q}_i)$ is the interdependent part of the production cost that is a function of all the other manufacturers' output levels $\bar{q}_i = (q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_m)$ and reflects the impact of the other manufacturers' production patterns on manufacturer i's cost. This interdependent part of the production cost may describe the competition for the resources, consumption of the homogeneous raw materials, etc.

We now explore additional qualitative properties of the vector function F that enters the variational inequality problem. Specifically, we show that F is monotone as well as Lipschitz continuous. These properties are fundamental in establishing the convergence of the algorithmic scheme in the subsequent section.

Lemma 1

Let $g(s,\rho)^T = (\lambda^+ P(s,\rho) - (\lambda^- + \rho_2)(1 - P(s,\rho)), s - d(\rho))$, where P is a probability distribution with the density function of $\mathcal{F}(x,\rho)$. Then $g(s,\rho)$ is monotone if and only if $d'(\rho) \leq -(4\alpha\mathcal{F})^{-1}(P + \alpha\frac{\partial P}{\partial \rho})^2$, where $\alpha = \lambda^+ + \lambda^- + \rho$.

Proof: In order to prove that $g(s,\rho)$ is monotone with respect to s and ρ, we only need to show that its Jacobian matrix is positive semidefinite, which will be the case if all eigenvalues of the symmetric part of the Jacobian matrix are nonnegative real numbers.

Let $\alpha = \lambda^+ + \lambda^- + \rho$, then the Jacobian matrix of g is

$$\nabla g(s,\rho) = \begin{bmatrix} \alpha\mathcal{F}(s,\rho) & -1 + P(s,\rho) + \alpha\frac{\partial P(s,\rho)}{\partial \rho} \\ 1 & -d'(\rho) \end{bmatrix}, \tag{27}$$

and its symmetric part is

$$\frac{1}{2}[\nabla g(s,\rho) + \nabla^T g(s,\rho)] = \begin{bmatrix} \alpha\mathcal{F}(s,\rho), & \frac{1}{2}\left(\alpha\frac{\partial P}{\partial \rho} + P(s,\rho)\right) \\ \frac{1}{2}\left(\alpha\frac{\partial P}{\partial \rho} + P(s,\rho)\right), & -d'(\rho) \end{bmatrix}. \tag{28}$$

The two eigenvalues of (28) are

$$\gamma_{min}(s,\rho) = \frac{1}{2}\left[(\alpha\mathcal{F} - d') - \sqrt{(\alpha\mathcal{F} - d')^2 + (\alpha\frac{\partial P}{\partial \rho} + P)^2 + 4\alpha\mathcal{F}d'}\right], \tag{29}$$

$$\gamma_{max}(s,\rho) = \frac{1}{2}\left[(\alpha\mathcal{F} - d') + \sqrt{(\alpha\mathcal{F} - d')^2 + (\alpha\frac{\partial P}{\partial \rho} + P)^2 + 4\alpha\mathcal{F}d'}\right]. \tag{30}$$

Since what is inside the square root in both (29) and (30) can be rewritten as

$$(\alpha\mathcal{F} + d')^2 + \left(\alpha\frac{\partial P}{\partial \rho} + P\right)^2$$

and can be seen as nonnegative, both eigenvalues are real. Furthermore, under the condition of the lemma, d' is non-positive, so the first item in (29) and (30) is nonnegative. The condition further implies that the second item in (29) and (30), the square root part, is not greater than the first item, which guarantees that both eigenvalues are nonnegative real numbers. □

The condition of Lemma 1 states that the expected demand function of a retailer is a nonincreasing function with respect to the demand price and its first order derivative has an upper bound.

Theorem 4: Monotonicity

The function that enters the variational inequality problem (20) is monotone, if the condition assumed in Lemma 1 is satisfied for each j; $j = 1,\dots,n$, and if the following conditions are also satisfied.

Suppose that the production cost functions $f_i; i = 1,...,m$, are additive, as defined in Definition 2, and that the $f_i^1; i = 1,...,m$, are convex functions. If the c_{ij} and c_j functions are convex, for all i, j, then the vector function F that enters the variational inequality (20) is monotone, that is,

$$\langle (F(X') - F(X''))^T, X' - X''\rangle \geq 0, \quad \forall X', X'' \in \mathcal{K}. \tag{31}$$

Proof: Let $X' = (Q', \rho_2')$, $X'' = (Q'', \rho_2'')$. Then, inequality (31) can been seen in the following deduction:

$$\langle (F(X') - F(X''))^T, X' - X''\rangle$$

$$
\begin{aligned}
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(Q')}{\partial q_{ij}} - \frac{\partial f_i(Q'')}{\partial q_{ij}}\right] \times \left[q'_{ij} - q''_{ij}\right] \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial c_j(Q')}{\partial q_{ij}} - \frac{\partial c_j(Q'')}{\partial q_{ij}}\right] \times \left[q'_{ij} - q''_{ij}\right] \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial c_{ij}(q'_{ij})}{\partial q_{ij}} - \frac{\partial c_{ij}(q''_{ij})}{\partial q_{ij}}\right] \times \left[q'_{ij} - q''_{ij}\right] \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n}[\lambda_j^+ P_j(\sum_{i=1}^{n} q'_{ij}, \rho'_{2j}) - \lambda_j^+ P_j(\sum_{i=1}^{m} q''_{ij}, \rho''_{2j})] \times [q'_{ij} - q''_{ij}] \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n}[-\lambda_j^-(1 - P_j(\sum_{i=1}^{m} q'_{ij}, \rho'_{2j})) + \lambda_j^-(1 - P_j(\sum_{i=1}^{m} q''_{ij}, \rho''_{2j}))] \times [q'_{ij} - q''_{ij}] \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n}[-\rho'_{2j}(1 - P_j(\sum_{i=1}^{m} q'_{ij}, \rho'_{2j})) + \rho''_{2j}(1 - P_j(\sum_{i=1}^{m} q''_{ij}, \rho''_{2j}))] \times [q'_{ij} - q''_{ij}] \\
&+ \sum_{j}^{n}[s'_j - d_j(\rho'_{2j}) - s''_j + d_j(\rho''_{2j})] \times [\rho'_{2j} - \rho''_{2j}] \\
&= (I) + (II) + (III) + (IV) + (V) + (VI) + (VII). \qquad (32)
\end{aligned}
$$

Since the f_i; $i = 1, ..., m$, are additive, and the f_i^1; $i = 1, ..., m$, are convex functions, one has that

$$(I) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i^1(Q')}{\partial q_{ij}} - \frac{\partial f_i^1(Q'')}{\partial q_{ij}}\right] \times \left[q'_{ij} - q''_{ij}\right] \geq 0. \qquad (33)$$

The convexity of c_j, for all j, and c_{ij}, for all i, j, gives, respectively,

$$(II) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial c_j(Q')}{\partial q_{ij}} - \frac{\partial c_j(Q')}{\partial q_{ij}}\right] \times \left[q'_{ij} - q''_{ij}\right] \geq 0 \qquad (34)$$

$$(III) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial c_{ij}(q'_{ij})}{\partial q_{ij}} - \frac{\partial c_{ij}(q''_{ij})}{\partial q_{ij}}\right] \times \left[q'_{ij} - q''_{ij}\right] \geq 0. \qquad (35)$$

Since the probability function P_j is an increasing function, for all j, hence, (IV) and (V) are greater than or equal to zero.

Let $s_j = \sum_{i=1}^{m} q_{ij}$. Then we have that

$$(IV) + (V) + (VI) + (VII) =$$

$$+\sum_{j=1}^{n}[\lambda_j^+ P_j(s'_j,\rho'_{2j})-\lambda_j^+ P_j(s''_j,\rho''_{2j})]\times[s'_j-s''_j]$$

$$+\sum_{j=1}^{n}[-\lambda_j^-(1-P_j(s'_j,\rho'_{2j}))+\lambda_j^-(1-P_j(s''_j,\rho''_{2j}))]\times[s'_j-s''_j]$$

$$\sum_{j=1}^{n}[-\rho'_{2j}(1-P_j(s'_j,\rho'_{2j}))+\rho''_{2j}(1-P_{2j}(s''_j,\rho''_{2j}))]\times[s'_j-s''_j]$$

$$+\sum_{j=1}^{n}[s'_j-d_j(\rho'_{2j})-s''_j+d_j(\rho''_{2j})]\times[\rho'_{2j}-\rho''_{2j}]. \tag{36}$$

Since for each j, applying Lemma 1, we can see that $g_j(s_j,\rho_{2j})$ is monotone, hence, (36) is nonnegative. Therefore, we conclude that (32) is nonnegative in $\mathcal{K}$. The proof is complete. □

If the conditions in Theorem 4 are slightly strengthened so that the the vector function enters into the variational inequality problem (20) is strictly monotone, then its solution is unique (See, e.g., Nagurney (1999)).

Theorem 5: Uniqueness

Suppose that the production cost functions $f_i; i=1,...,m$, are additive, as defined in Definition 2, and that the $f_i^1; i=1,...,m$, are strictly convex functions. If the c_{ij} and c_j functions are strictly convex, for all i,j, then the function that enters the variational inequality (20) has a unique solution in $\mathcal{K}$.

From Theorem 5 it follows that, under the above conditions, the equilibrium product shipment pattern between the manufacturers and the retailers, as well as the equilibrium price pattern at the retailers, is unique.

Theorem 6: Lipschitz Continuity

The function that enters the variational inequality problem (20) is Lipschitz continuous, that is,

$$\|F(X')-F(X'')\|\le L\|X'-X''\|,\quad \forall X',X''\in\mathcal{K},\ \text{with } L>0, \tag{37}$$

under the following conditions:

(i). Each f_i; $i = 1, ..., m$, is additive and has a bounded second order derivative;

(ii). The c_{ij} and c_j have bounded second order derivatives, for all i, j;

Proof: Since the probability function P_j is always less than or equal to 1, for each retailer j, the result is direct by applying a mid-value theorem from calculus to the vector function F that enters the variational inequality problem (20). $\square$

4. The Algorithm

In this Section, an algorithm is presented which can be applied to solve any variational inequality problem in standard form (see (20)), that is:

Determine $X^* \in \mathcal{K}$, satisfying:

$$\langle F(X^*)^T, X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{38}$$

The algorithm is guaranteed to converge provided that the function F that enters the variational inequality is monotone and Lipschitz continuous (and that a solution exists). The algorithm is the modified projection method of Korpelevich (1977). The statement of the modified projection method is as follows, where $\mathcal{T}$ denotes an iteration counter:

Modified Projection Method

Step 0: Initialization

Set $X^0 \in \mathcal{K}$. Let $\mathcal{T} = 1$ and let α be a scalar such that $0 < a \leq \frac{1}{L}$, where L is the Lipschitz continuity constant (cf. Korpelevich (1977)) (see (37)).

Step 1: Computation

Compute $\bar{X}^{\mathcal{T}}$ by solving the variational inequality subproblem:

$$\langle (\bar{X}^{\mathcal{T}} + aF(X^{\mathcal{T}-1}) - X^{\mathcal{T}-1})^T, X - \bar{X}^{\mathcal{T}} \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{39}$$

Step 2: Adaptation

Compute $X^{\mathcal{T}}$ by solving the variational inequality subproblem:

$$\langle (X^{\mathcal{T}} + aF(\bar{X}^{\mathcal{T}}) - X^{\mathcal{T}-1})^T, X - X^{\mathcal{T}} \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{40}$$

Step 3: Convergence Verification

If $\max |X_l^{\mathcal{T}} - X_l^{\mathcal{T}-1}| \leq \epsilon$, for all l, with $\epsilon > 0$, a prespecified tolerance, then stop; else, set $\mathcal{T} =: \mathcal{T} + 1$, and go to Step 1.

We now state the convergence result for the modified projection method for this model.

Theorem 7: Convergence

Assume that the function that enters the variational inequality (19) (or (20)) has at least one solution and satisfies the conditions in Theorem 4 and in Theorem 6. Then the modified projection method described above converges to the solution of the variational inequality (19) or (20).

Proof: According to Korpelevich (1977), the modified projection method converges to the solution of the variational inequality problem of the form (20), provided that the function F that enters the variational inequality is monotone and Lipschitz continuous and that a solution exists. Existence of a solution follows from Theorem 3. Monotonicity follows Theorem 4. Lipschitz continuity, in turn, follows from Theorem 6. □

We emphasize that, in view of the fact that the feasible set $\mathcal{K}$ underlying the supply chain network equilibrium model with random demands is the nonnegative orthant, the projection operation encountered in (39) and (40) takes on a very simple form for computational purposes. Indeed, the product shipments as well as the product prices at a given iteration in both (39) and in (40) can be exactly and computed in closed form. Hence, the modified projection method is, in the context of our problem, straightforward to implement. Of course, one still must determine the *step size* a, which is fixed, and which depends on the Lipschitz constant for the particular problem. We return to this point in the subsequent section in which we present the numerical examples. We note that variants and extensions of the Korpelevich method for the solution of montonone variational inequalities have been developed. In particular, we note the method of Khobotov (1987), which provides a rule for the determination of the step size which is allowed to vary (see also, Marcotte (1991), Solodov and Tseng (1996), Solodov and Svaiter (1999), and the references therein), but which also requires the selection of a parameter that is problem dependent.

5. Numerical Examples

In this Section, we apply the modified projection method to several numerical examples. The algorithm was implemented in FORTRAN and the computer system used was a DEC Alpha system located at the University of Massachusetts at Amherst. The convergence criterion used was that the absolute value of the product shipments and prices between two successive iterations differed by no more than 10^{-4}. The parameter a in the modified projection method (see (39) and (40)) was set to .01 for all the examples. The algorithm was initialized as follows for all the examples: the initial product shipments were set to zero whereas the initial demand prices at the retailers were set to one for all the retailers.

In all the examples, we assumed that the demands associated with the retail outlets followed a uniform distribution. Hence, we assumed that the random demand, $\hat{d}_j(\rho_{2j})$, of retailer j, is uniformly distributed in $[0, \frac{b_j}{\rho_{2j}}]$, $b_j > 0$; $j = 1, \ldots, n$. Therefore,

$$P_j(x, \rho_{2j}) = \frac{x\rho_{2j}}{b_j}, \tag{41}$$

$$\mathcal{F}_j(x, \rho_{2j}) = \frac{\rho_{2j}}{b_j}, \tag{42}$$

$$d_j(\rho_{2j}) = E(\hat{d}_j) = \frac{1}{2}\frac{b_j}{\rho_{2j}}; \quad j = 1, \ldots, n. \tag{43}$$

It is easy to verify that the expected demand function $d_j(\rho_{2j})$ associated with retailer j is a decreasing function of the price at the demand market.

Example 1

The first numerical supply chain example consisted of two manufacturers and two retailers, as depicted in Figure 2.

The data for this example were constructed for easy interpretation purposes. The production cost functions for the manufacturers were given by:

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 2q_1, \quad f_2(q) = 2.5q_2^2 + q_1q_2 + 2q_2.$$

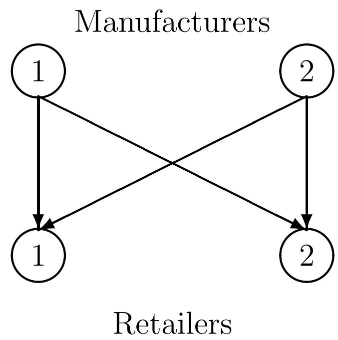

Figure 2: Supply Chain Network for Numerical Example 1 and its Variants

The transaction cost functions faced by the manufacturers and associated with transacting with the retailers were given by:

$$c_{ij}(q_{ij}) = .5q_{ij}^2 + 3.5q_{ij}, \quad \text{for } i = 1,2;\ j = 1,2.$$

The handling costs of the retailers, in turn, were given by:

$$c_1(Q) = .5(\sum_{i=1}^{2} q_{i1})^2, \quad c_2(Q) = .5(\sum_{i=1}^{2} q_{i2})^2.$$

The b_js were set to 10 for both retail outlets yielding probability distribution functions as in (41) and the expected demand functions as in (43). The weights (see (13)) associated with the excess supply and excess demand at the retailers were: $\lambda_j^+ = \lambda_j^- = 1$ for $j = 1,2$. Hence, we assigned equal weights for each retailer for excess supply and for excess demand.

The modified projection method converged in 5895 iterations and in a negligible amount of CPU time and yielded the following equilibrium pattern: the product shipments between the two manufacturers and the two retailers were: $Q^* : q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = .1590$ and the demand prices at the retailers were: $\rho_{21}^* = \rho_{22}^* = 15.2460$. It is easy to verify that the optimality/equilibrium conditions were satisfied with good accuracy.

Example 2: Variant 1 of Example 1

We then proceeded to construct a variant of Example 1 as follows. We increased the b_js associated with both retailers from 10 to 100 but kept the remainder of the data as in Example 1. In view of (43), this implies that the expected demand associated with each retailer increased. The structure of the supply chain remained as in Figure 2.

The modified projection method required 4330 iterations for convergence and a negligible amount of CPU time and yielded the new equilibrium product shipment pattern given by: $Q^* : q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = .7479$ and the new equilibrium demand price pattern given by: $\rho_{21}^* = \rho_{22}^* = 33.2017$.

Observe that with a higher b_j for each retailer, the product shipments from each manufacturer to each retailer increased since the expected demand increased at each outlet and the demand price at each outlet also increased.

Example 3: Variant 2 of Example 1

To construct Example 3, we kept the data as in Example 1, but now we increased the b_js even more than they were increased in Example 2. In particular, we now had $b_1 = b_2 = 1000$, which implies (cf. (43)) that the expected demand associated with the retailers was even higher than in the two preceding examples. Of course, the structure of the supply chain network remained as in Figure 2.

The modified projection method again converged, in 4345 iterations, and yielded the equilibrium product shipment pattern given by: $Q^* : q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 2.7093$ and the equilibrium demand price pattern given by: $\rho_{21}^* = \rho_{22}^* = 92.1003$.

Note that, in this example, the production outputs of the manufacturers increased since the demand at the retailers increased as did the demand prices for the product at the retailers.

Example 4

The fourth numerical example (as well as its subsequent variant) consisted of three manufacturers and two retailers. Hence, the supply chain network was now as depicted in Figure 3.

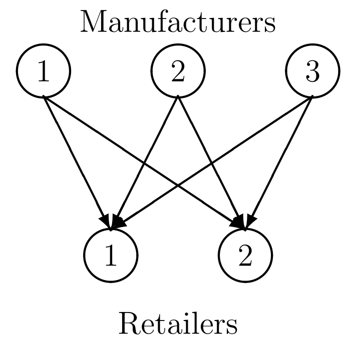

Figure 3: Supply Chain Network for Numerical Example 4 and its Variant

The data for this example were constructed from the data for Example 3, but we added the necessary functions for the third manufacturer resulting in the following functions:

The production cost functions for the manufacturers were given by:

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 2q_1, \quad f_2(q) = 2.5q_2^2 + q_1q_2 + 2q_2, \quad f_3(q) = .5q_3^2 + .5q_1q_3 + 2q_3.$$

Note that the production cost function associated with the third manufacturer was distinct from those of the other two manufacturers.

The transaction cost functions faced by the manufacturers and associated with transacting with the retailers were given by:

$$c_{ij}(q_{ij}) = .5q_{ij}^2 + 3.5q_{ij}, \quad \text{for } i = 1, 2, 3; j = 1, 2.$$

Hence, we retained the transaction cost functions utilized in the preceding three examples except that we now added new ones associated with the transactions between the new manufacturer and the two retailers. The handling costs of the retailers remained as in the preceding examples as did the expected demand functions. This example, hence, illustrates what may happen when a new manufacturer enters the market with lower production costs than the other manufacturers.

The modified projection method converged in 2122 iterations and an insignificant amount of CPU time and yielded the following equilibrium pattern: the product shipments between the three manufacturers and the two retailers were: $Q^* := q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 1.3432$, $q_{31}^* = q_{32}^* = 5.3729$. The equilibrium demand prices at the two retailers were: $\rho_{21}^* = \rho_{22}^* = 61.9623$.

Note that, in comparison to the results in Example 3, with the addition of a new manufacturer, the price charged at the retailer outlets was now lower, due to the competition, and the increased supply of the product.

Example 5: Variant 1 of Example 4

The fifth numerical example was constructed from the fourth with the data retained but with the following change: we now increased the weight associated with oversupply at all retail outlets from 1 to 10. Also, we set the weights associated with undersupply at all retail outlets to 0. Hence, we now had that $\lambda_j^+ = 10$ for $j = 1, 2$ and $\lambda_j^- = 0$ for $j = 1, 2$.

The modified projection method for this example required 2614 iterations for convergence and yielded the following new equilibrium product shipment pattern: $Q^* := q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 1.2303$, $q_{31}^* = q_{32}^* = 4.9211$ and the new equilibrium demand prices at the two retailers were: $\rho_{21}^* = \rho_{22}^* = 67.6419$.

Hence, when the penalty associated with excess supply increased and there was no penalty imposed on shortage by each retailer, each retailer reduced his order quantity. The price at each retailer increased (vis a vis that in Example 4) due to the higher probability of having a shortage (undersupply of the product).

Example 6

The sixth numerical example consisted of 3 manufacturers and 3 retailers. We retained the production cost, the transaction cost, and the demand functions as in Example 5 but now we added data for the third retailer. In particular, we assumed that the transaction costs associated with transacting with the new retailer were of the same form as given above for other manufacturer/retailer pairs. In regards to the probability and the demand functions

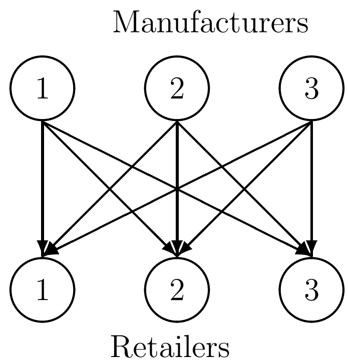

Figure 4: The Network Structure of Example 6

for the new retailer, we set $b_3 = 1000$. The weights associated with this retailer were $\lambda_3^+ = 10$ and $\lambda_3^- = 0$. Hence, these were the same as for the other two retailers in Example 5. The handling cost function associated with the new retailer was of the same form as for the other two retailers (and as in the preceding examples).

The supply chain was as depicted in Figure 4.

The modified projection method converged in 3062 iterations and required, as did the preceding examples, a negligible amount of CPU time, and yielded the equilibrium product shipment pattern: $Q^* := q_{11}^* = 1.3392, q_{12}^* = .8710, q_{13}^* = .8710, q_{21}^* = 1.3392, q_{22}^* = .8710, q_{23}^* = .8710$, $q_{31}^* = 4.0668, q_{32}^* = 5.0759, q_{33}^* = 5.0759$, and the equilibrium demand prices at the three retailers: $\rho_{21}^* = 74.0142$, $\rho_{22}^* = \rho_{23}^* = 73.2373$.

It is interesting to note that the total quantity that each retailer now ordered decreased from the amount ordered in Example 5. For example, here retailer 1 ordered $\sum_{i=1}^{3} q_{i1}^* = 7.3811$, whereas in Example 5 he had ordered (in equilibrium) the amount 6.8. This effect is reasonable since when there are more resources available (in the form of more manufacturers more producing a product) less has to be stored.

These examples illustrate the variety of scenarios that can be evaluated in regards to supply chain network problems with random demands. Indeed, one can vary the weights associated with the retailers, the number of manufacturers and/or retailers, as well as the parameters in the distribution (and demand) functions and, through the application of the computational procedure, evaluate the effects on the equilibrium product shipments and prices.

We emphasize the following features of the modified projection method of Korpelevich in the context or our supply chain model:

1. the method is easy to implement and results in closed form expressions that are efficiently solved using explicit formulae;

2. the method is robust and with the convergence tolerance used provided equilibrium solutions to good accuracy, and

3. the same step size, albeit small, was used for all the examples, and was selected without any precomputations; the step size was selected to be small to guarantee that it would satisfy the condition $0 < a \leq \frac{1}{L}$, as needed to guarantee convergence.

Of course, the method did require many iterations but the overall CPU time was negligible since the subproblems were so simple computationally. It would be interesting to investigate the application of extensions of the modified projection method with varying step sizes (including that of Khobotov's) to this (as well as other) supply chain network equilibrium models. Nevertheless, we agree with Solodov and Tseng (1996) in that the Korpelevich method "is a very practical method" for solving a variational inequality problem and, in particular, our supply chain model with random demands.

6. Summary and Conclusions

In this paper, we have proposed a theoretically rigorous framework for the modeling, qualitative analysis, and computation of solutions to supply chain network problems within an equilibrium context in the case of random demands associated with the retailers. The theoretical analysis is based on finite-dimensional variational inequality theory. Before the results obtained in the paper, the modeling and analysis of supply chain network problems using variational inequality theory assumed that all the underlying functions were known were certainty.

In particular, we assumed a supply chain consisting of competing manufacturers and competing retailers, each of whom seeks to maximize profits. The retailers are faced with random demands at their outlets and also penalize excess supply (inventory) and excess demand (shortage) at their particular outlets individually. We derived the governing equilibrium conditions and then showed that they satisfy a variational inequality problem. The variational inequality was then utilized to obtain, under reasonable conditions, existence of the equilibrium product and price pattern, as well as uniqueness. Moreover, we established additional properties of the function that enters the variational inequality that were then utilized to establish convergence of the proposed algorithmic scheme.

We then applied the computational procedure, which in the context of our model has the attractive feature that it yields subproblems in prices and product shipments that can be solved in closed form, to several numerical supply chain examples. The numerical examples illustrate the flexibility of the model.

This work establishes the foundations for decentralized and competitive supply chain network problems in the case of random demands within an equilibrium framework. Future research may include the modeling of random costs as well as the modeling of disequiibrium dynamics.

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