Supply Chain Networks and Electronic Commerce: A Theoretical Perspective

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April 2001; revised April 2002

Appears in *Netnomics* **4** (2002), pp. 187-220.

Abstract: In this paper, we develop a framework for the formulation, analysis, and computation of solutions to supply chain network problems in the presence of electronic commerce. Specifically, we consider manufacturers who are involved in the production of a homogeneous product and can now sell and have delivered the product not only to retailers but also directly to consumers. In addition, the manufacturers can transact with the retailers electronically. We assume that both the manufacturers and the retailers seek to maximize their profits, whereas the consumers take both the prices charged by the retailers and the manufacturers, along with the associated transaction costs, in making their consumption decisions. We identify the network structure of the problem, derive the equilibrium conditions, and establish the finite-dimensional variational inequality formulation. We then utilize variational inequality theory to obtain qualitative properties of the equilibrium pattern. In addition, we propose a continuous time adjustment process for the study of the disequilibrium dynamics and establish that the set of stationary points of the resulting *projected* dynamical system coincides with the set of solutions of the variational inequality problem. Finally, we apply an algorithm for the determination of equilibrium prices and product shipments in several supply chain examples. This paper synthesizes Business-to-Consumer (B2C) and Business-to-Business (B2B) decision-making in a supply chain context within the same framework.

1. Introduction

Electronic commerce (e-commerce), with the advent of the Internet Age, has had an immense effect on the manner in which businesses, as well as, consumers order goods and have them transported. The major portion of e-commerce transactions is in the form of business-to-business (B2B) with estimates ranging from approximately .1 trillion dollars to 1 trillion dollars in 1998 and with forecasts reaching as high as $4.8 trillion dollars in 2003 in the United States (see Federal Highway Administration (2000), Southworth (2000)). The business-to-consumer (B2C) component, on the other hand, has seen tremendous growth in recent years but its impact on the US retail activity is still relatively small. Nevertheless, this segment should grow to $80 billion per year (Southworth (2000)).

As noted by Handfield and Nichols (1999) and by the National Research Council (2000), the principal effect of business-to-business (B2B) commerce, estimated to be 90% of all e-commerce by value and volume, is in the creation of new and more profitable *supply chain networks*. Recall that a supply chain is a chain of relationships which synthesizes and integrates the movement of goods between suppliers, manufacturers, distributors, retailers, and consumers.

The topic of supply chain analysis is multidisciplinary by nature since it involves aspects of manufacturing, transportation and logistics, retailing/marketing, as well as economics. It has been the subject of a growing body of literature with researchers focusing both on the conceptualization of the underlying problems (see, e.g., Poirier (1996, 1999), Mentzer (2000), Bovet (2000)), due to the complexity of the problem and the numerous decision-makers, such as producers, retailers, or consumers involved in the transactions, as well as on the analytics (cf. Bramel and Simchi-Levi (1997), Stadtler and Kilger (2000), and Miller (2001) and the references therein).

The introduction of e-commerce has unveiled new opportunities in terms of research and practice in supply chain analysis and management (see, e.g., Kuglin and Rosenbaum (2001)). Indeed, the primary benefit of the Internet for business is its open access to potential suppliers and customers both within a particular country and past national boundaries. Consumers, on the other hand, may obtain goods, which they physically could not locate otherwise.

In this paper, we propose a theoretical framework for the study of supply chain networks with electronic commerce in the form of B2C and B2B transactions. The framework is sufficiently general to allow for the modeling, analysis, and computation of solutions to such problems. Our perspective is based on *Network Economics* (cf. Nagurney (1999)) in that we focus on the network interactions of the underlying decision-makers and on the underlying competitive processes. Moreover, we emphasize the equilibrium aspects of the problems rather than, simply, the optimization ones. Of course, we, nevertheless, assume that the decision-makers in the supply chain behave in some optimal fashion. The equilibrium perspective provides a valuable benchmark against which existing prices and product shipments can be compared against. For background on the economics of electronic commerce, see Whinston, Stahl, and Choi (1997).

We consider manufacturers who are involved in the production of a homogeneous product which can then be shipped to the retailers or the consumers directly or both. The manufacturers obtain a price for the product (which is endogenous) and seek to determine their optimal production and shipment quantities, given the production costs as well as the *transaction* costs associated with conducting business with the different retailers and demand markets. Note that we consider a transaction cost to be sufficiently general, for example, to include the transportation/shipping cost. On the other hand, in the case of an e-commerce link, the transaction costs can include the cost associated with the use of such a link, the congestion, etc.

The retailers, in turn, must agree with the manufacturers on the volume of shipments, either ordered physically or through the Internet, since they are faced with the handling cost associated with having the product in their retail outlet. In addition, they seek to maximize their profits with the price that the consumers being willing to pay for the product being endogenous.

Finally, in this supply chain, the consumers provide the "pull" in that, given the demand functions at the various demand markets, they determine their optimal consumption levels from the various retailers and manufacturers, subject both to the prices charged for the product as well as the cost of conducting the transaction (which, of course, may include the cost of transportation associated with obtaining the product from the manufacturer or retailer).

We establish that, in equilibrium, at which the manufacturers, the retailers, as well as the consumers, have reached optimality, given the competition, the structure of the supply chain network is that of a three-tiered network, with additional links connecting the top tier (the manufacturers) with the bottom tier (the demand markets) to represent e-commerce links and additional links from the top tier to the middle tier (the retailers) to also represent the e-commerce links. We then utilize the variational inequality formulation of the governing equilibrium conditions to obtain qualitative properties of the equilibrium pattern. In addition, we propose a continuous time adjustment process, formulate it as a projected dynamical system (see Nagurney and Zhang (1996)), and establish that the set of stationary points coincides with the set of solutions to the variational inequality problem. The dynamical system provides a means of studying the disequilibrium dynamics.

The paper is organized as follows. In Section 2, we present the supply chain network model with electronic commerce, derive the optimality conditions for each set of network agents or decision-makers, and then present the governing equilibrium conditions. We also derive the finite-dimensional variational inequality formulation of the problem. The model is an extension of the recently introduced supply chain network model of Nagurney, Dong, and Zhang (2002) to the case of e-commerce with B2C and B2B transactions.

In Section 3, we provide qualitative properties of the equilibrium pattern and establish the properties needed for proving convergence of the algorithm used for the numerical examples. In Section 4, we propose the projected dynamical system which describes the dynamic adjustment process associated with the various decision-makers. In Section 5, we outline the computational procedure, along with convergence results. The algorithm resolves the network problem into subproblems, each of which can be solved exactly and in closed form.

In Section 6, we apply the algorithm to numerical supply chain examples in order to determine the equilibrium product shipments and prices. We conclude the paper with a summary and suggestions for future research in Section 7.

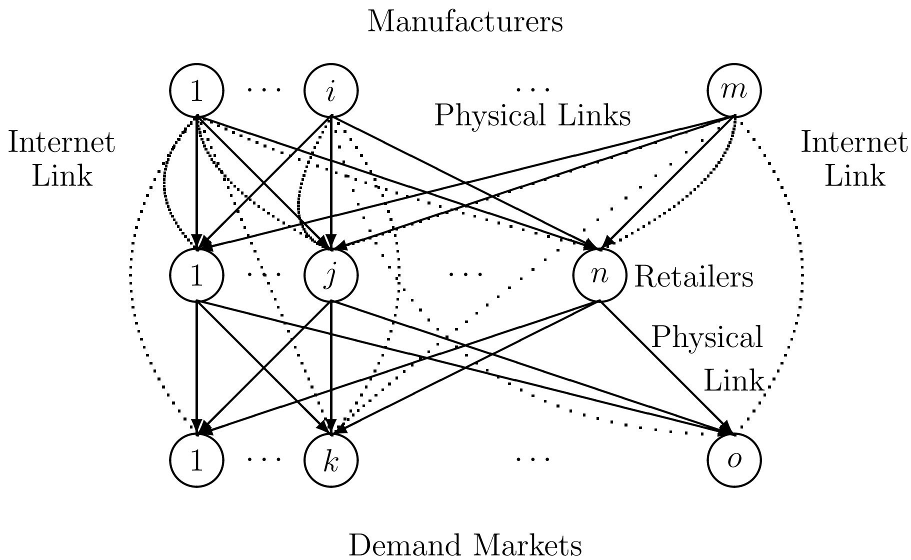

Figure 1: The Multitiered Network Structure of the Supply Chain with E-Commerce at Equilibrium

2. The Supply Chain Network Model with Electronic Commerce

In this Section, we develop the supply chain network model with manufacturers, retailers, and consumers in which the manufacturers can sell directly to the consumers at the demand markets through the Internet and can also conduct their business transactions with the retailers through the Internet. The depiction of the supply chain network at equilibrium, which we establish in this Section, is as depicted in Figure 1.

Specifically, we consider m manufacturers involved in the production of a homogeneous product which can then be purchased by n retailers and/or directly by the consumers located at the o demand markets. We denote a typical manufacturer by i, a typical retailer by j, and a typical demand market by k. Note that the manufacturers are located at the top tier of nodes of the network, the retailers at the middle tier, and the demand markets at the third or bottom tier of nodes.

The links in the supply chain network in Figure 1 include classical physical links as well as Internet links to allow for e-commerce.

Note that the introduction of e-commerce allows for “connections” that were, heretofore, not possible, such as allowing, for example, consumers to purchase a product directly from manufacturers. In order to conceptualize this B2C type of transaction, we construct a direct link from each top tier node to each bottom tier node. In addition, we consider the situation in which the manufacturers can now transact not only with the consumers directly but also with the retailers through the Internet. Hence, we also add an additional link between each top tier node and each middle tier node to reflect the possibility of Internet transactions between the manufacturers and the retailers. Thus, a manufacturer may now transact with a retailer through either a physical link or through an Internet link, or both.

We now describe the behavior of the various economic decision-makers represented by the three tiers of nodes in Figure 1. We first focus on the manufacturers. We then turn to the retailers and, subsequently, to the consumers.

The Behavior of the Manufacturers and their Optimality Conditions

Let q_i denote the nonnegative production output by manufacturer i. We group the production outputs of all manufacturers into the column vector $q \in R^m_+$. We assume that each manufacturer i is faced with a production cost function f_i, which can depend, in general, on the entire vector of production outputs, that is,

$$f_i = f_i(q), \quad \forall i. \tag{1}$$

In order to depict the allowable transactions of a typical manufacturer i with the consumers at the demand markets and with the retailers, we provide a graphical depiction in Figure 2. A manufacturer may transact with a retailer via a physical link, and/or via an Internet link. We denote the transaction cost associated with manufacturer i transacting with retailer j via link (also referred to as *mode*) l, where $l = 1$ denotes a physical link and $l = 2$ denotes an Internet link, by c_{ijl}. We denote the product shipment associated with manufacturer i, retailer j, and mode of transaction l by q_{ijl}, and we group these product shipments into the column vector $Q^1 \in R^{2mn}_+$. In addition, a manufacturer i may transact directly with consumers located at a demand market k with this transaction cost associated with the Internet transaction denoted by c_{ik} and the associated product shipment from man-

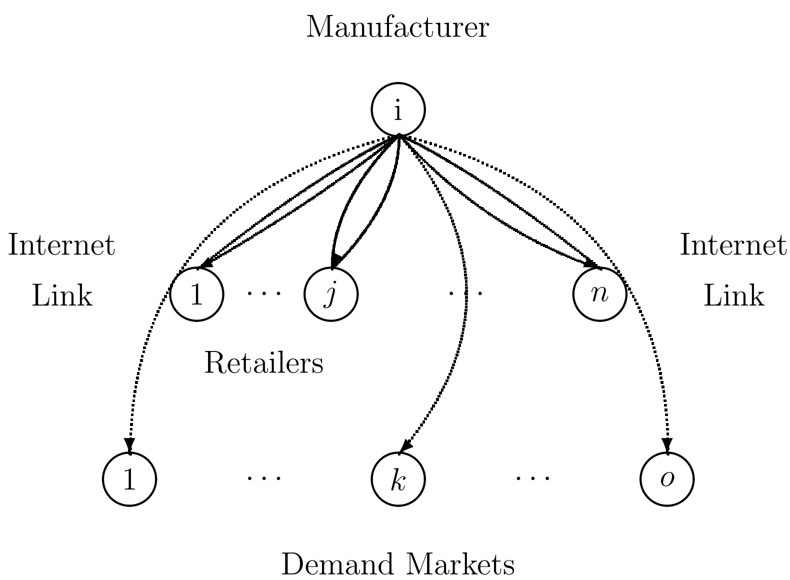

Figure 2: Network Structure of Manufacturer i's Transactions

ufacturer i to demand market k by q_{ik}. We group these product shipments into the column vector $Q^2 \in R_+^{mo}$.

We consider the situation in which the transaction cost between a manufacturer and retail pair as well as the transaction cost between a manufacturer and consumers at a demand market may depend upon the volume of transactions between each such pair, that is:

$$c_{ijl} = c_{ijl}(q_{ijl}), \quad \forall i, j, l, \tag{2a}$$

and

$$c_{ik} = c_{ik}(q_{ik}), \quad \forall i, k. \tag{2b}$$

The quantity of the product produced by manufacturer i must satisfy the following conservation of flow equation:

$$q_i = \sum_{j=1}^{n} \sum_{l=1}^{2} q_{ijl} + \sum_{k=1}^{o} q_{ik}, \tag{3}$$

which states that the quantity produced by manufacturer i is equal to the sum of the quantities shipped from the manufacturer to all retailers and to all demand markets.

The total costs incurred by a manufacturer i, thus, are equal to the sum of the manufacturer's production cost plus the total transaction costs. His revenue, in turn, is equal to the price that the manufacturer charges for the product (and the consumers are willing to pay) times the total quantity obtained/purchased of the product from the manufacturer by all the retail outlets and consumers at all demand markets. Let ρ^*_{1ijl} denote the price charged for the product by manufacturer i to retailer j who has transacted using mode l, and let ρ^*_{1ik} denote the price charged by manufacturer i for the product to consumers at demand market k. We, later, discuss how these prices are arrived at.

Noting the conservation of flow equations (3), we can express the criterion of profit maximization for manufacturer i as:

$$\text{Maximize} \quad \sum_{j=1}^{n}\sum_{l=1}^{2}\rho^*_{1ijl}q_{ijl} - f_i(Q^1,Q^2) - \sum_{j=1}^{n}\sum_{l=1}^{2}c_{ijl}(q_{ijl}) - \sum_{k=1}^{o}c_{ik}(q_{ik}) - \sum_{k=1}^{o}\rho^*_{1ik}q_{ik}, \qquad (4)$$

subject to $q_{ijl} \geq 0$, for all j, l, and $q_{ik} \geq 0$, for all k.

We assume that the manufacturers compete in a noncooperative fashion. Also, we assume that the production cost functions and the transaction cost functions for each manufacturer are continuous and convex. Given that the governing optimization/equilibrium concept underlying noncooperative behavior is that of Nash (1950, 1951), which states that each manufacturer will determine its optimal production quantity and shipments, given the optimal ones of the competitors, the optimality conditions for all manufacturers *simultaneously* can be expressed as the following inequality (cf. Bazaraa, Sherali, and Shetty (1996), Gabay and Moulin (1980); see also Dafermos and Nagurney (1987) and Nagurney (1999)):

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[\frac{\partial f_i(Q^{1*},Q^{2*})}{\partial q_{ijl}} + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \rho^*_{1ijl}\right] \times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\frac{\partial f_i(Q^{1*},Q^{2*})}{\partial q_{ik}} + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}} - \rho^*_{1ik}\right]\times[q_{ik} - q^*_{ik}] \geq 0, \quad \forall Q^1 \in R^{2mn}_+, \forall Q^2 \in R^{mo}_+. \quad (5)$$

The inequality (5), which is a *variational inequality* (cf. Nagurney (1999)) has a nice economic interpretation. In particular, from the first term we can infer that, if there is a positive shipment of the product transacted either in a classical manner or via the Internet from a manufacturer to a retailer, then the marginal cost of production plus the marginal

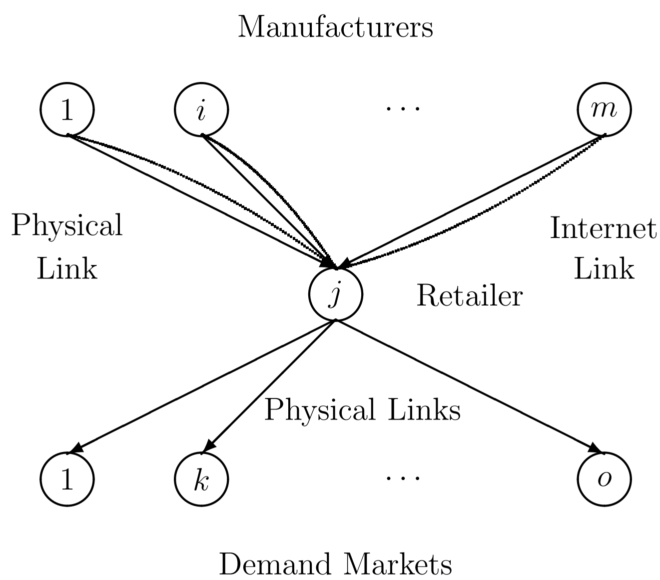

Figure 3: Network Structure of Retailer j's Transactions

cost of transacting must be equal to the price that the retailer is willing to pay for the product. If the marginal cost of production plus the marginal cost of transacting exceeds that price, then there will be zero volume of flow of the product on that link. The second term in (5) has a similar interpretation; in particular, there will be a positive volume of flow of the product from a manufacturer to a demand market if the marginal cost of production of the manufacturer plus the cost of transacting via the Internet for the manufacturer with the consumers is equal to the price the consumers are willing to pay for the product at the demand market.

The Behavior of the Retailers and their Optimality Conditions

The retailers, in turn, are involved in transactions both with the manufacturers since they wish to obtain the product for their retail outlets, as well as with the consumers, who are the ultimate purchasers of the product. Thus, a retailer conducts transactions both with the manufacturers as well as with the consumers. Refer to Figure 3 for a graphical depiction.

A retailer j is faced with what we term a *handling* cost, which may include, for example, the display and storage cost associated with the product. We denote this cost by c_j and, in the simplest case, we would have that c_j is a function of $\sum_{i=1}^{m} \sum_{l=1}^{2} q_{ijl}$, that is, the holding cost of a retailer is a function of how much of the product he has obtained from the various producers via the two different modes of transacting. However, for the sake of generality, and to enhance the modeling of competition, we allow the function to, in general, depend also on the amounts of the product held by other retailers and, therefore, we may write:

$$c_j = c_j(Q^1), \quad \forall j. \tag{6}$$

The retailers, in turn, also have associated transaction costs in regards to transacting with the manufacturers via either modal alternative. We denote the transaction cost associated with retailer j transacting with manufacturer i using mode l by $\hat{c}_{ijl}$ and we assume that the function can depend upon the manufacturer/retailer pair product shipment, that is,

$$\hat{c}_{ijl} = \hat{c}_{ijl}(q_{ijl}), \quad \forall i, j, l. \tag{7}$$

Let q_{jk} denote the amount of the product purchased/consumed by consumers located at demand market k from retailer j. We group these consumption quantities into the column vector $Q^3 \in R_{+}^{no}$.

The retailers associate a price with the product at their retail outlet, which is denoted by ρ_{2j}^*, for retailer j. This price, as we will show, will also be endogenously determined in the model. Assuming, as mentioned in the Introduction, that the retailers are also profit-maximizers, the optimization problem of a retailer j is given by:

$$\text{Maximize} \quad \rho_{2j}^* \sum_{k=1}^{o} q_{jk} - c_j(Q^1) - \sum_{i=1}^{m} \sum_{l=1}^{2} \hat{c}_{ijl}(q_{ijl}) - \sum_{i=1}^{m} \sum_{l=1}^{2} \rho_{1ijl}^* q_{ijl} \tag{8}$$

subject to:

$$\sum_{k=1}^{o} q_{jk} \leq \sum_{i=1}^{m} \sum_{l=1}^{2} q_{ijl}, \tag{9}$$

and the nonnegativity constraints: $q_{ijl} \geq 0$, and $q_{jk} \geq 0$, for all i, l and k. Objective function (8) expresses that the difference between the revenues minus the handling cost plus the transaction costs and the payout to the manufacturers should be maximized. Constraint (9) simply expresses that consumers cannot purchase more from a retailer than is held in stock.

We now consider the optimality conditions of the retailers assuming that each retailer is faced with the optimization problem (8), subject to (9), and the nonnegativity assumption on the variables. Here we also assume that the retailers compete in a noncooperative manner so that each maximizes his profits, given the actions of the other retailers. Note that, at this point, we consider that retailers seek to determine not only the optimal amounts purchased by the consumers from their specific retail outlet but, also, the amount that they wish to obtain from the manufacturers. In equilibrium, all the shipments between the tiers of network decision-makers will have to coincide.

Assuming that the handling cost for each retailer is continuous and convex as are the transaction costs, the optimality conditions for all the retailers satisfy the variational inequality:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[\frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \rho^*_{1ijl} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \gamma^*_j\right] \times \left[q_{ijl} - q^*_{ijl}\right] + \sum_{j=1}^{n}\sum_{k=1}^{o}\left[-\rho^*_{2j} + \gamma^*_j\right] \times \left[q_{jk} - q^*_{jk}\right]$$

$$+ \sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2} q^*_{ijl} - \sum_{k=1}^{o} q^*_{jk}\right] \times \left[\gamma_j - \gamma^*_j\right] \geq 0, \quad \forall Q^1 \in R^{2mn}_+, \forall Q^2 \in R^{no}_+, \forall \gamma \in R^n_+. \qquad (10)$$

Note that γ_j is the Lagrange multiplier associated with constraint (9) for retailer j and γ the column vector of all the retailers' multipliers. For further background on such a derivation, see Bertsekas and Tsitsiklis (1992). In this derivation, as in the derivation of inequality (5), we have not had the prices charged be variables. They become endogenous variables in the complete equilibrium model.

We now highlight the economic interpretation of the retailers' optimality conditions. From the second term in inequality (10), we have that, if consumers at demand market k purchase the product from a particular retailer j, that is, if the q^*_{jk} is positive, then the price charged by retailer j, ρ^*_{2j}, is precisely equal to γ^*_j, which, from the third term in the inequality, serves as the price to clear the market from retailer j. Also, note that, from the second term, we see that if no product is sold by a particular retailer, then the price associated with holding the product can exceed the price charged to the consumers. Furthermore, from the first term in inequality (10), we can infer that, if a manufacturer transacts with a retailer via a particular mode resulting in a positive flow of the product between the two, then the price

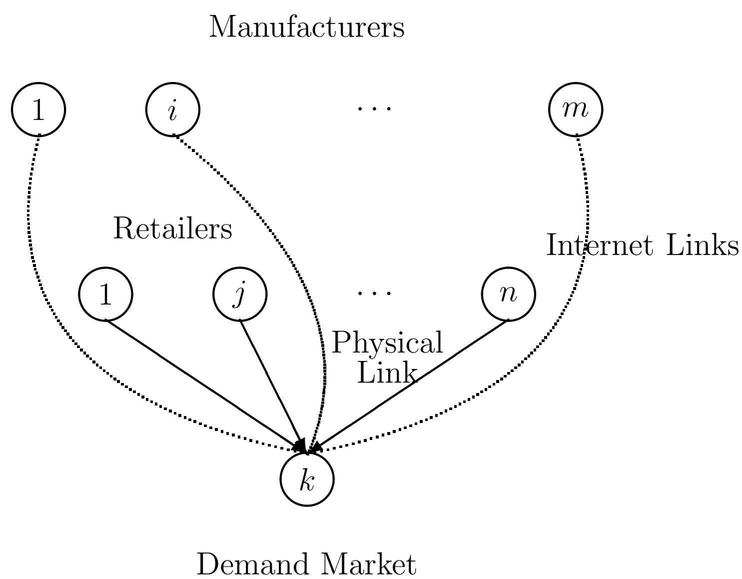

Figure 4: Network Structure of Consumers' Transactions at Demand Market k

γ_j^* is precisely equal to the retailer j's payment to the manufacturer, ρ_{1ijl}^*, plus his marginal cost of handling the product plus the retailer's marginal cost of transaction associated with transacting with the particular manufacturer.

The Consumers at the Demand Markets and the Equilibrium Conditions

We now describe the consumers located at the demand markets. The consumers take into account in making their consumption decisions not only the price charged for the product by the retailers and the manufacturers but also their transaction costs associated with obtaining the product. The consumers at the demand markets can transact either directly with the manufacturers through the Internet or physically with the retailers. A graphical depiction of consumers at a typical demand market k is given in Figure 4.

We let $\hat{c}_{jk}$ denote the transaction cost associated with obtaining the product by consumers at demand market k from retailer j and recall that q_{jk} is the amount of the product puchased (or flowing) between retailer j and consumers at demand market k. We assume that the transaction cost is continuous and of the general form:

$$\hat{c}_{jk} = \hat{c}_{jk}(Q^2, Q^3), \quad \forall j, k. \tag{11}$$

Also, we let $\hat{c}_{ik}$ denote the transaction cost, from the perspective of the consumers at demand market k, associated with manufacturer i. Here we assume that

$$\hat{c}_{ik} = \hat{c}_{ik}(Q^2, Q^3), \quad \forall i, k, \tag{12}$$

Hence, the cost of conducting a transaction with a manufacturer via the Internet can depend, in general, upon the volume of the product obtained via the Internet as well as the amount purchased from the retailers.

Let now ρ_{3k} denote the *generalized* price of the product as perceived by the consumers at demand market k. Further, denote the demand for the product at demand market k by d_k and assume, as given, the continuous demand functions:

$$d_k = d_k(\rho_3), \quad \forall k, \tag{13}$$

where ρ_3 is the o-dimensional column vector of generalized prices. Thus, according to (13), the demand of consumers for the product at a demand market depends, in general, not only on the price of the product at that demand market but also on the prices of the product at the other demand markets. Consequently, consumers at a demand market, in a sense, also compete with consumers at other demand markets.

The consumers take the price charged by the retailers for the product, which, recall was denoted by ρ^*_{2j} for retailer j, plus the transaction cost associated with obtaining the product, in making their consumption decisions. In addition, they take the price charged by a producer, ρ^*_{1ik}, plus that associated transaction cost into consideration.

The equilibrium conditions for consumers at demand market k, thus, take the form: for all retailers: $j; j = 1, \ldots, n$:

$$\rho^*_{2j} + \hat{c}_{jk}(Q^{2^*}, Q^{3^*}) \begin{cases} = \rho^*_{3k}, & \text{if} \quad q^*_{jk} > 0 \\ \geq \rho^*_{3k}, & \text{if} \quad q^*_{jk} = 0, \end{cases} \tag{14}$$

for all manufacturers i; $i = 1, \ldots, m$:

$$\rho^*_{1ik} + \hat{c}_{ik}(Q^{2^*}, Q^{3^*}) \begin{cases} = \rho^*_{3k}, & \text{if} \quad q^*_{ik} > 0 \\ \geq \rho^*_{3k}, & \text{if} \quad q^*_{ik} = 0, \end{cases} \tag{15}$$

and

$$d_k(\rho_3^*) \begin{cases} = \sum_{j=1}^{n} q_{jk}^* + \sum_{i=1}^{m} q_{ik}^*, & \text{if} \quad \rho_{3k}^* > 0 \\ \leq \sum_{j=1}^{n} q_{jk}^* + \sum_{i=1}^{m} q_{ik}^*, & \text{if} \quad \rho_{3k}^* = 0. \end{cases} \tag{16}$$

Conditions (14) state that consumers at demand market k will purchase the product from retailer j, if the price charged by the retailer for the product plus the transaction cost (from the perspective of the consumers) does not exceed the price that the consumers are willing to pay for the product. Conditions (15) state the analogue for the manufacturers and demand market. Condition (16), on the other hand, states that, if the price the consumers are will to pay for the product at a demand market is positive, then the quantity consumed by the consumers at the demand market is precisely equal to the demand. These conditions correspond to the well-known spatial price equilibrium conditions (cf. Samuelson (195), Takayama and Judge (1971), and Nagurney (1999) and the references therein).

In equilibrium, conditions (14), (15), and (16) will have to hold for all demand markets k, and these, in turn, can also be expressed as a variational inequality problem akin to (5) and (10) and given by: determine $(Q^{2^*}, Q^{3^*}, \rho_3^*) \in R^{mo+no+n}$, such that

$$\sum_{j=1}^{n} \sum_{k=1}^{o} \left[\rho_{2j}^* + \hat{c}_{jk}(Q^{2^*}, Q^{3^*}) - \rho_{3k}^*\right] \times \left[q_{jkl} - q_{jkl}^*\right]$$

$$+ \sum_{i=1}^{m} \sum_{k=1}^{n} \left[\rho_{1ik}^* + \hat{c}_{ik}(Q^{2^*}, Q^{3^*}) - \rho_{3k}^*\right] \times \left[q_{ik} - q_{ik}^*\right]$$

$$+ \sum_{k=1}^{o} \left[\sum_{j=1}^{n} q_{jk}^* + \sum_{i=1}^{m} q_{ik}^* - d_k(\rho_3^*)\right] \times \left[\rho_{3k} - \rho_{3k}^*\right] \geq 0, \quad \forall (Q^2, Q^3, \rho_3) \in R_+^{mo+no+n}. \tag{17}$$

Note that, in the context of the consumption decisions, we have utilized demand functions, rather than utility functions, as was the case for the manufacturers and the retailers, who were assumed to be faced with profit functions, which correspond to utility functions. Of course, demand functions can be derived from utility functions (cf. Arrow and Intrilligator (1982)). We expect the number of consumers to be much greater than that of the manufacturers and retailers and, hence, believe that the above formulation is the more natural and tractable one.

The Equilibrium Conditions of the Supply Chain

In equilibrium, the shipments of the product that the manufacturers ship to the retailers must be equal to the shipments that the retailers accept from the manufacturers. In addition the amounts of the product purchased by the consumers must be equal to the amounts sold by the retailers and directly to the consumers by the manufacturers. Furthermore, the equilibrium shipment and price pattern must satisfy the sum of the optimality conditions (5) and (10), and the conditions (17), in order to formalize the agreements between the tiers of the supply chain network.

We now state this formally.

Definition 1: Supply Chain Network Equilibrium with E-Commerce

The equilibrium state of the supply chain with electronic commerce is one where the flows between the tiers of the supply chain network coincide and the product shipments and prices satisfy the sum of the optimality conditions (5) and (10) and the equilibrium conditions (17).

We now establish the following:

Theorem 1: Variational Inequality Formulation

The equilibrium conditions governing the supply chain network model with electronic commerce are equivalent to the solution of the variational inequality problem given by:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[\frac{\partial f_i(Q^{1*},Q^{2*})}{\partial q_{ijl}}+\frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}}+\frac{\partial c_j(Q^{1*})}{\partial q_{ijl}}+\frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}}-\gamma^*_j\right]\times\left[q_{ijl}-q^*_{ijl}\right]$$
$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\frac{\partial f_i(Q^{1*},Q^{2*})}{\partial q_{ik}}+\frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}}+\hat{c}_{ik}(Q^{2*},Q^{3*})-\rho^*_{3k}\right]\times\left[q_{ik}-q^*_{ik}\right]$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\hat{c}_{jk}(Q^{2*},Q^{3*})+\gamma^*_j-\rho^*_{3k}\right]\times\left[q_{jk}-q^*_{jk}\right]+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2}q^*_{ijl}-\sum_{k=1}^{o}q^*_{jk}\right]\times\left[\gamma_j-\gamma^*_j\right]$$
$$+\sum_{k=1}^{o}\left[\sum_{j=1}^{n}q^*_{jk}+\sum_{i=1}^{m}q^*_{ik}-d_k(\rho^*_3)\right]\times\left[\rho_{3k}-\rho^*_{3k}\right]\geq 0,\quad\forall(Q^1,Q^2,Q^3,\gamma,\rho_3)\in R^{2mn+mo+no+n+o}_{+}. \tag{18}$$

Proof:

We first establish that the equilibrium conditions imply variational inequality (18). Indeed, the summation of inequalities (5), (10), and (17) yields, after algebraic simplification, the variational inequality (18).

We now establish the converse, that is, that a solution to variational inequality (18) satisfies conditions (5), (10), and (17) and is, hence, an equilibrium according to Definition 1.

To inequality (18), add the term: $\rho^*_{1ijl} + \rho^*_{1ijl}$ to the term in the first set of brackets, preceding the multiplication sign. Similarly, add the term: $-\rho^*_{1ik} + \rho^*_{1ik}$ to the term preceding the second multiplication sign, and, finally, add the term: $-\rho^*_{2j} + \rho^*_{2j}$ to the term preceding the third multiplication sign. Such "terms" do not change the value of the inequality since they are identically equal to zero, with the resulting inequality of the form:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[\frac{\partial f_i(Q^{1*},Q^{2*})}{\partial q_{ijl}} + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \gamma^*_j - \rho^*_{1ijl} + \rho^*_{1ijl}\right]\times\left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\frac{\partial f_i(Q^{1*},Q^{2*})}{\partial q_{ik}} + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(Q^{2*},Q^{3*}) - \rho^*_{3k} - \rho^*_{1ik} + \rho^*_{1ik}\right]\times\left[q_{ik} - q^*_{ik}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\hat{c}_{jk}(Q^{2*},Q^{3*}) + \gamma^*_j - \rho^*_{3k} - \rho^*_{2j} + \rho^*_{2j}\right]\times\left[q_{jk} - q^*_{jk}\right] + \sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2} q^*_{ijl} - \sum_{k=1}^{o} q^*_{jk}\right]$$

$$\times\left[\gamma_j - \gamma^*_j\right] \geq 0, \qquad \forall (Q^1,Q^2,Q^3,\gamma) \in R_+^{2mn+mo+no+n}, \tag{19}$$

which, in turn, can be rewritten as:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[\frac{\partial f_i(Q^{1*},Q^{2*})}{\partial q_{ijl}} + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \rho^*_{1ijl}\right]\times\left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\frac{\partial f_i(Q^{1*},Q^{2*})}{\partial q_{ik}} + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}} - \rho^*_{1ik}\right]\times\left[q_{ik} - q^*_{ik}\right]$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[\frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \rho^*_{1ijl} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \gamma^*_j\right]\times\left[q_{ijl} - q^*_{ijl}\right] + \sum_{j=1}^{n}\sum_{k=1}^{o}\left[-\rho^*_{2j} + \gamma^*_j\right]\times\left[q_{jk} - q^*_{jk}\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2}q_{ijl}^{*}-\sum_{k=1}^{o}q_{jk}^{*}\right]\times\left[\gamma_j-\gamma_j^{*}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\rho_{2j}^{*}+\hat{c}_{jk}(Q^{2*},Q^{3*})-\rho_{3k}^{*}\right]\times\left[q_{jk}-q_{jk}^{*}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\rho_{1ik}^{*}+\hat{c}_{ik}(Q^{2*},Q^{3*})-\rho_{3k}^{*}\right]\times[q_{ik}-q_{ik}^{*}]\geq 0,\quad\forall(Q^1,Q^2,Q^3,\gamma)\in R_{+}^{2mn+mo+no+n}. \tag{20}$$

But inequality (20) is equivalent to the price and product shipment pattern satisfying the sum of the conditions (5), (10), and (17). The proof is complete. □

For easy reference in the subsequent sections, variational inequality problem (18) can be rewritten in standard variational inequality form (cf. Nagurney (1999)) as follows:

$$\langle F(X^*)^T, X-X^*\rangle\geq 0,\quad\forall X\in\mathcal{K}\equiv R_{+}^{2mn+mo+no+n+o}, \tag{21}$$

where $X\equiv(Q^1,Q^2,Q^3,\gamma,\rho_3)$, and $F(X)\equiv(F_{ijl},F_{ik},F_{jk},F_j,F_k)_{i=1,\dots,m;j=1,\dots,n;l=1,2;k=1,\dots,o}$, and the specific components of F given by the functional terms preceding the multiplication signs in (18), respectively. The term $\langle\cdot,\cdot\rangle$ denotes the inner product in N-dimensional Euclidean space.

We now discuss how to recover the prices ρ_{1ijl}^*, for all i,j,l, and ρ_{2j}^*, for all j, from the solution of variational inequality (18). (In Section 5 we describe an algorithm for computing the solution.) Recall that, in the preceding discussions, we have noted that if $q_{jk}^*>0$, for some k and j, then ρ_{2j}^* is precisely equal to γ_j^*, which can be obtained from the solution of (18). The prices ρ_{1ijl}^*, in turn (cf. also (20)), can be obtained by finding a $q_{ijl}^*>0$, and then setting $\rho_{1ijl}^*=\left[\frac{\partial f(Q^{1*},Q^{2*})}{\partial q_{ijl}}+\frac{\partial c_{ijl}(q_{ijl}^*)}{\partial q_{ijl}}\right]$, or, equivalently, to $\left[\gamma_j^*-\frac{\partial c_j(Q^{1*})}{\partial q_{ijl}}-\frac{\partial\hat{c}_{ijl}(q_{ijl}^*)}{\partial q_{ijl}}\right]$, for all such i,j,l. The prices ρ_{1ik}^*, on the other hand, can be obtained by finding a $q_{ik}^*>0$ and setting $\rho_{1ik}^*=\left[\frac{\partial f_i(Q^{1*},Q^{2*})}{\partial q_{ik}}+\frac{\partial c_{ik}(q_{ik}^*)}{\partial q_{ik}}\right]$, or, equivalently, to $\left[\rho_{3k}^*-\hat{c}_{ik}(Q^{2*},Q^{3*})\right]$, for all such i,k.

We now construct the supply chain network in equilibrium (cf. Figure 1), using, as building blocks, the previously drawn networks in Figures 2 thorugh 4 corresponding, respectively, to the transactions of the manufacturers, the retailers, and the consumers. First, however, we need to establish the result that, in equilibrium, the sum of the product shipments to each retailer is equal to the sum of the product shipments out. Hence, the corresponding γ_j^*s will all be positive. This means that each retailer, assuming profit-maximization, only purchases from the producers the amount of the product that is actually consumed by the consumers. In order to establish this result, we utilize variational inequality (18). Clearly, we know that, if $\gamma_j^* > 0$, then the "market clears" for that retailer, that is, $\sum_{i=1}^m \sum_{l=1}^2 q_{ijl}^* = \sum_{k=1}^o q_{jk}^*$. Let us now consider the case where $\gamma_j^* = 0$ for some retailer j. From the first term in inequality (18), since the production cost functions, and the transaction cost functions and handling cost functions have been assumed to be convex, and assuming further, which is not unreasonable, that either the marginal cost of production or the marginal transaction costs or the marginal holding cost for each manufacturer/mode/retailer combination is strictly positive at equilibrium, then we know that $\frac{\partial f_i(Q^{1*},Q^{2*})}{\partial q_{ijl}} + \frac{\partial c_{ijl}(q_{ijl}^*)}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q_{ijl}^*)}{\partial q_{ijl}} > 0$, which implies that $q_{ijl}^* = 0$, and this holds for all i, j, l. It follows then from the third term in (18), that $\sum_{k=1}^o q_{jk}^* = 0$, and, hence, the market clears also in this case since the flow into a retailer is equal to the flow out and equal to zero. We have thus, established the following:

Corollary 1

The market for the product clears for each retailer in the supply chain network with e-commerce at equilibrium.

In Figure 1, we depict the structure of the supply chain network in equilibrium, consisting of all the manufacturers, all the retailers, and all the demand markets. Hence, we replicate Figure 2 for all manufacturers, Figure 3, for all retailers, and Figure 4 for all demand markets. These resulting networks represent the possible transactions of all the economic decision-makers. In addition, since there must be agreement between/among the transactors at equilibrium, the analogous links (and equilibrium flows on them) must coincide, yielding the network structure given in Figure 1.

Clearly, the special cases of our model in which there is only B2B commerce or only B2C commerce can be studied in our framework as well with a suitable reduction of the links and associated transaction costs and product shipments.

In this Section, we have proposed an equilibrium framework for the formulation of supply chain network problems with electronic commerce since we believe that the concept of equilibrium provides a valuable benchmark against which existing product shipments between tiers and prices at different tiers of the supply chain can be compared. In Section 4, we propose a dynamic adjustment process, which is then formulated as a projected dynamical system, whose set of stationary points coincides with the set of solutions to the variational inequality problem (18). The dynamical system provides a means of addressing the disequilibrium dynamics associated with a supply chain with multiple tiers.

3. Qualitative Properties

In this Section, we provide some qualitative properties of the solution to variational inequality (18). In particular, we derive existence and uniqueness results. We also investigate properties of the function F (cf. (21)) that enters the variational inequality of interest here.

Since the feasible set is not compact we cannot derive existence simply from the assumption of continuity of the functions. Nevertheless, we can impose a rather weak condition to guarantee existence of a solution pattern. Let

$$\mathcal{K}_b = \{(Q^1, Q^2, Q^3, \gamma, \rho_3) | 0 \leq Q^1 \leq b_1;\ 0 \leq Q^2 \leq b_2; 0 \leq Q^3 \leq b_3;\ 0 \leq \gamma \leq b_4;\ 0 \leq \rho_3 \leq b_5\}, \tag{22}$$

where $b = (b_1, b_2, b_3, b_4, b_5) \geq 0$ and $Q^1 \leq b_1; Q^2 \leq b_2; Q^3 \leq b_3; \gamma \leq b_4; \rho_3 \leq b_5$ means that $q_{ijl} \leq b_1$; $q_{ik} \leq b_2$; $q_{jk} \leq b_3$; $\gamma_j \leq b_4$; and $\rho_{3k} \leq b_5$ for all i, j, l, k. Then $\mathcal{K}_b$ is a bounded closed convex subset of $R^{2mn+mo+no+n+o}$. Thus, the following variational inequality

$$\langle F(X^b)^T, X - X^b \rangle \geq 0, \quad \forall X^b \in \mathcal{K}_b, \tag{23}$$

admits at least one solution $X^b \in \mathcal{K}_b$, from the standard theory of variational inequalities, since $\mathcal{K}_b$ is compact and F is continuous. Following Kinderlehrer and Stampacchia (1980) (see also Theorem 1.5 in Nagurney (1999)), we then have:

Theorem 2

Variational inequality (18) admits a solution if and only if there exists a $b > 0$, such that variational inequality (23) admits a solution in $\mathcal{K}_b$ with

$$Q^{1b} < b_1, \quad Q^{2b} < b_2, \quad Q^{3b} < b_3, \quad \gamma^b < b_4, \quad \rho_3^b < b_5. \tag{24}$$

Theorem 3: Existence

Suppose that there exist positive constants M, N, R with $R > 0$, such that:

$$\frac{\partial f_i(Q^1, Q^2)}{\partial q_{ijl}} + \frac{\partial c_{ijl}(q_{ijl})}{\partial q_{ijl}} + \frac{\partial c_j(Q^1)}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q_{ijl})}{\partial q_{ijl}} \geq M, \quad \forall Q^1 \text{ with } q_{ijl} \geq N, \quad \forall i, j, l, \tag{25}$$

$$\frac{\partial f_i(Q^1,Q^2)}{\partial q_{ik}} + \frac{\partial c_{ik}(q_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(Q^2,Q^3) \geq M, \quad \forall Q^2 \text{ with } q_{ik} \geq N, \quad \forall i,k,$$
$$\hat{c}_{jk}(Q^2,Q^3) \geq M, \quad \forall Q^3 \text{ with } q_{jk} \geq N, \quad \forall j,k,$$
$$d_k(\rho_3^*) \leq N, \quad \forall \rho \text{ with } \rho_{3k} > R, \quad \forall k. \tag{26}$$

Then variational inequality (18); equivalently, variational inequality (21), admits at least one solution.

Proof: Follows using analogous arguments as the proof of existence for Proposition 1 in Nagurney and Zhao (1993) (see also existence proof in Nagurney, Dong, and Zhang (2000)). □

Assumptions (25) and (26) are reasonable from an economics perspective, since when the product shipment between a manufacturer and demand market pair or a manufacturer and retailer is large, we can expect the corresponding sum of the associated marginal costs of production, handling, and transaction from either the manufacturer's or the retailer's perspectives as well as the transaction cost associated with the consumers, to exceed a positive lower bound. Moreover, in the case where the generalized price of the product as perceived by consumers at a demand market is high, we can expect that the demand for the product at the demand market to not exceed a positive bound.

We now recall the definition of an additive production cost introduced in Zhang and Nagurney (1996), which we will utilize as an assumption for establishing additional qualitative properties.

Definition 2: Additive Production Cost

Suppose that for each manufacturer i, the production cost f_i is additive, that is,

$$f_i(q) = f_i^1(q_i) + f_i^2(\bar{q}_i), \tag{27}$$

where $f_i^1(q_i)$ is the internal production cost that depends solely on the manufacturer's own output level q_i, which may include the production operation and the facility maintenance, etc., and $f_i^2(\bar{q}_i)$ is the interdependent part of the production cost that is a function of all the *other manufacturers' output levels* $\bar{q}_i = (q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_m)$ *and reflects the impact of the other manufacturers' production patterns on manufacturer i's cost. This interdependent part of the production cost may describe the competition for the resources, consumption of the homogeneous raw materials, etc.*

We now establish additional qualitative properties both of the function F that enters the variational inequality problem (cf. (21) and (18)), as well as uniqueness of the equilibrium pattern. Monotonicity and Lipschitz continuity of F will be utilized in Section 5 for proving convergence of the algorithmic scheme. Since the proofs of Theorems 4 and 5 are similar to the analogous proofs in Nagurney, Dong, and Zhang (2002) for the supply chain network model without e-commerce, they are omitted here.

Theorem 4: Monotonicity

Suppose that the production cost functions $f_i; i = 1, ..., m$, *are additive, as defined in Definition 2, and that the* $f_i^1; i = 1, ..., m$, *are convex functions. If the* c_{ijl}, c_j, *and* $\hat{c}_{ijl}$, *and* c_{ik} *functions are convex; the* $\hat{c}_{jk}$ *and the* $\hat{c}_{ik}$ *functions are monotone increasing, and the* d_k *functions are monotone decreasing functions of the generalized prices, for all* i, l, j, k, *then the vector function* F *that enters the variational inequality (21) is monotone, that is,*

$$\langle (F(X') - F(X''))^T, X' - X'' \rangle \geq 0, \quad \forall X', X'' \in R_+^{2mn+mo+no+n+o}. \tag{28}$$

Theorem 5: Strict Monotonicity

Assume all the conditions of Theorem 4. In addition, suppose that one of the families of convex functions $f_i^1; i = 1, ..., m$, $c_{ijl}; i = 1, ..., m$; $j = 1, ..., n$; $l = 1, 2$; c_j; $j = 1, ..., n$; $\hat{c}_{ijl}$; $i = 1, \ldots, m$; $j = 1, \ldots, n$; $l = 1, 2$; *and* c_{ik}; $i = 1, \ldots, m$; $k = 1, \ldots, o$, *is a family of strictly convex functions. Suppose also that* $\hat{c}_{ik}$; $i = 1, \ldots, m$; $k = 1, \ldots, o$; $\hat{c}_{jk}; j = 1, ..., n; k = 1, ..., o$, *and* -$d_k$; $k = 1, ..., o$, *are strictly monotone. Then, the vector function* F *that enters the variational inequality (21) is strictly monotone, with respect to* (Q^1, Q^2, Q^3, ρ_3), *that is, for any two* X', X'' *with* $(Q^{1'}, Q^{2'}, Q^{3'}, \rho_3') \neq (Q^{1''}, Q^{2''}, Q^{3''}, \rho_3'')$

$$\langle (F(X') - F(X''))^T, X' - X'' \rangle > 0. \tag{29}$$

Theorem 6: Uniqueness

Assuming the conditions of Theorem 5, there must be a unique shipment pattern (Q^{1*}, Q^{2*}, Q^{3*})*, and a unique generalized price vector* ρ_3^* *satisfying the equilibrium conditions of the supply chain. In other words, if the variational inequality (21) admits a solution, then that is the only solution in* (Q^1, Q^2, Q^3, ρ_3)*.*

Proof: Under the strict monotonicity result of Theorem 5, uniqueness follows from the standard variational inequality theory (cf. Kinderlehrer and Stanpacchia (1980)) □

Theorem 7: Lipschitz Continuity

The function that enters the variational inequality problem (21) is Lipschitz continuous, that is,

$$\|F(X') - F(X'')\| \leq L\|X' - X''\|, \quad \forall X', X'' \in \mathcal{K}, \text{ where } L > 0, \tag{30}$$

under the following conditions:

(i). Each f_i*;* $i = 1, ..., m$*, is additive and has a bounded second-order derivative;*

(ii). c_{ijl}*,* c_j*,* $\hat{c}_{ijl}$*, and* c_{ik} *have bounded second-order derivatives, for all* i, j, l, k*;*

(iii). $\hat{c}_{ik}$*,* $\hat{c}_{jk}$*, and* d_k *have bounded first-order derivatives.*

Proof: The result is direct by applying a mid-value theorem from calculus to the vector function F that enters the variational inequality problem (21). □

In the next Section, we utilize the Lipschitz continuity property in order to guarantee that the dynamic trajectories associated with the proposed continuous time adjustment process are well-defined. Lipschitz continuity, along with the monotonicity property of the function F, are utilized in Section 5 in order to establish convergence of the proposed therein algorithmic scheme.

4. The Dynamics

In this Section, we propose a dynamic adjustment process, formulated as a projected dynamical system. We then establish that the set of stationary points of the projected dynamical system coincides with the set of solutions of variational inequality (21), equivalently, variational inequality (18).

In particular, we now turn to describing the dynamics by which the manufacturers adjust their product shipments over time, the consumers adjust their consumption amounts based on the prices of the product at the demand markets, and the retailers operate between the two, except in the case of electronic commerece when the consumers at the demand markets can deal with the manufacturers directly. We also describe the dynamics by which the prices adjust over time. The dynamics are derived from the bottom tier of nodes on up since it is the demand for the product (and the corresponding prices) that actually drives the supply chain dynamics.

The Demand Market Price Dynamics

We begin by describing the dynamics underlying the prices of the product associated with the demand markets. Assume that the rate of change of the price ρ_{3k}, denoted by $\dot{\rho}_{3k}$, is equal to the difference between the demand at the demand market k, as a function of the demand market prices, and the amount available from the retailers and the manufacturers at the demand market. Hence, if the demand for the product at the demand market (at an instant in time) exceeds the amount available, the price at that demand market will increase; if the amount available exceeds the demand at the price, then the price at the demand market will decrease. Furthermore, it is guaranteed that the prices do not become negative. Consequently, the dynamics of the price ρ_{3k} associated with the product at demand market k can be expressed as:

$$\dot{\rho}_{3k} = \begin{cases} d_k(\rho_3) - \sum_{j=1}^{n} q_{jk} - \sum_{i=1}^{m} q_{ik}, & \text{if} \quad \rho_{3k} > 0 \\ \max\{0, d_k(\rho_3) - \sum_{j=1}^{n} q_{jk} - \sum_{i=1}^{m} q_{ik}\}, & \text{if} \quad \rho_{3k} = 0. \end{cases} \tag{31}$$

The Dynamics of the Product Shipments between the Retailers and the Demand Markets

The dynamics of the product shipments over the links joining the retailers to the demand markets are now described. Recall that there is a unit transaction cost $\hat{c}_{jk}$ associated with transacting between retailer j and the consumers at demand market k, where $\hat{c}_{jk}$ is given by (11) and can depend upon, in general, all the product shipments to all the demand markets. The rate of change of the product shipment q_{jk} is assumed to be equal to the difference between the price the consumers are willing to pay for the product at demand market k minus the unit transaction cost and the price charged for the product at the retail outlet. Note that here, without loss of generality, we refer to γ_j as a "price" associated with retailer j. This is not unreasonable since if there is any consumption at any demand market from retailer j, then we know (cf. (10)) that $\gamma_j^* = \rho_{2j}^*$. Moreover, in the variational inequality (18) it is the γs that appear as the variables.

Of course, one also must guarantee that these product shipments do not become negative. Hence, one may write:

$$\dot{q}_{jk} = \begin{cases} \rho_{3k} - \hat{c}_{jk}(Q^2, Q^3) - \gamma_j, & \text{if} \quad q_{jk} > 0 \\ \max\{0, \rho_{3k} - \hat{c}_{jk}(Q^2, Q^3) - \gamma_j\}, & \text{if} \quad q_{jk} = 0, \end{cases} \tag{32}$$

where $\dot{q}_{jk}$ denotes the rate of change of the product shipment q_{jk}.

Thus, according to (32), if the price the consumers are willing to pay for the product at a demand market exceeds the price the retailers charge for the product at the outlet plus the unit transaction cost (at an instant in time), then the volume of the product between that retail and demand market pair will increase; if the price charged by the retailer plus the transaction cost exceeds the price the consumers are willing to pay, then the volume of flow of the product between that pair will decrease.

The Dynamics of the Product Shipments between the Manufacturers and the Demand Markets

In Section 2, it was assumed that each manufacturer i is faced with a production cost f_i, which can depend, in general, upon all the product shipments from all the manufacturers to the retailers and demand markets. In addition, recall that c_{ik} is the transaction cost associated with manufacturer i transacting with demand market k, with the function being given by (2b). The consumers at the demand markets, in turn, are also faced with a transaction cost associated with transacting with a manufacturer directly. For manufacturer/demand market pair (i,k), this function is denoted by $\hat{c}_{ik}$ and, as in (12), can depend, in general, upon all the product shipments to all the demand markets from all the manufacturers or retailers.

Since each manufacturer is assumed to be a profit-maximizer according to (4), a fair price to charge the consumers at a demand market who have transacted directly via a manufacturer through an Internet link is to charge the manufacturer's marginal production cost plus its marginal transaction cost, which for a pair (i,k) would be equal to: $\frac{\partial f_i(Q^1,Q^2)}{\partial q_{ik}} + \frac{\partial c_{ik}(q_{ik})}{\partial q_{ik}}$. The consumers at demand market k also incur a unit transaction cost associated with transacting with manufacturer i. Thus, the following rate of change for the product shipments between the top tier of nodes and the bottom tier of nodes in the logistical network is proposed:

$$\dot{q}_{ik} = \begin{cases} \rho_{3k} - \frac{\partial f_i(Q^1,Q^2)}{\partial q_{ik}} - \frac{\partial c_{ik}(q_{ik})}{\partial q_{ik}} - \hat{c}_{ik}(Q^2,Q^3), & \text{if} \quad q_{ik} > 0 \\ \max\{0, \rho_{3k} - \frac{\partial f_i(Q^1,Q^2)}{\partial q_{ik}} - \frac{\partial c_{ik}(q_{ik})}{\partial q_{ik}} - \hat{c}_{ik}(Q^2,Q^3)\}, & \text{if} \quad q_{ik} = 0, \end{cases} \tag{33}$$

where $\dot{q}_{ik}$ denotes the rate of change of the product shipment q_{ik}.

Hence, according to (33), if the demand price at a demand market exceeds the marginal production cost plus the marginal transaction cost of the manufacturer associated with transacting via the Internet directly with the consumers and the consumers' transaction cost, then the volume of the product transacted via the Internet between the manufacturer/demand market pair will increase; if the demand price at the demand market is less than the above described marginal and unit costs, then the volume of product shipment between the pair will decrease.

The Dynamics of the Prices at the Retail Outlets

The prices for the product at the retail outlets, in turn, must reflect supply and demand conditions as well. In particular, assume that the price for the product associated with retail outlet j, γ_j, evolves over time according to:

$$\dot{\gamma}_j = \begin{cases} \sum_{k=1}^{o} q_{jk} - \sum_{i=1}^{m} \sum_{l=1}^{2} q_{ijl}, & \text{if} \quad \gamma_j > 0 \\ \max\{0, \sum_{k=1}^{o} q_{jk} - \sum_{i=1}^{m} \sum_{l=1}^{2} q_{ijl}\}, & \text{if} \quad \gamma_j = 0, \end{cases} \tag{34}$$

where $\dot{\gamma}_j$ denotes the rate of change of the price γ_j. Hence, if the amount of the product desired to be transacted by the consumers (at an instant in time) exceeds that available at the retail outlet, then the price at the retail outlet will increase; if the amount available is greater than that desired by the consumers, then the price at the retail outlet will decrease.

The Dynamics of Product Shipments between Manufacturers and Retailers

The dynamics underlying the product shipments between the manufacturers and the retailers are now described. As already noted, each manufacturer is faced with a production cost and transaction costs. Recall that the transaction cost associated with manufacturer i and retailer j transacting via mode l is denoted by c_{ijl} and is of the form (2a).

As noted in Section 2, the total costs incurred by a manufacturer i, thus, are equal to the sum of the manufacturer's production cost plus the total transaction costs. His revenue, in turn, with regard to the transactions associated with the retailers, is equal to the price that the manufacturer charges for the product to the retailers (and the retailers are willing to pay) times the quantity of the product obtained/purchased from the manufacturer by the retail outlets and by the consumers directly. Hence, a fair price for the product associated with a given manufacturer/retailer pair and transacted via a mode is equal to the manufacturer's corresponding marginal costs of production and transacting, that is to: $\frac{\partial f_i(Q^1,Q^2)}{\partial q_{ijl}} + \frac{\partial c_{ijl}(q_{ijl})}{\partial q_{ijl}}$.

Recall that a retailer j, in turn, is faced with a handling cost given by (6). A retailer j, on the other hand, ideally, would accept a product shipment from manufacturer i at a price that is equal to the price charged at the retail outlet for the product (and that the consumers are willing to pay) minus its marginal cost associated with handling the product. Now, since the product shipments sent from the manufacturers must be accepted by the retailers in order for the transactions to take place in the supply chain, we propose the following rate of change for the product shipments between the top tier of nodes and the middle tier:

$$\dot{q}_{ijl} = \begin{cases} \gamma_j - \frac{\partial f_i(Q^1,Q^2)}{\partial q_{ijl}} - \frac{\partial c_{ijl}(q_{ijl})}{\partial q_{ijl}} - \frac{\partial c_j(Q^1)}{\partial q_{ijl}} - \frac{\partial \hat{c}_{ijl}(q_{ijl})}{\partial q_{ijl}}, \text{ if } q_{ijl} > 0 \\ \max\{0, \\ \gamma_j - \frac{\partial f_i(Q^1,Q^2)}{\partial q_{ijl}} - \frac{\partial c_{ijl}(q_{ijl})}{\partial q_{ijl}} - \frac{\partial c_j(Q^1)}{\partial q_{ijl}} - \frac{\partial \hat{c}_{ijl}(q_{ijl})}{\partial q_{ijl}}\}, \text{ if } q_{ijl} = 0, \end{cases} \tag{35}$$

where $\dot{q}_{ijl}$ denote the rate of change of the product shipment between manufacturer i and retailer j transacted via mode l.

Following the above discussion, (35) states that the product shipment between a manufacturer/retailer pair via a transaction mode evolves according to the difference between the price charged for the product by the retailer and its marginal costs, and the price charged by the manufacturer (which, recall, assuming profit-maximizing behavior, was set to the marginal cost of production plus its marginal cost of transacting with the retailer via the mode). Here it is also guaranteed that the product shipments do not become negative as they evolve over time.

The Projected Dynamical System

Consider now the dynamic model in which the demand prices evolve according to (31) for all demand market prices k, the retail/demand market product shipments evolve according to (32) for all retailers/demand markets j, k, and the product shipments between the manufacturers and the demand markets evolve according to (33). The prices associated with the retailers, in turn, evolve according to (34) for all retailers j, and the product shipments between the manufacturers and retailers evolve over time according to (35) for all manufacturer/retailer/mode combinations i, j, l.

Let X and $F(X)$ be defined as following (21).

Then the dynamic model described by (35), (33), (32), (34), and (31) for all k, i, j, l can be rewritten as the projected dynamical system (PDS) (cf. Nagurney and Zhang (1996)) defined by the following initial value problem:

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X_0, \tag{36}$$

where $\Pi_{\mathcal{K}}$ is the projection operator of $-F(X)$ onto $\mathcal{K}$ at X and $X_0 = (Q^{1^0}, Q^{2^0}, Q^{3^0}, \gamma^0, \rho_3^0)$ is the initial point corresponding to the initial product shipments between the manufacturers and the retailers and the demand markets; the initial product shipments between the retailers and the demand markets; and the initial retailers' prices and the demand prices. Since the feasible set $\mathcal{K}$ underlying the dynamic supply chain is simply the nonnegative orthant, the projection operation is very simple. Indeed, it simply guarantees, through the use of the "max" term (cf. (31)–(35)), that the dynamic trajectory never yields negative values for the product flows and prices.

The dynamical system (36) is non-classical in that the right-hand side is discontinuous in order to guarantee that the constraints, which in the context of the above model are nonnegativity constraints on the variables, are not violated. Such dynamical systems were introduced by Dupuis and Nagurney (1993) and to date have been used to model a variety of applications ranging from dynamic traffic network problems to dynamic oligopoly problems (cf. Nagurney and Zhang (1996) and the references therein).

Stationary/Equilibrium Points

The following theorem states that the projected dynamical system evolves until it reaches a stationary point, that is, $\dot{X} = 0$, in which there is no change in the product shipments and prices, and that the stationary point coincides with the equilibrium point of the supply chain network model according to Definition 1. The notation "*" is utilized here to denote an equilibrium point, as was also done in Section 2, as well as a stationary point, since these are shown to be equivalent in Theorem 8 below.

Theorem 8: The Set of Stationary Points Coincides with the Set of Equilibrium Points

The set of stationary points of the projected dynamical system (36) coincides with the set of equilibrium points defined by Definition 1.

Proof: According to Dupuis and Nagurney (1993), the necessary and sufficient condition for X^* to be a stationary point of the PDS (36), that is, to satisfy:

$$\dot{X} = 0 = \Pi_{\mathcal{K}}(X^*, -F(X^*)), \tag{37}$$

is that $X^* \in \mathcal{K}$ solves the variational inequality problem:

$$\langle F(X^*)^T, X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{38}$$

where, in our problem, $F(X)$, X, and $\mathcal{K}$ are as defined following (21). Writing out (38) explicitly, we have that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{l=1}^{2} \left[\frac{\partial f_i(Q^{1^*}, Q^{2^*})}{\partial q_{ijl}} + \frac{\partial c_{ijl}(q_{ijl}^*)}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1^*})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q_{ijl}^*)}{\partial q_{ijl}} - \gamma_j^* \right]$$

$$\times \left[q_{ijl} - q_{ijl}^*\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\frac{\partial f_i(Q^{1*},Q^{2*})}{\partial q_{ik}} + \frac{\partial c_{ik}(q_{ik}^*)}{\partial q_{ik}} + \hat{c}_{ik}(Q^{2*},Q^{3*}) - \rho_{3k}^*\right] \times [q_{ik} - q_{ik}^*]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[\gamma_j^* + \hat{c}_{jk}(Q^{2*},Q^{3*}) - \rho_{3k}^*\right] \times \left[q_{jk} - q_{jk}^*\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}^* - \sum_{k=1}^{o} q_{jk}^*\right] \times \left[\gamma_j - \gamma_j^*\right]$$

$$+\sum_{k=1}^{o}\left[\sum_{j=1}^{n} q_{jk}^* + \sum_{i=1}^{m} q_{ik}^* - d_k(\rho_3^*)\right] \times [\rho_{3k} - \rho_{3k}^*] \geq 0, \quad (39)$$

$$\forall (Q^1, Q^2, Q^3, \gamma, \rho_3) \in \mathcal{K} = R_+^{2mn+mo+no+n+o}.$$

But variational inequality (39) is precisely the variational inequality (18) (and their corresponding $F(\cdot)$s, Xs, and $\mathcal{K}$s are one and the same), which, in turn, according to Theorem 1 coincides with $(Q^{1^*}, Q^{2^*}, Q^{3^*}, \gamma^*, \rho_3^*)$ being an equilibrium pattern according to Definition 1. The proof is complete. □

Hence, Theorem 8 establishes the linkage between the solution to the variational inequality problem (18) governing the static supply chain network model with e-commerce described in Section 2, and the stationary points of the dynamic supply chain model described by the projected dynamical system (36). Indeed, it shows that they are one and the same. Thus, once a stationary point of the dynamic supply chain model has been achieved, that point satisfies the equilibrium conditions, at which the manufacturers, retailers, and consumers have formalized their agreements and the shipments between the tiers coincide.

We now state the following theorem.

Theorem 9: Existence and Uniqueness of a Solution to the Initial Value Problem

Assume the conditions of Theorem 7. Then, for any $X_0 \in \mathcal{K}$, there exists a unique solution $X_0(t)$ to the initial value problem (36).

Proof: Lipschitz continuity of the function F is sufficient for the conclusion based on Theorem 2.5 in Nagurney and Zhang (1996).

Theorem 9 guarantees that, if the Lipschitz property is satisfied, then the disequilibrium dynamics associated with the proposed projected dynamical system model of the supply chain are well-defined. In other words, given an initial product shipment and price pattern, there exists a unique trajectory associated with (36). Note that this existence and uniqueness result is not the same as those given in Theorems 3 and 6, respectively, since the latter results are for the equilibrium or stationary point, rather than for the dynamic trajectories.

5. The Algorithm

In this Section, we consider the computation of solutions to variational inequality (18); equivalently, the stationary points of the projected dynamical system (36). The algorithm that we propose is the modified projection method of Korpelevich (1977), which is guaranteed to solve any variational inequality problem in standard form (see (21)), that is:

Determine $X^* \in \mathcal{K}$, satisfying:

$$\langle F(X^*)^T, X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K},$$

provided that the function F that enters the variational inequality is monotone and Lipschitz continuous (and that a solution exists).

The statement of the modified projection method is as follows, where $\mathcal{T}$ denotes an iteration counter:

Modified Projection Method

Step 0: Initialization

Set $X^0 \in \mathcal{K}$. Let $\mathcal{T} = 1$ and let α be a scalar such that $0 < \alpha \leq \frac{1}{L}$, where L is the Lipschitz continuity constant (cf. Korpelevich (1977)) (see (30)).

Step 1: Computation

Compute $\bar{X}^{\mathcal{T}}$ by solving the variational inequality subproblem:

$$\langle (\bar{X}^{\mathcal{T}} + \alpha F(X^{\mathcal{T}-1}) - X^{\mathcal{T}-1})^T, X - \bar{X}^{\mathcal{T}} \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{40}$$

Step 2: Adaptation

Compute $X^{\mathcal{T}}$ by solving the variational inequality subproblem:

$$\langle (X^{\mathcal{T}} + \alpha F(\bar{X}^{\mathcal{T}}) - X^{\mathcal{T}-1})^T, X - X^{\mathcal{T}} \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{41}$$

Step 3: Convergence Verification

If max $|X_l^{\mathcal{T}} - X_l^{\mathcal{T}-1}| \leq \epsilon$, for all l, with $\epsilon > 0$, a prespecified tolerance, then stop; else, set $\mathcal{T} =: \mathcal{T} + 1$, and go to Step 1.

We now give an explicit statement of the modified projection method for the solution of variational inequality problem (18) for the supply chain network equilibrium model with electronic commerce.

Modified Projection Method for the Solution of Variational Inequality (18)

Step 0: Initialization

Set $({Q^1}^0, {Q^2}^0, {Q^3}^0, \gamma^0, \rho_3^0) \in \mathcal{K}$. Let $\mathcal{T} = 1$ and set α such that $0 < \alpha \leq \frac{1}{L}$, where L is the Lipschitz constant for the problem.

Step 1: Computation

Compute $(\bar{Q^1}^{\mathcal{T}}, \bar{Q^2}^{\mathcal{T}}, \bar{Q^3}^{\mathcal{T}}, \bar{\gamma}^{\mathcal{T}}, \bar{\rho}_3^{\mathcal{T}}) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[\bar{q}_{ijl}^{\mathcal{T}} + \alpha(\frac{\partial f_i({Q^1}^{\mathcal{T}-1}, {Q^2}^{\mathcal{T}-1})}{\partial q_{ijl}} + \frac{\partial c_{ijl}(q_{ijl}^{\mathcal{T}-1})}{\partial q_{ijl}} + \frac{\partial c_j({Q^1}^{\mathcal{T}-1})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q_{ijl}^{\mathcal{T}-1})}{\partial q_{ijl}} - \gamma_j^{\mathcal{T}-1})\right.$$

$$\left. -q_{ijl}^{\mathcal{T}-1}\right] \times \left[q_{ijl} - \bar{q}_{ijl}^{\mathcal{T}}\right]$$

$$\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\bar{q}_{ik}^{\mathcal{T}} + \alpha(\frac{\partial f_i({Q^1}^{\mathcal{T}-1}, {Q^2}^{\mathcal{T}-1})}{\partial q_{ik}} + \frac{\partial c_{ik}(q_{ik}^{\mathcal{T}-1})}{\partial q_{ik}} + \hat{c}_{ik}({Q^2}^{\mathcal{T}-1}, {Q^3}^{\mathcal{T}-1}) - \rho_{3k}^{\mathcal{T}-1}) - q_{ik}^{\mathcal{T}-1}\right]$$

$$\times \left[q_{ik} - \bar{q}_{ik}^{\mathcal{T}}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\bar{q}_{jk}^{\mathcal{T}} + \alpha(\hat{c}_{jk}({Q^2}^{\mathcal{T}-1}, {Q^3}^{\mathcal{T}-1}) + \gamma_j^{\mathcal{T}-1} - \rho_{3k}^{\mathcal{T}-1}) - q_{jk}^{\mathcal{T}-1}\right] \times \left[q_{jk} - \bar{q}_{jk}^{\mathcal{T}}\right]$$

$$+\sum_{j=1}^{n}\left[\bar{\gamma}_j^{\mathcal{T}} + \alpha(\sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}^{\mathcal{T}-1} - \sum_{k=1}^{o} q_{jk}^{\mathcal{T}-1}) - \gamma_j^{\mathcal{T}-1}\right] \times \left[\gamma_j - \bar{\gamma}_j^{\mathcal{T}}\right]$$

$$+\sum_{k=1}^{o}\left[\bar{\rho}_{3k}^{\mathcal{T}} + \alpha(\sum_{j=1}^{n} q_{jk}^{\mathcal{T}-1} + \sum_{i=1}^{m} q_{ik}^{\mathcal{T}-1} - d_k(\rho_3^{\mathcal{T}-1})) - \rho_{3k}^{\mathcal{T}-1}\right] \times \left[\rho_{3k} - \bar{\rho}_{3k}^{\mathcal{T}}\right] \geq 0,$$

$$\forall (Q^1, Q^2, Q^3, \gamma, \rho_3) \in \mathcal{K}. \tag{42}$$

Step 2: Adaptation

Compute $(Q^{1^{\mathcal{T}}}, Q^{2^{\mathcal{T}}}, Q^{3^{\mathcal{T}}}, \gamma^{\mathcal{T}}, \rho_3^{\mathcal{T}}) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[q_{ijl}^{\mathcal{T}} + \alpha(\frac{\partial f_i(\bar{Q}^{1\mathcal{T}}, \bar{Q}^{2\mathcal{T}})}{\partial q_{ijl}} + \frac{\partial c_{ijl}(\bar{q}_{ijl}^{\mathcal{T}})}{\partial q_{ijl}} + \frac{\partial c_j(\bar{Q}^{1\mathcal{T}})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(\bar{q}_{ijl}^{\mathcal{T}})}{\partial q_{ijl}} - \bar{\gamma}_j^{\mathcal{T}}) - q_{ijl}^{\mathcal{T}-1}\right]$$

$$\times \left[q_{ijl} - q_{ijl}^{\mathcal{T}}\right]$$

$$\sum_{i=1}^{m}\sum_{k=1}^{0}\left[q_{ik}^{\mathcal{T}} + \alpha(\frac{\partial f_i(\bar{Q}^{1\mathcal{T}}, \bar{Q}^{2\mathcal{T}})}{\partial q_{ik}} + \frac{\partial c_{ik}(\bar{q}_{ik}^{\mathcal{T}})}{\partial q_{ik}} + \hat{c}_{ik}(\bar{Q}^{2\mathcal{T}}, \bar{Q}^{3\mathcal{T}}) - \rho_{3k}^{\mathcal{T}}) - q_{ik}^{\mathcal{T}-1}\right]$$

$$\times \left[q_{ik} - q_{ik}^{\mathcal{T}}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[q_{jk}^{\mathcal{T}} + \alpha(\hat{c}_{jk}(\bar{Q}^{2\mathcal{T}}, \bar{Q}^{3\mathcal{T}}) + \bar{\gamma}_j^{\mathcal{T}} - \bar{\rho}_{3k}^{\mathcal{T}}) - q_{jk}^{\mathcal{T}-1}\right] \times \left[q_{jk} - q_{jk}^{\mathcal{T}}\right]$$

$$+\sum_{j=1}^{n}\left[\gamma_j^{\mathcal{T}} + \alpha(\sum_{i=1}^{m}\sum_{l=1}^{2}\bar{q}_{ijl}^{\mathcal{T}} - \sum_{k=1}^{o}\bar{q}_{jk}^{\mathcal{T}}) - \gamma_j^{\mathcal{T}-1}\right] \times \left[\gamma_j - \gamma_j^{\mathcal{T}}\right]$$

$$+\sum_{k=1}^{o}\left[\rho_{3k}^{\mathcal{T}} + \alpha(\sum_{j=1}^{n}\bar{q}_{jk}^{\mathcal{T}} + \sum_{i=1}^{m}\bar{q}_{ik}^{\mathcal{T}} - d_k(\bar{\rho}_3^{\mathcal{T}})) - \rho_{3k}^{\mathcal{T}-1}\right] \times \left[\rho_{3k} - \rho_{3k}^{\mathcal{T}}\right] \geq 0,$$

$$\forall (Q^1, Q^2, Q^3, \gamma, \rho_3) \in \mathcal{K}. \tag{43}$$

Step 3: Convergence Verification

If $|q_{ijl}^{\mathcal{T}} - q_{ijl}^{\mathcal{T}-1}| \leq \epsilon$, $|q_{ik}^{\mathcal{T}} - q_{ik}^{\mathcal{T}-1}| \leq \epsilon$, $|q_{jk}^{\mathcal{T}} - q_{jk}^{\mathcal{T}-1}| \leq \epsilon$, $|\gamma_j^{\mathcal{T}} - \gamma_j^{\mathcal{T}-1}| \leq \epsilon$, $|\rho_{3k}^{\mathcal{T}} - \rho_{3k}^{\mathcal{T}-1}| \leq \epsilon$, for all $i = 1, \cdots, m$; $j = 1, \cdots, n$; $l = 1, 2$; $k = 1, \cdots, o$, with $\epsilon > 0$, a pre-specified tolerance, then stop; otherwise, set $\mathcal{T} := \mathcal{T} + 1$, and go to Step 1.

Note that the variational inequality subproblems (42) and (43) can be solved explicitly and in closed form since the feasible set is that of the nonnegative orthant. Indeed, they yield subproblems in the q_{ijl}, q_{ik}, q_{jk}, γ_j and ρ_{3k} variables $\forall i, j, l, k$.

We now state the convergence result for the modified projection method for this model.

Theorem 10: Convergence

Assume that the function that enters the variational inequality (18) (or (21)) has at least one solution and satisfies the conditions in Theorem 4 and in Theorem 7. Then the modified projection method described above converges to the solution of the variational inequality (18) or (21).

Proof: According to Korpelevich (1977), the modified projection method converges to the solution of the variational inequality problem of the form (18), provided that the function F that enters the variational inequality is monotone and Lipschitz continuous and that a solution exists. Existence of a solution follows from Theorem 3. Monotonicity follows Theorem 4. Lipschitz continuity, in turn, follows from Theorem 7. □

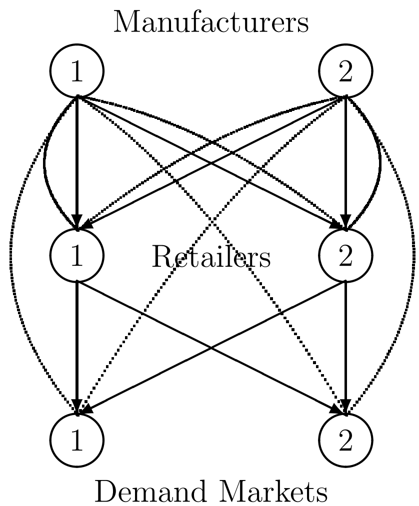

Figure 5: Supply Chain Network Structure for the Numerical Examples

6. Numerical Examples

In this Section, we apply the modified projection method to several numerical examples. The modified projection method was implemented in FORTRAN and the computer system used was a DEC Alpha system located at the University of Massachusetts at Amherst. The convergence criterion used was that the absolute value of the flows and prices between two successive iterations differed by no more than 10^{-4}. For the examples, α was set to .01 in the algorithm. The numerical examples had the network structure depicted in Figure 5 and consisted of two manufacturers, two retailers, and two demand markets, with both B2B and B2C transactions permitted.

Example 1

The data for the first example were constructed for easy interpretation purposes. The production cost functions for the manufacturers were given by:

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 2q_1, \quad f_2(q) = 2.5q_2^2 + q_1q_2 + 2q_2.$$

The transaction cost functions faced by the manufacturers and associated with transacting with the retailers using the physical link, that is, mode 1, were given by:

$$c_{111}(q_{111}) = .5q_{111}^2 + 3.5q_{111}, \quad c_{121}(q_{121}) = .5q_{121}^2 + 3.5q_{121},$$

$$c_{211}(q_{211}) = .5q_{211}^2 + 3.5q_{211}, \quad c_{221}(q_{221}) = .5q_{221}^2 + 3.5q_{221},$$

whereas the analogous transaction costs, but for mode 2, were given by:

$$c_{112}(q_{112}) = 1.5q_{112}^2 + 3q_{112}, \quad c_{122}(q_{122}) = 1.5q_{122}^2 + 3q_{122},$$

$$c_{212}(q_{212}) = 1.5q_{212}^2 + 3q_{212}, \quad c_{222}(q_{222}) = 1.5q_{222}^2 + 3q_{222},$$

The transaction costs of the manufacturers associated with dealing with the consumers at the demand markets via the Internet were given by:

$$c_{11}(q_{11}) = q_{11}^2 + 2q_{11}, \quad c_{12}(q_{12}) = q_{12}^2 + 2q_{12},$$

$$c_{21}(q_{21}) = q_{21}^2 + 2q_{21}, \quad c_{22}(q_{22}) = q_{22}^2 + 2q_{22}.$$

The handling costs of the retailers, in turn, were given by:

$$c_1(Q^1) = .5(\sum_{i=1}^{2}\sum_{l=1}^{2} q_{i1})^2, \quad c_2(Q^1) = .5(\sum_{i=1}^{2}\sum_{l=1}^{2} q_{i2})^2.$$

The transaction costs of the retailers associated with transacting with the manufacturers via mode 1 and mode 2 were, respectively, given by:

$$\hat{c}_{111}(q_{111}) = 1.5q_{111}^2 + 3q_{111}, \quad \hat{c}_{121}(q_{121}) = 1.5q_{121}^2 + 3q_{121},$$

$$\hat{c}_{211}(q_{211}) = 1.5q_{211}^2 + 3q_{211}, \quad \hat{c}_{221}(q_{221}) = 1.5q_{221}^2 + 3q_{221},$$

$$\hat{c}_{112}(q_{112}) = 1.5q_{112}^2 + 3q_{112}, \quad \hat{c}_{122}(q_{122}) = 1.5q_{122}^2 + 3q_{122},$$

$$\hat{c}_{212}(q_{212}) = 1.5q_{212}^2 + 3q_{212}, \quad \hat{c}_{222}(q_{222}) = 1.5q_{222}^2 + 3q_{222}.$$

The demand functions at the demand markets were:

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1000, \quad d_2(\rho_3) = -2\rho_{32} - 1.5\rho_{31} + 1000,$$

and the transaction costs between the retailers and the consumers at the demand markets (denoted for a typical pair by $\hat{c}_{jk}$ with the associated shipment by q_{jk} were given by:

$$\hat{c}_{11}(Q^2, Q^3) = q_{11} + 5, \quad \hat{c}_{12}(Q^2, Q^3) = q_{12} + 5, \quad \hat{c}_{21}(Q^2, Q^3) = q_{21} + 5, \quad \hat{c}_{22}(Q^2, Q^3) = q_{22} + 5,$$

whereas the transaction costs associated with transacting via the Internet for the consumers at the demand markets (denoted for a typical such pair by $\hat{c}_{ik}$ with the associated shipment of q_{ik}) were given by:

$$\hat{c}_{11}(Q^2, Q^3) = q_{11} + 1, \quad \hat{c}_{12}(Q^2, Q^3) = q_{12} + 1, \quad \hat{c}_{21}(Q^2, Q^3) = q_{21} + 1, \quad \hat{c}_{22}(Q^2, Q^3) = q_{22} + 1.$$

The modified projection method converged and yielded the following equilibrium pattern: the product shipments between the two manufacturers and the two retailers associated with the physical links, and with the Internet links, respectively, that is, with transacting via mode 1 and mode 2 were:

$$Q^{1^*} := q^*_{111} = q^*_{121} = q^*_{211} = q^*_{221} = 3.4611,$$

$$q^*_{112} = q^*_{122} = q^*_{212} = q^*_{222} = 2.3907.$$

The product shipments between the two manufacturers and the two demand markets with transactions conducted through the Internet were:

$$Q^{2^*} := q^*_{11} = q^*_{12} = q^*_{21} = q^*_{22} = 13.3033.$$

The product shipments (consumption volumes) between the two retailers and the two demand markets were:

$$Q^{3^*} := q^*_{11} = q^*_{12} = q^*_{21} = q^*_{22} = 5.8513.$$

The vector γ^*, which was equal to the prices charged by the retailers ρ^*_2, had components:

$$\gamma^*_1 = \gamma^*_2 = 263.9088,$$

and the demand prices at the demand markets were:

$$\rho^*_{31} = \rho^*_{32} = 274.7701.$$

It is easy to verify that the optimality/equilibrium conditions were satisfied with good accuracy.

The prices charged by the manufacturers were as follows and were recovered according to the discussion following variational inequality (25). The ρ^*_{1ijl}s were as follows for $l = 1$ and for $l = 2$, respectively: All ρ^*_{1ij1}s= 238.8218 and all ρ^*_{1ij2}s= 242.0329. All the ρ^*_{1ik}s were equal to 260.4673. These values were obtained in both ways as discussed following (21) and either manner yielded the same value for the corresponding price. Note that the price charged by the manufacturers to the consumers at the demand markets, approximately 260, was higher than the price charged to the retailers, regardless of the mode of transacting. The price charged to the retailers for the product transacted via the Internet, in turn, exceeded that charged using the classical physical manner.

Example 2: Variant of Example 1

We then modified Example 1 as follows: The production cost function for manufacturer 1 was now given by:

$$f_1(q) = 2.5q_1^2 + q_1 q_2 + 12q_1,$$

whereas the transaction costs for manufacturer 1 were now given by:

$$c_{11}(q_{11}) = q_{11}^2 + 3.5q_{11}, \quad c_{12}(q_{12}) = q_{12}^2 + 3.5q_{12}.$$

The remainder of the data was as in Example 1. Hence, both the production costs and the transaction costs increased for manufacturer 1.

The modified projection method converged and yielded the following equilibrium pattern: the product shipments between the two manufacturers and the two retailers associated with the physical links, and with the Internet links, respectively, that is, with transacting via mode 1 and mode 2 were:

$$Q^{1^*} := q^*_{111} = q^*_{121} = 3.3265, \quad q^*_{211} = q^*_{221} = 3.5408,$$

$$q^*_{112} = q^*_{122} = 2.3010, \quad q^*_{212} = q^*_{222} = 2.4438.$$

The product shipments between the two manufacturers and the two demand markets with transactions conducted through the Internet were:

$$Q^{2^*} := q_{11}^* = q_{12}^* = 12.5781, \quad q_{21}^* = q_{22}^* = 13.3638.$$

The product shipments (consumption volumes) between the two retailers and the two demand markets were:

$$Q^{3^*} := q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 5.8056.$$

The vector γ^* had components:

$$\gamma_1^* = \gamma_2^* = 264.1706,$$

and the demand prices at the demand markets were:

$$\rho_{31}^* = \rho_{32}^* = 274.9861.$$

The optimality/equilibrium conditions were, again, satisfied at the desired accuracy.

The ρ_{1ijl}^*s were as follows for $l = 1$ and for $l = 2$, respectively: The ρ_{11j1}^*s$= 239.5789$ for both j and the ρ_{11j2}^*s$= 242.6553$ for both j. For firm 2, on the other hand, $\rho_{12j1}^* = 238.9360$ for both j, whereas $\rho_{12j2}^* = 242.268$ for both j. The ρ_{11k}^*s were equal to 261.4085, for both k, whereas the ρ_{12k}^*s were equal to 260.6223 for both k. Note that these values were obtained in both ways as discussed following (21) and either manner yielded the same value for the corresponding price. Note that, again, the prices charged by the manufacturers to the consumers at the demand markets were higher than the prices charged to the retailers. Of course, the generalized demand price was, nevertheless, equal for all consumers at a given demand market. In fact, both in this and in the preceding example the equilibrium generalized demand prices were the same for each demand market.

Hence, manufacturer 1 now produced less than it did in Example 1, whereas manufacturer 2 increased its production output. The prices charged by the retailers to the consumers increased, as did the generalized price at the demand markets, with a decrease in the incurred demand.

Example 3: Variant of Example 2

We then modified Example 2 as follows: The data were identical to that in Example 2 except that we increased the demand function for demand market 1 as follows:

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 2000.$$

The modified projection method converged and yielded the following equilibrium pattern: the product shipments between the two manufacturers and the two retailers associated with the physical links, and with the Internet links, respectively, that is, with transacting via mode 1 and mode 2 were:

$$Q^{1^*} := q^*_{111} = q^*_{121} = 16.1444, \quad q^*_{211} = q^*_{221} = 16.4974,$$

$$q^*_{112} = q^*_{122} = 10.8463, \quad q^*_{212} = q^*_{222} = 11.0816.$$

The product shipments between the two manufacturers and the two demand markets with transactions conducted through the Internet were:

$$Q^{2^*} := q^*_{11} = 60.2397, \quad q^*_{12} = 0.0000, \quad q^*_{21} = 61.2103, \quad q^*_{22} = 0.0000.$$

The product shipments (consumption volumes) between the two retailers and the two demand markets were:

$$Q^{3^*} := q^*_{11} = 54.5788, \quad q^*_{12} = 0.0000, \quad q^*_{21} = 54.5788, \quad q^*_{22} = 0.0000,$$

the vector γ^*, which was equal to the prices charged by the retailers ρ^*_2, had components:

$$\gamma^*_1 = \gamma^*_2 = 825.1216,$$

and the demand prices at the demand markets were:

$$\rho^*_{31} = 884.694, \quad \rho^*_{32} = 0.0000.$$

It is easy to verify that the optimality/equilibrium conditions were satisfied with good accuracy.

The prices charged by the manufacturers were as follows and were, again, recovered according to the discussion following variational inequality (21).

The ρ^*_{1ijl}s were as follows for $l = 1$ and for $l = 2$, respectively: $\rho^*_{1111} = 719.1185 = \rho^*_{1121}$, $\rho^*_{1211} = 718.0597 = \rho^*_{1221}$, $\rho^*_{1112} = \rho^*_{1122} = 735.019$, $\rho^*_{1212} = \rho^*_{1222} = 734.3071$. The ρ^*_{111}s was equal to 823.4536, whereas the ρ^*_{121} was equal to 822.4830.

In this example, only the consumers at demand market 1 consume a positive amount. Indeed, there is no consumption of the product by consumers located at demand market 2.

7. Summary and Conclusions

In this paper, we have proposed a framework for the formulation, analysis, and computation of solutions to supply chain problems in the presence of electronic commerce in the form of B2B and B2C transactions. Specifically, we have proposed an equilibrium framework in which the prices associated with the manufacturers, the retailers, and the consumers are endogenous, as are the production, shipment, and consumption flows.

We formulated the optimization problems facing the manufacturers and the retailers, and identified the network structure of their transactions. We also established the network structure of the supply chain network in which the manufacturers' product shipments to the retailers must be in agreement with the shipments that the retailers accept. Moreover, the amounts purchased by the consumers must be in agreement with the amounts that the retailers accept from the manufacturers and that the manufacturers transact with the consumers directly through the Internet.

The methodology used for the formulation, qualitative analysis, as well as computation of the equilibrium prices and product shipments was that of finite-dimensional variational inequality theory. We established existence of an equilibrium pattern and also provided uniqueness results.

We then turned to the study of the of the disequilibrium dynamics and proposed a continuous time adjustment process, which was formulated as a projected dynamical system. We subsequently proved that the set of stationary points of the dynamical system coincides with the set of solutions of the variational inequality problem. We also showed that, under reasonable conditions, the trajectories of the dynamical system are well-defined.

Of course, there are many additional interesting questions that arise in regards to the study of the dynamics of supply chain interactions but, nevertheless, we hope that the work in this paper provides part of the foundation.

An algorithm was also proposed for computational purposes and convergence results given. Finally, we applied the computational procedure to several illustrative supply chain network examples.

Although we considered a supply chain consisting of manufacturers, retailers, and consumers at demand markets, the approach here should also be applicable to the case where distribution centers are included and with, suitable modifications, and/or where raw material suppliers are also incorporated. We plan to address such modeling issues in future research.

Acknowledgments

The authors are grateful to the anonymous referee for helpful comments and suggestions.

The research of the first and second authors was supported, in part, by NSF Grant No.: IIS-0002647 and, in part, by NSF Grant No.: CMS-0085720. The research of the third author was also supported by the former grant. This support is gratefully acknowledged.

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