Dynamic Supernetworks
for the
Integration of Social Networks and Supply Chains with Electronic Commerce:
Modeling and Analysis of Buyer-Seller Relationships with Computations

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Abstract: In this paper, we develop a dynamic supernetwork framework for the modeling and analysis of supply chains with electronic commerce that also includes the role that relationships play. Manufacturers are assumed to produce a homogeneous product and to sell it either through physical or electronic links to retailers and/or directly to consumers through electronic links. Retailers, in turn, can sell the product through physical links to consumers. Increasing relationship levels in our framework are assumed to reduce transaction costs as well as risk and to have some additional value for both sellers and buyers. Establishing those relationship levels incurs some costs that have to be borne by the decision-makers in the supernetwork, which is multilevel in structure and consists of the supply chain and the social network. The decision-makers, who are located at distinct tiers in the supernetwork, try to optimize their objective functions and are faced with multiple criteria including relationship-related ones and weight them according to their preferences. We establish the optimality conditions for the manufacturers, retailers, and consumers, derive the equilibrium conditions, and provide the variational inequality formulation. We then present the projected dynamical system, which describes the disequilibrium dynamics of the product transactions, relationship levels, and prices on the supernetwork, and whose set of stationary points coincides with the set of solutions of the variational inequality problem. We also illustrate the dynamic supernetwork model through several numerical examples, for which the explicit equilibrium patterns are computed.

1. Introduction

As Uzzi (1996, p. 674) stated, there is a "growing need to understand how social structure assists or impedes economic performance." In this paper, we attempt to contribute to this understanding by extending the fundamental supply chain network model with electronic commerce developed by Nagurney et al. (2002b) to construct a dynamic supernetwork model that explicitly integrates social network analysis with supply chain modelling and captures rigorously the role that relationship levels play. Hence, this work represents a new development in the synthesis and enhancement of theoretical foundations of economic decision-making combined with social constructs within a network setting. We choose to frame the ideas within a supply chain context due to the topical, multidisciplinary nature of the problem with its many associated practical applications.

Indeed, supply chain networks with electronic commerce (e-commerce) are a topic of growing interest. Nagurney et al. (2002b) developed a supply chain network model with tiers of decision-makers in the presence of e-commerce, where both business-to-business (B2B) and business-to-consumer (B2C) transactions were possible. Their approach is based on the supernetwork concept (cf. Nagurney and Dong (2002)) which captured the trade-offs associated with telecommunication versus transportation networks. Dong, Zhang, and Nagurney (2002) considered additional criteria to that of profit maximization (or cost minimization) that decision-makers in supply chains may take into account and introduced multicriteria decision-making into supply chain network modelling. Their network, however, did not include electronic commerce. Nagurney and Toyasaki (2003), in turn, developed a supernetwork model for supply chain decision-making with environmental criteria that also included the possibility of electronic commerce. Nagurney et al. (2002a) proposed a dynamic supply chain network model over a multilevel network which included the logistical, financial, and informational networks but did not consider e-commerce.

In this paper, we turn to the influence that relationships play in supply chains. Relationship issues surrounding supply chains have been a topic of high interest in the disciplines of sociology, marketing; specifically, relationship marketing, and economics. For example, embeddedness theory (cf. Granovetter (1985) and Uzzi (1996), among others) attempts to explain the effects that relationships play in different economic actions, including financial transactions (see, e.g., Uzzi (1998)). It also emphasizes the importance of the consideration of the effects of relationships in order to explain different phenomena that can be observed in reality such as the relevance of Keiretsu networks in Japan (cf. Lincoln, Gerlach, and Takahashi (1992)). Jones, Hesterly, and Borgatti (1997), in turn, stressed that it is necessary to further concretize the results of the embeddedness theory. They described the conditions under which interfirm coordination can emerge by integrating transaction cost economics and social network theory.

In the context of relationship marketing (cf. Ganesan (1994) and Bagozzi (1995)), on the other hand, researchers have tried to illuminate the motivation of sellers and buyers who actively seek relationships in the context of B2B (see, e.g. Wilson (1995)) or B2C commerce (see, e.g. Sheth and Parvatiyar (1995)). Different attempts to classify relationship structures have been made (see, e.g., Donaldson and O'Toole (2000)). Vertical relationships and relationship-specific investments (see, e.g., Williamson (1983), Joskow (1988), Crawford (1990), Vickers and Waterson (1991), and Muthoo (1998)) are also a topic of growing interest in economics. Indeed, according to Crawford (1990, p. 561), "Relationship-specific investment is considered to be investment whose returns depend on the continuation of the relationship." Economists are especially concerned about "determining the importance of the economic characteristics that characterize specific buyer and seller relationships and the role of transaction costs in determining the cost-minimizing governance structure for exchange" (Joskow (1988, p. 99)).

An article by Golicic, Foggin, and Mentzer (2003) introduced the concept of relationship magnitude and differentiated it from relationship type. Their paper is based on a literature review as well as an exploratory study conducted with company executives. Their research results indicate that different relationship magnitudes lead to different benefits and that different levels of relationship magnitudes can be achieved by putting more or less time and effort into the relationship. They recommend that firms should optimize their portfolio of relationships with other companies by pursuing different relationship levels depending on the expected costs and benefits. However, their paper is conceptual in nature whereas our approach is mathematical and computational and based on economic principles.

In particular, in this paper, we develop a theoretical framework, which builds upon the work of Nagurney and Dong (2002), Dong, Zhang, and Nagurney (2002), Nagurney et al. (2002a, b), and Nagurney, Dong, and Zhang (2002). Specifically, we develop a dynamic supernetwork model of supply chains with e-commerce that also includes the role that relationships play. The supernetwork model is multilevel in structure and includes both the supply chain network and the social network with flows on the former corresponding to product transactions and flows on the latter to relationship levels. Prices are associated with the nodes in the supernetwork which correspond to the different tiers of decision-makers.

Manufacturers are assumed to produce homogeneous products and to sell them either over physical or electronic links via the Internet to retailers and over electronic links to consumers. Retailers, in turn, can sell the products over physical links to consumers. Increasing relationship levels are assumed to reduce both transaction costs and risk and to have some additional value for both sellers and buyers. Establishing those relationship levels, which are assumed to be symmetric, incurs some costs that have to be borne by the sellers in the network. Hence, our framework allows for the measurement of relationship levels, unlike that of classical social network analysis, which focuses primarily on graphical analysis through the study of underlying network topologies. For additional references and background on social networks, relationships, and organizational and marketing issues, see the edited volume by Iacobucci (1996). For an overview of social network analysis see Freeman (2000) and Krackhardt (2000). Carley (2003), in turn, provides information about dynamic network analysis within a social network framework.

The decision-makers, who are located at distinct tiers in the supernetwork try to optimize their objective functions. They are faced with multiple criteria including relationship-related ones and weight them according to their preferences. This framework makes it possible to simulate different scenarios depending on how concerned (or not) about relationships the decision-makers are through variations in the associated production and transaction cost functions and weightings of the different criteria. Moreover, since the dynamic supernetwork model is also computable, it allows for the explicit computation of the levels of relationships between the decision-makers in the social network as well as product transactions associated with the supply chain network.

This paper is organized as follows. In Section 2, we develop the multilevel supernetwork model consisting of multiple tiers of decision-makers, acting on the supply chain and social networks. We describe their optimizing behavior, and establish the governing equilibrium conditions, whose solution yields the equilibrium product transactions, prices, and relationship levels. In Section 3, we then describe the disequilibrium dynamics of the product transactions, relationship levels, and prices and establish that the set of stationary points of the resulting projected dynamical system (cf. Nagurney and Zhang (1996) and Dupuis and Nagurney (1993)) coincides with the set of solutions of the variational inequality problem given in Section 2. We also provide some qualitative properties of the dynamic trajectories. In Section 4, we propose a discrete-time algorithm for the tracking of the dynamic trajectories and then apply it to solve several illustrative dynamic supernetwork numerical examples, that show that relationship levels increase if the decision-makers are more concerned about relationship value. Finally, in Section 5, we summarize our results and suggest directions for future research.

2. The Supernetwork Model with Integrated Supply Chain and Social Network Analysis

In this section, we develop the supernetwork model with manufacturers, retailers, and demand markets in which we explicitly integrate relationship levels between buyers and sellers and also include electronic commerce. In this section we focus on the presentation of the model within an equilibrium context, whereas in Section 3, we then present the disequilibrium dynamics. As Figure 1 shows, the supernetwork model consists of a bottom level network corresponding to the social network and the top level network corresponding to the supply chain network. The manufacturers can sell directly to the consumers at the demand markets through the Internet and can also conduct their transactions with retailers through the Internet, or physically (in the standard manner). Retailers are assumed to transact physically with consumers. Internet links are denoted in the supply chain and social network in Figure 1 by dotted arcs with the dotted arcs joining these two networks representing the synthesis of these two networks into a supernetwork. We will see below how these two networks are related further through the underlying functions associated with the nodes and links and corresponding flows.

In our model, it is assumed that m manufacturers are involved in the production of a homogeneous product which can then be purchased by n retailers and/or directly by the consumers located at o demand markets. A typical manufacturer is denoted by i, a typical retailer by j, and a typical demand market by k. The manufacturers are located at the top tier of nodes of the social and the supply chain networks, the retailers are associated with the middle tier, and the demand markets with the third or bottom tier of nodes.

The Behavior of the Manufacturers

Each manufacturer faces three criteria: the maximization of profit, the minimization of risk, and the maximization of the relationship value, which is a function of the relationship levels. This means that he tries to create a relationship value that is as high as possible taking the other criteria into consideration, subject to his individual weight assignment to this criterion. This reflects the fact that the relationship level per se has some value to the manufacturer as it, among other reasons, is likely to lead to future business.

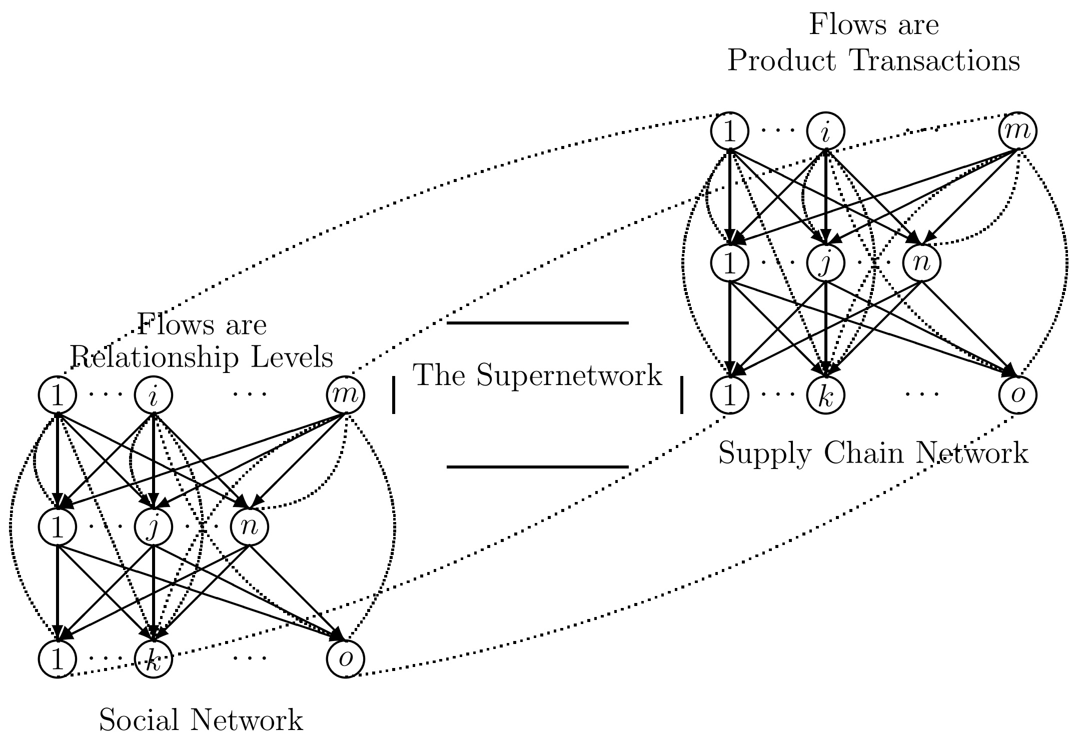

Figure 1: The Multilevel Supernetwork Structure of the Integrated Supply Chain / Social Network System

Let q_i denote the nonnegative production output by manufacturer i. Group the production outputs of all manufacturers into the column vector $q \in R^m_+$. Here it is assumed that each manufacturer i is faced with a production cost function f_i, which can depend, in general, on the entire vector of production outputs, that is,

$$f_i = f_i(q), \quad \forall i. \tag{1}$$

Let h_{ijl} denote the nonnegative level of relationship between manufacturer i and retailer j associated with mode of transaction l, where $l = 1$ denotes a physical transaction and $l = 2$ denotes a virtual transaction via the Internet. We assume that each manufacturer i is actively seeking to achieve a certain relationship level with a retailer and/or a demand market. Similarly, let h_{ik} denote the relationship level associated with a virtual transaction between manufacturer i and demand market k. The relationship levels are endogenously determined in the model. We group the h_{ijl}s for all manufacturer/retailer/mode combinations into the column vector $h^1 \in R^{2mn}_+$ and the h_{ik}s for all the manufacturer/demand market pairs into the column vector $h^2 \in R^{mo}_+$. We assume that these relationship levels lie in the range $[0, 1]$, with a relationship level of zero signifying no relationship and a relationship level of one signifying the strongest relationship.

In order to achieve a particular relationship level the manufacturer spends money (which may also be used as a proxy for time spent), for example, in the form of visits, gifts, and/or additional service, etc. We assume that a production cost function for the relationship levels that a manufacturer tries to achieve with retailers and consumers, respectively, exists. These production cost functions for relationship levels, denoted, respectively, by b_{ijl} and b_{ik} represent how much money a manufacturer i has to spend in order to achieve a certain relationship level with retailer j through a mode l transaction and/or demand market k. These relationship production cost functions will be distinct for each manufacturer/retailer/mode or manufacturer/demand market combination and will depend on several factors such as, for example, the willingness of retailers or demand markets to establish a relationship and the level of previous business relationships and private relationships that exist. The relationship production cost functions depend on the relationship level that the manufacturer wishes to achieve with retailer j or consumers at demand market k, that is,

$$b_{ijl} = b_{ijl}(h_{ijl}), \quad \forall i, j, l, \tag{2}$$

$$b_{ik} = b_{ik}(h_{ik}), \quad \forall i, k. \tag{3}$$

Note that in the supernetwork depicted in Figure 1 the relationship flows h^1 are associated with the links joining the manufacturers with the retailers on the social network, whereas the flows h^2 correspond to the flows on the links joining the manufacturers with the demand markets through Internet transactions in the social network.

As noted above, a manufacturer may transact with a retailer via a physical link, and/or via an Internet link. The nonnegative product transaction associated with manufacturer i, retailer j, and mode of transaction l is denoted by q_{ijl}. These product transactions are grouped into the column vector $Q^1 \in R^{2mn}_+$. A manufacturer i may also transact directly with consumers located at a demand market k with the associated nonnegative product transaction between manufacturer i and demand market k denoted by q_{ik}. These product transactions are grouped into the column vector $Q^2 \in R^{mo}_+$. Note that the product transactions Q^1 are associated with the links from the manufacturers to the retailers in the supply chain network in Figure 1. The product transactions Q^2, in turn, are associated with the links (and correspond to the flows) between the manufacturers and the demand markets in the supply chain network.

It is assumed that the transaction cost between a manufacturer and retailer pair via a particular mode of transaction as well as the transaction cost between a manufacturer and consumers at a demand market may depend upon the volume of transactions between each pair and also on the level of the relationship between them. If the level of relationship is higher, then we can expect the incurred transaction costs to be lower for several reasons, including reduced monitoring costs. We let c_{ijl} denote the transaction cost between manufacturer i associated with transactions with retailer j via mode l. Also, we let c_{ik} denote the transaction cost associated with the transaction between manufacturer i and demand market k. The transaction costs are as follows:

$$c_{ijl} = c_{ijl}(q_{ijl}, h_{ijl}), \quad \forall i, j, l \tag{4}$$

and

$$c_{ik} = c_{ik}(q_{ik}, h_{ik}), \quad \forall i, k. \tag{5}$$

We assume that the production cost and transaction cost functions (1) through (5) are convex and continuously differentiable.

The quantity of the product produced by manufacturer i must satisfy the following conservation of flow equation:

$$q_i = \sum_{j=1}^{n} \sum_{l=1}^{2} q_{ijl} + \sum_{k=1}^{o} q_{ik}, \tag{6}$$

which states that the quantity produced by manufacturer i is equal to the sum of the quantities transacted between the manufacturer and all retailers (via the two modes) and the demand markets.

Hence, the total costs incurred by manufacturer i are equal to the sum of the manufacturer's production cost plus the total transaction costs plus the costs that he incurs for establishing relationship levels. His revenue is the price that the manufacturer charges for the product (and the consumers and retailers are willing to pay) times the total quantity obtained/purchased of the product from the manufacturer by all the retailers and consumers at all demand markets. We denote the price actually charged for the product by manufacturer i to retailer j who has transacted using mode l by ρ^*_{1ijl}, and the price actually charged for the product by manufacturer i for the product to consumers at demand market k by ρ^*_{1ik}. How these prices are arrived at is discussed later in this section.

Noting the conservation of flow equation (6), and the production cost functions (1), we can express the manufacturers' production cost functions as $f_i(Q^1, Q^2)$, $\forall i$. The profit maximization problem, hence, faced by manufacturer i can be expressed as:

$$\text{Maximize} \quad \sum_{j=1}^{n} \sum_{l=1}^{2} \rho^*_{1ijl} q_{ijl} + \sum_{k=1}^{o} \rho^*_{1ik} q_{ik} - f_i(Q^1, Q^2) - \sum_{j=1}^{n} \sum_{l=1}^{2} c_{ijl}(q_{ijl}, h_{ijl}) - \sum_{k=1}^{o} c_{ik}(q_{ik}, h_{ik})$$

$$- \sum_{j=1}^{n} \sum_{l=1}^{2} b_{ijl}(h_{ijl}) - \sum_{k=1}^{o} b_{ik}(h_{ik}) \tag{7}$$

subject to:

$$q_{ijl} \geq 0, \quad \forall j, l, \quad q_{ik} \geq 0, \quad \forall k, \tag{8}$$

with

$$0 \leq h_{ijl} \leq 1, \quad \forall j, l, \quad 0 \leq h_{ik} \leq 1, \quad \forall k. \tag{9}$$

Note that in (7), the first two terms represent the revenue whereas the subsequent five terms represent the various costs.

In addition to the criterion of profit maximization, we also assume that each manufacturer is concerned with risk minimization. Here, for the sake of generality, we assume, as given, a risk function r_{ijl}, for manufacturer i transacting with retailer j via mode l, which is assumed to be continuous and convex and a function of not only the product transactions associated with the particular retailer but also the relationship with the particular retailer via the specific mode of transaction. A higher relationship level will reduce risk because trust reduces transactional uncertainty. The same is true for the risk function r_{ik} for manufacturer i transacting with demand market k through the Internet. Such risk functions are assumed to be continuous and convex and a function of not only the product transactions associated with the particular manufacturer/demand market but also of the relationship level of this pair.

Hence, we assume that

$$r_{ijl} = r_{ijl}(q_{ijl}, h_{ijl}), \quad \forall i, j, l, \tag{10}$$

$$r_{ik} = r_{ik}(q_{ik}, h_{ik}), \quad \forall i, k. \tag{11}$$

The second criterion faced by manufacturer i, thus, corresponds to risk minimization and can be expressed mathematically as:

$$\text{Minimize} \quad \sum_{j=1}^{n} \sum_{l=1}^{2} r_{ijl}(q_{ijl}, h_{ijl}) + \sum_{k=1}^{o} r_{ik}(q_{ik}, h_{ik}) \tag{12}$$

subject to:

$$q_{ijl} \geq 0, \quad \forall j, l, \quad q_{ik} \geq 0, \quad \forall k, \tag{13}$$

$$0 \leq h_{ijl} \leq 1, \quad \forall j, l, \quad 0 \leq h_{ik} \leq 1, \quad \forall k. \tag{14}$$

Finally, the manufacturer also tries to maximize the relationship value with the retailers and the demand markets. For the sake of generality, we assume, as given, a relationship value function v_{ijl}, for manufacturer i, retailer j, and mode l, which is assumed to be a function of the relationship level with the particular retailer via the specific mode of transaction. We also assume a relationship value function v_{ik}, for manufacturer i and demand market

k, which is assumed to be a function of the relationship level with the particular demand market k. A very simple relationship value function could be, for example, the sum of all the relationship levels. We assume that

$$v_{ijl} = v_{ijl}(h_{ijl}), \quad \forall i, j, l, \tag{15}$$

$$v_{ik} = v_{ik}(h_{ik}), \quad \forall i, k. \tag{16}$$

We assume that the value functions are continuously differentiable and concave.

The third criterion faced by manufacturer i, thus, corresponds to relationship value maximization and can be expressed mathematically as:

$$\text{Maximize} \quad \sum_{j=1}^{n}\sum_{l=1}^{2} v_{ijl}(h_{ijl}) + \sum_{k=1}^{o} v_{ik}(h_{ik}) \tag{17}$$

subject to:

$$0 \leq h_{ijl} \leq 1, \quad \forall j, l, \quad 0 \leq h_{ik} \leq 1, \quad \forall k. \tag{18}$$

A Manufacturer's Multicriteria Decision-Making Problem

It is assumed that manufacturer i assigns a nonnegative weight α_i to the risk generated and a nonnegative weight β_i to the relationship value. The weight associated with profit maximization is set equal to 1 and serves as the numeraire. The nonnegative weights measure the importance of risk and the relationship value and, in addition, transform these values into monetary values. We can construct a value function for each manufacturer (cf. Keeney and Raiffa (1993), Dong, Zhang, and Nagurney (2002), and the references therein) using a constant additive weight value function. Hence, the multicriteria decision-making problem for manufacturer i is transformed into:

$$\text{Maximize} \quad \sum_{j=1}^{n}\sum_{l=1}^{2} \rho_{1ijl}^{*} q_{ijl} + \sum_{k=1}^{o} \rho_{1ik}^{*} q_{ik} - f_i(Q^1, Q^2) - \sum_{j=1}^{n}\sum_{l=1}^{2} c_{ijl}(q_{ijl}, h_{ijl}) - \sum_{k=1}^{o} c_{ik}(q_{ik}, h_{ik})$$

$$- \sum_{j=1}^{n}\sum_{l=1}^{2} b_{ijl}(h_{ijl}) - \sum_{k=1}^{o} b_{ik}(h_{ik})$$

$$-\alpha_i\Big(\sum_{j=1}^{n}\sum_{l=1}^{2} r_{ijl}(q_{ijl}, h_{ijl}) + \sum_{k=1}^{o} r_{ik}(q_{ik}, h_{ik})\Big) + \beta_i\Big(\sum_{j=1}^{n}\sum_{l=1}^{2} v_{ijl}(h_{ijl}) + \sum_{k=1}^{o} v_{ik}(h_{ik})\Big) \tag{19}$$

subject to:

$$q_{ijl} \geq 0, \quad \forall j, l, \quad q_{ik} \geq 0, \quad \forall k, \tag{20}$$

$$0 \leq h_{ijl} \leq 1, \quad \forall j, l, \quad 0 \leq h_{ik} \leq 1, \quad \forall k. \tag{21}$$

The Optimality Conditions of Manufacturers

The manufacturers are assumed to compete in a noncooperative fashion. The governing optimization/equilibrium concept underlying noncooperative behavior is that of Nash (1950, 1951), which states, in this context, that each manufacturer will determine his optimal transactions, given the optimal ones of the competitors. The optimality conditions for all manufacturers can be simultaneously expressed as the following inequality (cf. Bazaraa, Sherali, and Shetty (1993), Gabay and Moulin (1980); see also Nagurney (1999)): determine $(Q^{1*}, Q^{2*}, h^{1*}, h^{2*}) \in \mathcal{K}_1$, such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[\frac{\partial f_i(Q^{1*},Q^{2*})}{\partial q_{ijl}} + \frac{\partial c_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial q_{ijl}} + \alpha_i \frac{\partial r_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial q_{ijl}} - \rho^*_{1ijl}\right] \times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\frac{\partial f_i(Q^{1*},Q^{2*})}{\partial q_{ik}} + \frac{\partial c_{ik}(q^*_{ik},h^*_{ik})}{\partial q_{ik}} + \alpha_i \frac{\partial r_{ik}(q^*_{ik},h^*_{ik})}{\partial q_{ik}} - \rho^*_{1ik}\right] \times \left[q_{ik} - q^*_{ik}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\frac{\partial c_{ik}(q^*_{ik},h^*_{ik})}{\partial h_{ik}} - \beta_i \frac{\partial v_{ik}(h^*_{ik})}{\partial h_{ik}} + \alpha_i \frac{\partial r_{ik}(q^*_{ik},h^*_{ik})}{\partial h_{ik}} + \frac{\partial b_{ik}(h^*_{ik})}{\partial h_{ik}}\right] \times \left[h_{ik} - h^*_{ik}\right]$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[\frac{\partial c_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial h_{ijl}} - \beta_i \frac{\partial v_{ijl}(h^*_{ijl})}{\partial h_{ijl}} + \alpha_i \frac{\partial r_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial h_{ijl}} + \frac{\partial b_{ijl}(h^*_{ijl})}{\partial h_{ijl}}\right] \times \left[h_{ijl} - h^*_{ijl}\right] \geq 0,$$

$$\forall (Q^1, Q^2, h^1, h^2) \in \mathcal{K}_1, \tag{22}$$

where

$$\mathcal{K}_1 \equiv \left[(Q^1, Q^2, h^1, h^2) \mid q_{ijl} \geq 0,\, q_{ik} \geq 0,\, 0 \leq h_{ijl} \leq 1,\, 0 \leq h_{ik} \leq 1,\, \forall i, j, l, k\right]. \tag{23}$$

The inequality (22), which is a variational inequality (cf. Nagurney (1999)) has a meaningful economic interpretation. From the first term we can see that, if there is a positive transaction of the product transacted either in a classical manner or via the Internet from a manufacturer to a retailer, then the marginal cost of production plus the marginal cost of transacting plus the weighted marginal cost of risk must be equal to the price that the retailer is willing to pay for the product. If that sum, in turn, exceeds the price then there will be no product transacted.

The second term in (22) states that there will be a positive flow of the product from a manufacturer to a demand market if the marginal cost of production of the manufacturer plus the marginal cost of transacting via the Internet for the manufacturer with consumers and the weighted marginal cost of risk is equal to the price the consumers are willing to pay for the product at the demand market.

The third and the fourth term in (22) show that if there is a positive relationship level (and that level is less than one) established then the marginal cost of establishing this level is equal to the marginal reduction in transaction costs plus the weighted marginal reduction in risk plus the marginal value of relationship for the manufacturer.

The Behavior of the Retailers

The retailers transact with the manufacturers in order to obtain the product, as well as with the consumers, who are the ultimate purchasers/buyers of the product. The retailers try to maximize profits and relationship values with manufacturers and consumers and to minimize their individual risk associated with their transactions.

Establishing relationship levels with manufacturers and consumers again incurs some costs. Let h_{jk} denote the relationship level between retailer j and demand market k. We group the relationship levels for all retailer/demand market pairs, which are assumed to be nonnegative and lying in the range 0 through 1 into the column vector $h^3 \in R^{no}_+$. These relationship levels are the flows on the links in the social network level of the supernetwork in Figure 1 joining the retailer nodes with the demand market nodes.

The cost function for the establishment of relationship levels with manufacturers and consumers is increasing with the relationship level as in the case of the manufacturers and can be expressed as follows:

$$\hat{b}_{ijl} = \hat{b}_{ijl}(h_{ijl}), \quad \forall i, j, l, \tag{24}$$

$$b_{jk} = b_{jk}(h_{jk}), \quad \forall j, k, \tag{25}$$

with $\hat{b}_{ijl}$ denoting the cost function associated with the relationship that retailer j associates with manufacturer i transacting via mode l and b_{jk} denotes the analogous cost function but associated with retailer j and demand market k.

As discussed in Nagurney et al. (2002b), a retailer is faced with a handling cost which may include, for example, the display and storage cost for the product. We denote this cost by c_j for retailer j and assume, for the sake of generality and to model competition, that this function may depend, in general, on the amounts of product transacted between all manufacturer/retailer/mode combinations, that is,

$$c_j = c_j(Q^1), \quad \forall j. \tag{26}$$

A retailer also faces some transaction costs. The transaction cost associated with retailer j transacting with manufacturer i using mode l is denoted by $\hat{c}_{ijl}$. It is assumed that the function depends upon the amount of the product transacted by the manufacturer/retailer pair via the mode and on the relationship level established between the pair, that is,

$$\hat{c}_{ijl} = \hat{c}_{ijl}(q_{ijl}, h_{ijl}), \quad \forall i, j, l. \tag{27}$$

Let q_{jk} denote the nonnegative amount of the product transacted by consumers located at demand market k from retailer j. We group all such product transactions between the retailers and demand markets into the column vector $Q^3 \in R^{no}_+$. These product transactions correspond to the flows on the links in the supply chain network in Figure 1 between the retailer nodes and the demand market nodes. The cost associated with transacting between retailer j and demand market k from the perspective of the retailer is denoted by c_{jk} and is expressed as

$$c_{jk} = \ c_{jk}(q_{jk}, h_{jk}), \quad \forall j, k. \tag{28}$$

We assume that the above cost functions (25) through (28) are all convex and continuously differentiable.

A retailer j associates a price of the product at his retail outlet, which is denoted by ρ^*_{2j}. This price, as will be shown, will also be endogenously determined in the model and will be, given a positive volume of transaction between a retailer and a demand market, related to a clearing-type price. Assuming that the retailers are profit-maximizers the criterion of profit maximization for retailer j is given by:

$$\text{Maximize} \quad \rho_{2j}^* \sum_{k=1}^{o} q_{jk} - c_j(Q^1) - \sum_{i=1}^{m}\sum_{l=1}^{2} \hat{c}_{ijl}(q_{ijl}, h_{ijl}) - \sum_{k=1}^{o} c_{jk}(q_{jk}, h_{jk}) - \sum_{i=1}^{m}\sum_{l=1}^{2} \rho_{1ijl}^* q_{ijl}$$

$$-\sum_{i=1}^{m}\sum_{l=1}^{2} \hat{b}_{ijl}(h_{ijl}) - \sum_{k=1}^{o} b_{jk}(h_{jk}) \tag{29}$$

subject to:

$$\sum_{k=1}^{o} q_{jk} \leq \sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl} \tag{30}$$

and the nonnegativity constraints:

$$q_{ijl} \geq 0, \quad \forall i, l, \quad q_{jk} \geq 0, \quad \forall k, \tag{31}$$

$$0 \leq h_{ijl} \leq 1, \quad \forall i, l, \quad 0 \leq h_{jk} \leq 1, \quad \forall k. \tag{32}$$

Objective function (29) expresses that the difference between the revenue minus the handling cost and the transaction costs in dealing with manufacturers and the demand markets and the costs for establishing relationship levels with manufacturers and demand markets and the payout to the manufacturers should be maximized. Constraint (30) states that consumers cannot purchase more from a retailer than is held in stock.

In addition to the criterion of profit maximization, we also assume that each retailer is concerned with risk minimization associated with dealing with the manufacturer and the demand markets. Here, for the sake of generality, we assume, as given, a risk function $\hat{r}_{ijl}$, for retailer j in dealing with manufacturer i through mode l, which is assumed to be continuous and convex and a function of both the amount of product transacted with the particular manufacturer and the relationship level with this manufacturer given by

$$\hat{r}_{ijl} = \hat{r}_{ijl}(q_{ijl}, h_{ijl}), \quad \forall i, j, l. \tag{33}$$

A risk function r_{jk} for retailer j associated with his transacting with consumers at demand market k is a function of not only the product transactions associated with the particular demand market but also of the relationship level with consumers at that demand market.

A higher relationship level will reduce risk because trust reduces transactional uncertainty. The risk function may be distinct for each retailer/demand market combination. The risk of retailer j associated with dealing with demand market k is expressed as

$$r_{jk} = r_{jk}(q_{jk}, h_{jk}), \quad \forall j, k. \tag{34}$$

A retailer j tries to minimize his total risk, that is, he faces the problem:

$$\text{Minimize} \quad \sum_{i=1}^{m}\sum_{l=1}^{2}\hat{r}_{ijl}(q_{ijl}, h_{ijl}) + \sum_{k=1}^{o} r_{jk}(q_{jk}, h_{jk}) \tag{35}$$

subject to:

$$q_{ijl} \geq 0, \quad \forall i, l, \quad q_{jk} \geq 0, \quad \forall k, \tag{36}$$

$$0 \leq h_{ijl} \leq 1, \quad \forall i, l, \quad 0 \leq h_{jk} \leq 1, \quad \forall k. \tag{37}$$

Finally, the retailer j also tries to maximize the relationship value with manufacturers and demand markets. Again, we assume, as given, a relationship value function $\hat{v}_{ijl}$, for retailer j in dealing with manufacturer i through transaction mode l, which is assumed to be continuously differentiable and concave and a function of the relationship level with the manufacturer given by

$$\hat{v}_{ijl} = \hat{v}_{ijl}(h_{ijl}), \quad \forall i, j, l. \tag{38}$$

A relationship function v_{jk} for retailer j associated with his transacting with consumers at demand market k is assumed to be continuously differentiable and concave and a function of the relationship level with consumers at that demand market. Again, a very simple example of a relationship value function would be the sum of all the relationship levels. The relationship value of retailer j associated with dealing with demand market k is expressed as

$$v_{jk} = v_{jk}(h_{jk}), \quad \forall j, k. \tag{39}$$

Hence, a retailer j also tries to maximize the total relationship value, that is, he faces the problem:

$$\text{Maximize} \quad \sum_{i=1}^{m}\sum_{l=1}^{2}\hat{v}_{ijl}(h_{ijl}) + \sum_{k=1}^{o} v_{jk}(h_{jk}) \tag{40}$$

subject to:

$$0 \leq h_{ijl} \leq 1, \quad \forall i, l, \quad 0 \leq h_{jk} \leq 1, \quad \forall k. \tag{41}$$

A Retailer's Multicriteria Decision-Making Problem

Retailer j assigns a nonnegative weight δ_j with the risk generated and a nonnegative weight γ_j with the relationship value. The weight associated with profit maximization is set equal to 1 and serves as the numeraire. This yields the following multicriteria decision-making problem for retailer j:

$$\text{Maximize} \quad \rho^*_{2j} \sum_{k=1}^{o} q_{jk} - c_j(Q^1) - \sum_{i=1}^{m} \sum_{l=1}^{2} \hat{c}_{ijl}(q_{ijl}, h_{ijl}) - \sum_{k=1}^{o} \sum_{l=1}^{2} c_{jk}(q_{jk}, h_{jk}) - \sum_{i=1}^{m} \sum_{l=1}^{2} \rho^*_{1ijl} q_{ijl}$$

$$- \sum_{i=1}^{m} \sum_{l=1}^{2} \hat{b}_{ijl}(h_{ijl}) - \sum_{k=1}^{o} b_{jk}(h_{jk}) - \delta_j (\sum_{i=1}^{m} \sum_{l=1}^{2} \hat{r}_{ijl}(q_{ijl}, h_{ijl}) + \sum_{k=1}^{o} r_{jk}(q_{jk}, h_{jk}))$$

$$+ \gamma_j (\sum_{i=1}^{m} \sum_{l=1}^{2} \hat{v}_{ijl}(h_{ijl}) + \sum_{k=1}^{o} v_{jk}(h_{jk})) \tag{42}$$

subject to:

$$\sum_{k=1}^{o} q_{jk} \leq \sum_{i=1}^{m} \sum_{l=1}^{2} q_{ijl} \tag{43}$$

$$q_{ijl} \geq 0 \quad \forall i, l, \quad q_{jk} \geq 0, \quad \forall k, \tag{44}$$

$$0 \leq h_{ijl} \leq 1, \quad \forall i, l, \quad 0 \leq h_{jk} \leq 1, \quad \forall k. \tag{45}$$

The Optimality Conditions of Retailers

Now we turn to the optimality conditions of the retailers. Each retailer faces the multicriteria decision-making problem (42), subject to (43), the nonnegativity assumption on the variables (44), and the assumptions for the relationship values (45). As in the case of manufacturers, we assume that the retailers compete in a noncooperative manner, given the actions of the other retailers. Retailers seek to determine the optimal transactions associated with the demand markets and with the manufacturers. In equilibrium, all the transactions between the tiers of network decision-makers will have to coincide, as we will see later in this section.

If one assumes that the handling, transaction cost, risk functions are continuously differentiable and convex, and that the relationship values are also continuously differentiable but concave, then the optimality conditions for all the retailers satisfy the variational inequality: determine $(Q^{1*}, Q^{3*}, h^{1*}, h^{3*}, \epsilon^*) \in \mathcal{K}_2$, such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[\frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \delta_j \frac{\partial \hat{r}_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial q_{ijl}} + \rho^*_{1ijl} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial q_{ijl}} - \epsilon^*_j\right] \times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\frac{\partial c_{jk}(q^*_{jk}, h^*_{jk})}{\partial q_{jk}} - \rho^*_{2j} + \epsilon^*_j + \delta_j \frac{\partial r_{jk}(q^*_{jk}, h^*_{jk})}{\partial q_{jk}}\right] \times \left[q_{jk} - q^*_{jk}\right]$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[\frac{\partial \hat{c}_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial h_{ijl}} - \gamma_j \frac{\partial \hat{v}_{ijl}(h^*_{ijl})}{\partial h_{ijl}} + \delta_j \frac{\partial \hat{r}_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial h_{ijl}} + \frac{\partial \hat{b}_{ijl}(h^*_{ijl})}{\partial h_{ijl}}\right] \times \left[h_{ijl} - h^*_{ijl}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\frac{\partial c_{jk}(q^*_{jk}, h^*_{jk})}{\partial h_{jk}} - \gamma_j \frac{\partial v_{jk}(h^*_{jk})}{\partial h_{jk}} + \delta_j \frac{\partial r_{jk}(q^*_{jk}, h^*_{jk})}{\partial h_{jk}} + \frac{\partial b_{jk}(h^*_{jk})}{\partial h_{jk}}\right] \times \left[h_{jk} - h^*_{jk}\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2} q^*_{ijl} - \sum_{k=1}^{o} q^*_{jk}\right] \times \left[\epsilon_j - \epsilon^*_j\right] \geq 0, \quad \forall (Q^1, Q^3, h^1, h^3, \epsilon) \in \mathcal{K}_2, \qquad (46)$$

where

$$\mathcal{K}_2 \equiv \left[(Q^1, Q^3, h^1, h^3, \epsilon) \mid q_{ijl} \geq 0, q_{jk} \geq 0, 0 \leq h_{ijl} \leq 1, 0 \leq h_{jk} \leq 1, \epsilon_j \geq 0, \; \forall i, j, l, k\right]. \quad (47)$$

Here ϵ_j is the Lagrange multiplier associated with constraint (43) for retailer j and ϵ denotes the column vector of all the retailers' multipliers, with * denoting the optimal value. Note that these Lagrange multipliers also have an interpretation as shadow prices.

The economic interpretation of the retailers' optimality conditions is very interesting. The first term in (46) shows that if there is a positive amount of product transacted between a manufacturer/retailer pair via a mode, that is, $q^*_{ijkl} > 0$, then the shadow price at the retailer, ϵ^*_j is equal to the price charged for the product plus the various marginal costs and the weighted marginal risk associated. In addition, the second term in (46), shows that, if consumers at demand market k purchase the product from a particular retailer j, which means, if the q^*_{jk} is positive, then the price charged by retailer j, ρ^*_{2j}, is equal to ϵ^*_j plus the marginal transaction costs in dealing with the demand market plus the weighted marginal costs for the risk that he has to bear. ϵ^*_j serves as the price to clear the market at retailer j, as we can see from the fifth term. One also obtains interpretations from (46) as to the economic conditions at which the relationship levels associated with retailers interacting with either the manufacturers or the demand markets will take on positive values.

The Consumers at the Demand Markets

When consumers make their decisions they do not only take into account the price charged for the product by the retailers and the manufacturers but also the transaction costs associated with obtaining the product.

As we noted earlier, the consumers at the demand markets can transact either with manufacturers through the Internet or physically with the retailers. The transaction cost associated with obtaining the product by consumers at demand market k from retailer j is denoted by $\hat{c}_{jk}$. It depends on the amount of the product transacted and the relationship level between retailer j and consumers at demand market k. It is assumed that the transaction cost is continuous and of the general form:

$$\hat{c}_{jk} = \hat{c}_{jk}(q_{jk}, h_{jk}), \quad \forall j, k. \tag{48}$$

Furthermore, let $\hat{c}_{ik}$ denote the transaction cost, from the perspective of the consumers at demand market k, associated with manufacturer i:

$$\hat{c}_{ik} = \hat{c}_{ik}(q_{ik}, h_{ik}), \quad \forall i, k. \tag{49}$$

Therefore, the cost of conducting a transaction with a manufacturer via the Internet depends on the volume of the product transacted via the Internet as well as on the relationship level between demand market k and manufacturer i.

Let ρ_{3k} denote the price of the product at demand market k. The demand for the product at demand market k is denoted by d_k. The demand functions are assumed to be continuous and given by:

$$d_k = d_k(\rho_3), \quad \forall k, \tag{50}$$

where $\rho_3 \in R^o_+$ is the o-dimensional column vector of nonnegative prices. The demand for the product at a demand market depends, in general, not only on the price of the product at that demand market but also on the prices of the product at the other demand markets.

The price charged by the retailers for the product, denoted by ρ_{2j}^*, plus the transaction cost associated with obtaining the product are considered by the consumers when they make their purchase decisions. They also take the prices charged by the producers, ρ_{1ik}^*, plus the associated transaction costs into consideration in making their buying decisions.

The Equilibrium Conditions for the Demand Markets

Hence, the equilibrium conditions for consumers at demand market k, take the form: for all retailers: j; $j = 1, ..., n$:

$$\rho_{2j}^* + \hat{c}_{jk}(q_{jk}^*, h_{jk}^*) \begin{cases} = \rho_{3k}^*, & \text{if} \quad q_{jk}^* > 0 \\ \geq \rho_{3k}^*, & \text{if} \quad q_{jk}^* = 0, \end{cases} \tag{51}$$

and for all manufacturers i; $i = 1, \ldots, m$:

$$\rho_{1ik}^* + \hat{c}_{ik}(q_{ik}^*, h_{ik}^*) \begin{cases} = \rho_{3k}^*, & \text{if} \quad q_{ik}^* > 0 \\ \geq \rho_{3k}^*, & \text{if} \quad q_{ik}^* = 0, \end{cases} \tag{52}$$

and

$$d_k(\rho_3^*) \begin{cases} = \sum_{j=1}^n q_{jk}^* + \sum_{i=1}^m q_{ik}^*, & \text{if} \quad \rho_{3k}^* > 0 \\ \leq \sum_{j=1}^n q_{jk}^* + \sum_{i=1}^m q_{ik}^*, & \text{if} \quad \rho_{3k}^* = 0. \end{cases} \tag{53}$$

Conditions (51) state that consumers at demand market k will purchase the product from retailer j, if the price charged by the retailer for the product plus the transaction cost does not exceed the price that the consumers are willing to pay for the product. Conditions (52) state that the equivalent holds for the manufacturers and the demand market. Furthermore, conditions (53), express that, if the price the consumers are willing to pay for the product at a demand market is positive, then the quantity purchased/consumed by the consumers at the demand market is precisely equal to the demand. These conditions correspond to the well-known spatial price equilibrium conditions (cf. Takayama and Judge (1971), Nagurney (1999), and the references therein).

In equilibrium, conditions (51) – (53) will have to hold for all demand markets k. These can be expressed as the inequality problem given by: determine $(Q^{2*}, Q^{3*}, \rho_3^*) \in R_+^{mo+no+o}$, such that

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\rho_{2j}^* + \hat{c}_{jk}(q_{jk}^*, h_{jk}^*) - \rho_{3k}^*\right] \times \left[q_{jk} - q_{jk}^*\right] + \sum_{i=1}^{m}\sum_{k=1}^{o}[\rho_{1ik}^* + \hat{c}_{ik}(q_{ik}^*, h_{ik}^*) - \rho_{3k}^*] \times [q_{ik} - q_{ik}^*]$$

$$+\sum_{k=1}^{o}\left[\sum_{j=1}^{n} q_{jk}^* + \sum_{i=1}^{m} q_{ik}^* - d_k(\rho_3^*)\right] \times [\rho_{3k} - \rho_{3k}^*] \geq 0, \tag{54}$$

$$\forall (Q^2, Q^3, \rho_3) \in R_+^{mo+no+o}. \tag{55}$$

In the context of the consumption decisions, we have utilized demand functions, whereas profit functions, which correspond to objective functions, were used in the case of the manufacturers and the retailers. Since we can expect the number of consumers to be much greater than that of the manufacturers and retailers we believe that such a formulation is more natural. Also, note that the relationship levels in (54) are assumed as given. They are endogenous to the integrated model as is soon revealed.

The Equilibrium Conditions of the Supernetwork

The equilibrium transaction, relationship level, and price pattern must satisfy the sum of the optimality conditions (22) and (46), and the conditions (54), in order to formalize the agreements between the tiers of the supply chain and social networks (see also Nagurney et al. (2002a, b)). In equilibrium, the transactions that the manufacturers make with the retailers must be equal to the transactions that the retailers accept from the manufacturers. Furthermore, the amounts of the product purchased by the demand markets must be equal to the amounts sold to them by the retailers and the manufacturers. Moreover, the relationship levels between buyers and sellers must coincide. We, hence, have the following:

Definition 1: Supernetwork Equilibrium

The equilibrium state of the supernetwork is one where the flows between the tiers of the supernetwork coincide and the product transactions, relationship levels, and prices satisfy the sum of the optimality conditions (22) and (46), and the equilibrium conditions (54).

We now give the variational inequality formulation of the equilibrium.

Theorem 1: Variational Inequality Formulation

The equilibrium conditions governing the supernetwork model are equivalent to the solution of the variational inequality problem given by: determine $(Q^{1*}, Q^{2*}, Q^{3*}, h^{1*}, h^{2*}, h^{3*}, \epsilon^*, \rho_3^*)$

$\in \mathcal{K}$ *satisfying*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\Big[\frac{\partial f_i(Q^{1*},Q^{2*})}{\partial q_{ijl}}+\frac{\partial c_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial q_{ijl}}+\frac{\partial c_j(Q^{1*})}{\partial q_{ijl}}+\frac{\partial \hat{c}_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial q_{ijl}}+\alpha_i\frac{\partial r_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial q_{ijl}}$$

$$+\delta_j\frac{\partial \hat{r}_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial q_{ijl}}-\epsilon^*_j\Big]\times\Big[q_{ijl}-q^*_{ijl}\Big]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\frac{\partial f_i(Q^{1*},Q^{2*})}{\partial q_{ik}}+\frac{\partial c_{ik}(q^*_{ik},h^*_{ik})}{\partial q_{ik}}+\hat{c}_{ik}(q^*_{ik},h^*_{ik})+\alpha_i\frac{\partial r_{ik}(q^*_{ik},h^*_{ik})}{\partial q_{ik}}-\rho^*_{3k}\right]\times[q_{ik}-q^*_{ik}]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\frac{\partial c_{jk}(q^*_{jk},h^*_{jk})}{\partial q_{jk}}+\hat{c}_{jk}(q^*_{jk},h^*_{jk})+\epsilon^*_j+\delta_j\frac{\partial r_{jk}(q^*_{jk},h^*_{jk})}{\partial q_{jk}}-\rho^*_{3k}\right]\times\Big[q_{jk}-q^*_{jk}\Big]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\frac{\partial c_{jk}(q^*_{jk},h^*_{jk})}{\partial h_{jk}}-\gamma_j\frac{\partial v_{jk}(h^*_{jk})}{\partial h_{jk}}+\delta_j\frac{\partial r_{jk}(q^*_{jk},h^*_{jk})}{\partial h_{jk}}+\frac{\partial b_{jk}(h^*_{jk})}{\partial h_{jk}}\right]\times\Big[h_{jk}-h^*_{jk}\Big]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\frac{\partial c_{ik}(q^*_{ik},h^*_{ik})}{\partial h_{ik}}-\beta_i\frac{\partial v_{ik}(h^*_{ik})}{\partial h_{ik}}+\alpha_i\frac{\partial r_{ik}(q^*_{ik},h^*_{ik})}{\partial h_{ik}}+\frac{\partial b_{ik}(h^*_{ik})}{\partial h_{ik}}\right]\times[h_{ik}-h^*_{ik}]$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\Big[\frac{\partial c_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial h_{ijl}}+\frac{\partial \hat{c}_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial h_{ijl}}-\beta_i\frac{\partial v_{ijl}(h^*_{ijl})}{\partial h_{ijl}}-\gamma_j\frac{\partial \hat{v}_{ijl}(h^*_{ijl})}{\partial h_{ijl}}+\alpha_i\frac{\partial r_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial h_{ijl}}$$

$$+\delta_j\frac{\partial \hat{r}_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial h_{ijl}}+\frac{\partial b_{ijl}(h^*_{ijl})}{\partial h_{ijl}}+\frac{\partial \hat{b}_{ijl}(h^*_{ijl})}{\partial h_{ijl}}\Big]\times\Big[h_{ijl}-h^*_{ijl}\Big]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2}q^*_{ijl}-\sum_{k=1}^{o}q^*_{jk}\right]\times\Big[\epsilon_j-\epsilon^*_j\Big]+\sum_{k=1}^{o}\left[\sum_{j=1}^{n}q^*_{jk}+\sum_{i=1}^{m}q^*_{ik}-d_k(\rho^*_3)\right]\times[\rho_{3k}-\rho^*_{3k}]\geq 0,$$

$$\forall(Q^1,Q^2,Q^3,h^1,h^2,h^3,\epsilon,\rho_3)\in\mathcal{K}, \qquad (56)$$

where

$$\mathcal{K}\equiv\Big[(Q^1,Q^2,Q^3,h^1,h^2,h^3,\epsilon,\rho_3)|q_{ijl}\geq 0,q_{ik}\geq 0,q_{jk}\geq 0,0\leq h_{ij}\leq 1,0\leq h_{ik}\leq 1,$$

$$0\leq h_{jk}\leq 1,\epsilon_j\geq 0,\rho_{3k}\geq 0,\forall i,j,l,k\Big]. \qquad (57)$$

Proof: Follows using similar arguments as the proof of Theorem 1 in Nagurney et al. (2002b). □

Remark

In Figure 2 we show the supernetwork at equilibrium. Note that the equilibrium product transactions, relationship levels, and prices now appear on the supernetwork. Of course, if certain equilibrium relationship levels or product transactions associated with particular links are precisely equal to zero, one can remove the corresponding links from the supernetwork. Moreover, the size of the flows, in effect, provides us with the strength of the respective links and gives us information which is not obtainable through standard social network analysis. Hence, the supernetwork model developed here, in effect, also provides us with the optimal integrated supply chain and social network designs.

We now discuss how to recover the prices: ρ^*_{2j}, for all j; ρ^*_{1ijl}, for all i, j, l, and ρ^*_{1ik}, for all i, k, from the solution of variational inequality (56). In Section 4 we describe an algorithm for the computation of the solution $(Q^{1*}, Q^{2*}, Q^{3*}, h^{1*}, h^{2*}, h^3, \epsilon^*, \rho^*_3)$.

Recall that, in the preceding discussions, we have noted that if $q^*_{jk} > 0$, for some j and k, then ρ^*_{2j} is precisely equal to $\frac{\partial c_{jk}(q^*_{jk},h^*_{jk})}{\partial q_{jk}} + \epsilon^*_j + \delta_j \frac{\partial r_{jk}(q^*_{jk},h^*_{jk})}{\partial q_{jk}}$, with ϵ^*_j being obtained from the solution of (56). The prices ρ^*_{1ijl}, in turn (cf. also (22)), can be obtained by finding a $q^*_{ijl} > 0$, and then setting $\rho^*_{1ijl} = \left[\frac{\partial f_i(Q^{1*},Q^{2*})}{\partial q_{ijl}} + \frac{\partial c_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial q_{ijl}} + \alpha_i \frac{\partial r_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial q_{ijl}}\right]$, or, equivalently (see (46)), to $\left[-\frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} - \delta_j \frac{\partial \hat{r}_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial q_{ijl}} - \frac{\partial \hat{c}_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial q_{ijl}} + \epsilon^*_j\right]$, for all such i, j, l.

The prices ρ^*_{1ik}, on the other hand, can be obtained (see (22)) by finding a $q^*_{ik} > 0$ and setting $\rho^*_{1ik} = \left[\frac{\partial f_i(Q^{1*},Q^{2*})}{\partial q_{ik}} + \frac{\partial c_{ik}(q^*_{ik},h^*_{ik})}{\partial q_{ik}} + \alpha_i \frac{\partial r_{ik}(q^*_{ik},h^*_{ik})}{\partial q_{ik}}\right]$, or, equivalently (cf. (54)), to $[-\hat{c}_{ik}(q^*_{ik}, h^*_{ik}) + \rho^*_{3k}]$, for all such i, k.

What is important to realize (and can be easily proved) is that under the above pricing mechanism, the optimality conditions (22) and (46) as well as the equilibrium conditions (54) also each hold separately.

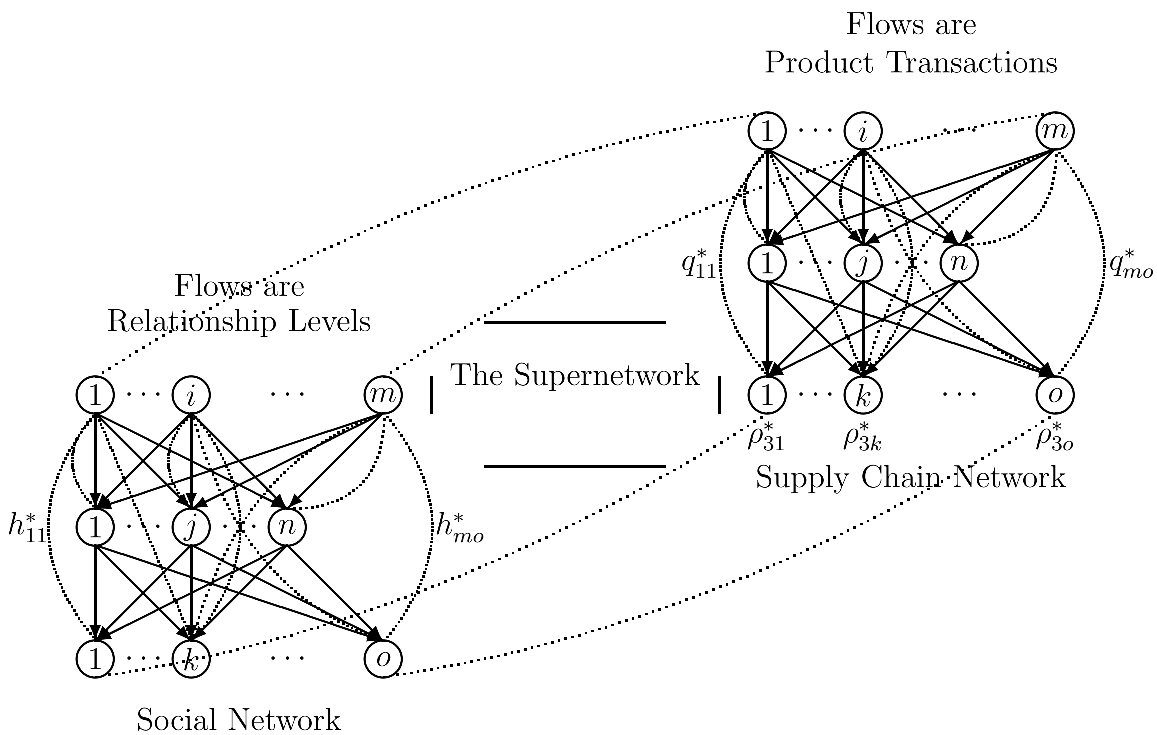

Figure 2: The Supernetwork at Equilibrium

For easy reference, and use in the subsequent sections, variational inequality problem (56) can be rewritten in standard variational inequality form (cf. Nagurney (1999)) as follows:

$$\langle F(X^*)^T, X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{58}$$

where $X \equiv (Q^1, Q^2, Q^3, h^1, h^2, h^3, \epsilon, \rho_3)$, and

$$F(X) \equiv (F_{ijl}, F_{ik}, F_{jk}, \hat{F}_{ijl}, \hat{F}_{ik}, \hat{F}_{jk}, F_j, F_k)_{i=1,\ldots,m; j=1,\ldots,n; l=1,2; k=1,\ldots,o}$$

and the specific components of F given by the functional terms preceding the multiplication signs in (56), respectively. The term $\langle \cdot, \cdot \rangle$ denotes the inner product in N-dimensional Euclidean space.

3. The Disequilibrium Dynamics

In this section, we utilize the definitions of X and $F(X)$ given in Section 2 to present a dynamic supernetwork model which integrates the social network and the supply chain network and describes the time evolution of the product transactions, the relationship levels, as well as the prices over the supernetwork of Figure 1 until the equilibrium pattern (cf. Figure 2) is achieved. Importantly, the dynamic model is formulated as a projected dynamical system whose set of stationary points coincides with the set of solutions of the variational inequality problem (58), which, in turn, are equilibria of the supernetwork, according to Theorem 1. Hence, here we provide the disequilibrium dynamics of the flows on the links on the supernetwork as well as the prices associated with the demand markets and the shadow prices at the retailer level.

In particular, we consider the projected dynamical system (cf. Nagurney and Zhang (1996) and Nagurney et al. (2002a, b)) defined by the initial value problem:

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X_0, \tag{59}$$

where $\Pi_{\mathcal{K}}$ denotes the projection of $-F(X)$ onto $\mathcal{K}$ at X and X_0 is equal to the point corresponding to the initial product transactions, relationship levels, shadow prices, and demand market prices.

The trajectory of (59) describes the dynamic evolution of the product transactions on the supply chain network as well as the relationship levels on the social network along with the demand market prices and, of course, the Lagrange multipliers or shadow prices associated with the retailers. From (56), (58), and (59) we can observe the following. Beginning with the demand market nodes at the bottom of the supply chain network, we note that the demand market prices evolve according to the difference between the demand at the market (as a function of the prices at the demand markets at that time) and the amount of the product transactions with the projection operator guaranteeing that the prices do not take on negative values. Similarly, the Lagrange multipliers/shadow prices associated with the retailers (and, hence, the middle tier of nodes on the supply chain level of the supernetwork) evolve according to the difference between the sum of the product transacted with the demand markets and that obtained from the manufacturers. Again, the projection operation guarantees that these prices do not become negative.

The relationship levels, in turn, evolve on the social network level of the supernetwork according to the difference of the sum of the corresponding weighted value functions and the sum of the various marginal transaction cost and weighted marginal risk functions. Finally, the product transactions evolve on the supply chain network links according to the difference between the characteristic price (either the shadow price or the demand market price) and the marginal production cost and various marginal transaction and handling costs plus the weighted marginal risk cost functions (associated with the particular transaction). These flows are also guaranteed to not assume negative values due to the projection operation.

We emphasize that the projection operation in this context is very simple since the constraint set consists of simply nonnegativity constraints and the box-type constraints on the relationship levels.

The following theorem provides the crucial linkage between the set of stationary points of the projected dynamical system (59) and the set of solutions of variational inequality (56).

Theorem 2: The Set of Stationary Points of the Projected Dynamical System Coincides with the Set of Solutions to the Variational Inequality Problem

The set of stationary points of the projected dynamical system (59) coincides with the set of solutions of variational inequality (56).

Proof: According to Dupuis and Nagurney (1993), the necessary and sufficient condition for X^* to be a stationary point of the projected dynamical system (59), that is, to satisfy:

$$\dot{X} = 0 = \Pi_{\mathcal{K}}(X^*, -F(X^*)), \tag{60}$$

is that $X^* \in \mathcal{K}$ solves the variational inequality problem:

$$\langle F(X^*)^T, X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{61}$$

where, in our problem, X and $F(X)$ are defined following (58). But variational inequality (59) is precisely (56) which, in turn, according to Theorem 1 coincides with $(Q^{1*}, Q^{2*}, Q^{3^*}, h^{1*}, h^{2*}, h^{3*}, \epsilon^*, \rho_3^*)$ being an equilibrium pattern according to Definition 1. □

Hence, once a stationary point of the dynamic supernetwork model is reached that point satisfies the equilibrium conditions of the supernetwork at which the manufacturers, retailers, and demand markets have formalized their agreements and the product transactions and relationship levels between tiers of the supply chain network and the social network, respectively, coincide.

We now state the following:

Theorem 3: Existence and Uniqueness of a Solution to the Initial Value Problem

Assume that $F(X)$ is Lipschitz continuous, that is,

$$\|F(X') - F(X'')\| \leq L\|X' - X''\|, \quad \forall X', X'' \in \mathcal{K}, \text{ where } L > 0. \tag{62}$$

Then, for any $X_0 \in \mathcal{K}$, there exists a unique solution $X_0(t)$ to the initial value problem (59).

Proof: Lipschitz continuity of the function F is sufficient for the result following Theorem 2.5 in Nagurney and Zhang (1996). □

A similar result was obtained for a dynamic supply chain network model with electronic commerce in Nagurney et al. (2002b). However that model was not a multicriteria one and, moreover, had no social network component. Theorem 3 is essential since it demonstrates that if the Lipschitz property is satisfied, then the dynamic trajectories are well-defined.

Also, we note that under suitable conditions on the underlying functions (see also Nagurney et al. (2002a, b) and Nagurney and Dong (2002), Zhang and Nagurney (1995, 1996)), one can obtain stability results for the supernetwork.

4. The Algorithm and Numerical Examples

Observe that the projected dynamical system given by (59) is a continuous time adjustment process but for computational purposes, a discrete-time analogue is needed. Here, we propose a discrete-time algorithm, an Euler-type method, which is a special case of the general iterative scheme proposed by Dupuis and Nagurney (1993). Its statement is as follows: At iteration τ compute

$$X_\tau = P_{\mathcal{K}}(X_{\tau-1} - a_{\tau-1}F(X_{\tau-1})), \tag{63}$$

where $P_{\mathcal{K}}$ denotes the projection operator in the Euclidean sense (see Nagurney (1999)) onto the closed convex set $\mathcal{K}$ and $F(X)$ is defined following (58)). The sequence of positive terms $\{a_\tau\}$ is discussed below.

Specifically, the complete statement of the method in the context of the dynamic supernetwork model is as follows:

Step 0: Initialization Step

Set $(Q^1_0, Q^2_0, Q^3_0, h^1_0, h^2_0, h^3_0, \epsilon_0, \rho_{30}) \in \mathcal{K}$ and set the sequence $\{a_\tau\}$ with $a_\tau > 0$ for all τ and $\sum_\tau a_\tau = \infty$, as $\tau \to \infty$. The $\{a_\tau\}$ sequence must satisfy these conditions for convergence (see additional convergence results in Dupuis and Nagurney (1993)).

Step 1: Computation Step

Compute $(Q^{1^\tau}, Q^{2^\tau}, Q^{3^\tau}, h^{1^\tau}, h^{2^\tau}, h^{3^\tau}, \epsilon^\tau, \rho_3^\tau) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\sum_{i=1}^m \sum_{j=1}^n \sum_{l=1}^2 \Bigg[q^\tau_{ijl} + a_\tau \Big(\frac{\partial f_i(Q^{1^{\tau-1}}, Q^{2^{\tau-1}})}{\partial q_{ijl}} + \frac{\partial c_{ijl}(q^{\tau-1}_{ijl}, h^{\tau-1}_{ijl})}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1^{\tau-1}})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q^{\tau-1}_{ijl}, h^{\tau-1}_{ijl})}{\partial q_{ijl}}$$

$$+\alpha_i \frac{\partial r_{ijl}(q^{\tau-1}_{ijl}, h^{\tau-1}_{ijl})}{\partial q_{ijl}} + \delta_j \frac{\partial \hat{r}_{ijl}(q^{\tau-1}_{ijl}, h^{\tau-1}_{ijl})}{\partial q_{ijl}} - \epsilon^{\tau-1}_j \Big) - q^{\tau-1}_{ijl} \Bigg] \times \Big[q_{ijl} - q^\tau_{ijl} \Big]$$

$$+ \sum_{i=1}^m \sum_{k=1}^o \Bigg[q^\tau_{ik} + a_\tau \Big(\frac{\partial f_i(Q^{1^{\tau-1}}, Q^{2^{\tau-1}})}{\partial q_{ik}} + \frac{\partial c_{ik}(q^{\tau-1}_{ik}, h^{\tau-1}_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(q^{\tau-1}_{ik}, h^{\tau-1}_{ik}) + \alpha_i \frac{\partial r_{ik}(q^{\tau-1}_{ik}, h^{\tau-1}_{ik})}{\partial q_{ik}}$$

$$-\rho^{\tau-1}_{3k} \Big) - q^{\tau-1}_{ik} \Bigg] \times [q_{ik} - q^\tau_{ik}]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[q_{jk}^{\tau}+a_{\tau}(\frac{\partial c_{jk}(q_{jk}^{\tau-1},h_{jk}^{\tau-1})}{\partial q_{jk}}+\hat{c}_{jk}(q_{jk}^{\tau-1},h_{jk}^{\tau-1})+\epsilon_{j}^{\tau-1}+\delta_{j}\frac{\partial r_{jk}(q_{jk}^{\tau-1},h_{jk}^{\tau-1})}{\partial q_{jk}}-\rho_{3k}^{\tau-1})-q_{jk}^{\tau-1}\right]$$

$$\times\left[q_{jk}-q_{jk}^{\tau}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[h_{jk}^{\tau}+a_{\tau}(\frac{\partial c_{jk}(q_{jk}^{\tau-1},h_{jk}^{\tau-1})}{\partial h_{jk}}-\gamma_{j}\frac{\partial v_{jk}(h_{jk}^{\tau-1})}{\partial h_{jk}}+\delta_{j}\frac{\partial r_{jk}(q_{jk}^{\tau-1},h_{jk}^{\tau-1})}{\partial h_{jk}}+\frac{\partial b_{jk}(h_{jk}^{\tau-1})}{\partial h_{jk}})-h_{jk}^{\tau-1}\right]$$

$$\times\left[h_{jk}-h_{jk}^{\tau}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[h_{ik}^{\tau}+a_{\tau}(\frac{\partial c_{ik}(q_{ik}^{\tau-1},h_{ik}^{\tau-1})}{\partial h_{ik}}-\beta_{i}\frac{\partial v_{ik}(h_{ik}^{\tau-1})}{\partial h_{ik}}+\alpha_{i}\frac{\partial r_{ik}(q_{ik}^{\tau-1},h_{ik}^{\tau-1})}{\partial h_{ik}}+\frac{\partial b_{ik}(h_{ik}^{\tau-1})}{\partial h_{ik}})-h_{ik}^{\tau-1}\right]$$

$$\times\left[h_{ik}-h_{ik}^{\tau}\right]$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[h_{ijl}^{\tau}+a_{\tau}(\frac{\partial c_{ijl}(q_{ijl}^{\tau-1},h_{ijl}^{\tau-1})}{\partial h_{ijl}}+\frac{\partial \hat{c}_{ijl}(q_{ijl}^{\tau-1},h_{ijl}^{\tau-1})}{\partial h_{ijl}}-\beta_{i}\frac{\partial v_{ijl}(h_{ijl}^{\tau-1})}{\partial h_{ijl}}-\gamma_{j}\frac{\partial \hat{v}_{ijl}(h_{ijl}^{\tau-1})}{\partial h_{ijl}}+\right.$$

$$\left.\alpha_{i}\frac{\partial r_{ijl}(q_{ijl}^{\tau-1},h_{ijl}^{\tau-1})}{\partial h_{ijl}}+\delta_{j}\frac{\partial \hat{r}_{ijl}(q_{ijl}^{\tau-1},h_{ijl}^{\tau-1})}{\partial h_{ijl}}+\frac{\partial b_{ijl}(h_{ijl}^{\tau-1})}{\partial h_{ijl}}+\frac{\partial \hat{b}_{ijl}(h_{ijl}^{\tau-1})}{\partial h_{ijl}})-h_{ijl}^{\tau-1}\right]\times\left[h_{ijl}-h_{ijl}^{\tau}\right]$$

$$+\sum_{j=1}^{n}\left[\epsilon_{j}^{\tau}+a_{\tau}(\sum_{i=1}^{m}\sum_{l=1}^{2}q_{ijl}^{\tau-1}-\sum_{k=1}^{o}\ q_{jk}^{\tau-1})-\epsilon_{j}^{\tau-1}\right]\times\left[\epsilon_{j}-\epsilon_{j}^{\tau}\right]$$

$$+\sum_{k=1}^{o}\left[\rho_{3k}^{\tau}+a_{\tau}(\sum_{j=1}^{n}q_{jk}^{\tau-1}+\sum_{i=1}^{m}q_{ik}^{\tau-1}-d_{k}(\rho_{3k}^{\tau-1}))-\rho_{3k}^{\tau-1}\right]\times\left[\rho_{3k}-\rho_{3k}^{\tau}\right]\geq 0,$$

$$\forall(Q^{1},Q^{2},Q^{3},h^{1},h^{2},h^{3},\epsilon,\rho_{3})\in\mathcal{K}. \tag{64}$$

Step 2: Convergence Verification

If $|q_{ijl}^{\tau}-q_{ijl}^{\tau-1}|\leq e$, $|q_{ik}^{\tau}-q_{ik}^{\tau-1}|\leq e$, $|q_{jk}^{\tau}-q_{jk}^{\tau-1}|\leq e$, $|h_{ijl}^{\tau}-h_{ijl}^{\tau-1}|\leq e$, $|h_{ik}^{\tau}-h_{ik}^{\tau-1}|\leq e$, $|h_{jk}^{\tau}-h_{jk}^{\tau-1}|\leq e$, $|\epsilon_{j}^{\tau}-\epsilon_{j}^{\tau-1}|\leq e$, $|\rho_{3k}^{\tau}-\rho_{3k}^{\tau-1}|\leq e$, for all $i=1,\ldots,m$; $j=1,\ldots,n$; $l=1,2$; $k=1,\ldots,o$, with $e>0$, a pre-specified tolerance, then stop; otherwise, set $\tau:=\tau+1$, and go to Step 1.

Due to the simplicity of the feasible set $\mathcal{K}$ the solution of (64) is accomplished exactly and in closed form as follows:

Computation of the Product Transactions

At iteration τ compute the q^{τ}_{ijl}s according to:

$$q^{\tau}_{ijl} = \max\{0, q^{\tau-1}_{ijl} - a_{\tau}(\frac{\partial f_i(Q^{1^{\tau-1}}, Q^{2^{\tau-1}})}{\partial q_{ijl}} + \frac{\partial c_{ijl}(q^{\tau-1}_{ijl}, h^{\tau-1}_{ijl})}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1^{\tau-1}})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q^{\tau-1}_{ijl}, h^{\tau-1}_{ijl})}{\partial q_{ijl}}$$

$$+\alpha_i \frac{\partial r_{ijl}(q^{\tau-1}_{ijl}, h^{\tau-1}_{ijl})}{\partial q_{ijl}} + \delta_j \frac{\partial \hat{r}_{ijl}(q^{\tau-1}_{ijl}, h^{\tau-1}_{ijl})}{\partial q_{ijl}} - \epsilon^{\tau-1}_j)\}, \quad \forall i, j, l. \tag{65}$$

At iteration τ compute the q^{τ}_{ik}s according to:

$$q^{\tau}_{ik} = \max\{0, q^{\tau-1}_{ik} - a_{\tau}(\frac{\partial f_i(Q^{1^{\tau-1}}, Q^{2^{\tau-1}})}{\partial q_{ik}} + \frac{\partial c_{ik}(q^{\tau-1}_{ik}, h^{\tau-1}_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(q^{\tau-1}_{ik}, h^{\tau-1}_{ik})$$

$$+\alpha_i \frac{\partial r_{ik}(q^{\tau-1}_{ik}, h^{\tau-1}_{ik})}{\partial q_{ik}} - \rho^{\tau-1}_{3k})\}, \quad \forall i, k. \tag{66}$$

Also, at iteration τ compute the q^{τ}_{jk}s according to:

$$q^{\tau}_{jk} = \max\{0, q^{\tau-1}_{jk} - a_{\tau}(\frac{\partial c_{jk}(q^{\tau-1}_{jk}, h^{\tau-1}_{jk})}{\partial q_{jk}} + \hat{c}_{jk}(q^{\tau-1}_{jk}, h^{\tau-1}_{jk}) + \epsilon^{\tau-1}_j + \delta_j \frac{\partial r_{jk}(q^{\tau-1}_{jk}, h^{\tau-1}_{jk})}{\partial q_{jk}} - \rho^{\tau-1}_{3k})\},$$

$$\forall j, k. \tag{67}$$

Computation of the Relationship Levels

At iteration τ compute the h^{τ}_{ijl}s according to:

$$h^{\tau}_{ijl} = \min\{1, \max\{0, h^{\tau-1}_{ijl} - a_{\tau}(\frac{\partial c_{ijl}(q^{\tau-1}_{ijl}, h^{\tau-1}_{ijl})}{\partial h_{ijl}} + \frac{\partial \hat{c}_{ijl}(q^{\tau-1}_{ijl}, h^{\tau-1}_{ijl})}{\partial h_{ijl}} - \beta_i \frac{\partial v_{ijl}(h^{\tau-1}_{ijl})}{\partial h_{ijl}} -$$

$$\gamma_j \frac{\partial \hat{v}_{ijl}(h^{\tau-1}_{ijl})}{\partial h_{ijl}} + \alpha_i \frac{\partial r_{ijl}(q^{\tau-1}_{ijl}, h^{\tau-1}_{ijl})}{\partial h_{ijl}} + \delta_j \frac{\partial \hat{r}_{ijl}(q^{\tau-1}_{ijl}, h^{\tau-1}_{ijl})}{\partial h_{ijl}} + \frac{\partial b_{ijl}(h^{\tau-1}_{ijl})}{\partial h_{ijl}} + \frac{\partial \hat{b}_{ijl}(h^{\tau-1}_{ijl})}{\partial h_{ijl}})\}\}, \quad \forall i, j, l. \tag{68}$$

At iteration τ compute the h^{τ}_{ik}s according to:

$$h^{\tau}_{ik} = \min\{1, \max\{0, h^{\tau-1}_{ik} - a_{\tau}(\frac{\partial c_{ik}(q^{\tau-1}_{ik}, h^{\tau-1}_{ik})}{\partial h_{ik}} - \beta_i \frac{\partial v_{ik}(h^{\tau-1}_{ik})}{\partial h_{ik}} + \alpha_i \frac{\partial r_{ik}(q^{\tau-1}_{ik}, h^{\tau-1}_{ik})}{\partial h_{ik}}$$

$$+\frac{\partial b_{ik}(h_{ik}^{\tau-1})}{\partial h_{ik}})\}\}, \quad \forall i, k. \tag{69}$$

Also, at iteration τ compute the h_{jk}^{τ}s according to:

$$h_{jk}^{\tau} = \min\{1, \max\{0, h_{jk}^{\tau-1} - a_{\tau}(\frac{\partial c_{jk}(q_{jk}^{\tau-1}, h_{jk}^{\tau-1})}{\partial h_{jk}} - \gamma_j \frac{\partial v_{jk}(h_{jk}^{\tau-1})}{\partial h_{jk}} + \delta_j \frac{\partial r_{jk}(q_{jk}^{\tau-1}, h_{jk}^{\tau-1})}{\partial h_{jk}}$$

$$+\frac{\partial b_{jk}(h_{jk}^{\tau-1})}{\partial h_{jk}})\}\}, \quad \forall j, k. \tag{70}$$

Computation of the Shadow Prices

The shadow prices, ϵ_j^{τ}, in turn, are computed at iteration τ explicitly according to:

$$\epsilon_j^{\tau} = \max\{0, \epsilon_j^{\tau-1} - a_{\tau}(\sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}^{\tau-1} - \sum_{k=1}^{o} q_{jk}^{\tau-1})\}, \quad \forall j. \tag{71}$$

Computation of the Demand Market Prices

The demand market prices, ρ_{3k}^{τ}, are computed according to:

$$\rho_{3k}^{\tau} = \max\{0, \rho_{3k}^{\tau-1} - a_{\tau}(\sum_{j=1}^{n} q_{jk}^{\tau-1} + \sum_{i=1}^{m} q_{ik}^{\tau-1} - d_k(\rho_{3k}^{\tau-1}))\}, \quad \forall k. \tag{72}$$

Now we describe the above discrete-time adjustment process in the context of the multilevel network in Figure 1. The product transactions are computed in the supply chain network of the supernetwork according to (65), (66), and (67). The demand market prices are also computed in the supply chain network according to (72) whereas the shadow prices are computed according to (71). The relationship levels, in turn, are computed in the social network level of the supernetwork as in (68), (69), and (70), respectively.

Equation (65) shows that in order to compute the new product transactions between a manufacturer/retailer pair via a particular mode at an iteration τ, which also can be interpreted as a time period, the information required consists of: the relationship level between the manufacturer and the retailer (via that mode) from the preceding iteration as well as all the product transactions associated with the manufacturers in the preceding time period and the shadow price of the retailer. This type of computation can be done simultaneously for all manufacturer/retailer/mode combinations. According to (66), in order to compute the new product transactions between a manufacturer/demand market pair at a time period, the information required consists of: the relationship level between the manufacturer and the demand market from the preceding iteration as well as all the product transactions associated with the manufacturers and the demand market price at the demand market at the preceding iteration. In the case of a retailer/demand market pair, according to (67), only the product transaction and relationship level associated with this pair in the preceding iteration as well as the shadow price of that retailer and the demand market price of that demand market in the preceding iteration are needed to compute the new product transaction between the retailer/demand market pair. The above-described computations can be done simultaneously for all manufacturer/demand market pairs and for all retailer/demand market pairs, respectively.

For the computation of the shadow prices associated with the second tier of the supply chain component of the supernetwork at a given iteration, for a given retailer, the shadow price of the retailer from the previous period is needed as well as the product transactions to and from that retailer in the previous period (cf. (71)).

For the computation of the demand market price at a given iteration according to (72), in turn, the demand market prices for the product at all the demand markets from the preceding iteration and the product transactions at the preceding iteration from the retailers and the manufacturers to the particular demand markets are necessary (cf. (72)).

Finally, the social network requires at a given iteration (or time period) for the computation of the relationship levels between a manufacturer and retailer via a mode (cf. (68)), the product transactions from the manufacturer to the particular retailer at the preceding iteration, as well as the relationship levels between the manufacturer and the retailer from the preceding iteration, via the specific mode of transaction. This type of computation can also be done simultaneously for all combinations of manufacturer/retailer/modes. The analogues hold true for the manufacturer/demand market (cf. (69)) and the retailer/demand market-pairs (cf. (70)) relationship level computations.

According to the discrete-time adjustment process described above, the process is initialized with a vector of product transactions, relationship levels, and prices. The vector components can all be set to zero, which means that, at the beginning, there are no production transactions, no relationship levels, and the prices for the product are zero. The new product transactions, the new relationship levels, and the new prices, associated with the corresponding links and nodes of the supply chain network, are computed on the supply chain network at each iteration whereas the relationship levels are associated with the links of the social network and updated there at each iteration. The dynamic supernetwork system will then evolve according to the discrete-time adjustment process (65) through (72) until a stationary/equilibrium point is achieved. Moreover, from (65) through (72) one can conclude that once the convergence tolerance has been reached (and, hence, these differences are approximately zero) then the equilibrium conditions according to Definition 1 are satisfied; equivalently, a stationary point of the projected dynamical system (59) and also a solution to variational inequality (56) are achieved.

4.1 Numerical Examples

In this subsection, we apply the Euler-type method to several numerical examples. The algorithm was implemented in FORTRAN and the computer system used was a Sun system located at the University of Massachusetts at Amherst. The convergence criterion used was that the absolute value of the flows and prices between two successive iterations differed by no more than 10^{-4}. For the examples, the sequence $\{a_\tau\}$ was set to $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\}$ in the algorithm since this sequence satisfies the conditions needed for convergence (see also Nagurney and Zhang (1996)). The numerical examples had the supernetwork structure depicted in Figure 3 and consisted of two manufacturers, two retailers, and two demand markets, with both B2B and B2C transactions permitted.

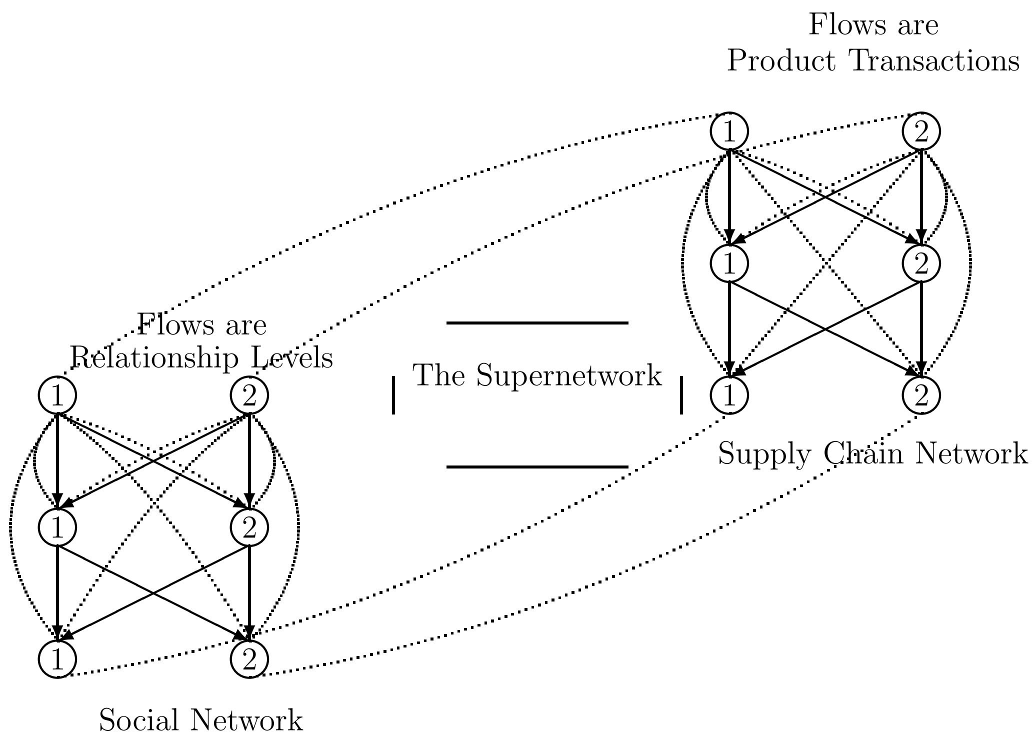

Figure 3: The Supernetwork Structure of the Numerical Examples

Example 1

The data for the first example were constructed for easy interpretation purposes. The production cost functions (cf. (1) and (6)) for the manufacturers were given by:

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 2q_1, \quad f_2(q) = 2.5q_2^2 + q_1q_2 + 2q_2.$$

The transaction cost functions faced by the manufacturers (see (4)) and associated with transacting with the retailers using the physical link, that is, mode 1, were given by:

$$c_{ij1}(q_{ij1}, h_{ij1}) = .5q_{ij1}^2 + 3.5q_{ij1} - h_{ij1}, \quad \forall i, j,$$

whereas the analogous transaction costs, but for mode 2, were given by:

$$c_{ij2}(q_{ij2}, h_{ij2}) = 1.5q_{ij2}^2 + 3q_{ij2} - .5h_{ij2}, \quad \forall i, j.$$

The transaction costs of the manufacturers associated with dealing with the consumers at the demand markets via the Internet (cf. (5)) were given by:

$$c_{ik}(q_{ik}, h_{ik}) = q_{ik}^2 + 2q_{ik} - 2h_{ik}, \quad \forall i, k.$$

The handling costs of the retailers (see (26)), in turn, were given by:

$$c_1(Q^1) = .5(\sum_{i=1}^{2}\sum_{l=1}^{2} q_{i1})^2, \quad c_2(Q^1) = .5(\sum_{i=1}^{2}\sum_{l=1}^{2} q_{i2})^2.$$

The transaction costs of the retailers associated with transacting with the manufacturers via the modes (see (27)) were given by:

$$\hat{c}_{ijl}(q_{ijl}, h_{ijl}) = 1.5q_{ijl}^2 + 3q_{ijl}, \quad \forall i, j, l.$$

The demand functions at the demand markets (see (50)) were:

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1000, \quad d_2(\rho_3) = -2\rho_{32} - 1.5\rho_{31} + 1000,$$

and the transaction costs between the retailers and the consumers at the demand markets (denoted for a typical pair by $\hat{c}_{jk}$ with the associated transaction by q_{jk} (cf. (48)) were given by:

$$\hat{c}_{jk}(q_{jk}, h_{jk}) = q_{jk} - h_{jk} + 5, \quad \forall j, k,$$

whereas the transaction costs associated with transacting via the Internet for the consumers at the demand markets (denoted for a typical such pair by $\hat{c}_{ik}$ with the associated transaction q_{ik} (cf. (49)) were given by:

$$\hat{c}_{ik}(q_{ik}, h_{ik}) = q_{ik} + 1, \quad \forall i, k.$$

The relationship value functions (see (15), (16), and (39)) were given by:

$$v_{ijl}(h_{ijl}) = h_{ijl}, \quad \forall i, j, l; \quad v_{ik}(h_{ik}) = h_{ik}, \quad \forall i, k; \quad v_{jk}(h_{jk}) = h_{jk}, \quad \forall j, k.$$

The relationship cost functions (cf. (2), (3), and (25)), in turn, were given by:

$$b_{ijl}(h_{ijl}) = 2h_{ijl} + 1, \quad \forall i, j, l; \quad b_{ik}(h_{ik}) = h_{ik} + 1, \quad \forall i, k; \quad b_{jk}(h_{jk}) = h_{jk} + 1, \quad \forall j, k.$$

We set all other functions equal to zero. In addition, we assumed that the weights associated with the risk functions were all equal to zero.

In this example, we assigned a weight equal to 1 for all the relationship values for all the manufacturers and retailers.

The Euler method converged in 369 iterations and yielded the following equilibrium pattern: the product transactions between the two manufacturers and the two retailers associated with the physical links, and with the Internet links, respectively, that is, with transacting via mode 1 and mode 2 were:

$$Q^{1^*} := q^*_{111} = q^*_{121} = q^*_{211} = q^*_{221} = 3.4622; \quad q^*_{112} = q^*_{122} = q^*_{212} = q^*_{222} = 2.3914.$$

The product transactions between the two manufacturers and the two demand markets with transactions conducted through the Internet were:

$$Q^{2^*} := q^*_{11} = q^*_{12} = q^*_{21} = q^*_{22} = 13.3016.$$

The product transactions between the two retailers and the two demand markets were:

$$Q^{3*} := q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 5.8521.$$

The vector ϵ^* had components:

$$\epsilon_1^* = \epsilon_2^* = 263.9186,$$

and the demand prices at the demand markets were:

$$\rho_{31}^* = \rho_{32}^* = 274.7686.$$

All the relationship levels h_{ij1}^*, h_{jk}^*, and h_{ik}^*, for all i, j, l, k were identically equal to 0. The relationship levels h_{ij2}^*, however, were all equal to 1. This means that manufacturers established the strongest relationships (via the social networks) with the retailers through the Internet. Hence, in effect, the supernetwork in equilibrium consists of the supply chain network and the links on the social network joining the manufacturers with the retailers through the Internet. In the next example we increase the weight associated with the relationship values associated with the manufacturers.

It is easy to verify that the optimality/equilibrium conditions were satisfied with good accuracy.

Example 2

We then modified Example 1 as follows: The data were identical to that in Example 1 except that we increased the relationship weight β_i from 1 to 10 for the two manufacturers. The Euler method converged in 744 iterations and yielded the following new equilibrium product transaction pattern:

$$Q^{1*} := q_{111}^* = q_{121}^* = q_{211}^* = q_{221}^* = 3.4791; \quad q_{112}^* = q_{122}^* = q_{212}^* = q_{222}^* = 2.4027,$$

and

$$Q^{2*} := q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 13.2790.$$

The computed equilibrium shadow price vector now had components:

$$\epsilon_1^* = \epsilon_2^* = 264.1087,$$

and the new equilibrium demand market prices were:

$$\rho_{31}^* = \rho_{32}^* = 274.7666.$$

The new equilibrium relationship levels were as follows:

$$h_{ijl}^* = 1, \quad \forall i, j, l; \quad h_{jk}^* = .2179, \quad \forall j, k; \quad h_{ik}^* = 0, \quad \forall i, k.$$

With the increase in weights associated with the manufacturers' relationship levels, note that now the relationship levels between manufacturers and the retailers for both modes of transaction were at the highest levels, that is, all were equal to 1. In addition, the relationship levels between retailers and the demand markets increased and this may be due to the fact that since the product transactions increased it made sense for the retailers to increase their relationship levels since, in view, of the transaction cost functions (which are decreasing in the relationship levels), these costs would be reduced. Indeed, all the product transactions increased (relative to those obtained in Example 1), except for the transactions associated with B2C commerce.

Hence, the social network component (in equilibrium) in this example is much denser than that in Example 1 since now we have positive equilibrium relationship levels not only on the Internet links between manufacturers and retailers but also on the physical links between manufacturers and retailers as well as on the links on the social network representing retailers transacting with the demand markets. Thus, the social network component (cf. Figures 2 and 3) of the equilibrium supernetwork for this example has links not only in the social network joining the manufacturers with the retailers but also on the links connecting retailers with the demand markets.

Example 3

We then modified Example 2 as follows: The data were identical to that in Example 2 except that we now further increased the weights for the manufacturers associated with the relationships and now had $\beta_1 = \beta_2 = 20$.

Again, the Euler method converge and yielded a new equilibrium pattern in 1276 iterations. Specifically, the new equilibrium product transaction pattern was:

$$Q^{1*} := q^*_{111} = q^*_{121} = q^*_{211} = q^*_{221} = 3.4904; \quad q^*_{112} = q^*_{122} = q^*_{212} = q^*_{222} = 2.4102;$$

$$Q^{2*} := q^*_{11} = q^*_{12} = q^*_{21} = q^*_{22} = 13.2790,$$

and

$$Q^{3*} := q^*_{11} = q^*_{21} = q^*_{21} = q^*_{22} = 5.8696.$$

The new equilibrium price pattern was:

$$\epsilon^*_1 = \epsilon^*_2 = 264.2347$$

and

$$\rho^*_{31} = \rho^*_{32} = 274.7649.$$

The computed new equilibrium relationship level pattern was the same as for Example 2 except that now the relationship levels between retailers and demand markets all increased:

$$h^*_{jk} = .3700, \quad \forall j, k.$$

This makes sense since the relationship levels that were already at level 1 could not increase more (since they are already at their upper bounds) even with an increase in weight.

The network topology of the supernetwork in equilibrium for this example will be that obtained for Example 2.

5. Summary and Conclusions

In this paper, we developed and analyzed a dynamic supernetwork framework consisting of a supply chain network, which allows for physical as well as electronic transactions, and a social network. The decision-makers are: manufacturers, retailers, as well as consumers associated with the demand markets. We modeled their behavior, which involved multicriteria decision-making and included not only profit functions but also risk and value relationship functions, each weighted accordingly by the appropriate decision-maker.

We established the optimality conditions for the manufacturers and the retailers, along with the equilibrium conditions, and provided the variational inequality formulation. We then presented the disequilibrium dynamics that describe the time evolution of the product transactions, the relationship levels, as well as the prices over the supernetwork until the equilibrium pattern is achieved. We showed that the set of stationary points of the projected dynamical system coincides with the set of solutions of the variational inequality problem and discussed the meaning of the equilibrium conditions from an economic perspective. Furthermore, we established certain qualitative properties of the dynamic trajectories. In addition, we proposed a discrete-time approximation to the continuous time adjustment process. In this process, the computation of the product transactions and the prices takes place on the supply chain network whereas the computation of the relationship levels takes place on the social network component of the supernetwork. Finally, we applied the algorithm to several numerical examples that illustrated that the relationship levels increase if the weights that decision-makers put on relationship levels increase until an upper bound on the levels is attained.

This work adds to the foundations of the integration of social networks with other complex networks and identifies also the levels of relationship (in terms of flows) as well as the flows of product transactions. Moreover, through the computation of the equilibrium patterns, we obtain, in a sense, the optimal designs of the supernetworks since those links with zero flows at equilibrium can be eliminated.

Further research may include empirical applications, extensions of this work to financial networks, as well as other critical network infrastructures.

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