The Evolution and Emergence of Integrated Social and Financial Networks with Electronic Transactions: A Dynamic Supernetwork Theory for the Modeling, Analysis, and Computation of Financial Flows and Relationship Levels

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Abstract: In this paper, we propose a rigorous dynamic supernetwork theory for the integration of social networks with financial networks with intermediation in the presence of electronic transactions. We consider decision-makers with sources of funds, financial intermediaries, as well as demand markets for the various financial products. Through a multilevel supernetwork framework consisting of the financial network and the social network we model the multicriteria decision-making behavior of the various decision-makers, which includes the maximization of net return, the maximization of relationship values, and the minimization of risk. Increasing relationship levels in our framework are assumed to reduce transaction costs as well as risk and to have some additional value for the decision-makers. We explore the dynamic evolution of the financial flows, the associated product prices, as well as the relationship levels on the supernetwork until an equilibrium pattern is achieved. We provide some qualitative properties of the dynamic trajectories, under suitable assumptions, and propose a discrete-time algorithm which is then applied to track the evolution of the relationship levels over time as well as the financial flows and prices. The equilibrium pattern yields, as a byproduct, the emergent structure of the social and financial networks since it identifies not only which pairs of nodes will have flows but also the size of the flows, i.e., the relationship levels and the financial transactions.

1. Introduction

The literature on social networks is considerable and growing (cf. Freeman (2000) and Krackhardt (2000)) but it has developed largely in the discipline of sociology without much emphasis on the economic aspects of human decision-making including those surrounding financial transactions. The literature on financial networks, in turn, as noted in Nagurney and Siokos (1997), dates to the classical work of Quesnay (1758) who depicted the circular flow of financial funds in an economy as a graph. It has yielded a wealth of models, supported by a variety of methodological tools (cf. Nagurney (2003) and the references therein), that aim to not only capture the economic behavior of the various decision-makers involved in the allocation and uses of funds but also allow for the computation of the optimal/equilibrium financial flows and prices.

In this paper, our goal is to provide a theory for the integration of social networks with financial networks that explores the evolution of financial transactions over time coupled with the evolution of relationships between the financial decision-makers; that is, those with sources of funds, the intermediaries, as well as the consumers of the various financial products. In our model, the decision-makers can choose between the possibility of physical as well as electronic transactions along the lines proposed by Nagurney and Ke (2003). In addition, they can, within a multicriteria decision-making context, also select their relationship levels which, in turn, affect the transaction costs and the risk and are assumed to hold some additional value. Establishing (and maintaining) the relationship levels, however, incurs some costs that need to be borne by the decision-makers. Hence, our framework, unlike much of the literature in social networks, considers flows in the form of relationship levels between tiers of decision-makers.

The role that relationships play in financial networks has gained increasing attention and has been studied analytically as well as empirically in several different contexts. Ghatak (2002) as well as Anthony (1997) described the role of social networks in the context of micro-financing. Boot and Thakor (2000), in turn, developed a model to determine the effect of an increase of interbank competition and capital market competition on relationship lending. Sharpe (1990), Petersen and Rajan (1994), Berger and Udell (1995) as well as Petersen and Rajan (1995) dealt with the connection between relationships and lending.

Petersen and Rajan (1994) and Arrow (1998) concluded that bank-firm ties are more critical to lending markets than classical theory suggests, and that social relationships and networks affect who gets capital and at what cost. Wilner's (2000) findings concerning the optimal pricing, lending, and renegotiation strategies for companies with relationships such that one company depends strongly on another one are compatible with those in the relationship-lending literature. Jackson (2003) surveyed the recent literature on network formation with a focus on game theoretic models as well as networks of relationships and applications. He also emphasized that networks of relationships play an important role in many economic situations.

Uzzi (1997) and DiMaggio and Louch (1998), basing their research on the sociological research on lending, suggested that social relationships and networks affect personal and corporate financial dealings. Uzzi (1999) pointed out that firms are more likely to get loans and to receive lower interest rates on loans if social relationships and network ties exist. Burt (2000) along with cited references overviews the theoretical foundations and related research of the network structure of social capital in various fields.

Informal/social networks also play an important role in controlling access to finance. As Garmaise and Moskowitz (2002a) described, the market participants may resolve information asymmetries by purchasing nearby properties, trading properties with long income histories, and avoiding transactions with informed professional brokers. Empirically, Garmaise and Moskowitz (2002b), Zumpano, Elder and Baryla (1996), Jud and Frew (1986), and Janssen and Jobson (1980), demonstrated that realtors as service intermediaries, who do not supply finance themselves, can facilitate their clients' access to finance via repeated informal relationships with lenders.

Carnevali (1996) and Ormerod and Smith (2001), in turn, suggested that noneconomic (that is, not easily priced) factors such as social relations, locations, and shared history can influence economic transactions. Social networks can, hence, also facilitate the access to financial services for consumers. Indeed, the above-noted research has empirically demonstrated that informal networks play an important role in controlling access to finance. However, there is no theoretical modeling as to the behavior of the various decision-makers, including the financial intermediaries in that literature and no insights as to how the market equilibrium is affected by such informal, that is, social networks.

In this paper, we develop a dynamic supernetwork model of financial networks with intermediation and electronic transactions that also includes the role that relationships play in this context. As in the paper by Gulpinar, Rustem, and Settergren (2003) and the book by Rustem and Howe (2002) we consider the role that risk and transaction costs play in financial decisions. The supernetwork model is multilevel in structure and includes both the financial network with intermediation and the social network with flows on the former corresponding to the financial flows (and prices at the nodes) and the flows on the latter to the relationship levels. The decision-makers are associated with the nodes of the supernetwork. The functions associated with the nodes and links of the financial network may depend on the flows not only of that network but also on the flows on the social network and, vice versa. Hence, the supernetwork represents an integration of the two networks which have historically been studied theoretically as separate network systems. The sources of financial funds as well as the financial intermediaries are multicriteria decision-makers and seek to maximize their net return, to minimize their risk, and to maximize the value of relationships, with individual weightings of these criteria.

As noted earlier, this work is based on the paper by Nagurney and Ke (2003) who proposed a model for decision-making in financial networks with intermediation and electronic transactions. We extend that model by explicitly including the role that relationship levels play. As in the paper by Wakolbinger and Nagurney (2004), that describes the role of relationships in supply chain networks, decision-makers in a given tier of the supernetwork can decide on the relationship levels that they want to achieve with decision-makers associated with the other tiers of the network. Establishing/maintaining a certain relationship level induces some costs, but may also lower the risk and the transaction costs. Unlike the paper by Nagurney and Ke (2003), we use general risk functions as suggested in the paper by Nagurney and Cruz (2004), but focus on a single country setting. For additional background on supernetworks and other applications, see the book by Nagurney and Dong (2002). Nagurney et al. (2002a) developed a dynamic multilevel network model to model the evolution of supply chain networks. The multilevel network therein consisted of a logistical network, a financial network, and also an information network (but no social network to capture relationship levels).

This paper is organized as follows. In Section 2, we develop the multilevel supernetwork model consisting of multiple tiers of decision-makers acting on the financial and the social networks. We describe their optimizing behavior, and establish the governing equilibrium conditions along with the corresponding variational inequality formulation.

In Section 3, we describe the disequilibrium dynamics of the financial flows, prices, and relationship levels as they evolve over time and formulate the dynamics as a projected dynamical system (cf. Nagurney and Zhang (1996), Nagurney and Ke (2003), and Nagurney and Cruz (2004)). We show that the set of stationary points of the projected dynamical system coincides with the set of solutions to the derived variational inequality problem. In addition, we establish a stability analysis result for the supernetwork system.

In Section 4, we present a discrete-time algorithm to approximate (and track) the financial flow, price, and relationship level trajectories over time until the equilibrium values are reached. We then apply the discrete-time algorithm in Section 5 to several numerical examples to further illustrate the supernetwork model. We conclude with Section 6, in which we summarize our results and suggest possibilities for future research.

2. The Supernetwork Model Integrating Social Networks and Financial Networks with Intermediation and with Electronic Transactions

In this Section, we develop the supernetwork model consisting of the integration of the financial network and the social network in which the decision-makers are those with sources of funds, the financial intermediaries, as well as the consumers associated with the demand markets. Here we describe the model in an equilibrium context, whereas in Section 3, we provide the disequilibrium dynamics and the evolution of the financial flows, the prices, as well as the relationship levels between tiers of decision-makers over time.

In the model, the sources of funds can transact directly electronically with the consumers through the Internet and can also conduct their financial transactions with the intermediaries either physically or electronically. The intermediaries, in turn, can transact with the consumers either physically in the standard manner or electronically as described in Nagurney and Ke (2003).

The depiction of the supernetwork is given in Figure 1. As this figure illustrates, the supernetwork is comprised of the social network, which is the bottom level network, and the financial network, which is the top level network. Internet links to denote the possibility of electronic financial transactions are denoted in the figure by dotted arcs. In addition, dotted arcs/links are used to depict the integration of the two networks into a supernetwork. Subsequently, we describe the interrelationships between the financial and social networks through the functional forms and flows on the links.

As in the model of Nagurney and Ke (2003), we assume that there are m agents or decision-makers with sources of financial funds who seek to allocate their financial resources among a portfolio of financial instruments. They can transact physically as well as electronically with distinct n financial intermediaries and/or directly and electronically with the consumers at the o demand markets. The financial intermediaries transact with the source agents to obtain financial funds and then reallocate/convert those funds into financial products associated with the o demand markets.

The supernetwork in Figure 1 consists of a social and a financial network. Both networks consist of three tiers of decision-makers. In the top tier are the decision-makers with sources

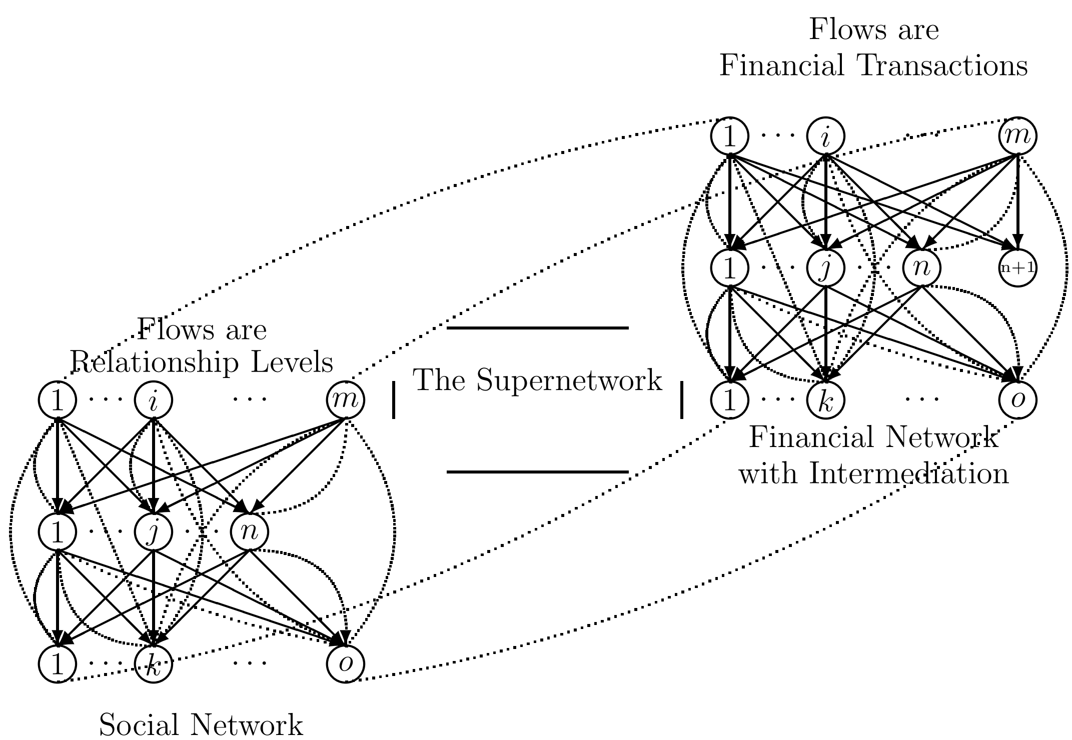

Figure 1: The Multilevel Supernetwork Structure of the Integrated Financial Network / Social Network System

of funds, with a typical source agent denoted by i and associated with node i. The middle tier of nodes in each of the two networks consists of the intermediaries, with a typical intermediary denoted by j and associated with node j in the network. As in the model of Nagurney and Ke (2003), we allow for the possibility of source agents not investing their funds (or a portion thereof), which we represent by the node $n + 1$ at the middle tier of nodes in the financial network only. The bottom tier of nodes in both the social and in the financial networks consists of the demand markets, with a typical demand market denoted by k and corresponding to node k. Each demand market is associated with a particular financial product.

The supernetwork in Figure 1 includes classical physical links as well as Internet links to allow for electronic financial transactions. Electronic transactions are possible between the source agents and the intermediaries, the source agents and the demand markets as well as the intermediaries and the demand markets. Physical transactions can occur between the source agents and the intermediaries and between the intermediaries and the demand markets.

We now turn to the description of the behavior of the various economic decision-makers, i.e., the source agents, the financial intermediaries, and the demand markets.

The Behavior of the Source Agents and their Optimality Conditions

As described above, a source agent i may transact with an intermediary j via a physical or via an electronic link. A source agent i may also transact directly with consumers at demand market k. In the model, we let l denote the type of transaction with $l = 1$ representing a physical transaction and $l = 2$ denoting an electronic transaction via the Internet. S^i denotes the nonnegative amount of funds that source agent i holds.

The quantity of financial funds transacted between source agent i and intermediary j through mode l is denoted by q_{ijl}. We group all these financial flows into the column vector $Q^1 \in R^{2mn}_+$. The quantity of financial funds transacted with demand market k and the associated funds flow is denoted by q_{ik}. We group all these financial flows into the column vector $Q^2 \in R^{mo}_+$. The vector of flows Q^1 is associated with the links on the financial network component of the supernetwork in Figure 1 joining the top tier nodes with the middle tier nodes. The vector of flows Q^2, in turn, is associated with the links in the financial network joining the top tier nodes with the bottom tier nodes (corresponding to the demand markets).

For each source agent i the amount of funds transacted either electronically and/or physically cannot exceed his financial holdings. Hence, the following conservation of flow equation must hold:

$$\sum_{j=1}^{n}\sum_{l=1}^{2} q_{ijl} + \sum_{k=1}^{o} q_{ik} \leq S^i, \quad \forall i. \tag{1}$$

In Figure 1 the slack associated with constraint (1) for source agent i is represented as the flow on the link joining node i with the non-investment node $n+1$.

Furthermore, let h_{ijl} denote the nonnegative level of the relationship between source agent i and intermediary j associated with mode of transaction l and let h_{ik} denote the nonnegative relationship level associated with the virtual mode of transaction between source agent i and demand market k. Each source agent i may actively try to achieve a certain relationship level with an intermediary and/or a demand market. We group the h_{ijl}s for all source agent/intermediary/mode combinations into the column vector $h^1 \in R_+^{2mn}$ and the h_{ik}s for all the source agent/demand market pairs into the column vector $h^2 \in R_+^{mo}$. Moreover, we assume that these relationship levels take on a value that lies in the range $[0, 1]$. No relationship is indicated by a relationship level of zero and the strongest possible relationship is indicated by a relationship level of one. In the supernetwork depicted in Figure 1 the relationship flows are associated with the links in the social network component of the supernetwork. Specifically, the vector of flows h^1 corresponds to flows on the links between the source agents and the intermediaries and the vector of flows h^2 corresponds to the flows on the links between the source agents and the demand markets on the social network. The relationship levels, along with the financial flows, are endogenously determined in the model.

The source agent may spend money, for example, in the form of gifts and/or additional time/service in order to achieve a particular relationship level. The production cost functions for relationship levels are denoted by b_{ijl} and b_{ik} and represent, respectively, how much money a source agent i has to spend in order to achieve a certain relationship level with intermediary j transacting through mode l or in order to achieve a certain relationship level with demand market k. These relationship production cost functions are distinct for each source agent/intermediary/mode combination and for each source agent/demand market combination. Their specific functional forms may be influenced by such factors as the willingness of intermediaries or demand markets to establish/maintain a relationship and the level of previous business relationships and private relationships that exist.

The relationship production cost function is assumed, hence, to be a function of the relationship level between the source agent and intermediary j transacting via mode l or with the consumers at demand market k, that is,

$$b_{ijl} = b_{ijl}(h_{ijl}), \quad \forall i, j, l, \tag{2}$$

$$b_{ik} = b_{ik}(h_{ik}), \quad \forall i, k. \tag{3}$$

We assume that these functions are convex and continuously differentiable.

We further assume that the transaction cost between a source agent/ intermediary/transaction-type (ijl) combination, which is denoted by c_{ijl}, as well as, the transaction cost between a source agent and a demand market pair (ik), denoted by c_{ik}, depend on the volume of financial transactions between the particular pair via the particular mode and on the relationship level between them, that is:

$$c_{ijl} = c_{ijl}(q_{ijl}, h_{ijl}), \quad \forall i, j, l, \tag{4}$$

and

$$c_{ik} = c_{ik}(q_{ik}, h_{ik}), \quad \forall i, k. \tag{5}$$

These functions are also assumed to be convex and continuously differentiable.

We denote the price obtained by source agent i from intermediary j by transacting via mode l by ρ^*_{1ijl} and the price associated with source agent i transacting electronically with demand market k by ρ^*_{1ik}. Later, we will discuss how these prices are arrived at. The source agent i faces total costs that equal the sum of the total transaction costs plus the costs that he incurs for establishing relationship levels. As the source agent i tries to maximize his net revenue he faces the following maximization problem:

$$\text{Maximize} \quad \sum_{j=1}^{n} \sum_{l=1}^{2} \rho^*_{1ijl} q_{ijl} + \sum_{k=1}^{o} \rho^*_{1ik} q_{ik} - \sum_{j=1}^{n} \sum_{l=1}^{2} c_{ijl}(q_{ijl}, h_{ijl}) - \sum_{k=1}^{o} c_{ik}(q_{ik}, h_{ik})$$

$$-\sum_{j=1}^{n}\sum_{l=1}^{2} b_{ijl}(h_{ijl}) - \sum_{k=1}^{o} b_{ik}(h_{ik}) \tag{6}$$

subject to:

$$q_{ijl} \geq 0, \quad \forall j, l, \quad q_{ik} \geq 0, \quad \forall k, \tag{7}$$

$$0 \leq h_{ijl} \leq 1, \quad \forall j, l, \quad 0 \leq h_{ik} \leq 1, \quad \forall k, \tag{8}$$

and the constraint (1) for source agent i.

The first two terms in (6) represent the revenue. The next four terms represent the various costs. The constraints are comprised of the conservation of flow equation (1), the nonnegativity assumptions on the financial flows (cf. (7)) and on the relationship levels with the latter being bounded from above by the value one (cf. (8)).

Furthermore, it is reasonable to assume that each source agent tries to minimize risk. Here, for the sake of generality, and as in the paper by Nagurney and Cruz (2003), we assume, as given, a risk function for source agent i, dealing with intermediary j via mode l, denoted by r_{ijl}, and a risk function for source agent i dealing with demand market k denoted by r_{ik}. These functions depend not only on the quantity of the financial flow transacted between the pair of nodes (and via a particular mode) but also on the corresponding relationship level. These risk functions are as follows:

$$r_{ijl} = r_{ijl}(q_{ijl}, h_{ijl}), \quad \forall i, j, l, \tag{9}$$

$$r_{ik} = r_{ik}(q_{ik}, h_{ik}), \quad \forall i, k, \tag{10}$$

where r_{ijl} and r_{ik} are assumed to be convex and continuously differentiable.

Hence, source agent i also faces an optimization problem associated with his desire to minimize the total risk and corresponding to:

$$\text{Minimize} \quad \sum_{j=1}^{n}\sum_{l=1}^{2} r_{ijl}(q_{ijl}, h_{ijl}) + \sum_{k=1}^{o} r_{ik}(q_{ik}, h_{ik}) \tag{11}$$

subject to:

$$q_{ijl} \geq 0, \quad \forall j, l, \quad q_{ik} \geq 0, \quad \forall k, \tag{12}$$

$$0 \leq h_{ijl} \leq 1, \quad \forall j, l, \quad 0 \leq h_{ik} \leq 1, \quad \forall k. \tag{13}$$

In addition, the source agent also tries to maximize the relationship value generated by interacting with other decision-makers in the network. Here, v_{ijl} denotes the relationship value function for source agent i, intermediary j, and mode l and v_{ijl} is assumed to be a function of the relationship level of i with intermediary j transacting via mode l. Similarly, v_{ik} denotes the relationship value function for source agent i and demand market k. It is assumed to be a function of the relationship level with the particular demand market k such that

$$v_{ijl} = v_{ijl}(h_{ijl}), \quad \forall i, j, l, \tag{14}$$

$$v_{ik} = v_{ik}(h_{ik}), \quad \forall i, k. \tag{15}$$

We assume that the value functions are continuously differentiable and concave.

Hence, source agent i is also faced with the optimization problem representing the maximization of the total value of his relationships expressed mathematically as:

$$\text{Maximize} \quad \sum_{j=1}^{n} \sum_{l=1}^{2} v_{ijl}(h_{ijl}) + \sum_{k=1}^{o} v_{ik}(h_{ik}) \tag{16}$$

subject to:

$$0 \leq h_{ijl} \leq 1, \quad \forall j, l, \quad 0 \leq h_{ik} \leq 1, \quad \forall k. \tag{17}$$

We can now construct the multicriteria decision-making problem facing a source agent which allows him to weight the criteria of net revenue maximization (cf. (6)), risk minimization (cf. (11)), and total relationship value maximization (see (16)) in an individual manner. Source agent i's multicriteria decision-making objective function is denoted by U^i. Assume that source agent i assigns a nonnegative weight α_i to the risk generated and a nonnegative weight β_i to the relationship value. The weight associated with net revenue maximization serves as the numeraire and is set equal to 1. The nonnegative weights measure the importance of risk and the total relationship value and, in addition, transform these values into monetary units. We can now construct a value function for each source agent (cf. Keeney and Raiffa (1993), Dong, Zhang, and Nagurney (2002), and the references therein) using a constant additive weight value function. Therefore, the multicriteria decision-making problem of source agent i can be expressed as:

$$\text{Maximize } U^i = \sum_{j=1}^{n} \sum_{l=1}^{2} \rho_{1ijl}^{*} q_{ijl} + \sum_{k=1}^{o} \rho_{1ik}^{*} q_{ik} - \sum_{j=1}^{n} \sum_{l=1}^{2} c_{ijl}(q_{ijl}, h_{ijl}) - \sum_{k=1}^{o} c_{ik}(q_{ik}, h_{ik})$$

$$-\sum_{j=1}^{n}\sum_{l=1}^{2} b_{ijl}(h_{ijl}) - \sum_{k=1}^{o} b_{ik}(h_{ik}) - \alpha_i(\sum_{j=1}^{n}\sum_{l=1}^{2} r_{ijl}(q_{ijl}, h_{ijl}) + \sum_{k=1}^{o} r_{ik}(q_{ik}, h_{ik}))$$

$$+\beta_i(\sum_{j=1}^{n}\sum_{l=1}^{2} v_{ijl}(h_{ijl}) + \sum_{k=1}^{o} v_{ik}(h_{ik})) \tag{18}$$

subject to:

$$q_{ijl} \geq 0, \quad \forall j, l, \quad q_{ik} \geq 0, \quad \forall k, \tag{19}$$

$$0 \leq h_{ijl} \leq 1, \quad \forall j, l, \quad 0 \leq h_{ik} \leq 1, \quad \forall k, \tag{20}$$

and the constraint (1) for source agent i.

The first six terms on the right-hand side of the equal sign in (18) represent the net revenue which is to be maximized, the next two terms represent the weighted total risk which is to be minimized and the last two terms represent the weighted total relationship value, which is to be maximized. We can observe that such an objective function is in concert with those used in classical portfolio optimization (see Markowitz (1952, 1959)) but substantially more general to reflect specifically the additional criteria, notably, that of total relationship value maximization.

Under the above assumed and imposed assumptions on the underlying functions, the optimality conditions for all source agents *simultaneously* can be expressed as the following inequality (cf. Bazaraa, Sherali, and Shetty (1993), Gabay and Moulin (1980); see also Nagurney (1999)): determine $(Q^{1*}, Q^{2*}, h^{1*}, h^{2*}) \in \mathcal{K}_1$, such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[\frac{\partial c_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial q_{ijl}} + \alpha_i \frac{\partial r_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial q_{ijl}} - \rho^*_{1ijl}\right] \times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\frac{\partial c_{ik}(q^*_{ik}, h^*_{ik})}{\partial q_{ik}} + \alpha_i \frac{\partial r_{ik}(q^*_{ik}, h^*_{ik})}{\partial q_{ik}} - \rho^*_{1ik}\right] \times [q_{ik} - q^*_{ik}]$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[\frac{\partial c_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial h_{ijl}} + \frac{\partial b_{ijl}(h^*_{ijl})}{\partial h_{ijl}} - \beta_i \frac{\partial v_{ijl}(h^*_{ijl})}{\partial h_{ijl}} + \alpha_i \frac{\partial r_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial h_{ijl}}\right] \times \left[h_{ijl} - h^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\frac{\partial c_{ik}(q^*_{ik}, h^*_{ik})}{\partial h_{ik}} + \frac{\partial b_{ik}(h^*_{ik})}{\partial h_{ik}} - \beta_i \frac{\partial v_{ik}(h^*_{ik})}{\partial h_{ik}} + \alpha_i \frac{\partial r_{ik}(q^*_{ik}, h^*_{ik})}{\partial h_{ik}}\right] \times [h_{ik} - h^*_{ik}] \geq 0,$$

$$\forall (Q^1, Q^2, h^1, h^2) \in \mathcal{K}_1, \tag{21}$$

where

$$\mathcal{K}_1 \equiv \Big[(Q^1, Q^2, h^1, h^2) \mid q_{ijl} \geq 0, \; q_{ik} \geq 0, \; 0 \leq h_{ijl} \leq 1, \; 0 \leq h_{ik} \leq 1, \; \forall i, j, l, k, \text{and (1) holds}\Big]. \tag{22}$$

Inequality (21) is actually a variational inequality (cf. Kinderlehrer and Stampacchia (1980) and Nagurney (1999) and the references therein).

The Behavior of the Financial Intermediaries and their Optimality Conditions

The financial intermediaries also seek to maximize their net revenue and their total relationship values with the source agents and the demand markets while simultaneously minimizing the risk associated with their transactions. The financial intermediaries transact with the source agents in order to obtain the financial funds, as well as with the consumers at the demand markets, who are the ultimate users of the funds in the form of the financial products. Let q_{jkl} denote the quantity of the financial product transacted between intermediary j and demand market k by mode l. We group these financial quantities/flows into the column vector $Q^3 \in R^{2no}_+$. The intermediaries convert the incoming financial flows Q^1 into the outgoing financial products Q^3. Note that the Q^3 are associated with the links of the financial network component of the supernetwork (cf. Figure 1) that connect the demand markets with the intermediaries.

As in the case of source agents, the intermediaries have to bear some costs to establish and maintain relationship levels with source agents and with the consumers. We denote the relationship level between intermediary j and demand market k transacting through mode l by h_{jkl}. We group the relationship levels for all intermediary/demand market pairs into the column vector $h^3 \in R^{no}_+$. We assume that the relationship levels are nonnegative and that they may assume a value from 0 through 1. These relationship levels represent the flows between the intermediaries and the demand market nodes in the social network level of the supernetwork in Figure 1.

Let $\hat{b}_{ijl}$ denote the cost function associated with the relationship between intermediary j and source agent i transacting via mode l and let b_{jkl} denote the analogous cost function but associated with intermediary j, demand market k, and mode l. Note that these functions are from the perspective of the intermediary (whereas (2) and (3) are from the perspective of the source agents). These cost functions are a function of the relationship levels (as in the case of the source agents) and are given by:

$$\hat{b}_{ijl} = \hat{b}_{ijl}(h_{ijl}), \quad \forall i, j, l, \tag{23}$$

$$b_{jkl} = b_{jkl}(h_{jkl}), \quad \forall j, k, l. \tag{24}$$

Intermediary j is faced with a transaction cost when transacting with the source agents. We denote the transaction cost associated with intermediary j transacting with source agent i using mode l by $\hat{c}_{ijl}$. We assume that the function depends on the financial flow and the relationship level between intermediary j and source agent i transacting through mode l, that is,

$$\hat{c}_{ijl} = \hat{c}_{ijl}(q_{ijl}, h_{ijl}), \quad \forall i, j, l. \tag{25}$$

Intermediaries also have to bear transaction costs in transacting with consumers at demand market k via mode l. This cost is a function of the amount of transactions between intermediary j and demand market k via transaction mode l as well as the relationship level between them, that is,

$$c_{jkl} = c_{jkl}(q_{jkl}, h_{jkl}), \quad \forall j, k, l. \tag{26}$$

In addition, intermediary j is faced with a handling/conversion cost as discussed in Nagurney et al. (2002b), which may include, for example, the cost of converting the incoming financial flows into the financial products at the demand markets. We denote this cost by c_j. For the sake of generality, we assume that c_j is a function of all the transactions between source agents and intermediaries. Hence, we have that:

$$c_j = c_j(Q^1), \quad \forall j. \tag{27}$$

All the above functions (23) – (27) are assumed to be convex and continuously differentiable.

Let ρ^*_{2jkl} denote the price associated with the financial product transacted between intermediary j and demand market k via mode l. Later in this section, we discuss how this price is obtained. The optimization problem faced by an intermediary which reflects net revenue maximization is given by:

$$\text{Maximize} \quad \sum_{k=1}^{o}\sum_{l=1}^{2}\rho^*_{2jkl}q_{jkl} - c_j(Q^1) - \sum_{i=1}^{m}\sum_{l=1}^{2}\hat{c}_{ijl}(q_{ijl}, h_{ijl}) - \sum_{k=1}^{o}\sum_{l=1}^{2} c_{jkl}(q_{jkl}, h_{jkl})$$

$$-\sum_{i=1}^{m}\sum_{l=1}^{2}\rho^*_{1ijl}q_{ijl} - \sum_{i=1}^{m}\sum_{l=1}^{2}\hat{b}_{ijl}(h_{ijl}) - \sum_{k=1}^{o}\sum_{l=1}^{2}b_{jkl}(h_{jkl}) \quad (28)$$

subject to:

$$\sum_{k=1}^{o}\sum_{l=1}^{2} q_{jkl} \leq \sum_{i=1}^{m}\sum_{l=1}^{2}q_{ijl} \quad (29)$$

$$q_{ijl} \geq 0, \quad \forall i, l, \quad q_{jkl} \geq 0, \quad \forall k, l, \quad (30)$$

$$0 \leq h_{ijl} \leq 1, \quad \forall i, l, \quad 0 \leq h_{jkl} \leq 1, \quad \forall k, l. \quad (31)$$

Constraint (29) guarantees that each intermediary does not reallocate more financial flows than he has available. Constraints (30) and (31) guarantee that the financial flows and relationship levels are nonnegative (from the perspective of the intermediary) and that the levels of the relationships do not exceed one.

In addition, we assume that each intermediary is also concerned with risk minimization. For the sake of generality, we assume, as given, a risk function $\hat{r}_{ijl}$, for intermediary j in transacting with source agent i through mode l and a risk function r_{jkl} for intermediary j associated with his transacting with consumers at demand market k through mode l. The risk functions are assumed to be continuous and convex and a function of the amount transacted with the particular source agent or demand market and the relationship level with this source agent or demand market. A higher relationship level can be expected to reduce risk since trust reduces transactional uncertainty. The risk functions may be distinct for each intermediary/source agent/mode and each intermediary/demand market/mode combination and are given, respectively, by:

$$\hat{r}_{ijl} = \hat{r}_{ijl}(q_{ijl}, h_{ijl}), \quad \forall i, j, l, \quad (32)$$

$$r_{jkl} = r_{jkl}(q_{jkl}, h_{jkl}), \quad \forall j, k, l. \quad (33)$$

Since a financial intermediary j is assumed to minimize his total risk, he is also faced with the optimization problem given by:

$$\text{Minimize} \quad \sum_{i=1}^{m}\sum_{l=1}^{2}\hat{r}_{ijl}(q_{ijl},h_{ijl}) + \sum_{k=1}^{o}\sum_{l=1}^{2} r_{jkl}(q_{jkl},h_{jkl}) \tag{34}$$

subject to:

$$q_{ijl} \geq 0, \quad \forall i,l, \quad q_{jkl} \geq 0, \quad \forall k,l, \tag{35}$$

$$0 \leq h_{ijl} \leq 1, \quad \forall i,l, \quad 0 \leq h_{jkl} \leq 1, \quad \forall k,l. \tag{36}$$

As in the case of the source agents, intermediary j also tries to maximize his relationship values associated with the source agents and with the demand markets. We assume, as given, a relationship value function $\hat{v}_{ijl}$ for intermediary j in dealing with source agent i through transaction mode l and a relationship value function v_{jkl} for intermediary j associated with his transacting with consumers at demand market k through mode l. The relationship value functions are assumed to be continuously differentiable and concave. They are assumed to be functions of the corresponding relationship levels and given, respectively, by

$$\hat{v}_{ijl} = \hat{v}_{ijl}(h_{ijl}), \quad \forall i,j,l, \tag{37}$$

$$v_{jkl} = v_{jkl}(h_{jkl}), \quad \forall j,k,l. \tag{38}$$

Finally, financial intermediary j tries to maximize his total relationship value, given mathematically by the optimization problem:

$$\text{Maximize} \quad \sum_{i=1}^{m}\sum_{l=1}^{2}\hat{v}_{ijl}(h_{ijl}) + \sum_{k=1}^{o}\sum_{l=1}^{2} v_{jkl}(h_{jkl}) \tag{39}$$

subject to:

$$0 \leq h_{ijl} \leq 1, \quad \forall i,l, \quad 0 \leq h_{jkl} \leq 1, \quad \forall k,l. \tag{40}$$

We are now ready to construct the multicriteria decision-making problem faced by an intermediary which combines with appropriate individual weights the criteria of net revenue maximization given by (28); risk minimization, given by (34), and total relationship value maximization, given by (39). In particular, we let intermediary j assign a nonnegative weight

δ_j to the total risk and a nonnegative weight γ_j to the total relationship value. The weight associated with net revenue maximization is set equal to 1 and serves as the numeraire (as in the case of the source agents). Let U^j denote the multicriteria objective function associated with intermediary j with his multicriteria decision-making problem expressed as:

$$\text{Maximize} \quad U^j = \sum_{k=1}^{o}\sum_{l=1}^{2} \rho^*_{2jkl} q_{jkl} - c_j(Q^1) - \sum_{i=1}^{m}\sum_{l=1}^{2} \hat{c}_{ijl}(q_{ijl}, h_{ijl}) - \sum_{k=1}^{o}\sum_{l=1}^{2} c_{jkl}(q_{jkl}, h_{jkl})$$

$$-\sum_{i=1}^{m}\sum_{l=1}^{2} \rho^*_{1ijl} q_{ijl} - \sum_{i=1}^{m}\sum_{l=1}^{2} \hat{b}_{ijl}(h_{ijl}) - \sum_{k=1}^{o}\sum_{l=1}^{2} b_{jkl}(h_{jkl}) - \delta_j(\sum_{i=1}^{m}\sum_{l=1}^{2} \hat{r}_{ijl}(q_{ijl}, h_{ijl}) + \sum_{k=1}^{o}\sum_{l=1}^{2} r_{jkl}(q_{jkl}, h_{jkl}))$$

$$+\gamma_j(\sum_{i=1}^{m}\sum_{l=1}^{2} \hat{v}_{ijl}(h_{ijl}) + \sum_{k=1}^{o}\sum_{l=1}^{2} v_{jkl}(h_{jkl})) \tag{41}$$

subject to:

$$\sum_{k=1}^{o}\sum_{l=1}^{2} q_{jkl} \leq \sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl} \tag{42}$$

$$q_{ijl} \geq 0, \quad \forall i, l, \quad q_{jkl} \geq 0, \quad \forall k, l, \tag{43}$$

$$0 \leq h_{ijl} \leq 1, \quad \forall i, l, \quad 0 \leq h_{jkl} \leq 1, \quad \forall k, l. \tag{44}$$

We assume that the financial intermediaries can compete, with the governing optimality/equilibrium concept underlying noncooperative behavior being that of Nash (1950, 1951). The optimality/equilibrium conditions (under the above imposed assumptions on the underlying functions) for all financial intermediaries *simultaneously* can be expressed as: determine $(Q^{1*}, Q^{3*}, h^{1*}, h^{3*}, \epsilon^*) \in \mathcal{K}_2$, such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[\frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \delta_j \frac{\partial \hat{r}_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial q_{ijl}} + \rho^*_{1ijl} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial q_{ijl}} - \epsilon^*_j\right] \times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[\frac{\partial c_{jkl}(q^*_{jkl}, h^*_{jkl})}{\partial q_{jkl}} - \rho^*_{2jkl} + \epsilon^*_j + \delta_j \frac{\partial r_{jkl}(q^*_{jkl}, h^*_{jkl})}{\partial q_{jkl}}\right] \times \left[q_{jkl} - q^*_{jkl}\right]$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[\frac{\partial \hat{c}_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial h_{ijl}} - \gamma_j \frac{\partial \hat{v}_{ijl}(h^*_{ijl})}{\partial h_{ijl}} + \delta_j \frac{\partial \hat{r}_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial h_{ijl}} + \frac{\partial \hat{b}_{ijl}(h^*_{ijl})}{\partial h_{ijl}}\right] \times \left[h_{ijl} - h^*_{ijl}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[\frac{\partial c_{jkl}(q^*_{jkl}, h^*_{jkl})}{\partial h_{jkl}} - \gamma_j \frac{\partial v_{jkl}(h^*_{jkl})}{\partial h_{jkl}} + \delta_j \frac{\partial r_{jkl}(q^*_{jkl}, h^*_{jkl})}{\partial h_{jkl}} + \frac{\partial b_{jkl}(h^*_{jkl})}{\partial h_{jkl}}\right] \times \left[h_{jkl} - h^*_{jkl}\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2}q_{ijl}^{*}-\sum_{k=1}^{o}\sum_{l=1}^{2}\ q_{jkl}^{*}\right]\times\left[\epsilon_{j}-\epsilon_{j}^{*}\right]\geq 0, \quad \forall(Q^{1},Q^{3},h^{1},h^{3},\epsilon)\in\mathcal{K}_{2}, \tag{45}$$

where

$$\mathcal{K}_{2}\equiv\left[(Q^{1},Q^{3},h^{1},h^{3},\epsilon)\mid q_{ijl}\geq 0, q_{jkl}\geq 0, 0\leq h_{ijl}\leq 1, 0\leq h_{jkl}\leq 1, \epsilon_{j}\geq 0,\ \forall i,j,l,k\right], \tag{46}$$

ϵ_j denotes the Lagrange multiplier associated with constraint (42), and ϵ is the column vector of all the intermediaries' Lagrange multipliers. These Lagrange multipliers can also be interpreted as shadow prices. Indeed, according to the fifth term in (45), ϵ_j^* serves as the price to "clear the market" at intermediary j.

Inequality (45) provides us with conditions under which optimal virtual and/or physical financial transactions between intermediaries and source agents occur and optimal conditions under which virtual transactions between source agents and demand markets occur. Furthermore, it formulates the optimality conditions under which the relationship levels associated with intermediaries interacting with either the source agents or the demand markets will take on positive values; in other words, a relationship exists.

The Consumers at the Demand Markets and their Equilibrium Conditions

Of course, the consumers at the demand markets can only buy as many financial products as the former decision-makers decide to sell. When making their decisions, consumers at the demand markets consider the price charged as well as the transaction costs associated with obtaining the financial products.

Let ρ_{3k}^* denote the price of the financial product at demand market k and group the demand market prices into the column vector $\rho_3 \in R_+^o$. Let d_k denote the demand for the product at demand market k. Demand functions are continuous and of the general form:

$$d_k = d_k(\rho_3), \quad \forall k. \tag{47}$$

Note that we allow the demand for a product to depend, in general upon the prices of all the financial products (associated with the demand markets).

Let $\hat{c}_{jkl}$ denote the transaction cost associated with obtaining the financial product at demand market k from intermediary j via mode l. We assume that the transaction cost is continuous and of the general form:

$$\hat{c}_{jkl} = \hat{c}_{jkl}(Q^2, Q^3, h^2, h^3), \quad \forall j, k, l. \tag{48}$$

Hence, the cost of transacting between an intermediary and a demand market via a specific mode, from the perspective of the consumers, can depend upon the volume of financial flows transacted either physically and/or electronically from intermediaries as well as from source agents and the associated relationship levels. The generality of this cost function structure enables the modeling of competition on the demand side as does the generality of the demand functions (47). Moreover, it allows for information exchange between the consumers at the demand markets who may inform one another as to their relationship levels which, in turn, can affect the transaction costs.

In addition, let $\hat{c}_{ik}$ denote the transaction cost associated with obtaining the financial product at demand market k from source agent i. We assume that this transaction cost function is continuous and of the general form:

$$\hat{c}_{ik} = \hat{c}_{ik}(Q^2, Q^3, h^2, h^3), \quad \forall i, k. \tag{49}$$

Hence, the cost of conducting a transaction with a source agent via the Internet depends on the volume of the product transacted either physically and/or electronically from intermediaries as well as from source agents and on the relationship levels associated with the demand markets.

The equilibrium conditions for demand market k, take the form: for all intermediaries: j; $j = 1, ..., n$ and all modes l; $l = 1, 2$:

$$\rho^*_{2jkl} + \hat{c}_{jkl}(Q^{2*}, Q^{3*}, h^{2*}, h^{3*}) \begin{cases} = \rho^*_{3k}, & \text{if} \quad q^*_{jkl} > 0 \\ \geq \rho^*_{3k}, & \text{if} \quad q^*_{jkl} = 0, \end{cases} \tag{50}$$

and for all source agents i; $i = 1, ..., m$:

$$\rho^*_{1ik} + \hat{c}_{ik}(Q^{2*}, Q^{3*}, h^{2*}, h^{3*}) \begin{cases} = \rho^*_{3k}, & \text{if} \quad q^*_{ik} > 0 \\ \geq \rho^*_{3k}, & \text{if} \quad q^*_{ik} = 0. \end{cases} \tag{51}$$

In addition, we must have that

$$d_k(\rho_3^*) \begin{cases} = \sum_{j=1}^{n} \sum_{l=1}^{2} q_{jkl}^* + \sum_{i=1}^{m} q_{ik}^*, & \text{if} \quad \rho_{3k}^* > 0 \\ \leq \sum_{j=1}^{n} \sum_{l=1}^{2} q_{jkl}^* + \sum_{i=1}^{m} q_{ik}^*, & \text{if} \quad \rho_{3k}^* = 0. \end{cases} \tag{52}$$

According to condition (50) consumers at demand market k will purchase the financial product from intermediary j, transacted via mode l, if the price charged by the intermediary plus the transaction cost (from the perspective of the consumers) is not higher than the price that the consumers are willing to pay for the product. Furthermore, according to condition (51) the analogue is true for the case of electronic transactions with the source agents. If the price that the consumers are willing to pay for a financial product is positive, then the quantity of the product at the demand market is precisely equal to the demand as can be seen from condition (52). Note that in the above conditions, the relationship levels are at their equilibrium values, denoted by the superscipt *. Recall also that we associate a specific product with each demand market and, hence, the relationship level between a source agent or a financial intermediary and a demand market is actually with the consumers at the demand market. Below we integrate the financial network and the social network components and also construct the supernetwork in equilibrium.

Conditions (50), (51), and (52) must hold for all demand markets in equilibrium. Hence, this leads to the inequality problem given by: determine $(Q^{2^*}, Q^{3^*}, \rho_3^*) \in R^{2no+mo+n}$, such that

$$\sum_{j=1}^{n} \sum_{k=1}^{o} \sum_{l=1}^{2} \left[\rho_{2jkl}^* + \hat{c}_{jkl}(Q^{2^*}, Q^{3^*}, h^{2*}, h^{3*}) - \rho_{3k}^*\right] \times \left[q_{jkl} - q_{jkl}^*\right]$$

$$+ \sum_{i=1}^{m} \sum_{k=1}^{o} \left[\rho_{1ik}^* + \hat{c}_{ik}(Q^{2^*}, Q^{3^*}, h^{2*}, h^{3*}) - \rho_{3k}^*\right] \times [q_{ik} - q_{ik}^*]$$

$$+ \sum_{k=1}^{o} \left[\sum_{l=1}^{2} \sum_{j=1}^{n} q_{jkl}^* + \sum_{i=1}^{m} q_{ik}^* - d_k(\rho_3^*)\right] \times [\rho_{3k} - \rho_{3k}^*] \geq 0, \quad \forall (Q^2, Q^3, \rho_3) \in R_+^{mo+2no+n}. \tag{53}$$

For further background on such a result, see Nagurney and Ke (2001).

The Equilibrium Conditions of the Supernetwork Integrating the Financial Network and the Social Network

In equilibrium, the financial transaction, the relationship level, and the price pattern must satisfy the sum of the optimality conditions (21) and (45), and the equilibrium conditions (53) in order to formalize the agreements between tiers of the supernetwork. The financial flows that the source agents transact with the intermediaries must be equal to those that the intermediaries accept from the source agents. The amounts that the consumers at the demand markets accept must be equal to the volume that the source agents and the intermediaries transact with the demand markets. In addition, the relationship levels between the appropriate tiers of the supernetwork must coincide.

Definition 1: Supernetwork Equilibrium Associated with the Integration of Social and Financial Networks with Intermediation and Electronic Transactions

The equilibrium state of the supernetwork is one where the financial and relationship levels between tiers coincide and the financial flows, the relationship levels, and the prices satisfy the sum of conditions (21), (45), and (53).

We now establish the following:

Theorem 1: Variational Inequality Formulation

The equilibrium conditions governing the supernetwork integrating the financial network with the social network are equivalent to the solution of the variational inequality given by: determine $(Q^{1*}, Q^{2*}, Q^{3*}, h^{1*}, h^{2*}, h^{3*}, \epsilon^*, \rho_3^*) \in \mathcal{K}$ *satisfying*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\Bigg[\frac{\partial c_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial q_{ijl}} + \alpha_i \frac{\partial r_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial q_{ijl}}$$

$$+\delta_j \frac{\partial \hat{r}_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial q_{ijl}} - \epsilon^*_j\Bigg] \times \Big[q_{ijl} - q^*_{ijl}\Big]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\frac{\partial c_{ik}(q^*_{ik}, h^*_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(Q^{2^*}, Q^{3^*}, h^{2*}, h^{3*}) + \alpha_i \frac{\partial r_{ik}(q^*_{ik}, h^*_{ik})}{\partial q_{ik}} - \rho^*_{3k}\right] \times [q_{ik} - q^*_{ik}]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[\frac{\partial c_{jkl}(q^*_{jkl},h^*_{jkl})}{\partial q_{jkl}}+\hat{c}_{jkl}(Q^{2^*},Q^{3^*},h^{2*},h^{3*})+\epsilon^*_j+\delta_j\frac{\partial r_{jkl}(q^*_{jkl},h^*_{jkl})}{\partial q_{jkl}}-\rho^*_{3k}\right]\times\left[q_{jkl}-q^*_{jkl}\right]$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[\frac{\partial c_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial h_{ijl}}+\frac{\partial \hat{c}_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial h_{ijl}}-\beta_i\frac{\partial v_{ijl}(h^*_{ijl})}{\partial h_{ijl}}-\gamma_j\frac{\partial \hat{v}_{ijl}(h^*_{ijl})}{\partial h_{ijl}}+\alpha_i\frac{\partial r_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial h_{ijl}}\right.$$

$$\left.+\delta_j\frac{\partial \hat{r}_{ijl}(q^*_{ijl},h^*_{ijl})}{\partial h_{ijl}}+\frac{\partial b_{ijl}(h^*_{ijl})}{\partial h_{ijl}}+\frac{\partial \hat{b}_{ijl}(h^*_{ijl})}{\partial h_{ijl}}\right]\times\left[h_{ijl}-h^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\frac{\partial c_{ik}(q^*_{ik},h^*_{ik})}{\partial h_{ik}}-\beta_i\frac{\partial v_{ik}(h^*_{ik})}{\partial h_{ik}}+\alpha_i\frac{\partial r_{ik}(q^*_{ik},h^*_{ik})}{\partial h_{ik}}+\frac{\partial b_{ik}(h^*_{ik})}{\partial h_{ik}}\right]\times[h_{ik}-h^*_{ik}]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[\frac{\partial c_{jkl}(q^*_{jkl},h^*_{jkl})}{\partial h_{jkl}}-\gamma_j\frac{\partial v_{jkl}(h^*_{jkl})}{\partial h_{jkl}}+\delta_j\frac{\partial r_{jkl}(q^*_{jkl},h^*_{jkl})}{\partial h_{jkl}}+\frac{\partial b_{jkl}(h^*_{jkl})}{\partial h_{jkl}}\right]\times\left[h_{jkl}-h^*_{jkl}\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2}q^*_{ijl}-\sum_{k=1}^{o}\sum_{l=1}^{2}\ q^*_{jkl}\right]\times\left[\epsilon_j-\epsilon^*_j\right]+\sum_{k=1}^{o}\left[\sum_{j=1}^{n}\sum_{l=1}^{2}q^*_{jkl}+\sum_{i=1}^{m}q^*_{ik}-d_k(\rho^*_3)\right]\times[\rho_{3k}-\rho^*_{3k}]\geq 0,$$

$$\forall(Q^1,Q^2,Q^3,h^1,h^2,h^3,\epsilon,\rho_3)\in\mathcal{K}, \tag{54}$$

where

$$\mathcal{K}\equiv\Big[(Q^1,Q^2,Q^3,h^1,h^2,h^3,\epsilon,\rho_3)|q_{ijl}\geq 0,q_{ik}\geq 0,q_{jkl}\geq 0,0\leq h_{ijl}\leq 1,0\leq h_{ik}\leq 1,$$

$$0\leq h_{jkl}\leq 1,\epsilon_j\geq 0,\rho_{3k}\geq 0,\forall i,j,l,k,\ and\,(1)\ holds\Big]. \tag{55}$$

Proof: Follows using similar arguments as the proof of Theorem 1 in Nagurney et al. (2002b) and Nagurney and Ke (2003). □

For easy reference in the subsequent sections, variational inequality problem (54) can be rewritten in standard variational inequality form (cf. Nagurney (1999)) as follows:

$$\langle F(X^*)^T,X-X^*\rangle\geq 0,\quad \forall X\in\mathcal{K}, \tag{56}$$

where $X\equiv(Q^1,Q^2,Q^3,h^1,h^2,h^3,\epsilon,\rho_3)$,

$$F(X)\equiv(F_{ijl},F_{ik},F_{jkl},\hat{F}_{ijl},\hat{F}_{ik},\hat{F}_{jkl},F_j,F_k)_{i=1,\ldots,m;j=1,\ldots,n;l=1,2;k=1,\ldots,o},$$

and the specific components of F given by the functional terms preceding the multiplication signs in (54), respectively. The term $\langle \cdot, \cdot \rangle$ denotes the inner product in N-dimensional Euclidean space.

We now show how to recover the prices: ρ^*_{2jkl}, for all j, k, l; ρ^*_{1ijl}, for all i, j, l, and ρ^*_{1ik}, for all i, k, from the solution of variational inequality (54). In Section 4 we describe an algorithm for the computation of the equilibrium solution $(Q^{1*}, Q^{2*}, Q^{3*}, h^{1*}, h^{2*}, h^3, \epsilon^*, \rho^*_3)$ and then apply it to several numerical examples.

Note from (45) that if $q^*_{jkl} > 0$, for some j, k and l, then ρ^*_{2jkl} is equal to $\frac{\partial c_{jkl}(q^*_{jkl}, h^*_{jkl})}{\partial q_{jkl}} + \epsilon^*_j + \delta_j \frac{\partial r_{jkl}(q^*_{jkl}, h^*_{jkl})}{\partial q_{jkl}}$, with ϵ^*_j being obtained from the solution of (54). Equivalently, from (53), we have that, for $q^*_{jkl} > 0$, $\rho^*_{2jkl} = \rho^*_{3k} - \hat{c}_{jkl}(Q^{2*}, Q^{3*}, h^{2*}, h^{3*})$. It follows from (21), in turn, that if $q^*_{ijl} > 0$, for some i, j, l then the price ρ^*_{1ijl} is equal to $\rho^*_{1ijl} = \left[\frac{\partial c_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial q_{ijl}} + \alpha_i \frac{\partial r_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial q_{ijl}}\right]$, or, equivalently, (cf. (45)), to $\left[-\frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} - \delta_j \frac{\partial \hat{r}_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial q_{ijl}} - \frac{\partial \hat{c}_{ijl}(q^*_{ijl}, h^*_{ijl})}{\partial q_{ijl}} + \epsilon^*_j\right]$. Furthermore if $q^*_{ik} > 0$ for some i, k then the price ρ^*_{1ik} is equal to (see (21)) $\rho^*_{1ik} = \left[\frac{\partial c_{ik}(q^*_{ik}, h^*_{ik})}{\partial q_{ik}} + \alpha_i \frac{\partial r_{ik}(q^*_{ik}, h^*_{ik})}{\partial q_{ik}}\right]$, or, equivalently (cf. (53)), to $\left[-\hat{c}_{ik}(Q^{2^*}, Q^{3^*}, h^{2*}, h^{3*}) + \rho^*_{3k}\right]$ (for all such i, k).

Under the above pricing mechanism, the optimality conditions (21) and (45) as well as the equilibrium conditions (53) also hold separately (as well as for each individual decision-maker).

In Figure 2, we display the supernetwork in equilibrium in which the equilibrium financial flows, relationship levels, and prices now appear. Note that, if the equilibrium values of the flows (be they financial or relationship levels) on links are identically equal to zero, then those links can effectively be removed from the supernetwork (in equilibrium). Moreover, the size of the equilibrium flows represent the "strength" of respective links (as discussed also in the social network/supply chain network equilibrium model of Wakolbinger and Nagurney (2004)). Thus, the supernetwork model developed here also provides us with the *emergent* integrated social and financial network structures. In the next section, we discuss the dynamic evolution of the financial flows, relationship levels, and prices until this equilibrium is achieved.

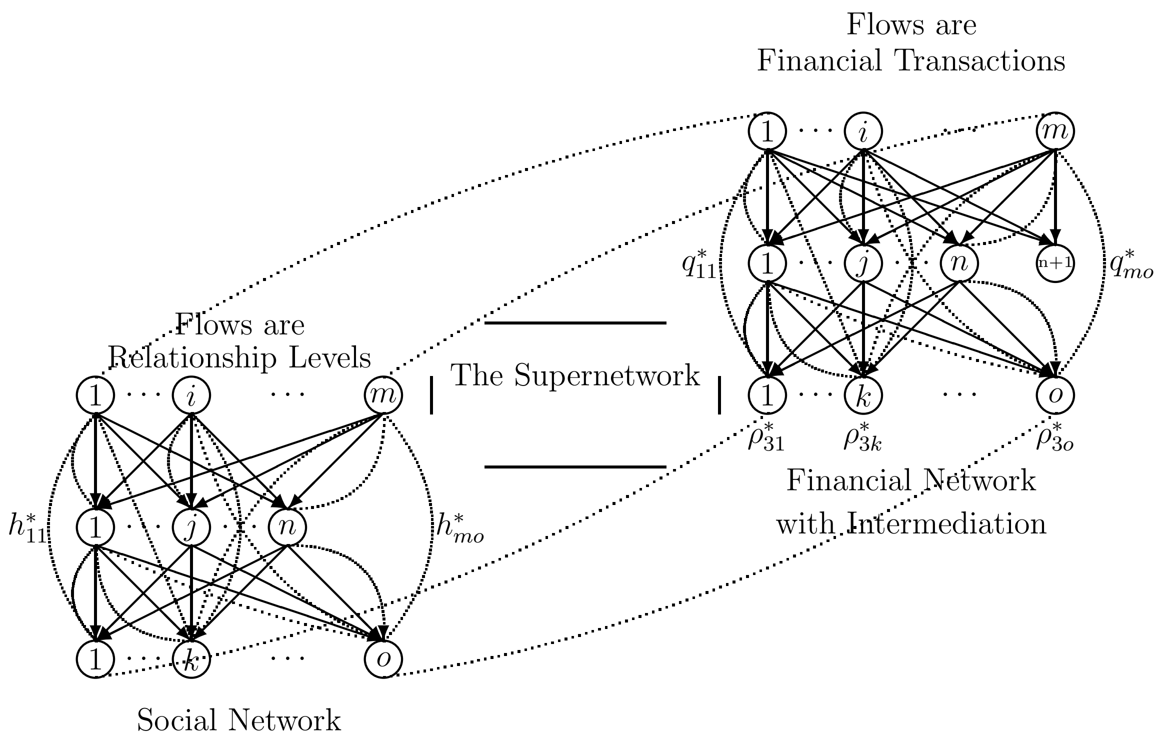

Figure 2: The Supernetwork at Equilibrium

3. The Dynamics

In this section, we describe the dynamics associated with the supernetwork model developed in Section 2 and formulate the corresponding dynamic model as a projected dynamical system (cf. Nagurney and Zhang (1996), Nagurney et al. (2002a, b), Nagurney and Ke (2003), Nagurney and Cruz (2004)). In particular, we describe the disequilibrium dynamics of the financial flows, the relationship levels, as well as the prices.

First, we establish some precursors to the derivation of the dynamics.

Precursors to the Derivation of the Dynamics of the Financial Flows

Recall that, as discussed in Section 2, source agent i is faced with a multicriteria decision-making problem in which he tries to maximize his net revenue, maximize his total relationship value, and to minimize his risk, with the latter two objective functions being weighted accordingly. We will ignore, for the time being, the constraints faced by a source agent. We denote the gradient of source agent i's multicriteria objective function U^i (cf. (18)) with respect to the vector of variables q_i where $q_i \equiv \{q_{i11}, \ldots, q_{in2}, q_{i1}, \ldots, q_{io}\}$ by $\nabla_{q_i} U^i$. It represents source agent i's idealized direction and can be expressed as

$$\nabla_{q_i} U^i = (\frac{\partial U^i}{\partial q_{i11}}, \ldots, \frac{\partial U^i}{\partial q_{in2}}, \frac{\partial U^i}{\partial q_{i1}}, \ldots, \frac{\partial U^i}{\partial q_{io}}).$$

The jl-th component of $\nabla_{q_i} U^i$ is given by:

$$\left[\rho_{1ijl} - \frac{\partial c_{ijl}(q_{ijl}, h_{ijl})}{\partial q_{ijl}} - \alpha_i \frac{\partial r_{ijl}(q_{ijl}, h_{ijl})}{\partial q_{ijl}}\right], \quad \text{for } j = 1, \ldots, n; l = 1, 2, \tag{57}$$

and the $2n + k$-th component of $\nabla_{q_i} U^i$ is given by:

$$\left[\rho_{1ik} - \frac{\partial c_{ik}(q_{ik}, h_{ik})}{\partial q_{ik}} - \alpha_i \frac{\partial r_{ik}(q_{ik}, h_{ik})}{\partial q_{ik}}\right], \quad \text{for } k = 1, \ldots, o. \tag{58}$$

Note that in considering the multicriteria objective function (18) we treat the top-tier prices out of equilibrium since these prices also adjust over time and provide price signals.

Intermediary j also tries to maximize his net revenue and to maximize his total relationship value and to minimize his total risk with the latter two objective functions also being weighted accordingly. For the time being, we also ignore his underlying constraints. Hence, we can express intermediary j's idealized direction by the gradient of his multicriteria objective function U^j (cf. (41)), denoted by $\nabla_{q_j} U^j$, where

$$\nabla_{q_j} U^j = (\frac{\partial U^j}{\partial q_{1j1}}, \ldots, \frac{\partial U^j}{\partial q_{mj2}}, \frac{\partial U^j}{\partial q_{j11}}, \ldots, \frac{\partial U^j}{\partial q_{jo2}})$$

and q_j denotes the vector with components: $\{q_{1j1}, \ldots, q_{mj2}, q_{j11}, \ldots q_{jo2}\}$. Component il of $\nabla_{q_j} U^j$ is given by:

$$\left[-\rho_{1ijl} - \frac{\partial \hat{c}_{ijl}(q_{ijl}, h_{ijl})}{\partial q_{ijl}} - \frac{\partial c_j(Q^1)}{\partial q_{ijl}} - \delta_j \frac{\partial \hat{r}_{ijl}(q_{ijl}, h_{ijl})}{\partial q_{ijl}}\right], \quad \text{for } i = 1, \ldots, m; l = 1, 2. \quad (59)$$

Component $2m + kl$, in turn, is given by:

$$\left[\rho_{2jkl} - \frac{\partial c_{jkl}(q_{jkl}, h_{jkl})}{\partial q_{jkl}} - \delta_j \frac{\partial r_{jkl}(q_{jkl}, h_{jkl})}{\partial q_{jkl}}\right], \quad \text{for } k = 1, \ldots, o; l = 1, 2. \quad (60)$$

Source agent i and intermediary j must agree on the value of q_{ijl} and must respond also to the price signal ϵ_j associated with intermediary j in order for a transaction to take place. Hence, summing up the expression (57) and (59), with the response to the price signal, we get a combined force (see also Nagurney and Dong (2002)), which, after algebraic simplification, results in:

$$\epsilon_j - \frac{\partial c_{ijl}(q_{ijl}, h_{ijl})}{\partial q_{ijl}} - \frac{\partial c_j(Q^1)}{\partial q_{ijl}} - \frac{\partial \hat{c}_{ijl}(q_{ijl}, h_{ijl})}{\partial q_{ijl}} - \alpha_i \frac{\partial r_{ijl}(q_{ijl}, h_{ijl})}{\partial q_{ijl}} - \delta_j \frac{\partial \hat{r}_{ijl}(q_{ijl}, h_{ijl})}{\partial q_{ijl}}. \quad (61)$$

Hence, the financial transaction/flow q_{ijl} responds to the difference between the price signal at intermediary j and the marginal costs and weighted marginal risks as can be seen from expression (61).

From the consumers' perspective, an idealized direction is one where the flow of the financial product between an intermediary/demand market pair (j, k) transacted via mode l can be expressed as:

$$\rho_{3k} - \rho_{2jkl} - \hat{c}_{jkl}(Q^2, Q^3, h^2, h^3), \quad \text{for } l = 1, 2. \quad (62)$$

Furthermore, the equivalent but between a source agent i and demand market k can be expressed as:

$$\rho_{3k} - \rho_{1ik} - \hat{c}_{ik}(Q^2, Q^3, h^2, h^3). \quad (63)$$

Again, the constraints are ignored in (62) and (63).

The consumers at demand market k, on the other hand, must agree with the intermediary j concerning the financial flow q_{jkl}; otherwise, a transaction does not take place. Hence, we may add expressions (60) and (62). Furthermore, we note that they must also respond to the price signals at the intermediaries. This results in a combined force of:

$$\rho_{3k} - \epsilon_j - \frac{\partial c_{jkl}(q_{jkl}, h_{jkl})}{\partial q_{jkl}} - \hat{c}_{jkl}(Q^2, Q^3, h^2, h^3) - \delta_j \frac{\partial r_{jkl}(q_{jkl}, h_{jkl})}{\partial q_{jkl}}. \tag{64}$$

Hence, the flow of the financial product between an intermediary/demand market pair transacted via mode l responds to the difference between the price for the product at the demand market and the price at the particular intermediary plus the sum of the marginal and unit costs and the weighted marginal risk as can be seen from expression (64).

We note that the source agent i and demand market k must also agree on a price and, hence, we can achieve the combined force by adding the expressions (58) and (63), which yields:

$$\rho_{3k} - \frac{\partial c_{ik}(q_{ik}, h_{ik})}{\partial q_{ik}} - \hat{c}_{ik}(Q^2, Q^3, h^2, h^3) - \alpha_i \frac{\partial r_{ik}(q_{ik}, h_{ik})}{\partial q_{ik}}. \tag{65}$$

The flow of the financial product between a source agent/demand market pair responds to the difference between the price for the product at the demand market, the sum of the marginal and unit costs and the weighted marginal risk as can be seen from expression (65).

We now turn to the relationship levels. If we ignore the constraints of a source agent, the gradient of source agent i's objective function with respect to the vector of variables h_i, is denoted by $\nabla_{h_i} U^i$. It represents source agent i's idealized direction with respect to the relationship levels and can be expressed as

$$\nabla_{h_i} U^i = (\frac{\partial U^i}{\partial h_{i11}}, \ldots, \frac{\partial U^i}{\partial h_{in2}}, \frac{\partial U^i}{\partial h_{i1}}, \ldots, \frac{\partial U^i}{\partial h_{io}}).$$

The jl-th component of $\nabla_{h_i} U^i$ is given by:

$$\left[\beta_i \frac{\partial v_{ijl}(h_{ijl})}{\partial h_{ijl}} - \frac{\partial c_{ijl}(q_{ijl}, h_{ijl})}{\partial h_{ijl}} - \frac{\partial b_{ijl}(h_{ijl})}{\partial h_{ijl}} - \alpha_i \frac{\partial r_{ijl}(q_{ijl}, h_{ijl})}{\partial h_{ijl}}\right], \quad \text{for } j = 1, \ldots, n; l = 1, 2, \tag{66}$$

and the $2n + k$-th component of $\nabla_{h_i} U^i$ given by:

$$\left[\beta_i \frac{\partial v_{ik}(h_{ik})}{\partial h_{ik}} - \frac{\partial c_{ik}(q_{ik}, h_{ik})}{\partial h_{ik}} - \frac{\partial b_{ik}(h_{ik})}{\partial h_{ik}} - \alpha_i \frac{\partial r_{ik}(q_{ik}, h_{ik})}{\partial h_{ik}}\right], \quad \text{for } k = 1, \ldots, o. \tag{67}$$

Recall that intermediary j also tries to maximize his net revenue and his total relationship value and to minimize his risk, with his individual weightings of these criteria. Again, for the time being, we ignore the underlying constraints, and, hence, we can express financial intermediary j's idealized direction with respect to his relationship levels by the gradient of his multicriteria objective function U^j, denoted by $\nabla_{h_j} U^j$, where

$$\nabla_{h_j} U^j = \left(\frac{\partial U^j}{\partial h_{1j1}}, \ldots, \frac{\partial U^j}{\partial h_{mj2}}, \frac{\partial U^j}{\partial h_{j11}}, \ldots, \frac{\partial U^j}{\partial h_{jo2}}\right)$$

with component il given by:

$$\left[\gamma_j \frac{\partial \hat{v}_{ijl}(h_{ijl})}{\partial h_{ijl}} - \frac{\partial \hat{c}_{ijl}(q_{ijl}, h_{ijl})}{\partial h_{ijl}} - \frac{\partial \hat{b}_{ijl}(h_{ijl})}{\partial h_{ijl}} - \delta_j \frac{\partial \hat{r}_{ijl}(q_{ijl}, h_{ijl})}{\partial h_{ijl}}\right], \quad \text{for } i = 1, \ldots, m; l = 1, 2, \tag{68}$$

and with component $2m + kl$ given by:

$$\left[\gamma_j \frac{\partial v_{jkl}(h_{jkl})}{\partial h_{jkl}} - \frac{\partial c_{jkl}(q_{jkl}, h_{jkl})}{\partial h_{jkl}} - \frac{\partial b_{jkl}(h_{jkl})}{\partial h_{jkl}} - \delta_j \frac{\partial r_{jkl}(q_{jkl}, h_{jkl})}{\partial h_{jkl}}\right],$$
$$\text{for } k = 1, \ldots, o; l = 1, 2. \tag{69}$$

Source agent i and intermediary j must agree on a relationship level h_{ijl} associated with mode l transactions. Hence, summing up the expression (66) and (68), we get a combined force (see also Nagurney and Dong (2002)), which reduces to:

$$\beta_i \frac{\partial v_{ijl}(h_{ijl})}{\partial h_{ijl}} + \gamma_j \frac{\partial \hat{v}_{ijl}(h_{ijl})}{\partial h_{ijl}} - \frac{\partial c_{ijl}(q_{ijl}, h_{ijl})}{\partial h_{ijl}} - \frac{\partial \hat{c}_{ijl}(q_{ijl}, h_{ijl})}{\partial h_{ijl}} - \frac{\partial b_{ijl}(h_{ijl})}{\partial h_{ijl}}$$
$$- \frac{\partial \hat{b}_{ijl}(h_{ijl})}{\partial h_{ijl}} - \alpha_i \frac{\partial r_{ijl}(q_{ijl}, h_{ijl})}{\partial h_{ijl}} - \delta_j \frac{\partial \hat{r}_{ijl}(q_{ijl}, h_{ijl})}{\partial h_{ijl}}. \tag{70}$$

The relationship level/flow h_{ijl} responds to the difference between the weighted relationship value for source agent i, intermediary j, and mode l, and the sum of the marginal costs and the weighted marginal risks as can be seen from expression (70).

The relationship level/flow h_{ik}, in turn, responds to the difference between the weighted relationship value for source agent i and the sum of the marginal costs and weighted marginal risks as can be seen from expression (67).

Finally, the relationship flow h_{jkl} responds to the difference between the weighted relationship value for intermediary j and the sum of the marginal costs and weighted marginal risks as can be seen from expression (69).

The Dynamics of the Financial Flows between the Source Agents and the Intermediaries and Demand Markets

We now present the dynamics of the financial flows between the source agents and the intermediaries and the source agents and the demand markets in which we explicitly include the underlying constraints.

Recall the projection operator $\Pi_\kappa(x, v)$ (see also Nagurney and Zhang (1996)) defined as:

$$\Pi_\kappa(x, v) = \lim_{\delta \to 0} \frac{P_\kappa(x + \delta v) - x}{\delta}, \tag{71}$$

where P_κ is the norm projection given by

$$P_\kappa(x) = \text{argmin}_{x' \in \kappa} \|x' - x\|. \tag{72}$$

Let $F_i \equiv (F_{i11}, \ldots, F_{in2}, F_{i1}, \ldots, F_{io})$ where F_{ijl} is minus the term in expression (61) and F_{ik} is minus the expression in (65) (see also following (56)). The dynamics for the vector of financial flows q_i associated with source agent i can be mathematically expressed as:

$$\dot{q}_i = \Pi_{K_i}(q_i, -F_i), \tag{73}$$

where K_i is the feasible set consisting of constraint (1) for source agent i and the nonnegativity assumptions on the elements of q_i. According to expression (72) the financial flows associated with a source agent will evolve according to (61) and (65), while, at the same time, it is ensured that they do not become negative and the constraint (1) is not violated for the source agent. If we define $K \equiv \prod_{i=1}^m K_i$ then the evolution of all the financial transactions of the source agents is given by (71) where $\kappa = K$, and, in this case, $x = (Q^1, Q^2)$, with $v = (F_1, \ldots, F_m)$.

The Dynamics of the Financial Products between the Financial Intermediaries and the Demand Markets

We now turn to the description of the dynamics of the financial products between the financial intermediaries and the demand markets. The rate of change of the financial product q_{jkl} transacted via mode l evolves according to (64). Of course, it must also be guaranteed that these financial flows do not become negative, that is, satisfy the constraints. Therefore, we may write:

$$\dot{q}_{jkl} = \begin{cases} \rho_{3k} - \epsilon_j - \frac{\partial c_{jkl}(q_{jkl},h_{jkl})}{\partial q_{jkl}} - \hat{c}_{jkl}(Q^2,Q^3,h^2,h^3) - \delta_j \frac{\partial r_{jkl}(q_{jkl},h_{jkl})}{\partial q_{jkl}}, & \text{if} \quad q_{jkl} > 0 \\ \max\{0, \rho_{3k} - \epsilon_j - \frac{\partial c_{jkl}(q_{jkl},h_{jkl})}{\partial q_{jkl}} - \hat{c}_{jkl}(Q^2,Q^3,h^2,h^3) - \delta_j \frac{\partial r_{jkl}(q_{jkl},h_{jkl})}{\partial q_{jkl}}\}, & \text{if} \quad q_{jkl} = 0. \end{cases} \tag{74}$$

Here, $\dot{q}_{jkl}$ denotes the rate of change of the financial product flow q_{jkl}.

As can be seen from expression (74) the volume of the financial product transacted via mode l between financial intermediary j and demand market k will increase if the price the consumers are willing to pay for the financial product at the demand market is higher than the price the financial intermediaries charge plus the marginal transaction cost (from the perspective of the intermediaries) and the unit transaction cost (at an instant in time) plus the weighted marginal risk associated with the intermediary/market/mode combination. If the price the consumers are willing to pay is lower, then the volume of financial product between that financial intermediary and demand market pair will decrease.

The Dynamics of the Relationship Levels between the Source Agents and the Financial Intermediaries

Now the dynamics of the relationship levels between the source agents and the intermediaries are described. The rate of change of the relationship level h_{ijl} evolves according to (70). Again, one must also guarantee that the relationship levels do not become negative.

Moreover, they may not exceed the level equal to one. Hence, we can immediately write:

$$\dot{h}_{ijl} = \begin{cases} \beta_i \frac{\partial v_{ijl}(h_{ijl})}{\partial h_{ijl}} + \gamma_j \frac{\partial \hat{v}_{ijl}(h_{ijl})}{\partial h_{ijl}} - \frac{\partial c_{ijl}(q_{ijl},h_{ijl})}{\partial h_{ijl}} - \frac{\partial \hat{c}_{ijl}(q_{ijl},h_{ijl})}{\partial h_{ijl}} - \frac{\partial b_{ijl}(h_{ijl})}{\partial h_{ijl}} & \\ -\frac{\partial \hat{b}_{ijl}(h_{ijl})}{\partial h_{ijl}} - \alpha_i \frac{\partial r_{ijl}(q_{ijl},h_{ijl})}{\partial h_{ijl}} - \delta_j \frac{\partial \hat{r}_{ijl}(q_{ijl},h_{ijl})}{\partial h_{ijl}}, & \text{if } 0 < h_{ijl} < 1 \\ \min\{1, \max\{0, \beta_i \frac{\partial v_{ijl}(h_{ijl})}{\partial h_{ijl}} + \gamma_j \frac{\partial \hat{v}_{ijl}(h_{ijl})}{\partial h_{ijl}} - \frac{\partial c_{ijl}(q_{ijl},h_{ijl})}{\partial h_{ijl}} - \frac{\partial \hat{c}_{ijl}(q_{ijl},h_{ijl})}{\partial h_{ijl}} & \\ -\frac{\partial b_{ijl}(h_{ijl})}{\partial h_{ijl}} - \frac{\partial \hat{b}_{ijl}(h_{ijl})}{\partial h_{ijl}} - \alpha_i \frac{\partial r_{ijl}(q_{ijl},h_{ijl})}{\partial h_{ijl}} - \delta_j \frac{\partial \hat{r}_{ijl}(q_{ijl},h_{ijl})}{\partial h_{ijl}}\}\}, & \text{otherwise,} \end{cases} \tag{75}$$

where $\dot{h}_{ijl}$ denotes the rate of change of the relationship level h_{ijl}.

This shows that if the sum of the weighted relationship values for the source agent and the intermediary are higher than the total marginal costs plus the total weighted marginal risk, then the level of relationship between that financial source agent and intermediary pair will increase. If it is lower, the relationship value will decrease.

The Dynamics of the Relationship Levels between the Source Agents and the Demand Markets

Here we describe the dynamics of the relationship levels between the source agents and the demand markets. The rate of change of the relationship level h_{ik} is assumed to evolve according to (67). One also must guarantee that these relationship levels do not become negative (nor higher than one). Hence, one may write:

$$\dot{h}_{ik} = \begin{cases} \beta_i \frac{\partial v_{ik}(h_{ik})}{\partial h_{ik}} - \frac{\partial c_{ik}(q_{ik},h_{ik})}{\partial h_{ik}} - \frac{\partial b_{ik}(h_{ik})}{\partial h_{ik}} - \alpha_i \frac{\partial r_{ik}(q_{ik},h_{ik})}{\partial h_{ik}}, & \text{if } 0 < h_{ik} < 1 \\ \min\{1, \max\{0, \beta_i \frac{\partial v_{ik}(h_{ik})}{\partial h_{ik}} - \frac{\partial c_{ik}(q_{ik},h_{ik})}{\partial h_{ik}} - \frac{\partial b_{ik}(h_{ik})}{\partial h_{ik}} - \alpha_i \frac{\partial r_{ik}(q_{ik},h_{ik})}{\partial h_{ik}}\}\}, & \text{otherwise,} \end{cases} \tag{76}$$

where $\dot{h}_{ik}$ denotes the rate of change of the relationship level h_{ik}. This shows that if the weighted relationship value for the source agent is higher than the total marginal costs plus the total weighted marginal risk, then the level of relationship between that financial source agent and demand market pair will increase. If it is lower, the relationship value will decrease. Of course, the bounds on the relationship levels must also hold.

The Dynamics of the Relationship Levels between the Financial Intermediaries and the Demand Markets

The dynamics of the relationship levels between the financial intermediaries and demand markets are now described. The rate of change of the relationship level product h_{jkl} transacted via mode l is assumed to evolve according to (69), where, of course, one also must guarantee that the relationship levels do not become negative nor exceed one. Hence, one may write:

$$\dot{h}_{jkl} = \begin{cases} \gamma_j \frac{\partial v_{jkl}(h_{jkl})}{\partial h_{jkl}} - \frac{\partial c_{jkl}(q_{jkl},h_{jkl})}{\partial h_{jkl}} - \frac{\partial b_{jkl}(h_{jkl})}{\partial h_{jkl}} - \delta_j \frac{\partial r_{jkl}(q_{jkl},h_{jkl})}{\partial h_{jkl}}, & \text{if } 0 < h_{jkl} < 1, \\ \min\{1, \max\{0, \gamma_j \frac{\partial v_{jkl}(h_{jkl})}{\partial h_{jkl}} - \frac{\partial c_{jkl}(q_{jkl},h_{jkl})}{\partial h_{jkl}} - \frac{\partial b_{jkl}(h_{jkl})}{\partial h_{jkl}} - \delta_j \frac{\partial r_{jkl}(q_{jkl},h_{jkl})}{\partial h_{jkl}}\}\}, & \text{otherwise,} \end{cases} \quad (77)$$

where $\dot{h}_{jkl}$ denotes the rate of change of the relationship level h_{jkl}. Expression (77) reveals that if the weighted relationship value for the intermediary with the demand market is higher than the total marginal costs plus the total weighted marginal risk, then the level of relationship between that intermediary and demand market pair will increase. If it is lower, the relationship value will decrease.

The Dynamics of the Prices at the Demand Market

The price ρ_{3k} associated with the financial product at demand market k cannot become negative. Its dynamics can be expressed as:

$$\dot{\rho}_{3k} = \begin{cases} d_k(\rho_3) - \sum_{j=1}^{n} \sum_{l=1}^{2} q_{jkl} - \sum_{i=1}^{m} q_{ik}, & \text{if} \quad \rho_{3k} > 0 \\ \max\{0, d_k(\rho_3) - \sum_{j=1}^{n} \sum_{l=1}^{2} q_{jkl} - \sum_{i=1}^{m} q_{ik}\}, & \text{if} \quad \rho_{3k} = 0, \end{cases} \quad (78)$$

where $\dot{\rho}_{3k}$ denotes the rate of change of the price ρ_{3k}. If the demand for the financial product at the demand market (at an instant in time) is higher than the amount available, the price at that demand market will increase; if it is lower, it will decrease.

The Dynamics of the Prices at the Financial Intermediaries

The prices at the financial intermediaries reflect supply and demand conditions. The price associated with financial intermediary j, ϵ_j, develops according to:

$$\dot{\epsilon}_j = \begin{cases} \sum_{k=1}^{o} \sum_{l=1}^{2} q_{jkl} - \sum_{i=1}^{m} \sum_{l=1}^{2} q_{ijl}, & \text{if} \quad \epsilon_j > 0 \\ \max\{0, \sum_{k=1}^{o} \sum_{l=1}^{2} q_{jkl} - \sum_{i=1}^{m} \sum_{l=1}^{2} q_{ijl}\}, & \text{if} \quad \epsilon_j = 0, \end{cases} \quad (79)$$

where $\dot{\epsilon}_j$ denotes the rate of change of the price ϵ_j.

Therefore, if the amount of the product that consumers (at an instant in time) wish to transact is higher than the amount available at the financial intermediary, the price at the financial intermediary will increase; if it is lower, then the price will decrease.

The Projected Dynamical System

We now turn to stating the complete dynamic model. In the dynamic model the flows evolve according to the mechanisms described above; specifically, the financial flows from the source agents evolve according to (73) for all source agents i. The financial flows from the financial intermediaries to the demand markets evolve according to (74) for all financial intermediaries j, demand markets k, and modes l. The relationship levels between source agents and financial intermediaries for all modes l evolve according to (75), the relationship levels between source agents i and demand markets k evolve according to (76), and the relationship levels between financial intermediaries j and demand markets k for all modes l evolve according to (77). Furthermore, the prices associated with the intermediaries evolve according to (79) for all intermediaries j, and the demand market prices evolve according to (78) for all k.

If we let X and $F(X)$ be defined following (56) then the dynamic model described by (73) - (79) for all i, j, k, l can be rewritten as the *projected dynamical system* (PDS) (cf. Nagurney and Zhang (1996)) defined by the following initial value problem:

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X_0, \tag{80}$$

where $\Pi_{\mathcal{K}}$ denotes the projection of $-F(X)$ onto $\mathcal{K}$ at X and X_0 is equal to the point corresponding to the initial financial product transactions, relationship levels, shadow prices, and the demand market prices.

The trajectory of (80) describes the dynamic evolution of the relationship levels on the social network, the financial product transactions on the financial network, the demand market prices and the Lagrange multipliers or shadow prices associated with the intermediaries. The projection operation guarantees the constraints underlying the supernetwork system are not violated. Recall that the constraint set $\mathcal{K}$ consists not only of the conservation of flow constraints (cf. (1)) associated with the source agents but also the nonnegativity constraints associated with all the financial flows, the prices, as well as the relationships levels. Moreover, the relationship levels are assumed to not exceed the value of one.

Theorem 2: The Set of Stationary Points of the Projected Dynamical System Coincides with the Set of Solutions to the Variational Inequality Problem

The set of stationary points of the projected dynamical system (80) coincides with the set of solutions of variational inequality (54).

Proof: According to Dupuis and Nagurney (1993), the necessary and sufficient condition for X^* to be a stationary point of the projected dynamical system (80), that is, to satisfy:

$$\dot{X} = 0 = \Pi_{\mathcal{K}}(X^*, -F(X^*)), \tag{81}$$

is that $X^* \in \mathcal{K}$ solves the variational inequality problem:

$$\langle F(X^*)^T, X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{82}$$

where, in our problem, X and $F(X)$ are defined following (56). Hence, this is exactly variational inequality (54). According to Theorem 1 variational inequality (54) is the same as $(Q^{1*}, Q^{2*}, Q^{3^*}, h^{1*}, h^{2*}, h^{3*}, \epsilon^*, \rho_3^*)$ which is an equilibrium pattern according to Definition 1. $\square$

Theorem 2 shows that the solution to the variational inequality problem (54) governing the equilibrium of the supernetwork model developed in Section 2 and the stationary points of the projected dynamical system (80) which describes the dynamic analogue of the supernetwork model are one and the same. Therefore, a stationary point of the dynamic supernetwork model satisfies also the supernetwork equilibrium conditions as defined in Definition 1.

We now state the following:

Theorem 3: Existence and Uniqueness of a Solution to the Initial Value Problem

Assume that $F(X)$ is Lipschitz continuous, that is,

$$\|F(X') - F(X'')\| \leq L\|X' - X''\|, \quad \forall X', X'' \in \mathcal{K}, \text{ where } L > 0. \tag{83}$$

Then, for any $X_0 \in \mathcal{K}$, there exists a unique solution $X_0(t)$ to the initial value problem (80).

Proof: Lipschitz continuity of the function F is sufficient for the result following Theorem 2.5 in Nagurney and Zhang (1996). □

Under suitable conditions on the underlying functions (see also Nagurney et al. (2002a, b) and Nagurney and Dong (2002), Zhang and Nagurney (1995, 1996)), one can obtain stability results for the supernetwork. A similar result was obtained for a supply chain network model with electronic commerce and relationship levels in Wakolbinger and Nagurney (2004).

4. The Algorithm

In this section, we propose the Euler method for the computation of solutions to variational inequality (54); equivalently, the stationary points of the projected dynamical system (80). The Euler method is a special case of the general iterative scheme introduced by Dupuis and Nagurney (1993) for the solution of projected dynamical systems. Besides providing a solution to variational inequality problem (54), this algorithm also yields a time discretization of the continuous-time adjustment process of the projected dynamical system (80). This discretization may also be interpreted as a discrete-time adjustment process. Conditions for convergence of this algorithm are given in Dupuis and Nagurney (1993) and in Nagurney and Zhang (1996). In Section 5, we apply this algorithm to several numerical examples.

The Euler Method

The statement of the Euler method is the following: At iteration τ compute

$$X_{\tau} = P_{\mathcal{K}}(X_{\tau-1} - \alpha_{\tau-1} F(X_{\tau-1})), \tag{84}$$

where $P_{\mathcal{K}}$ denotes the projection operator in the Euclidean sense (cf. (72) and Nagurney (1999)) onto the closed convex set $\mathcal{K}$ and $F(X)$ is defined following (56)). We discuss the sequence of positive terms $\{\alpha_{\tau}\}$ below.

The complete statement of the method in the context of the dynamic supernetwork model is as follows:

Step 0: Initialization Step

Set $(Q^1_0, Q^2_0, Q^3_0, h^1_0, h^2_0, h^3_0, \epsilon_0, \rho_{30}) \in \mathcal{K}$ and set the sequence $\{\alpha_{\tau}\}$ with $\alpha_{\tau} > 0$ for all τ and $\sum_{\tau=1}^{\infty} \alpha_{\tau} = \infty$, as $\tau \rightarrow \infty$. The $\{\alpha_{\tau}\}$ sequence must satisfy these conditions for convergence (see additional convergence results in Dupuis and Nagurney (1993)).

Step 1: Computation Step

Compute $(Q^{1^\tau}, Q^{2^\tau}, Q^{3^\tau}, h^{1^\tau}, h^{2^\tau}, h^{3^\tau}, \epsilon^{\tau}, \rho_3^{\tau}) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\Big[q_{ijl}^{\tau}+a_{\tau}(\frac{\partial c_{ijl}(q_{ijl}^{\tau-1},h_{ijl}^{\tau-1})}{\partial q_{ijl}}+\frac{\partial c_{j}(Q^{1^{\tau-1}})}{\partial q_{ijl}}+\frac{\partial \hat{c}_{ijl}(q_{ijl}^{\tau-1},h_{ijl}^{\tau-1})}{\partial q_{ijl}}$$

$$+\alpha_i\frac{\partial r_{ijl}(q_{ijl}^{\tau-1},h_{ijl}^{\tau-1})}{\partial q_{ijl}}+\delta_j\frac{\partial \hat{r}_{ijl}(q_{ijl}^{\tau-1},h_{ijl}^{\tau-1})}{\partial q_{ijl}}-\epsilon_j^{\tau-1})-q_{ijl}^{\tau-1}\Big]\times\Big[q_{ijl}-q_{ijl}^{\tau}\Big]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\Big[q_{ik}^{\tau}+a_{\tau}(\frac{\partial c_{ik}(q_{ik}^{\tau-1},h_{ik}^{\tau-1})}{\partial q_{ik}}+\hat{c}_{ik}(Q^{2^{\tau-1}},Q^{3^{\tau-1}},h^{2^{\tau-1}},h^{3^{\tau-1}})+\alpha_i\frac{\partial r_{ik}(q_{ik}^{\tau-1},h_{ik}^{\tau-1})}{\partial q_{ik}}$$

$$-\rho_{3k}^{\tau-1})-q_{ik}^{\tau-1}\Big]\times[q_{ik}-q_{ik}^{\tau}]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\Big[q_{jkl}^{\tau}+a_{\tau}(\frac{\partial c_{jkl}(q_{jkl}^{\tau-1},h_{jkl}^{\tau-1})}{\partial q_{jkl}}+\hat{c}_{jkl}(Q^{2^{\tau-1}},Q^{3^{\tau-1}},h^{2^{\tau-1}},h^{3^{\tau-1}})+\epsilon_j^{\tau-1}+\delta_j\frac{\partial r_{jkl}(q_{jkl}^{\tau-1},h_{jkl}^{\tau-1})}{\partial q_{jkl}}$$

$$-\rho_{3k}^{\tau-1})-q_{jkl}^{\tau-1}\Big]\times\Big[q_{jkl}-q_{jkl}^{\tau}\Big]$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\Big[h_{ijl}^{\tau}+a_{\tau}(\frac{\partial c_{ijl}(q_{ijl}^{\tau-1},h_{ijl}^{\tau-1})}{\partial h_{ijl}}+\frac{\partial \hat{c}_{ijl}(q_{ijl}^{\tau-1},h_{ijl}^{\tau-1})}{\partial h_{ijl}}-\beta_i\frac{\partial v_{ijl}(h_{ijl}^{\tau-1})}{\partial h_{ijl}}-\gamma_j\frac{\partial \hat{v}_{ijl}(h_{ijl}^{\tau-1})}{\partial h_{ijl}}+$$

$$\alpha_i\frac{\partial r_{ijl}(q_{ijl}^{\tau-1},h_{ijl}^{\tau-1})}{\partial h_{ijl}}+\delta_j\frac{\partial \hat{r}_{ijl}(q_{ijl}^{\tau-1},h_{ijl}^{\tau-1})}{\partial h_{ijl}}+\frac{\partial b_{ijl}(h_{ijl}^{\tau-1})}{\partial h_{ijl}}+\frac{\partial \hat{b}_{ijl}(h_{ijl}^{\tau-1})}{\partial h_{ijl}})-h_{ijl}^{\tau-1}\Big]\times\Big[h_{ijl}-h_{ijl}^{\tau}\Big]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\Big[h_{ik}^{\tau}+a_{\tau}(\frac{\partial c_{ik}(q_{ik}^{\tau-1},h_{ik}^{\tau-1})}{\partial h_{ik}}-\beta_i\frac{\partial v_{ik}(h_{ik}^{\tau-1})}{\partial h_{ik}}+\alpha_i\frac{\partial r_{ik}(q_{ik}^{\tau-1},h_{ik}^{\tau-1})}{\partial h_{ik}}+\frac{\partial b_{ik}(h_{ik}^{\tau-1})}{\partial h_{ik}})-h_{ik}^{\tau-1}\Big]$$

$$\times[h_{ik}-h_{ik}^{\tau}]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\Big[h_{jkl}^{\tau}+a_{\tau}(\frac{\partial c_{jkl}(q_{jkl}^{\tau-1},h_{jkl}^{\tau-1})}{\partial h_{jkl}}-\gamma_j\frac{\partial v_{jkl}(h_{jkl}^{\tau-1})}{\partial h_{jkl}}+\delta_j\frac{\partial r_{jkl}(q_{jkl}^{\tau-1},h_{jkl}^{\tau-1})}{\partial h_{jkl}}+\frac{\partial b_{jkl}(h_{jkl}^{\tau-1})}{\partial h_{jkl}})-h_{jkl}^{\tau-1}\Big]$$

$$\times\Big[h_{jkl}-h_{jkl}^{\tau}\Big]$$

$$+\sum_{j=1}^{n}\Big[\epsilon_j^{\tau}+a_{\tau}(\sum_{i=1}^{m}\sum_{l=1}^{2}q_{ijl}^{\tau-1}-\sum_{k=1}^{o}\sum_{l=1}^{2}q_{jkl}^{\tau-1})-\epsilon_j^{\tau-1}\Big]\times\Big[\epsilon_j-\epsilon_j^{\tau}\Big]$$

$$+\sum_{k=1}^{o}\Big[\rho_{3k}^{\tau}+a_{\tau}(\sum_{j=1}^{n}\sum_{l=1}^{2}q_{jkl}^{\tau-1}+\sum_{i=1}^{m}q_{ik}^{\tau-1}-d_k(\rho_{3k}^{\tau-1}))-\rho_{3k}^{\tau-1}\Big]\times[\rho_{3k}-\rho_{3k}^{\tau}]\geq 0,$$

$$\forall(Q^1,Q^2,Q^3,h^1,h^2,h^3,\epsilon,\rho_3)\in\mathcal{K}. \quad (85)$$

Step 2: Convergence Verification

If $|q_{ijl}^{\tau} - q_{ijl}^{\tau-1}| \leq e$, $|q_{ik}^{\tau} - q_{ik}^{\tau-1}| \leq e$, $|q_{jkl}^{\tau} - q_{jkl}^{\tau-1}| \leq e$, $|h_{ijl}^{\tau} - h_{ijl}^{\tau-1}| \leq e$, $|h_{ik}^{\tau} - h_{ik}^{\tau-1}| \leq e$, $|h_{jkl}^{\tau} - h_{jkl}^{\tau-1}| \leq e$, $|\epsilon_j^{\tau} - \epsilon_j^{\tau-1}| \leq e$, $|\rho_{3k}^{\tau} - \rho_{3k}^{\tau-1}| \leq e$, for all $i = 1, \ldots, m$; $j = 1, \ldots, n$; $l = 1, 2$; $k = 1, \ldots, o$, with $e > 0$, a pre-specified tolerance, then stop; otherwise, set $\tau := \tau + 1$, and go to Step 1.

Due to the simplicity of the feasible set $\mathcal{K}$ the solution of (85) is accomplished exactly and in closed form. In (85) the variational subproblem of the variables (Q^1, Q^2) can be solved using exact equilibration (cf. Dafermos and Sparrow (1969), Nagurney (1999)). The other variables can be obtained using the following explicit formulae:

Computation of the Financial Product Transactions

At iteration τ compute the q_{jkl}^{τ}s according to:

$$q_{jkl}^{\tau} = \max\{0, q_{jkl}^{\tau-1} - a_{\tau}(\frac{\partial c_{jkl}(q_{jkl}^{\tau-1}, h_{jkl}^{\tau-1})}{\partial q_{jkl}} + \hat{c}_{jkl}(Q^{2^{\tau-1}}, Q^{3^{\tau-1}}, h^{2\tau-1}, h^{3\tau-1}) + \epsilon_j^{\tau-1} +$$

$$\delta_j \frac{\partial r_{jkl}(q_{jkl}^{\tau-1}, h_{jkl}^{\tau-1})}{\partial q_{jkl}} - \rho_{3kl}^{\tau-1})\}, \forall j, k, l. \quad (86)$$

Computation of the Relationship Levels

At iteration τ compute the h_{ijl}^{τ}s according to:

$$h_{ijl}^{\tau} = \min\{1, \max\{0, h_{ijl}^{\tau-1} - a_{\tau}(\frac{\partial c_{ijl}(q_{ijl}^{\tau-1}, h_{ijl}^{\tau-1})}{\partial h_{ijl}} + \frac{\partial \hat{c}_{ijl}(q_{ijl}^{\tau-1}, h_{ijl}^{\tau-1})}{\partial h_{ijl}} - \beta_i \frac{\partial v_{ijl}(h_{ijl}^{\tau-1})}{\partial h_{ijl}} -$$

$$\gamma_j \frac{\partial \hat{v}_{ijl}(h_{ijl}^{\tau-1})}{\partial h_{ijl}} + \alpha_i \frac{\partial r_{ijl}(q_{ijl}^{\tau-1}, h_{ijl}^{\tau-1})}{\partial h_{ijl}} + \delta_j \frac{\partial \hat{r}_{ijl}(q_{ijl}^{\tau-1}, h_{ijl}^{\tau-1})}{\partial h_{ijl}} + \frac{\partial b_{ijl}(h_{ijl}^{\tau-1})}{\partial h_{ijl}} + \frac{\partial \hat{b}_{ijl}(h_{ijl}^{\tau-1})}{\partial h_{ijl}})\}\}, \quad \forall i, j, l. \quad (87)$$

Furthermore, at iteration τ compute the h_{ik}^{τ}s according to:

$$h_{ik}^{\tau} = \min\{1, \max\{0, h_{ik}^{\tau-1} - a_{\tau}(\frac{\partial c_{ik}(q_{ik}^{\tau-1}, h_{ik}^{\tau-1})}{\partial h_{ik}} - \beta_i \frac{\partial v_{ik}(h_{ik}^{\tau-1})}{\partial h_{ik}} + \alpha_i \frac{\partial r_{ik}(q_{ik}^{\tau-1}, h_{ik}^{\tau-1})}{\partial h_{ik}}$$

$$+\frac{\partial b_{ik}(h_{ik}^{\tau-1})}{\partial h_{ik}})\}\}, \quad \forall i,k. \tag{88}$$

At iteration τ compute the h_{jkl}^{τ}s satisfying:

$$h_{jkl}^{\tau}=\min\{1,\max\{0,h_{jkl}^{\tau-1}-a_{\tau}(\frac{\partial c_{jkl}(q_{jkl}^{\tau-1},h_{jkl}^{\tau-1})}{\partial h_{jkl}}-\gamma_j\frac{\partial v_{jkl}(h_{jkl}^{\tau-1})}{\partial h_{jkl}}+\delta_j\frac{\partial r_{jkl}(q_{jkl}^{\tau-1},h_{jkl}^{\tau-1})}{\partial h_{jkl}}$$

$$+\frac{\partial b_{jkl}(h_{jkl}^{\tau-1})}{\partial h_{jkl}})\}\}, \quad \forall j,k,l. \tag{89}$$

Computation of the Shadow Prices

At iteration τ compute the ϵ_j^{τ}s thus:

$$\epsilon_j^{\tau}=\max\{0,\epsilon_j^{\tau-1}-a_{\tau}(\sum_{i=1}^{m}\sum_{l=1}^{2}q_{ijl}^{\tau-1}-\sum_{k=1}^{o}\sum_{l=1}^{2}q_{jkl}^{\tau-1})\}, \quad \forall j. \tag{90}$$

Computation of the Demand Market Prices

Finally, at iteration τ compute the demand market prices, the ρ_{3k}^{τ}s, according to:

$$\rho_{3k}^{\tau}=\max\{0,\rho_{3k}^{\tau-1}-a_{\tau}(\sum_{j=1}^{n}\sum_{l=1}^{2}q_{jkl}^{\tau-1}+\sum_{i=1}^{m}q_{ik}^{\tau-1}-d_k(\rho_{3k}^{\tau-1}))\}, \quad \forall k. \tag{91}$$

As one can see in the discrete-time adjustment process(es) described above the process is initialized with a vector of financial flows, relationship levels, and prices. For example, the relationship levels may be set to zero (and the same holds for the prices, initially). The financial flows, shadow prices, and the demand market prices are computed in the financial network of the supernetwork. The financial product transactions between intermediaries and demand markets are computed according to (86). The relationship levels are computed in the social network of the supernetwork according to (87), (88), and (89), respectively. Finally, the shadow prices are computed according to (90) and the demand market prices are computed according to (91).

The dynamic supernetwork system will then evolve according to the discrete-time adjustment process (86) through (91) until a stationary/equilibrium point of the projected dynamical system (80) (equivalently, and a solution to variational inequality (54)) is achieved.

Once the convergence tolerance has been reached then the equilibrium conditions according to Definition 1 are satisfied as one can see from (86) through (91).

5. Numerical Examples

In this Section, we apply the discrete-time algorithm (the Euler method) to several simple, but illustrative, numerical examples. The algorithm was implemented in FORTRAN and the computer system used was a Sun system located at the University of Massachusetts at Amherst. For the solution of the induced network subproblems in (Q^1, Q^2) we utilized the exact equilibration algorithm (see Dafermos and Sparrow (1969), Nagurney (1999), and Nagurney and Ke (2001)). The other subproblems in the Q^3, ϵ, and the ρ_3 variables were solved exactly and in closed form as described in Section 4.

The convergence criterion used was that the absolute value of the financial flows, relationship levels, and prices between two successive iterations differed by no more than 10^{-4}. For the examples, the sequence $\{\alpha_\tau\}$ was set to $\{\alpha_\tau\} = \{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\}$, which is of the form required by the algorithm for convergence. The algorithm provides a time discretization of the dynamic trajectories and converges (under reasonable conditions) to a stationary point of the projected dynamical system; equivalently, to the solution of the corresponding variational inequality problem, yielding, hence, the equilibrium financial flow, relationship level, and price pattern.

We initialized the algorithm as follows: we set $q_{ij1} = \frac{S^i}{n}$ for each source agent i and all intermediaries j. All the other variables were initialized to zero.

Example 1

The first as well as the subsequent examples consisted of two source agents, two intermediaries, and two demand markets, as depicted in Figure 3. Hence, for simplicity, we assumed that no electronic transactions were allowed.

The data for the first example were constructed for easy interpretation purposes. The financial holdings of the two source agents were: $S^1 = 20$ and $S^2 = 20$. We assumed that the risk functions for both the source agents and the financial intermediaries consisted of variance-covariance matrices and these were equal to the identity matrices (cf. Nagurney

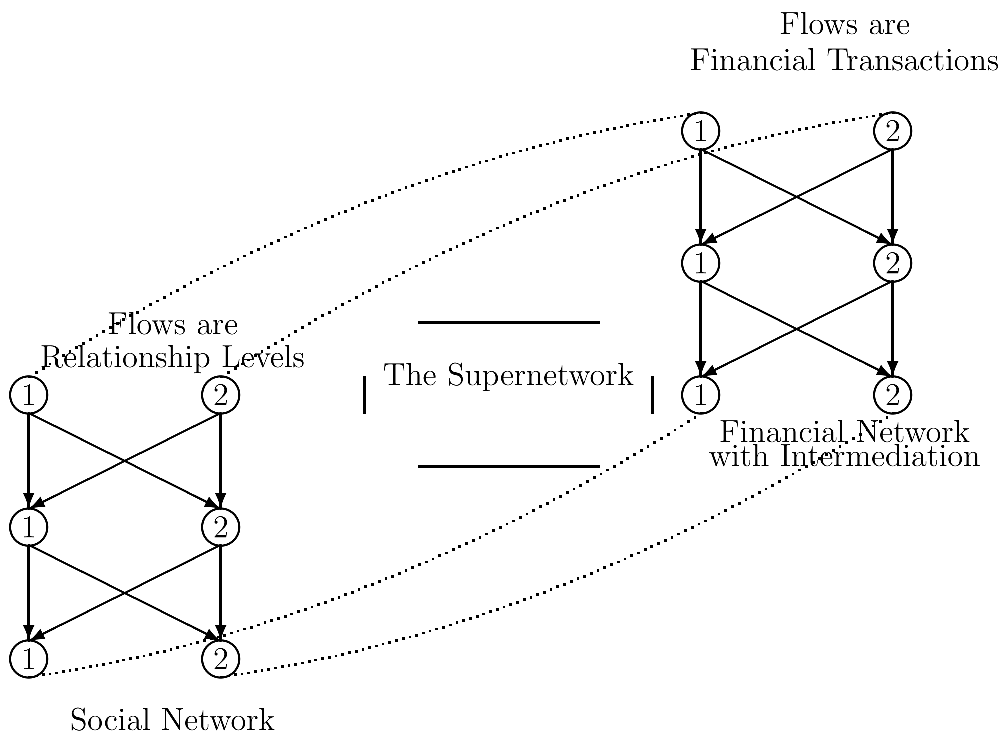

Figure 3: The Supernetwork Structure of Numerical Examples

and Ke (2003)).

The transaction cost functions faced by the source agents associated with transacting with the intermediaries (cf. (4)) were given by:

$$c_{ij1}(q_{ij1}, h_{ij1}) = .5q_{ijl}^2 + 3.5q_{ijl} - h_{ij1}, \quad \text{for } i = 1, 2; j = 1, 2.$$

The handling costs of the intermediaries, in turn (see (27)), were given by:

$$c_j(Q^1) = .5(\sum_{i=1}^{2} q_{ij1})^2, \quad \text{for } j = 1, 2.$$

The transaction costs of the intermediaries associated with transacting with the source agents were (cf. (25)) given by:

$$\hat{c}_{ij1}(q_{ij1}, h_{ijl}) = 1.5q_{ijl}^2 + 3q_{ijl}, \quad \text{for } i = 1, 2; j = 1, 2.$$

The demand functions at the demand markets (refer to (47)) were:

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1000, \quad d_2(\rho_3) = -2\rho_{32} - 1.5\rho_{31} + 1000,$$

and the transaction costs between the intermediaries and the consumers at the demand markets (see (48)) were given by:

$$\hat{c}_{jk1}(Q^2, Q^2, h^2, h^3) = q_{jk1} - h_{jk1} + 5, \quad \text{for } j = 1, 2; k = 1, 2.$$

We assumed for this and the subsequent examples that the transaction costs as perceived by the intermediaries and associated with transacting with the demand markets were all zero, that is, $c_{jkl}(q_{jkl}) = 0$, for all j, k, l.

The relationship value functions (14) and (38) were given by:

$$v_{ij1}(h_{ij1}) = h_{ij1}, \quad \forall i, j; \quad v_{jk1}(h_{jk1}) = h_{jk1}, \quad \forall j, k,$$

with all other relationship value functions being set equal to zero.

The relationship cost functions (cf. (2) and (24)) were as follows:

$$b_{ijl}(h_{ijl}) = 2h_{ijl} + 1, \quad \forall i, j, l = 1; \quad b_{jkl}(h_{jk1}) = h_{jk1} + 1, \quad \forall j, k.$$

We set the weights associated with the risk functions and the relationship values to one.

The Euler method converged and yielded the following equilibrium financial flow pattern:

$$Q^{1*} := q^*_{111} = q^*_{121} = q^*_{211} = q^*_{221} = 1.00,$$
$$Q^{3*} := q^*_{111} = q^*_{121} = q^*_{211} = q^*_{221} = 1.00.$$

There was slack associated with the source agent's financial transactions and, in fact, 18 units of financial flows were not allocated to any financial intermediary from each source agent. Hence, each source agent chose to not invest a substantial portion of its financial holdings.

Note that since there were no electronic transactions and, hence, only physical ones, the above vectors are only associated with the transaction costs on the physical links, as are the resulting equilibrium financial flows.

The equilibrium relationship levels were all equal to zero.

The vector of shadow prices ϵ^* had components: $\epsilon^*_1 = \epsilon^*_2 = 14.5808$, and the computed demand prices at the demand markets were: $\rho^*_{31} = \rho^*_{32} = 285.1427$.

It is easy to verify that the optimality/equilibrium conditions were satisfied with good accuracy.

Example 2

We then constructed the following variant of Example 1. We kept the data identical to that in Example 1 except that now we increased the weight associated with the relationship levels of the two source agents from 1 to 10.

The Euler method converged and yielded the identical financial flow equilibrium pattern to that obtained for Example 1, that is:

$$Q^{1*} := q^*_{111} = q^*_{121} = q^*_{211} = q^*_{221} = 1.00,$$

$$Q^{3*} := q^*_{111} = q^*_{121} = q^*_{211} = q^*_{221} = 1.00.$$

Now, however, the relationship levels associated with the source agents' transactions increased from their values of zero in Example 1 to new equilibrium levels of one. Hence, whereas before there were no relationships (and, in effect, the social network component of the supernetwork could be entirely eliminated from the Figure 2 analogue of Figure 3), in Example 2, the relationship levels between the source agents and the financial intermediaries were at their highest possible levels (recall that the relationship levels cannot exceed the value of one). The vector ϵ^* remained unchanged and the equilibrium demand prices ρ_3^* also remained unchanged from their values in Example 1.

Example 3

We then modified Example 2 as follows: The data were identical to that in Example 2 except that now we modified the demand function at the first demand market as follows:

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1100.$$

Hence, the demand associated with the first product increased.

The Euler method yielded the following new equilibrium pattern:

$$Q^{1*} := q^*_{111} = q^*_{121} = q^*_{211} = q^*_{221} = 1.0449,$$

$$Q^{3*} := q^*_{111} = 1.3952, \quad q^*_{121} = 0.6680, \quad q^*_{211} = 1.3952, \quad q^*_{221} = 0.6680.$$

Both source agents did not invest 17.9101 units of their financial holdings.

The vector of shadow prices ϵ^* had components: $\epsilon_1^* = \epsilon_2^* = 14.8486$, and the demand prices at the demand markets were: $\rho^*_{31} = 397.9481$, $\rho^*_{32} = 200.8729$.

The relationship levels remained as in Example 2.

It is worth noting that in this, as in the preceding examples, the constraint (1) did not hold tightly for each source agent, that is, not all the financial holdings were allocated.

6. Summary and Conclusions

In this paper, we developed a supernetwork model that integrated financial networks with intermediation with social networks in which relationship levels were made explicit. Both networks had three tiers of decision-makers, consisting of: the sources of financial funds, the financial intermediaries, and the consumers associated with the demand markets for the financial products. We allowed for physical as well as electronic transactions between the decision-makers in the supernetwork. The relationship levels were allowed to affect not only the risk but also the transaction costs (by reducing them, in general) but did have associated costs. Moreover, we allowed for multicriteria decision-making behavior in which the source agents as well as the financial intermediaries were permitted to weight, in an individual fashion, their objective functions of net revenue maximization, total risk minimization, and total relationship value maximization.

We modeled the supernetwork in equilibrium, in which the financial flows between the tiers as well as the relationship levels coincide and established the variational inequality formulation of the governing equilibrium conditions. We then proposed the underlying dynamics and the continuous-time adjusrtment process(es) and constructed its projected dynamical system representation. We established that the set of stationary points of the projected dynamical system coincides with the set of solutions of the variational inequality problem. We also provided conditions under which the dynamic trajectories of the financial flows, relationship levels, and prices are well-defined.

We proposed a discrete-time algorithm to approximate the continuous-time adjustment process and applied it to several simple numerical examples for completeness and illustrative purposes. The framework developed here further advances the work in financial equilibrium modeling and analysis, especially within a network context by explicitly considering the effect of relationship levels in financial networks and by establishing the optimal relationship levels as well as financial transaction quantities and prices. Finally, it also gives us insight into the optimal designs of the supernetworks.

It would be very interesting to apply the theories herein to actual case studies.

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