Supply Chain Networks
with
Multicriteria Decision-Makers

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Abstract: This paper presents a theoretical framework for supply chain network modeling, analysis, and computation in the presence of competition, in which all the decision-makers, who are located at distinct tiers of the network, are multicriteria decision-makers. In particular, in this framework, transportation time and transportation cost are explicit criteria and these can depend on the service level provided. The optimality conditions of the different tiers of decision-makers, consisting of manufacturers, retailers, and consumers are derived, as well as the equilibrium conditions of the integrated model. The variational inequality formulation of the governing equilibrium conditions is then utilized to obtain qualitative properties as well as a computational procedure for the determination of the equilibrium product shipments between the tiers of the network, the prices, as well as the service levels.

1. Introduction

In this paper, we develop a theoretical framework for multicriteria decision-making in multitiered supply chain networks. The framework handles competition among the decision-makers who consist of manufacturers, retailers, and consumers. The endogenous variables include the product shipments between the network tiers, the prices of the product at the various nodes of the supply chain network, as well as the service levels provided. The work signficantly generalizes the model of Nagurney, Zhang, and Dong (2001) by introducing another tier of decision-makers in the form of retailers and by allowing all tiers to be multicriteria decision-makers. Furthermore, it generalizes the model of Nagurney, Dong, and Zhang (2001), which considered three tiers of decision-makers but the decision-makers were exclusively faced with single criteria to optimize. Moreover, it formalizes the incorporation of *service* levels.

We emphasize that the topic of supply chain analysis is interdisciplinary by nature since it involves manufacturing, transportation and logistics, as well as retailing/marketing. It has been the subject of a growing body of literature (cf. Stadtler and Kilger (2000) and the references therein) with the associated research being both conceptual in nature (see, e.g., Poirier (1996, 1999), Mentzer (2000), Bovet (2000)), due to the complexity of the problem and the numerous agents, such as manufacturers, retailers, or consumers involved in the transactions, as well as analytical (cf. Bramel and Simchi-Levi (1997) and Miller (2001) and the references therein). Daganzo (1999), for example, examined logistics systems in an integrated way and showed how to find rational structures for such systems. Indeed, many researchers, in addition to, practitioners, have described the various networks that underly supply chain analysis. However, the framework considered has been primarily that of single objective optimization. In this paper, in contrast, we focus on both the multicriteria and the multitiered aspects of supply chain analysis with an eye towards understanding the underlying behavior of the various decision-makers and the ultimate equilibrium patterns.

In particular, in this paper, we consider a comprehensive supply chain network for a homogeneous product that is comprised of competitive manufacturers, retailers, and consumers. It should be noted that, although manufacturers, retailers, and consumers are selected here to be the focal decision-makers, the framework of the model presented in this paper is sufficiently general to include other levels of decision-makers such as owners of distribution centers, for example. Moreover, the model formulates multicriteria decision-making as regards the manufacturers, the retailers, as well as the consumers. To-date, except for the recent work of Nagurney, Zhang, and Dong (2001), the subject and theory of multicriteria decision-making in a supply chain context has focused exclusively on either the production side or on the consumption side, and, typically, has considered only a single decision-maker (see, e.g., Chankong and Haimes (1983), Yu (1985), Keeney and Raiffa (1993) and the references therein). For references on multicriteria decision-making in transportation networks, in general, see Nagurney and Dong (2001) and the references therein.

Specifically, we assume that the manufacturers are involved in the production of a homogeneous product which is then shipped to the retailers. The manufacturers seek to determine their optimal production and shipment quantities, given the production costs and the transaction costs associated with conducting business with the different retailers as well as the prices that the retailers are willing to pay for the product and shipment alternative combinations. The manufacturers are assumed to have two objectives and these are profit maximization and market share maximization with the weight associated with the latter criterion being distinct for each manufacturer.

The retailers, in turn, seek to determine the "optimal" quantity of the product to obtain from each of the manufacturers, that is, the "most-economic" shipment pattern in terms of both the amount and the shipment means selected, as well as the "optimal" service level. In particular, the retailers have three criteria and these are: the maximization of profit, the minimization of transportation time, and the maximization of service level. Each of the retailers can weight these criteria in a different manner. The shipment alternatives are represented by links characterized by specific transportation cost and transportation time functions. Hence, a specific shipment option or link may have a low associated transportation time but a high transportation cost, whereas another may have a high associated transportation time and a low cost. A higher price may be charged to have the products shipped faster to the retailers or, in contrast, manufacturers may accept a lower price if they select a slower and, presumably, cheaper, shipping mode. The service level indicates the percentage of time that the product is not out of stock. Usually, a higher service level implies a greater stock size on average, and thus, larger order quantities and less frequent order times. In other words, with a higher service level, the stock (inventory) holding cost goes up, and the transportation time and cost go down. Therefore, the service level set by each retailer significantly impacts his holding cost as well as his transportation time and cost.

Finally, in this supply chain network model, the consumers provide the "pull" in that, given the demand functions at the various demand markets, they determine their optimal consumption levels from the various retailers subject to the prices charged for the product as well as the transportation time and transportation cost associated with obtaining the product. The consumers are classified into different classes according to various characteristics that may include, for example: geographic location, consumption behavior, travel pattern, profession or business type, and income level. Different classes of consumers may weight time and cost associated with obtaining the product in an individual manner. It is worth noting that the service level at a retailer may affect the transportation time and transportation cost of the consumers who purchase at the retailer outlet. This is because when customers come to shop at a store and the product they wish to purchase is out of stock, they have, in effect, wasted their time and have incurred an associated cost.

We establish that the equilibrium conditions governing the state at which the manufacturers, the retailers, as well as the consumers, have reached optimality can be formulated as a variational inequality. We then utilize the derived variational inequality to obtain qualitative properties of the equilibrium product shipment, price, and service level pattern. We also propose an algorithm for the computation of the equilibrium pattern and give conditions for convergence.

The paper is organized as follows. In Section 2, we present the supply chain network model with competition, derive the optimality conditions for the distinct tiers of decision-makers, and provide the equilibrium conditions, which are then shown to be equivalent to a finite-dimensional variational inequality problem. In Section 3, we establish certain qualitative properties of the multicriteria, multitiered supply chain network model, in particular, the existence and uniqueness of an equilibrium, under reasonable assumptions. In Section 4, we describe the computational procedure, along with convergence results. In Section 5, we summarize our results and present the conclusions.

2. The Multicriteria, Multitiered Supply Chain Network Model

In this Section, we develop the supply chain network model with manufacturers, retailers, and consumers located at the demand markets. As mentioned in the Introduction, these decision-makers are multicriteria ones. We first focus on the manufacturers. We then turn to the retailers and, subsequently, to the consumers. The integrated model is then constructed along with the variational inequality formulation of the governing equilibrium conditions.

The Manufacturers

We assume that a homogeneous product is produced by m manufacturers, is shipped to n retailers, and is consumed by o different classes of consumers as represented by distinct demand markets. We denote a typical manufacturer by i, a typical retailer by j, and a typical consumer class by k. Each manufacturer can ship the product to each retailer using one (or more) of r possible shipment alternatives, which can represent mode/route alternatives. We denote a typical shipment alternative by l. Let q_{ijl} denote the quantity of the product produced by manufacturer i and shipped to retailer j via shipment alternative l. For simplicity of presentation, we now define certain vectors. Let $q_i = \sum_{j=1}^{n} \sum_{l=1}^{r} q_{ijl}$ be the total production of manufacturer i and let q be the m-dimensional vector with components: $\{q_1, \ldots, q_m\}$. Let $q_j = \sum_{i=1}^{m} \sum_{l=1}^{r} q_{ijl}$ denote the product shipments to retailer j from all the manufacturers. Finally, let Q^1 denote the mnr-dimensional vector of all the product shipments from all the manufacturers to all the retailers, with components: $\{q_{111}, \ldots, q_{mnr}\}$. We assume that all the vectors are column vectors.

Each manufacturer i is assumed to be faced with a production cost function f_i, which may depend, in general, on the entire vector of production outputs, that is,

$$f_i = f_i(q), \quad \forall i. \tag{1}$$

We associate with manufacturer i, who is transacting with retailer j and using shipment alternative l, a transaction cost c_{1ijl}, where

$$c_{1ijl} = c_{1ijl}(q_{ijl}), \quad \forall j, l. \tag{2}$$

To help fix ideas, and in order to facilitate the ultimate construction of the supply chain

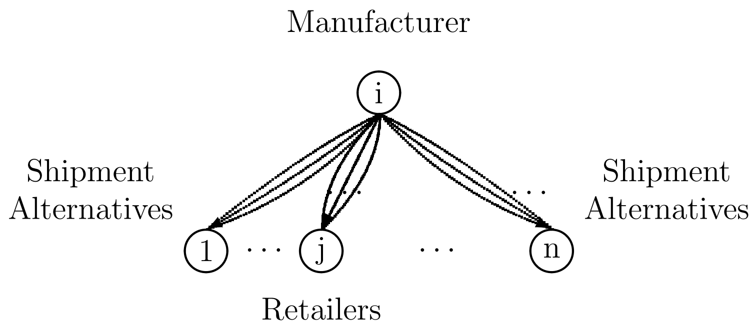

Figure 1: Network Structure of Manufacturer i's Transactions

network in equilibrium, we depict the manufacturers and the retailers as nodes and the shipment alternatives between the manufacturers and the retailers as links, where recall that since we assume that there are r shipment alternatives available, there are r links joining each pair of nodes (i, j). See Figure 1 for the graphical depiction for a specific manufacturer i.

The total costs incurred by a manufacturer i, thus, are equal to the sum of the manufacturer's production cost plus his transaction costs. His revenue, in turn, is equal to the sum over all the retailers and shipment alternatives of the price of the product charged to each retailer and associated with a particular shipment alternative times the quantity. If we let ρ^*_{1ijl} denote the (endogenous) price charged for the product by manufacturer i to retailer j and associated with shipment alternative l, we can express the criterion of profit maximization for manufacturer i as:

$$\text{Maximize} \quad \sum_{j=1}^{n}\sum_{l=1}^{r}\rho^*_{1ijl}q_{ijl} - f_i(q) - \sum_{j=1}^{n}\sum_{l=1}^{r}c_{1ijl}(q_{ijl}), \tag{3}$$

subject to $q_{ijl} \geq 0$, for all j, l. We discuss how the supply price ρ^*_{1ijl} is determined later in this Section.

In addition to the criterion of profit maximization, we also assume that each manufacturer seeks to maximize his production output in an endeavor to gain market share. Therefore, the second criterion of each manufacturer i can be expressed mathematically as:

$$\text{Maximize} \quad \sum_{j=1}^{n} \sum_{l=1}^{r} q_{ijl} \tag{4}$$

subject to $q_{ijl} \geq 0$, for all j, l.

A Manufacturer i's Multicriteria Decision-Making Problem

We now describe how we construct a value function associated with the two criteria facing each manufacturer, that is, profit maximization and output maximization. Each manufacturer i associates a nonnegative weight α_i with the output maximization criterion, with the weight associated with the profit maximization criterion serving as the numeraire and being set equal to 1. Hence, we can construct a value function for each manufacturer (cf. Fishburn (1970), Chankong and Haimes (1983), Yu (1985), Keeney and Raiffa (1993)) using a constant additive weight value function. Consequently, the multicriteria decision-making problem for manufacturer i is transformed into:

$$\text{Maximize} \quad \sum_{j=1}^{n} \sum_{l=1}^{r} \rho_{1ijl}^{*} q_{ijl} - f_i(q) - \sum_{j=1}^{n} \sum_{l=1}^{r} c_{1ijl}(q_{ijl}) + \alpha_i \sum_{j=1}^{n} \sum_{l=1}^{r} q_{ijl}, \tag{5}$$

subject to:

$$q_{ijl} \geq 0, \forall j, l.$$

The Retailers

The retailers, in turn, are positioned in the supply chain so as to have transactions with both their suppliers, that is, with the m manufacturers as well as with their customers, that is, the o classes of consumers. Hence, the network structure of a particular retailer's transactions is as depicted in Figure 2.

Let Q_{jk} denote the amount of the product purchased by the kth class of consumer at the retail outlet j. Let the vector Q^2 consist of all the consumption quantities of the product by the consumers and be the no-dimensional vector with components: $\{Q_{11}, \ldots, Q_{no}\}$. Let ρ_{2j}^{*} denote the retail price of retailer j. This price, as we will show, will be endogenously

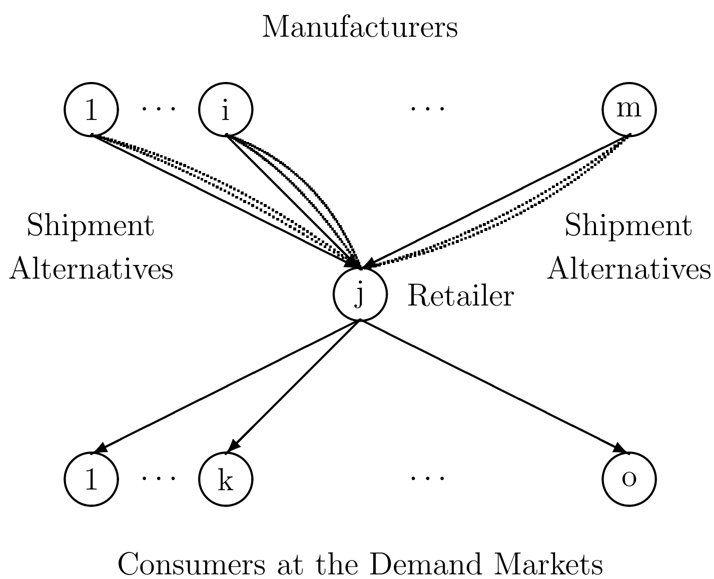

Figure 2: Network Structure of Retailer j's Transactions

determined in the integrated model. Then, the total revenue of retailer j is given by the expression

$$\rho_{2j}^{*} \sum_{k=1}^{o} Q_{jk}. \tag{6}$$

The costs that a typical retailer j is faced with include the price of obtaining the product from the manufacturers, the transaction cost, which includes the transportation cost and which is denoted by c_{2ijl}, and the holding cost, h_j, for carrying, displaying, and maintaining the product stock. Since a retailer must decide upon an ordering system in terms of the order quantity and the order time (order point), the service level, denoted by s_j, is one of the main factors, financially, in addition to the ordering cost and carrying cost. Here we define the service level to be the percentage of time that the product is in stock when consumers come to buy it (cf. Arnold (2000)). Usually, a higher service level implies a greater stock size, on the average, and, thus, larger order quantities and less frequent order times (Beamon (1998)).

We assume that the holding cost depends, in general, on j's shipment pattern q_j, and his service level s_j, that is,

$$h_j = h_j(s_j, q_j). \tag{7}$$

Retailer j's transaction cost associated with obtaining the product from manufacturer i via transportation alternative l, in turn, is given by

$$c_{2ijl} = c_{2ijl}(s_j, q_j). \tag{8}$$

We denote the transportation time of shipping the product from manufacturer i to retailer j via alternative l by t_{ijl}, which, in general, may depend upon q_j as well as on the service level s_j as discussed above. Therefore, we have that

$$t_{ijl} = t_{ijl}(s_j, q_j). \tag{9}$$

A Retailer j's Multicriteria Decision-Making Problem

In a multicriteria decision-making setting, we assume that retailer j has three criteria. They are: to maximize the profit, to minimize the total transportation time in getting the product from the manufacturers, and to maximize the service level. As mentioned earlier, by increasing his service level s_j, retailer j is, indeed, in a sense, improving his service reputation, since the customers shopping at retail outlet j have a lower chance of experiencing a stockout of the product when retailer j adopts a higher service level. The model assumes that retailer j associates with his value function a weight of 1 to his profit attribute in dollar value, a weight of β_{1j} to his transportation time attribute, as the conversion rate of time to dollar value, and a weight of β_{2j} to his service level attribute, as the conversion rate of service reputation to dollar value.

Therefore, the multicriteria decision-making problem for retailer j; $j = 1, ..., n$, can be transformed into:

$$\begin{aligned} \text{Maximize} \quad & \rho^*_{2j} \sum_{k=1}^{o} Q_{jk} - \beta_{1j} \sum_{i=1}^{m} \sum_{l=1}^{r} t_{ijl}(s_j, q_j) + \beta_{2j} s_j \\ & - \sum_{i=1}^{m} \sum_{l=1}^{r} \rho^*_{1ijl} q_{ijl} - \sum_{i=1}^{m} \sum_{l=1}^{r} c_{2ijl}(s_j, q_j) - h_j(s_j, q_j) \end{aligned} \tag{10}$$

subject to

$$\sum_{k=1}^{o} Q_{jk} \leq \sum_{i=1}^{m} \sum_{l=1}^{r} q_{ijl}, \quad (\gamma_j) \tag{11}$$

$$s_j \leq 1, \quad (\eta_j) \tag{12}$$

and

$$q_{ijl} \geq 0, \quad Q_{jk} \geq 0, \quad s_j \geq 0, \quad \forall i, l, k,$$

where γ_j is the Lagrange multiplier associated with inequality (11) and η_j is the Lagrange multiplier associated with inequality (12).

The objective function (10) represents a value function for retailer j, with β_{1j} having the interpretation as the conversion rate of time into dollar value and β_{2j} having the interpretation as the conversion rate of service level into dollar value. Constraint (11) expresses that consumers cannot purchase more from a retailer than is held in stock. Constraint (12) and the nonnegativity constraint on the level of service specify that the service level is set between 0 to 100 percent.

Subsequently, we derive the optimality conditions for this problem.

The Consumers at the Demand Markets

We now describe the consumers located at the demand markets. We assume that there are o classes of consumers, with a typical class denoted by k. The consumers are classified into different classes according to such characteristics as geographic location, consumption behavior, travel pattern, profession and income level, etc. As such, each class of consumer takes into account in making his consumption decision not only the different retail prices at the retail stores, but also the transportation time and the transportation cost required to obtain the product, as well as the service levels provided at the retail stores. Therefore, the consumers in this model are multiclass and multicriteria decision-makers. Let C_{jk} denote the transportation cost incurred to class k consumers for purchasing the product from retailer j, and let T_{jk} denote the corresponding transportation time. In general, these functions may depend upon the quantity of the product purchased at retailer j by consumer class k, as well as upon the service level at retailer j. Indeed, one can expect that the higher the service level at retailer j, the lower the procurement effort needed to obtain the product, and, thus, the lower the associated time and cost.

We assume that the transportation cost and transportation time are in the following form:

$$C_{jk} = C_{jk}(s_j, Q_{jk}), \quad \forall j, k \tag{13}$$

$$T_{jk} = T_{jk}(s_j, Q_{jk}), \quad \forall j, k. \tag{14}$$

In addition, we assume that each class of consumer perceives the transportation cost and the transportation time associated with obtaining the product in an individual manner, and weights these three criteria accordingly. Let λ_{1k} and λ_{2k} denote, respectively, the nonnegative weights associated with transportation cost and transportation time by class k consumers. Therefore, their value function for the *generalized* retail price associated with the kth class of consumer to obtain the product at retailer j is expressed as

$$\rho_{2j}^* + \lambda_{1k} C_{jk}(s_j, Q_{jk}) + \lambda_{2k} T_{jk}(s_j, Q_{jk}), \tag{15}$$

that is, the true price of the product from the perspective of consumers of class k is the price charged for the product at the particular retail outlet plus the "effective" price associated with obtaining the product which is composed of the weighted transportation cost plus the weighted transportation time.

Let now ρ_{3k} denote the generalized price of the product as perceived by class k and group the generalized demand prices into the vector $\rho_3 \in R_+^o$. Further, denote the demand of class k by d_k, and assume continuous demand functions given by:

$$d_k = d_k(\rho_3), \quad \forall k. \tag{16}$$

The Multicriteria Equilibrium Conditions for Consumers of Class k

We now present the multicriteria equilibrium conditions for consumers of class k. In particular, in equilibrium, we must have the following conditions holding:

$$\rho_{2j}^* + \lambda_{1k} C_{jk}(s_j^*, Q_{jk}^*) + \lambda_{2k} T_{jk}(s_j^*, Q_{jk}^*) \begin{cases} = \rho_{3k}^*, & \text{if} \quad Q_{jk}^* > 0 \\ \geq \rho_{3k}^*, & \text{if} \quad Q_{jk}^* = 0, \end{cases} \tag{17}$$

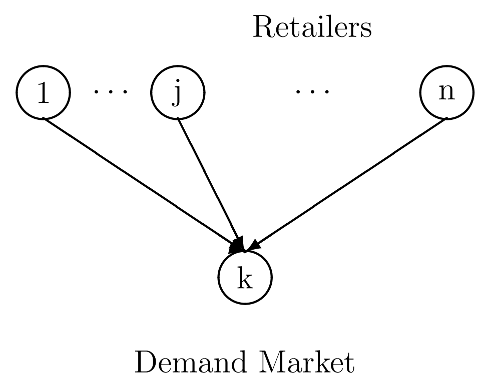

Figure 3: Network Structure of Consumers' Transactions at Demand Market k

and

$$d_k(\rho_3^*)\begin{cases} = \sum_{j=1}^{n} Q_{jk}^*, & \text{if} \quad \rho_{3k}^* > 0 \\ \leq \sum_{j=1}^{n} Q_{jk}^*, & \text{if} \quad \rho_{3k}^* = 0. \end{cases} \tag{18}$$

Conditions (17) state that a member of consumer class k will purchase the product from retailer j, if the generalized retail price at retailer j as perceived by consumers of class k does not exceed the demand price that consumers of class k are willing to pay for the product. Conditions (18) state, in turn, that if the demand price that consumers of class k are willing to pay for the product is positive, then the quantities purchased of the product from the retailers will be precisely equal to the demand of consumers of class k for the product. These conditions correspond to the well-known spatial price equilibrium conditions (cf. Samuelson (1952), Takayama and Judge (1971) and Nagurney (1999) and the references therein).

In Figure 3, the network of transactions between the retailers and the consumers at demand market k is depicted. Each demand market is represented by a node and the transactions, as previously, by links.

The Integrated Model

We now present the integrated supply chain network model with multicriteria manufacturers, retailers, and consumers, which synthesizes the optimality conditions of all the manufacturers and all the retailers, and the equilibrium conditions of the consumers at all the demand markets. In order to obtain an equilibrium of the multicriteria supply chain network system, we must have that the optimality conditions of the manufacturers and the retailers as well as the equilibrium conditions of the consumers are satisfied *simultaneously*.

The Optimality Conditions of the Manufacturers

The manufacturers are assumed to compete in a noncooperative manner in the sense of Cournot (1838) and Nash (1950, 1951), seeking to determine their own optimal production and shipment quantities. If the production cost functions for each manufacturer is continuously differentiable and convex, as is each manufacturer's transaction cost function, then the optimality conditions (see Dafermos and Nagurney (1987), Nagurney (1999) and Gabay and Moulin (1980)) take the form of a variational inequality problem given by: Determine $Q^{1^*} \in R_+^{mnr}$, such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{r}\left[\frac{\partial f_i(q^*)}{\partial q_{ijl}}+\frac{\partial c_{1ijl}(q^*_{ijl})}{\partial q_{ijl}}-\alpha_i-\rho^*_{1ijl}\right]\times\left[q_{ijl}-q^*_{ijl}\right]\geq 0, \quad \forall Q^1\in R_+^{mnr}. \qquad (19)$$

The optimality conditions as expressed by (19) have a nice economic interpretation, which is that a manufacturer will ship a positive amount of the product to a retailer via a particular shipment alternative (and the flow on the corresponding link will be positive) if the price that the retailer is willing to pay for the product, ρ^*_{1ijl}, is precisely equal to the manufacturer's marginal production and transaction costs associated with that retailer and shipment alternative, discounted by the market share weight. If the manufacturer's marginal production and transaction costs discounted by the market share weight exceed the price that the retailer is willing to pay for the product and shipment alternative, then the flow on the transaction link will be zero. Hence, expressed alternatively, we can say that the price ρ^*_{1ijl} must be precisely equal to the marginal production cost plus the marginal transaction cost discounted by the market share weight, if there is a positive shipment of the product between the manufacturer and retailer pair using the particular shipment alternative.

The Optimality Conditions of the Retailers

We now consider the optimality conditions of the retailers assuming that each retailer is faced with the optimization problem (10), (11) and (12). Here, we also assume that the retailers compete in a noncooperative manner so that each maximizes his profits, given the actions of the other retailers. Note that, at this point, we consider that retailers seek to determine not only the optimal amounts purchased by the consumers from their specific retail outlet but, also, the amount that they wish to obtain from the manufacturers, their shipment pattern via the alternatives, as well as their service levels. In equilibrium, all the shipments between the tiers of decision-makers will have to coincide.

Assuming that all the holding costs (7), the retailer transaction costs (8), and the transportation times for shipping the product from the manufacturers to the retailers (9) are all continuous and convex, then the optimality conditions for all the retailers satisfy the variational inequality: Determine nonnegative $(Q^{1^*}, Q^{2^*}, \gamma^*, \eta^*, s^*)$ satisfying (12) such that:

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\gamma_j^* - \rho_{2j}^*\right]\times\left[Q_{jk} - Q_{jk}^*\right]$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{r}\left[\frac{\partial h_j(s_j^*, q_j^*)}{\partial q_{ijl}} + \rho_{1ijl}^* + \frac{\partial c_{2ijl}(s_j^*, q_j^*)}{\partial q_{ijl}} + \beta_{1j}\frac{\partial t_{ijl}(s_j^*, q_j^*)}{\partial q_{ijl}} - \gamma_j^*\right]\times\left[q_{ijl} - q_{ijl}^*\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{r} q_{ijl}^* - \sum_{k=1}^{o} Q_{jk}^*\right]\times\left[\gamma_j - \gamma_j^*\right] + \sum_{j=1}^{n}\left[1 - s_j^*\right]\times\left[\eta_j - \eta_j^*\right]$$

$$+\sum_{j=1}^{n}\left[\frac{\partial h_j(s_j^*, q_j^*)}{\partial s_j} + \sum_{i=1}^{m}\sum_{l=1}^{r}\frac{\partial c_{2ijl}(s_j^*, q_j^*)}{\partial s_j} + \sum_{i=1}^{m}\sum_{l=1}^{r}\beta_{1j}\frac{\partial t_{ijl}(s_j^*, q_j^*)}{\partial s_j} + \eta_j^* - \beta_{2j}\right]\times\left[s_j - s_j^*\right] \geq 0, \quad (20)$$

for all nonnegative $Q^1, Q^2, s, \gamma, \eta$ satisfying (12), where recall that γ_j is the Lagrange multiplier for (11) and η_j is the Lagrange multiplier for (12). For further background on such a derivation, see Bertsekas and Tsitsiklis (1992). In this derivation, as in the derivation of inequality (19), we have not had the prices charged be variables. They become endogenous variables in the integrated model.

We now highlight the economic interpretation of the retailers' optimality conditions. The third term in inequality (20) reveals that the Lagrange multiplier γ_j^* can be interpreted as the market clearing price at retailer j's outlet, that is, if γ_j^* is positive, then, at equilibrium, the total inflow of the product should equal the total outflow at retailer j. We now turn to the first term in inequality (20). The first term implies that, if consumers of class k purchase the product at retailer j, that is, $Q_{jk}^* > 0$, then the price charged by retailer j, ρ_{2j}^*, should be equal to γ_j^*, which is the price to clear the market at retailer j. From the second term in (20), in turn, we see that if retailer j adopts transportation alternative l to ship the product from manufacturer i, that is, $q_{ijl}^* > 0$, then the total marginal cost for obtaining the product from manufacturer i via alternative l should be equal to γ_j^*, the price to clear market at retailer j. On the other hand, if this marginal cost for obtaining the product from manufacturer i via alternative l is greater than the market clearing price at retailer j, then retailer j will not use alternative l to ship the product from manufacturer i. The fourth term in (20) suggests that η_j^* is the equilibrium shadow price for the deviation of the service level, s_j^*, from one hundred percent, since η_j is defined to be the Lagrange multiplier for constraint (12). Finally, the fifth term in (20) argues that in order to have a positive service level s_j^* at retailer j, the sum of the generalized marginal handling cost, which includes the marginal holding cost, the marginal transaction cost, the weighted transportation time (converted to its dollar value), and the shadow price for the deviation of service level from being one hundred percent, should be equal to the weight β_{2j} assigned to the service level. This is because this term should economically measure the importance of increasing service level against other marginal costs.

The Equilibrium Conditions of the Consumers

The equilibrium conditions associated with the consumers at the demand markets are that the systems of equalities and inequalities (17) and (18) must hold for all pairs of retailers and consumer classes. This is equivalent to the satisfaction of the variational inequality given by: $(Q^{2^*}, \rho_3^*) \in R_+^{no+o}$ must satisfy:

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\rho_{2j}^* + \lambda_{1k}C_{jk}(s_j^*, Q_{jk}^*) + \lambda_{2k}T_{jk}(s_j^*, Q_{jk}^*) - \rho_{3k}^*\right] \times \left[Q_{jk} - Q_{jk}^*\right]$$

$$+\sum_{k=1}^{o}\left[\sum_{j=1}^{n} Q_{jk}^* - d_k(\rho_{3k}^*)\right] \times [\rho_{3k} - \rho_{3k}^*] \geq 0, \quad \forall (Q^2, \rho_3) \in R_+^{no+o}. \tag{21}$$

Note that, in the context of the consumption decisions, we have utilized demand functions, rather than utility functions, as was the case for the manufacturers and the retailers. Of course, demand functions can be derived from utility functions (cf. Arrow and Intrilligator (1982)). We assume that the number of consumers to be much greater than that of the manufacturers and retailers and, hence, it is not appropriate to treat them individually.

The Equilibrium Conditions of the Supply Chain

In equilibrium, we must have that the optimality conditions for all manufacturers, as expressed by inequality (19), the optimality conditions for all retailers, as expressed by inequality (20), and the equilibrium conditions of the consumers, as expressed by inequality (21) must hold simultaneously.

Hence, the shipments that the manufacturers ship to the retailers must, in turn, be the shipments that the retailers accept from the manufacturers. In addition, the amounts of the product purchased by the consumers must be equal to the amounts sold by the retailers.

Thus, we have the following definition:

Definition 1: Supply Chain Equilibrium

A product shipment, price, and service level pattern $(Q^{1^*}, Q^{2^*}, \gamma^*, \rho_3^*, s^*, \eta^*) \in \mathcal{K}$*, where* $\mathcal{K} \equiv R_+^{mnr} \times R_+^{no} \times R_+^{n} \times R_+^{o} \times [0,1]^n \times R_+^n$ *is said to be a supply chain equilibrium if it satisfies the optimality conditions for all manufacturers, for all retailers, and for all consumers, given, respectively, by (19), (20), and (21), simultaneously.*

We then, immediately, through the summation of the inequalities (19), (20), and (21), after algebraic simplification, have the following:

Theorem 1: Variational Inequality Formulation

A supply chain network is in equilibrium, according to Definition 1 if and only if it satisfies the variational inequality problem: Determine $Q^{1^*}, Q^{2^*}, \gamma^*, \rho_3^*, s^*, \eta^*) \in \mathcal{K}$*, such that:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{r}\left[\frac{\partial f_i(q^*)}{\partial q_{ijl}} + \frac{\partial c_{1ijl}(q_{ijl}^*)}{\partial q_{ijl}} - \alpha_i + \frac{\partial h_j(s_j^*, q_j^*)}{\partial q_{ijl}} + \frac{\partial c_{2ijl}(s_j^*, q_j^*)}{\partial q_{ijl}} + \beta_{1j}\frac{\partial t_{ijl}(s_j^*, q_j^*)}{\partial q_{ijl}} - \gamma_j^*\right]$$

$$\times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\gamma^*_j + \lambda_{1k}C_{jk}(s^*_j, Q^*_{jk}) + \lambda_{2k}T_{jk}(s^*_j, Q^*_{jk}) - \rho^*_{3k}\right] \times \left[Q_{jk} - Q^*_{jk}\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{r} q^*_{ijl} - \sum_{k=1}^{o} Q^*_{jk}\right] \times \left[\gamma_j - \gamma^*_j\right] + \sum_{k=1}^{o}\left[\sum_{j=1}^{n} Q^*_{jk} - d_k(\rho^*_{3k})\right] \times \left[\rho_{3k} - \rho^*_{3k}\right]$$

$$+\sum_{j=1}^{n}\left[\frac{\partial h_j(s^*_j, q^*_j)}{\partial s_j} + \sum_{i=1}^{m}\sum_{l=1}^{r}\frac{\partial c_{2ijl}(s^*_j, q^*_j)}{\partial s_j} + \sum_{i=1}^{m}\sum_{l=1}^{r}\beta_{1j}\frac{\partial t_{ijl}(s^*_j, q^*_j)}{\partial s_j} + \eta^*_j - \beta_{2j}\right] \times \left[s_j - s^*_j\right]$$

$$+\sum_{j=1}^{n}\left[1 - s^*_j\right] \times \left[\eta_j - \eta^*_j\right] \geq 0, \quad \forall (Q^1, Q^2, \gamma, \rho_3, s, \eta) \in \mathcal{K}, \tag{22a}$$

where γ is the n-dimensional column vector with component j given by γ_j.

For easy reference in the subsequent sections, variational inequality problem (22a) can be rewritten in standard variational inequality form (cf. Nagurney (1999)) as follows:

$$\langle F(X^*)^T, X - X^*\rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{22b}$$

where $X \equiv (Q^1, Q^2, \gamma, \rho_3, s, \eta)$ and

$$F(X) \equiv (F^1_{ijl}, F^2_{jk}, F^3_j, F^4_k, F^5_j, F^6_j)_{i=1,\ldots,m, j=1,\ldots,n, k=1,\ldots,o, l=1,\ldots,r},$$

where the terms of F correspond to the terms preceding the multiplication signs in inequality (22a), and $\langle\cdot,\cdot\rangle$ denotes the inner product in N Euclidean dimensional space.

In Figure 4, we present the multitiered network structure of the supply chain in equilibrium. The network consists of all the manufacturers, all the retailers, and all the demand markets as depicted, respectively, by the top tier of nodes, the middle tier of nodes, and, finally, the bottom tier of nodes in Figure 4. In order to construct this network, Figure 1 was replicated for all the manufacturers; Figure 2 for all the retailers, and Figure 3 for all the demand markets. The supply chain network, hence, represents all the possible transactions that can take place. In addition, since there must be agreement between/among the transactors at equilibrium, the analogous links and the corresponding flows on them must coincide.

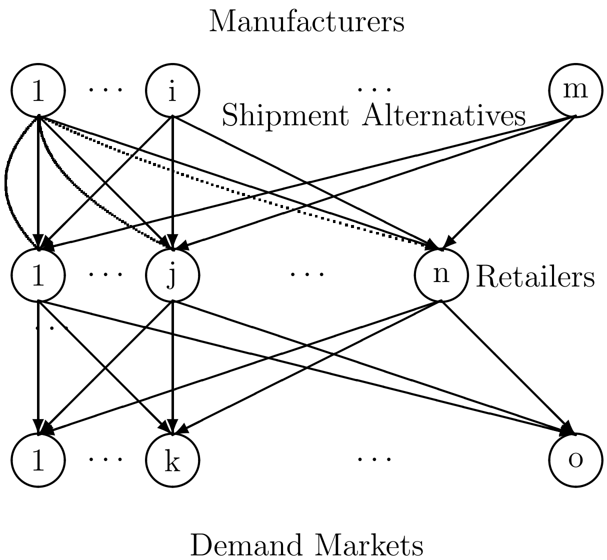

Figure 4: The Multitiered Network Structure of the Supply Chain with Multicriteria Decision-Makers

The vectors of prices ρ_1^*, γ^*, and ρ_3^* are associated with the respective tiers of nodes on the network, whereas the components of the vector of the equilibrium product shipments Q^{1*} correspond to the flows on the links joining the manufacturer nodes with the retailer nodes. The components of ρ_1^* and ρ_2^* can be determined as discussed following (19) and (20), respectively, whereas the components of ρ_3^* are explicit in the solution of the variational inequality (22a). The components of the vector of equilibrium product shipments Q^{2*}, in turn, correspond to the flows on the links joining the retailer nodes with the demand market nodes.

3. Qualitative Properties

In this Section, we provide some qualitative properties of the solution to variational inequality (22). In particular, we derive existence and uniqueness results. We also investigate properties of the function F (cf. (22b)) that enters the variational inequality of interest here.

Since the feasible set is not compact, we cannot derive existence simply from the assumption of the continuity of the functions. Nevertheless, we can impose a rather weak condition to guarantee existence of a solution pattern. Let

$$\mathcal{K}_b \equiv \{(Q^1, Q^2, \gamma, \rho_3, s, \eta)$$

$$|0 \leq Q^1 \leq b_1;\ 0 \leq Q^2 \leq b_2;\ 0 \leq \gamma \leq b_3; 0 \leq \rho_3 \leq b_4;\ 0 \leq s \leq 1;\ 0 \leq \eta \leq b_6\}, \tag{23}$$

where $b = (b_1, b_2, b_3, b_4, 1, b_6) \geq 0$ and $Q^1 \leq b_1; Q^2 \leq b_2; \gamma \leq b_3; \rho_3 \leq b_4; s \leq 1; \eta \leq b_6$, mean that each of the right-hand sides is a uniform upper bound for all the components of the corresponding vectors. Then $\mathcal{K}_b$ is a bounded closed convex subset of $\mathcal{K} = R^{mnr}_+ \times R^{no}_+ \times R^n_+ \times R^o_+ \times [0,1]^n \times R^n_+$. Thus, the following variational inequality

$$\langle F(X^b)^T, X - X^b\rangle \geq 0, \quad \forall X^b \in \mathcal{K}_b, \tag{24}$$

admits at least one solution $X^b \in \mathcal{K}_b$, from the standard theory of variational inequalities, since $\mathcal{K}_b$ is compact and F is continuous. Following Kinderlehrer and Stampacchia (1980) (see also Theorem 1.5 in Nagurney (1999)), we then have:

Theorem 2

Variational inequality (22) admits a solution if and only if there exists a $b > 0$, such that variational inequality (24) admits a solution in $\mathcal{K}_b$ with

$$Q^1 < b_1; Q^2 < b_2; \gamma < b_3; \rho_3 < b_4; s \leq 1; \eta < b_6. \tag{25}$$

Theorem 3: Existence

Suppose that there exist positive constants M, N, R with $R > 0$, such that:

$$\frac{\partial f_i(q)}{\partial q_{ijl}} + \frac{\partial c_{1ijl}(q_{ijl})}{\partial q_{ijl}} + \frac{\partial h_j(s_j, q_j)}{\partial q_{ijl}} + \frac{\partial c_{2ijl}(s_j, q_j)}{\partial q_{ijl}} + \beta_{1j}\frac{\partial t_{ijl}(s_j, q_j)}{\partial q_{ijl}} \geq M,$$

$$\forall q_{ijl} \qquad \text{with } q_{ijl} \geq N, \quad \forall i, j, l, \tag{26}$$

$$\lambda_{1k}C_{jk}(s_j, Q_{jk}) + \lambda_{2k}T_{jk}(s_j, Q_{jk}) \geq M, \quad \forall Q^2 \quad \text{with} \quad Q_{jk} \geq N, \quad \forall j, k, \tag{27}$$

$$d_k(\rho_3) \leq N, \quad \forall \rho_3 \quad \text{with} \quad \rho_{3k} > R, \quad \forall k. \tag{28}$$

Then, variational inequality (22) admits at least one solution.

Proof: Follows using analogous arguments as the proof of existence for Proposition 1 in Nagurney and Zhao (1993) (see also existence proof in Nagurney, Zhang and Dong (2001)). □

Assumptions (26), (27), and (28) are reasonable from an economics perspective. In particular, according to (26), when the product shipment between a manufacturer and a retailer via a certain shipment alternative is large, we can expect the corresponding sum of the marginal costs associated with the production, the shipment, and the holding of the product and the marginal time to exceed a positive lower bound. The rationale of assumption (27), in turn, can be seen through the following: if the amount of the product purchased by class k consumers at retailer j is large, the transportation cost and the transportation time associated with obtaining the the product at the retailer can also be expected to exceed a lower bound. Moreover, according to assumption (28), if the generalized price of the product as perceived by a consumer class is high, we can expect that the demand for the product by that class will be bounded from aove at that market.

We now recall the concept of additive production cost, which was introduced by Zhang and Nagurney (1996) in the stability analysis of dynamic spatial oligopolies, and has also been employed in the qualitative analysis by Nagurney, Zhang and Dong (2001) for the study of spatial economic networks with multicriteria tiers of producers and consumers. Additive production costs will be assumed in Theorems 4, 5, and 6.

Definition 2: Additive Production Cost

Suppose that for each manufacturer i, the production cost f_i is additive, that is

$$f_i(q) = f_i^1(q_i) + f_i^2(\bar{q}_i), \tag{29}$$

where $f_i^1(q_i)$ is the internal production cost that depends solely on the manufacturer's own output level q_i, which may include the production operation and the facility maintenance, etc., and $f_i^2(\bar{q}_i)$ is the interdependent part of the production cost that is a function of all the other manufacturer' output levels $\bar{q}_i = (q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_m)$ and reflects the impact of the other manufacturers' production patterns on manufacturer i's production cost. This interdependent part of the production cost may describe the competition for the resources, the cost of the raw materials, etc.

We now establish additional qualitative properties of the function F that enters the variational inequality problem, as well as uniqueness of the equilibrium pattern. Monotonicity and Lipschitz continuity of F will be utilized in the subsequent section for proving convergence of the algorithmic scheme.

Theorem 4: Monotonicity

Suppose that the production cost functions $f_i; i = 1, ..., m$, are additive, as defined in Definition 2, and that the $f_i^1; i = 1, ..., m$, are convex functions. In addition, suppose that
(i). the c_{1ijl}, h_j, c_{2ijl} and t_{ijl} are all convex functions in the shipment q_{ijl}, forall i, j, l;
(ii). the C_{jk}, T_{jk} are monotone increasing functions with respect to Q_{jk}, $\forall j, k$;
(iii). the d_k are monotone decreasing functions of the prices ρ_{3k}, for all k; and, finally,
(iv). the h_j, is a family of increasing convex function of the service levels s_j, $\forall j$, while the retailer transaction costs c_{2ijl} and the transportation times t_{ijl} are a family of decreasing and concave functions of the service levels s_j for all i, j, l.
Then the vector function F that enters the variational inequality (22b) is monotone, that is,

$$\langle (F(X') - F(X''))^T, X' - X'' \rangle \geq 0, \quad \forall X', X'' \in \mathcal{K} \equiv R_+^{mnr} \times R_+^{no} \times R_+^n \times R_+^o \times [0,1]^n \times R_+^n. \tag{30}$$

Proof: Let $X' = (q', Q', \gamma', \rho_3', s', \eta')$, $X'' = (q'', Q'', \gamma'', \rho_3'', s'', \eta'')$. Then, inequality (30) can been seen in the following deduction:

$$
\begin{aligned}
&\langle (F(X^1) - F(X^2))^T, X^1 - X^2 \rangle \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{r}\left[\frac{\partial f_i^1(q')}{\partial q_{ijl}} - \frac{\partial f_i^1(q'')}{\partial q_{ijl}}\right] \times \left[q'_{ijl} - q''_{ijl}\right] \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{r}\left[\frac{\partial c_{1ijl}(q'_{ijl})}{\partial q_{ijl}} - \frac{\partial c_{1ijl}(q''_{ijl})}{\partial q_{ijl}}\right] \times \left[q'_{ijl} - q''_{ijl}\right] \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{r}\left[\frac{\partial h_j(s'_j, q'_j)}{\partial q_{ijl}} - \frac{\partial h_j(s''_j, q''_j)}{\partial q_{ijl}}\right] \times \left[q'_{ijl} - q''_{ijl}\right] \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{r}\left[\frac{\partial c_{2ijl}(s'_j, q'_j)}{\partial q_{ijl}} - \frac{\partial c_{2ijl}(s''_j, q''_j)}{\partial q_{ijl}}\right] \times \left[q'_{ijl} - q''_{ijl}\right] \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{r}\left[\beta_{1j}\frac{\partial t_{ijl}(s'_j, q'_j)}{\partial q_{ijl}} - \beta_{1j}\frac{\partial t_{ijl}(s''_j, q''_j)}{\partial q_{ijl}}\right] \times \left[q'_{ijl} - q''_{ijl}\right] \\
&+ \sum_{j=1}^{n}\sum_{k=1}^{o}\left[\lambda_{1k} C_{jk}(s'_j, Q'_{jk}) - \lambda_{1k} C_{jk}(s''_j, Q''_{jk})\right] \times \left[Q'_{jk} - Q''_{jk}\right] \\
&+ \sum_{j=1}^{n}\sum_{k=1}^{o}\left[\lambda_{2k} T_{jk}(s'_j, Q'_{jk}) - \lambda_{2k} T_{jk}(s''_j, Q''_{jk})\right] \times \left[Q'_{jk} - Q''_{jk}\right] \\
&+ \sum_{k=1}^{o}\left[-d_k(\rho'_{3k}) + d_k(\rho''_{3k})\right] \times \left[\rho'_{3k} - \rho''_{3k}\right] \\
&+ \sum_{j=1}^{n}\left[\frac{\partial h_j(s'_j, q'_j)}{\partial s_j} - \frac{\partial h_j(s''_j, q''_j)}{\partial s_j}\right] \times \left[s'_j - s''_j\right] \\
&+ \sum_{i=1}^{m}\sum_{l=1}^{r}\left[\frac{\partial c_{2ijl}(s'_j, q'_j)}{\partial s_j} - \frac{\partial c_{2ijl}(s''_j, q''_j)}{\partial s_j} + \beta_{1j}\frac{\partial t_{ijl}(s'_j, q'_j)}{\partial s_j} - \beta_{1j}\frac{\partial t_{ijl}(s''_j, q''_j)}{\partial s_j}\right] \times \left[s'_j - s''_j\right] \\
&= (I) + (II) + (III) + (IV) + (V) + (VI) + (VII) + (VIII) + (IX) + (X). \qquad (31)
\end{aligned}
$$

Since the f_i; $i = 1, ..., m$, are additive, and the f_i^1; $i = 1, ..., m$, are convex functions, one has

$$
(I) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{r}\left[\frac{\partial f_i^1(q^{1'})}{\partial q_{ijl}} - \frac{\partial f_i^1(q^{1''})}{\partial q_{ijl}}\right] \times \left[q'_{ijl} - q''_{ijl}\right] \geq 0. \qquad (32)
$$

The nonnegativity of (II), (III), (IV), and (V) follows directly from assumption (i) that the corresponding functions are convex in q_{ijl}. Assumption (ii) implies that terms (VI) and (VII) are nonnegative. As assumed in (iii), since the demand functions are decreasing in consumption price, we have that term (VIII) is nonnegative. Finally, the nonnegativity of terms (IX) follows from asssumption (iv) that h_j is convex with respect to the service level s_j, and nonnegativity of (X) can been seen through this assumption that c_{2ijl}, t_{ijl} are concave in s_j.

Therefore, we see that under the conditions of the theorem, the right-hand side of (31) is nonnegative. The proof is complete. □

The strict monotonicity of the vector function F can be ensured with a slightly stronger condition than assumed in Theorem 4.

Theorem 5: Strict Monotonicity

Assume all the conditions of Theorem 4. In addition, suppose that
(i). one of the families of the vector functions c_{1ijl}, h_j, c_{2ijl}, *or* t_{ijl} *is strictly convex in shipment* q_{ijl};
(ii). either C_{jk} *or* T_{jk} *is strictly monotone increasing with respect to* Q_{jk};
(iii). the d_k *functions are strictly monotone decreasing functions of the prices* ρ_{3k}, *for all* j, k; *and, finally,*
(iv). either the holding costs, $h_j, \forall j$, *are increasing and strictly convex functions of the service levels* $s_j, \forall j$, *or one of the transaction cost functions* $c_{2ijl}, \forall i, j, l$, *and* $t_{ijl}, \forall i, j, l$, *is a family of decreasing and strictly concave functions of the service levels* $s_j, \forall j$.
Then the vector function F *that enters the variational inequality (22b) is strictly monotone, with respect to* (Q^1, Q^2, ρ_3, s), *that is, for any two* X', X'' *with* $(Q^{1'}, Q^{2'}, \rho_3', s') \neq (Q^{1''}, Q^{2''}, \rho_3'', s'')$:

$$\langle (F(X') - F(X''))^T, X' - X'' \rangle > 0. \tag{33}$$

Proof: For any two X', X'' with $(Q^{1'}, Q^{2'}, \rho_3', s') \neq (Q^{1''}, Q^{2''}, \rho_3'', s'')$, we must have at least one of the following four cases:

(i). $Q^{1'} \neq Q^{1''}$,

(ii). $Q^{2'} \neq Q^{2''}$,

(iii). $\rho_3' \neq \rho_3''$,

(iv). $s' \neq s''$.

Under the condition of the theorem, if (i) holds true, then, at the right-hand side of (31), at least one of (I), (II), (III), (IV) and (V) is positive. If (ii) is true, then either (VI) or (VII) is positive. In the case of (iii), (VIII) is positive. Finally, in the case of (iv), then either (IX) or (X) is positive. Hence, we can conclude that the right-hand side of (31) is greater than zero. The proof is complete. □

Theorem 5 has an important implication for the uniqueness of equilibrium production shipments, Q^{1^*}, the retailer shipments, Q^{2^*}, the prices at the demand markets, ρ_3^*, and the service levels, s^*, set at the retailer stores, s. We note also that no guarantee of a unique γ^* and η^* can be generally expected at the equilibrium.

Theorem 6: Uniqueness

Assuming the condition of Theorem 5, there must be a unique production shipment pattern Q^{1^*}, a unique retail shipment (consumption) pattern Q^{2^*}, a unique generalized price vector ρ_3^*, and a unique service vector s^* satisfying the equilibrium conditions of the multitiered, multicriteria supply chain network. In other words, if the variational inequality (22) admits a solution, that should be the only solution in Q^1, Q^2, ρ_3, s.

Proof: Under the strict monotonicity result of Theorem 5, uniqueness follows from the standard variational inequality theory (cf. Kinderlehrer and Stampacchia (1980)). □

Theorem 7: Lipschitz Continuity

The function that enters the variational inequality problem (22) is Lipschitz continuous, that is,

$$\|F(X') - F(X'')\| \leq L\|X' - X''\|, \quad \forall X', X'' \in \mathcal{K}, \tag{34}$$

under the following conditions:

(i). c_{1ijl}, h_j, c_{2ijl} *and* t_{ijl} *have bounded second-order derivatives, for all* i, j, l, *with respect to* q*;*

(ii). C_{jk}, T_{jk}, $\forall j, k$, *have bounded second-order derivatives, with respect to* Q_{jk}.

(iii). d_k *have bounded second-order derivatives, for all* k*;*

(iv). h_j, $\forall j; c_{2ijl}$, $\forall i, j, k; t_{ijl}$, $\forall i, j, k$, *have bounded second-order partial derivatives with respect to* s.

Proof: The result is direct by applying a mid-value theorem from calculus to the vector function F that enters the variational inequality problem (22). □

4. The Algorithm

In this Section, we present the modified projection method of Korpelevich (1977) which can be applied to solve the variational inequality problem in standard form (see (22b)), that is:

Determine $X^* \in \mathcal{K}$, satisfying:

$$\langle F(X^*)^T, X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}.$$

The algorithm is guaranteed to converge provided that the function F that enters the variational inequality is monotone and Lipschitz continuous (and that a solution exists).

The statement of the modified projection method applied to the solution of our variational inequality problem is as follows, where $\mathcal{T}$ denotes an iteration counter:

Step 0: Initialization

Set $(Q^{1^0}, Q^{2^0}, \gamma^0, \rho_3^0, s^0, \eta^0) \in \mathcal{K}$. Let $\mathcal{T} = 1$ and let ξ be a scalar such that $0 < \xi \leq \frac{1}{L}$, where L is the Lipschitz continuity constant (cf. Korpelevich (1977)) (see (34)).

Step 1: Computation

Compute $(\bar{Q}^{1^{\mathcal{T}}}, \bar{Q}^{2^{\mathcal{T}}}, \bar{\gamma}^{\mathcal{T}}, \bar{\rho}_3^{\mathcal{T}}, \bar{s}^{\mathcal{T}}, \bar{\eta}^{\mathcal{T}})$ by solving the variational inequality subproblem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{r}\Bigg[\bar{q}_{ijl}^{\mathcal{T}} + \xi\left(\frac{\partial f_i(q^{\mathcal{T}-1})}{\partial q_{ijl}} + \frac{\partial c_{1ijl}(q_{ijl}^{\mathcal{T}-1})}{\partial q_{ijl}} - \alpha_i + \frac{\partial h_j(s_j^{\mathcal{T}-1}, q_j^{\mathcal{T}-1})}{\partial q_{ijl}}\right.$$

$$\left. + \frac{\partial c_{2ijl}(s_j^{\mathcal{T}-1}, q_j^{\mathcal{T}-1})}{\partial q_{ijl}} + \beta_{1j}\frac{\partial t_{ijl}(s_j^{\mathcal{T}-1}, q_j^{\mathcal{T}-1})}{\partial q_{ijl}} - \gamma_j^{\mathcal{T}-1}\right) - q_{ijl}^{\mathcal{T}-1}\Bigg] \times \left[q_{ijl} - \bar{q}_{ijl}^{\mathcal{T}}\right]$$

$$+ \sum_{j=1}^{n}\sum_{k=1}^{o}\Big[\bar{Q}_{jk}^{\mathcal{T}} + \xi\left(\gamma_j^{\mathcal{T}-1} + \lambda_{1k}C_{jk}(s_j^{\mathcal{T}-1}, Q_{jk}^{\mathcal{T}-1})\right.$$

$$\left. + \lambda_{2k}T_{jk}(s_j^{\mathcal{T}-1}, Q_{jk}^{\mathcal{T}-1}) - \rho_{3k}^{\mathcal{T}-1}\right) - Q_{jk}^{\mathcal{T}-1}\Big] \times \left[Q_{jk} - \bar{Q}_{jk}^{\mathcal{T}}\right]$$

$$+ \sum_{j=1}^{n}\left[\bar{\gamma}_j^{\mathcal{T}} + \xi\left(\sum_{i=1}^{m}\sum_{l=1}^{r} q_{ijl}^{\mathcal{T}-1} - \sum_{k=1}^{o} Q_{jk}^{\mathcal{T}-1}\right) - \gamma^{\mathcal{T}_j-1}\right] \times \left[\gamma_j - \bar{\gamma}_j^{\mathcal{T}}\right]$$

$$+\sum_{k=1}^{o}\left[\bar{\rho}_{3k}^{\mathcal{T}}+\xi\left(\sum_{j=1}^{n}Q_{jk}^{\mathcal{T}-1}-d_k(\rho_{3k}^{\mathcal{T}-1})\right)-\rho_{3k}^{\mathcal{T}-1}\right]\times\left[\rho_{3k}-\bar{\rho}_{3k}^{\mathcal{T}}\right]$$

$$+\sum_{j=1}^{n}\left[\bar{s}_j^{\mathcal{T}}+\xi\left(\frac{\partial h_j(s_j^{\mathcal{T}-1},q_j^{\mathcal{T}-1})}{\partial s_j}+\sum_{i=1}^{m}\sum_{l=1}^{r}\frac{\partial c_{2ijl}(s_j^{\mathcal{T}-1},q_j^{\mathcal{T}-1})}{\partial s_j}\right.\right.$$

$$\left.\left.+\sum_{i=1}^{m}\sum_{l=1}^{r}\beta_{1j}\frac{\partial t_{ijl}(s_j^{\mathcal{T}-1},q_j^{\mathcal{T}-1})}{\partial s_j}+\eta_j^{\mathcal{T}-1}-\beta_{2j}\right)-s_j^{\mathcal{T}-1}\right]\times\left[s_j-\bar{s}_j^{\mathcal{T}}\right]$$

$$+\sum_{j=1}^{n}\left[\bar{\eta}_j^{\mathcal{T}}+\xi(1-s_j^{\mathcal{T}-1})-\eta_j^{\mathcal{T}-1}\right]\times\left[\eta_j-\bar{\eta}_j^{\mathcal{T}}\right]\geq 0,\qquad\forall(Q^1,Q^2,\gamma,\rho_3,s,\eta)\in\mathcal{K}.\qquad(35)$$

Step 2: Adaptation

Compute $({Q^1}^{\mathcal{T}},{Q^2}^{\mathcal{T}},\gamma^{\mathcal{T}},\rho_3^{\mathcal{T}},s^{\mathcal{T}},\eta^{\mathcal{T}})$ by solving the variational inequality subproblem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{r}\left[q_{ijl}^{\mathcal{T}}+\xi\left(\frac{\partial f_i(\bar{q}^{\mathcal{T}})}{\partial q_{ijl}}+\frac{c_{1ijl}(\bar{q}_{ijl}^{\mathcal{T}})}{\partial q_{ijl}}+\frac{\partial h_j(\bar{s}_j^{\mathcal{T}},\bar{q}_j^{\mathcal{T}})}{\partial q_{ij}}-\alpha_i+\frac{\partial c_{2ijl}(\bar{s}_j^{\mathcal{T}},\bar{q}_j^{\mathcal{T}})}{\partial q_{ijl}}\right.\right.$$

$$\left.\left.+\beta_{1j}\frac{\partial t_{ijl}(\bar{s}_j^{\mathcal{T}},\bar{q}_j^{\mathcal{T}})}{\partial q_{ijl}}-\bar{\gamma}_j^{\mathcal{T}}\right)-q_{ijl}^{\mathcal{T}-1}\right]\times\left[q_{ijl}-q_{ijl}^{\mathcal{T}}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[Q_{jk}^{\mathcal{T}}+\xi\left(\bar{\gamma}_j^{\mathcal{T}}+\lambda_{1k}C_{jk}(\bar{s}_j^{\mathcal{T}},\bar{Q}_{jk}^{\mathcal{T}})+\lambda_{2k}T_{jk}(\bar{s}_j^{\mathcal{T}},\bar{Q}_{jk}^{\mathcal{T}})-\bar{\rho}_{3k}^{\mathcal{T}}\right)-Q_{jk}^{\mathcal{T}-1}\right]\times\left[Q_{jk}-Q_{jk}^{\mathcal{T}}\right]$$

$$+\sum_{j=1}^{n}\left[\gamma_j^{\mathcal{T}}+\xi\left(\sum_{i=1}^{m}\sum_{l=1}^{r}\bar{q}_{ijl}^{\mathcal{T}}-\sum_{k=1}^{o}\bar{Q}_{jk}^{\mathcal{T}}\right)-\gamma_j^{\mathcal{T}-1}\right]\times\left[\gamma_j-\gamma_j^{\mathcal{T}}\right]$$

$$+\sum_{k=1}^{o}\left[\rho_{3k}^{\mathcal{T}}+\xi\left(\sum_{j=1}^{n}\bar{Q}_{jk}^{\mathcal{T}}-d_k(\bar{\rho}_{3k}^{\mathcal{T}})\right)-\rho_{3k}^{\mathcal{T}-1}\right]\times\left[\rho_{3k}-\rho_{3k}^{\mathcal{T}}\right]$$

$$+\sum_{j=1}^{n}\left[s_j^{\mathcal{T}}+\xi\left(\frac{\partial h_j(\bar{s}_j^{\mathcal{T}},\bar{q}_j^{\mathcal{T}})}{\partial s_j}+\sum_{i=1}^{m}\sum_{l=1}^{r}\frac{\partial c_{2ijl}(\bar{s}_j^{\mathcal{T}},\bar{q}_j^{\mathcal{T}})}{\partial s_j}\right.\right.$$

$$\left.\left.+\sum_{i=1}^{m}\sum_{l=1}^{r}\beta_{1j}\frac{\partial t_{ijl}(\bar{s}_j^{\mathcal{T}},\bar{q}_j^{\mathcal{T}})}{\partial s_j}+\bar{\eta}_j^{\mathcal{T}}-\beta_{2j}\right)-s_j^{\mathcal{T}-1}\right]\times\left[s_j-s_j^{\mathcal{T}}\right]$$

$$+\sum_{j=1}^{n}\left[\eta_j^{\mathcal{T}}+\xi\left(1-\bar{s}_j^{\mathcal{T}}\right)-\eta_j^{\mathcal{T}}\right]\times\left[\eta_j-\eta_j^{\mathcal{T}}\right]\geq 0,\qquad\forall(Q^1,Q^2,\gamma,\rho_3,s,\eta)\in\mathcal{K}.\qquad(36)$$

Note that the variational inequality subproblems (35) and (36) can be solved explicitly and in closed form since the feasible set is that of the nonnegative orthant and a box constraints.

We now state the convergence result for the modified projection method for this model.

Theorem 8: Convergence

Assume that the function that enters the variational inequality (22) has at least one solution and satisfies the conditions in Theorem 4 and in Theorem 7. Then the modified projection method described above converges to the solution of the variational inequality.

Proof: According to Korpelevich (1977), the modified projection method converges to the solution of the variational inequality problem of the form (23), provided that the function F that enters the variational inequality is monotone and Lipschitz continuous and that a solution exists. Existence of a solution follows from Theorem 3. Monotonicity follows Theorem 4. Lipschitz continuity, in turn, follows from Theorem 7. □

5. Summary and Conclusions

This paper develops a mathematical model for the study of multitiered supply chain networks with multicriteria decision-makers in the presence of competition. The model accommodates manufacturers, retailers, and consumers in different tiers of the supply chain network. The manufacturers or suppliers are the first-tier decision-makers in the network and are faced with a bicriterion objective function consisting of profit maximization and market share maximization. They compete with one another in the production and the shipment of the homogeneous product to the retailers in an oligopolistic manner.

The retailers are located in the second tier of the supply chain network and seek to procure the product from the various manufacturers at the lowest possible prices and to have the product shipped in a timely manner by selecting the appropriate shipment alternatives. The retailers then sell the product to the consumers at the retail prices and manage their inventory stock according to the service levels they set. The retailers are faced with threee criteria which are: the maximization of profit, the minimization of transportation time needed to procure the product, and the maximization of service level. Each retailer assigns weights to each of these criteria.

At the bottom tier of the supply chain network are the multiple classes of consumers. Each class of consumer takes into account in making the consumption decisions not only the different retail prices at the retail stores, but also the transportation time and transportation cost required to obtain the product.

The optimality conditions for the decision-makers at each tier of the network are derived and are interpreted economically. These conditions are then integrated into a single variational inequality formulation that governs the equilibrium conditions of the entire supply chain network. The analytic properties of the variational inequality formulation are investigated. In particular, we show that, under reasonable conditions, there exists a unique production and shipment pattern for the manufacturers, a unique set of service levels for the retailers, a unique consumption pattern of the multiclass consumers, and a unique demand price pattern. A computational procedure is also proposed for the determination of the equilibrium shipment, price, and service level pattern, along with convergence results.

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