Chapter 1

Supernetworks

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1.1 Background

The Information Age with the increasing availability of new computer and communication technologies, along with the Internet, have transformed the ways in which individuals work, travel, and conduct their daily activities, with profound implications for existing and future decision-making. Indeed, the decision-making process itself has been altered due to the addition of alternatives and options which were not, heretofore, possible or even feasible. The boundaries for decision-making have been redrawn as individuals may work from home and utilize telecommuting options or purchase products from work online. Managers can now locate raw materials and other inputs from suppliers via telecommunication networks in order to maximize profits while simultaneously ensuring timely delivery of finished goods. Financing for businesses can be now obtained online. Individuals, in turn, can obtain information about products from their homes and make their purchasing decisions accordingly.

The Internet, as a telecommunications and information network, par excellence, has impacted individuals, organizations, institutions, as well as businesses and societies in a way that that no other network system has in history, due to its speed of communications and its global reach. Moreover, the intertwining of such a communication network with other network systems such as transportation networks, energy networks, as well as a variety of economic networks, including financial networks, brings new challenges and opportunities for the conceptualization, modeling, and analysis of complex decision-making.

The reality of many of today's networks, notably, transportation and telecommunication networks, include: a large-scale nature and complexity, increasing congestion, alternative behaviors of users of the networks, as well as interactions among the networks themselves. The decisions made by the users of the networks, in turn, affect not only the users themselves but others, as well, in terms of profits and costs, timeliness of deliveries, the quality of the environment, etc.

In this chapter, the foundations and applications of the theory of *supernetworks* are described. **Super** networks may be thought of as networks that are *above and beyond* existing networks, which consist of nodes, links, and flows, with nodes corresponding to locations in space, links to connections in the form of roads, cables, etc., and flows to vehicles, data, etc. Supernetworks are, first and foremost, a mathematical formalization that allows the calculation of both static and dynamic equilibria/optima of complex networks with respect to the flows, which can include information, goods, persons, and prices. The flows are associated with the relevant technology that describes the generalized costs associated with using the networks along with the associated decision-making. In addition, supernetworks integrate existing unimodal network systems by providing a structure above and beyond the component networks. Supernetworks are conceptual in scope, graphical in perspective, and, with the accompanying theory, predictive in nature.

In particular, the supernetwork framework, captures, in a unified fashion, decision-making facing a variety of economic agents (decision-makers) including consumers and producers as well as distinct intermediaries in the context of today's networked economy. The decision-making process may entail weighting trade-offs associated with the use of transportation versus telecommunication networks. The behavior of the individual agents is modeled as well as their interactions on the complex network systems with the goal of identifying the resulting flows.

For definiteness, Table 1.1 presents some basic *classical* networks and the associated nodes, links, and flows. A *classical* network is one in which the nodes correspond to physical locations in space and the links to physical connections between the nodes.

The topic of networks and their management dates to ancient times with examples including the publicly provided Roman road network and the "time of day" chariot policy, whereby chariots were banned from the ancient city of Rome at particular times of day (see Banister and Button [3]). The formal study of networks, consisting of nodes, links, and flows, in turn, involves: how to model such applications (as well as numerous other ones) as mathematical entities, how to study the models qualitatively, and how to design algorithms to solve the resulting models effectively. The study of networks is necessarily interdisciplinary in nature due to their breadth of appearance and is based on scientific techniques from applied mathematics, computer science, engineering,

Table 1.1: Examples of Classical Networks

Network System	**Nodes**	**Links**	**Flows**
Transportation			
Urban	Intersections, Homes, Places of Work	Roads	Autos
Air	Airports	Airline Routes	Planes
Rail	Railyards	Railroad Track	Trains
Communication	Computers	Cables	Messages
	Satellites	Radio	Messages
	Phone Exchanges	Cables, Microwaves	Voice, Video
Manufacturing and Logistics	Distribution Points, Processing Points	Routes Assembly Line	Parts, Products
Energy	Pumping Stations	Pipelines	Water
	Plants	Pipelines	Gas, Oil

and economics. Network models and tools which are widely used by businesses, industries, as well as governments today (cf. Ahuja, Magnanti, and Orlin [2], Nagurney and Siokos [63], Nagurney ([43], [44]), Guenes and Pardalos [30], and the references therein).

Basic examples of network problems include: *the shortest path problem*, in which one seeks to determine the most efficient path from an origin node to a destination node; *the maximum flow problem*, in which one wishes to determine the maximum flow that one can send from an origin node to a destination node, given that there are capacities on the links that cannot be exceeded, and *the minimum cost flow problem*, where there are both costs and capacities associated with the links and one must satisfy the demands at the destination nodes, given supplies at the origin nodes, at minimal total cost associated with shipping the flows, and subject to not exceeding the arc capacities. Applications of all these problems are found in telecommunications and transportation.

Supernetworks may be comprised of such networks as transportation, telecommunication, logistical and financial networks, among others. They may be *multitiered* as when they formalize the study of supply chain networks with electronic commerce and financial networks with intermediation and electronic transactions. They may also be *multilevel* as when they capture the explicit interactions of distinct networks, including logistical, financial, social, and informational as in the case of dynamic supply chains. Furthermore, decision-makers on supernetworks may be faced with multiple criteria and, hence, the study of supernetworks also includes the study of multicriteria decision-making. In Table 1.2, some examples of supernetworks are given, which highlight the telecommunication aspect. We will overview several of these later in this chapter.

Table 1.2: Examples of Supernetworks

Supply Chain Networks with Electronic Commerce
Financial Networks with Electronic Transactions
Telecommuting/Commuting Networks
Teleshopping versus Shopping Networks
Reverse Supply Chains with Electronic Recycling
Energy Networks/Power Grids
Dynamic Knowledge Networks
Generalized Social Networks

In particular, the supernetwork framework allows one to formalize the alternatives available to decision-makers, to model their individual behavior, typically, characterized by particular criteria which they wish to optimize, and to, ultimately, compute the flows on the supernetwork. Hence, the concern is with human decision-making and how the supernetwork concept can be utilized to crystallize and inform in this dimension.

Below the theme of supernetworks is further elaborated upon and, in particular, the origins of the concept and the term *supernetworks* identified.

1.2 The Origins of Supernetworks

In this part of the chapter, a discussion of the three foundational classes of networks: transportation, telecommunication, and economic and financial networks is given. Such networks have served not only as the basis for the origins of the term *supernetwork*, but, also, they arise as critical subnetworks in the applications that are relevant to decision-making in the Information Age today.

1.2.1 Transportation Networks

Transportation networks are complex network systems in which the decisions of the individual travelers affect the efficiency and productivity of the entire network system. Transportation networks, as noted in Table 1.1, come in many forms: urban networks, freight networks, airline networks, etc. The "supply" in such a network system is represented by the network topology and the underlying cost characteristics, whereas the "demand" is represented by the users of the network system, that is, the travelers.

In 1972, Dafermos [16] demonstrated, through a formal model, how a *multiclass* traffic network could be cast into a single-class traffic network through the construction of an expanded (and *abstract*) network consisting of as many copies of the original network as there were classes. She identified the origin/destination pairs, demands, link costs, and flows on the abstract network. The applications of such networks she stated, "arise not only in street networks where vehicles of different types share the same roads (e.g., trucks and passenger cars) but also in other types of transportation networks (e. g., telephone networks).” Hence, she not only recognized that abstract networks could be used to handle multimodal transportation networks but also telecommunication networks! Moreover, she considered both user-optimizing and system-optimizing behavior, terms which she had coined with Sparrow in a paper in 1969 (see [23]). Beckmann [7] had earlier noted the potential relevance of network equilibrium (also referred to as user-optimization) in the context of communication networks.

In 1976, Dafermos [17] proposed an integrated traffic network equilibrium model in which one could visualize and formalize the entire transportation planning process (consisting of origin selection, or destination selection, or both, in addition to route selection, in an optimal fashion) as path choices over an appropriately constructed *abstract* network. The genesis and formal treatment of decisions more complex than route choices as *path* choices on abstract networks, that is, supernetworks, were, hence, reported as early as 1972 and 1976.

The importance and wider relevance of such abstract networks in decision-making, with a focus on transportation planning were accentuated through the term “hypernetwork” used by Sheffi [76], and Sheffi and Daganzo ([78], [79]), which was later retermed as “supernetwork” by Sheffi [77].

The recognition and appropriate construction of *abstract* networks was pivotal in that it allowed for the incorporation of transportation-related decisions (where as noted by Dafermos [16], transportation applied also to communication networks) which were not based solely on route selection in a classical sense, that is, what route should one take from one’s origin, say, place of residence, to one’s destination, say, place of employment. Hence, abstract networks, with origins and destinations corresponding to appropriately defined nodes, links connecting nodes having associated disutilities (costs), and paths comprised of directed links connecting the origins and destinations, could capture such travel alternatives as not simply just a route but, also, the “mode” of travel, that is, for example, whether one chose to use private or public transportation. Furthermore, with the addition of not only added abstract links and paths, but abstract origin and destination nodes as well one could include the selection of such locational decisions as the origins and destinations themselves within the same decision-making framework.

For example, in order to fix ideas, in Figure 1.1, a supernetwork topology for an example of a simple mode/route choice problem is presented. In this example, it is recognized, at the outset, that the routes underlying the different modes may be distinct and, hence, rather than making copies of the network according to [16], the supernetwork construction is done with the path choices directly on the supernetwork itself.

In the network in Figure 1.1, decision-makers in the form of travelers seek to determine their “best” paths from the origin node 1 to the destination node 4, where a path consists of both the selection of the mode of travel as well as the route of travel. The first link, which connects node 1 to node 4, corresponds to the use of public transit, and there is only one route choice using this mode of

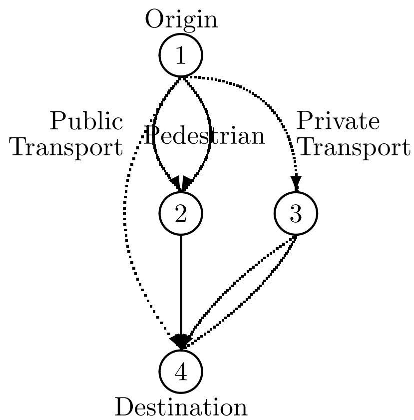

Figure 1.1: Example Mode and Route Choice Supernetwork Topology

travel. On the other hand, if one selects private transportation (typically, the automobile), one could take either of two routes: with the first route consisting of the first link joining nodes 1 and 3 and then the link joining node 3 to node 4, and the second route consisting of the second link joining nodes 1 and 3 and then onto node 4. Finally, one could choose either of two pedestrian routes to travel from node 1 to node 4, with the pedestrian routes differing by their second component links.

Additional references to supernetworks and transportation can be found in the book by Nagurney and Dong [51].

1.2.2 Telecommunication Networks

We now turn to a discussion of the use of the term "supernetworks" in the context of telecommunication networks. Denning [24], in the *American Scientist*, continued his discussion of the internal structure of computer networks, which had appeared in a volume of the same journal earlier that year, and emphasized how "protocol software can be built as a series of layers. Most of this structure is hidden from the users of the network." Denning then raised the question, "What should the users see?" In the article, he answered this question in the context of the then National Science Foundation's Advanced Scientific Computing Initiative to make national supercomputer centers accessible to the entire scientific community. Denning said that such a system would be a network of networks, that is, a "supernetwork," and a powerful tool for science. Interestingly, he emphasized the importance of location-independent naming, so that if a physical location of a resource would change, none of the supporting programs or files would need to be edited or recompiled. His view of supernetworks, hence, is in concert with that of ours in that nodes do not need to correspond to locations in space and may have an abstract association.

Schubert, Goebel, and Cercone [75] had earlier used the term in the context of knowledge representation as follows: “In the network approach to knowledge representation, concepts are represented as nodes in a network. Networks are compositional: a node in a network can be some other network, and the same subnetwork can be a subnetwork of several larger supernetworks,...”

In 1997, the Illinois Bar Association (see [31]) considered the following to be an accepted definition of the Internet: “the Internet is a supernetwork of computers that links together individual computers and computer networks located at academic, commercial, government and military sites worldwide, generally by ordinary local telephone lines and long-distance transmission facilities. Communications between computers or individual networks on the Internet are achieved through the use of standard, nonproprietary protocols.” The reference to the Internet as a supernetwork was also made by Fallows [27] who stated in *The Atlantic Monthly* that “The Internet is the supernetwork that links computer networks around the world.”

In his keynote address to the Internet/Telecom 95 Conference (see [83]), Mr. Vinton G. Cerf, the co-developer of the computer networking protocal, TCP/IP, used for the Internet, noted that at that time there were an estimated 23 million users of the Internet, and that vast quantities of the US Internet traffic “pass through internet MCI’s backbone.” Mr. Cerf then noted that “Just a few months back, MCI rolled out a supernetwork for the National Science Foundation known as the very broadband network service or VNBS...VBNS is being used as an experimental platform for developing new national networking applications.”

1.2.3 Economic and Financial Networks

The concept of a network in economics was implicit as early as in the classical work of Cournot [15], who not only seems to have first explicitly stated that a competitive price is determined by the intersection of supply and demand curves, but had done so in the context of two spatially separated markets in which the cost of transporting the good between markets was considered. Pigou [66] also studied a network system in the setting of a transportation network consisting of two routes and noted that the “system-optimized” solution was distinct from the “user-optimized” solution.

Nevertheless, the first instance of an abstract network or supernetwork in the context of economic applications, was actually due to Quesnay [68], who, in 1758, visualized the circular flow of funds in an economy as a network. Since that very early contribution there have been numerous economic and financial models that have been constructed over abstract networks. Dafermos and Nagurney [22], for example, identified the isomorphism between traffic network equilibrium problems and spatial price equilibrium problems, whose development had been originated by Samuelson [73] (who, interestingly, focused on the bipartite network structure of the spatial price equilibrium problem).

Zhao [87] (see also [88] and [89]) identified the general economic equilibrium problem known as Walrasian price equilibrium as a network equilibrium problem over an abstract network with very simple structure. The structure consisted of a single origin/destination pair of nodes and single links joining the two nodes. This structure was then exploited for computational purposes. A variety of abstract networks in economics were studied in the book by Nagurney (1999), which also contains extensive references to the subject.

Nagurney [44] used the term "supernetworks" in her essay in which she stated that "The interactions among transportation networks, telecommunication networks, as well as financial networks is creating supernetworks ..." .

1.3 Characteristics of Supernetworks

Supernetworks are a conceptual and analytical formalism for the study of a variety of decision-making problems on networks. Hence, their characteristics include characteristics of the foundational networks. The characteristics of today's networks include: large-scale nature and complexity of network topology; congestion; alternative behavior of users of the network, which may lead to paradoxical phenomena, and the interactions among networks themselves such as in transportation versus telecommunications networks. Moreover, policies surrounding networks today may have a major impact not only economically but also socially.

Large-Scale Nature and Complexity
Many of today's networks are characterized by both a large-scale nature and complexity of the underlying network topology. In Chicago's Regional Transportation Network, there are 12,982 nodes, 39,018 links, and 2,297,945 origin/destination (O/D) pairs (see [4]), whereas in the Southern California Association of Governments model there are 3,217 origins and/or destinations, 25,428 nodes, and 99,240 links, plus 6 distinct classes of users (cf. [86].

In terms of the size of existing telecommunications networks, AT&T's domestic network had 100,000 origin/destination pairs in 2000 (cf. Resende [70]). In AT&T's detail graph applications in which nodes are phone numbers and edges are calls, there were 300 million nodes and 4 billion edges in 1999 (cf. Abello, Pardalos, and Resende [1]).

Congestion
Congestion is playing an increasing role in not only transportation networks but also in telecommunication networks. For example, in the case of transportation networks in the United States alone, congestion results in $100 billion in lost productivity, whereas the figure in Europe is estimated to be $150 billion. The number of cars is expected to increase by 50% by 2010 and to double by 2030 (see [44]).

In terms of the Internet, according to Internet World Stats [32], there were over 800 million Internet users as of December 2004, with the usage growth globally between 2000-2004 being 125.2%. As individuals increasingly access the Internet through wireless communication such as through handheld computers and cellular phones, experts fear that the heavy use of airwaves will create ad-

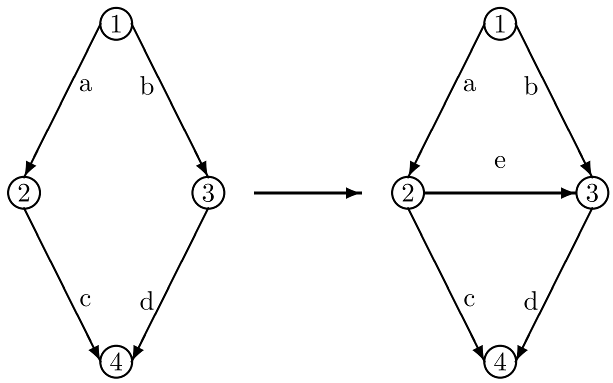

Figure 1.2: The Braess Network Example

ditional bottlenecks and congestion that could impede the further development of the technology.

System-Optimization versus User-Optimization
In many of today's networks, not only is congestion a characteristic feature leading to nonlinearities, but the behavior of the users of the networks themselves may be that of noncooperation. For example, in the case of urban transportation networks, travelers select their routes of travel from an origin to a destination so as to minimize their own travel cost or travel time, which although "optimal" from an individual's perspective (user-optimization) may not be optimal from a societal one (system-optimization) where one has control over the flows on the network and, in contrast, seeks to minimize the total cost in the network and, hence, the total loss of productivity. Consequently, in making any kind of policy decisions in such networks one must take into consideration the users of the particular network. Indeed, this point is vividly illustrated through a famous example known as the Braess paradox, in which it is assumed that the underlying behavioral principle is that of user-optimization. In the Braess network (cf. [10]), the addition of a new road with no change in the travel demand results in all travelers in the network incurring a higher travel cost and, hence, being worse off!

The Braess's paradox is now recalled. For easy reference, see the two networks depicted in Figure 1.2.

Example 1: Braess's Paradox

Assume a network as the first network depicted in Figure 1.2 in which there are four links: a, b, c, d; four nodes: 1, 2, 3, 4; and a single O/D pair $w_1 = (1, 4)$. There are, hence, two paths available to travelers between this O/D pair: $p_1 = (a, c)$ and $p_2 = (b, d)$.

The user link travel cost functions are given by:

$$c_a(f_a) = 10f_a, \quad c_b(f_b) = f_b + 50, \quad c_c(f_c) = f_c + 50, \quad c_d(f_d) = 10f_d.$$

Assume a fixed travel demand $d_{w_1} = 6$.

In the case of user-optimization, no traveler has any incentive to switch his path and this state is characterized by the property (for a complete description, see the next section), that all used paths connecting each O/D pair have equal and minimal travel costs (or times). Note that the user cost on a path is the sum of the user costs on the links that make up the path.

It is easy to verify that the equilibrium path flows that satisfy this condition (as well as the conservation of flow equations relating the nonnegative path flows to the travel demand) are:

$$x^*_{p_1} = 3, \quad x^*_{p_2} = 3;$$

the equilibrium link flows (incurred by the equilibrium path flows) are:

$$f^*_a = 3, \quad f^*_b = 3, \quad f^*_c = 3, \quad f^*_d = 3;$$

with associated equilibrium path travel costs:

$$C_{p_1} = c_a + c_c = 83, \quad C_{p_2} = c_b + c_d = 83.$$

Assume now that, as depicted in Figure 1.2, a new link "e," joining node 2 to node 3, is added to the original network, with user cost $c_e(f_e) = f_e + 10$. The addition of this link creates a new path $p_3 = (a, e, d)$ that is available to the travelers. Assume that the travel demand d_{w_1} remains at 6 units of flow. Note that the original flow distribution pattern $x_{p_1} = 3$ and $x_{p_2} = 3$ is no longer an equilibrium pattern, since at this level of flow, the cost on path p_3, denoted by C_{p_3}, is equal to 70. Hence, users from paths p_1 and p_2 would switch to path p_3.

The equilibrium flow pattern on the new network is:

$$x^*_{p_1} = 2, \quad x^*_{p_2} = 2, \quad x^*_{p_3} = 2;$$

with equilibrium link flows:

$$f^*_a = 4, \quad f^*_b = 2, \quad f^*_c = 2, \quad f^*_e = 2, \quad f^*_d = 4;$$

and with associated equilibrium path travel costs:

$$C_{p_1} = 92, \quad C_{p_2} = 92, \quad C_{p_3} = 92.$$

Indeed, one can verify that any reallocation of the path flows would yield a higher travel cost on a path.

Note that the travel cost increased for every user of the network from 83 to 92!

The increase in travel cost on the paths is due, in part, to the fact that in this network two links are shared by distinct paths and these links incur an increase in flow and associated cost. Hence, Braess's paradox is related to the underlying topology of the networks. It has been proven, however, that the addition of a path connecting an O/D pair that shares no links with the original

O/D pair will never result in Braess's paradox for that O/D pair (cf. Dafermos and Nagurney [21]).

In the next section, we will provide the system-optimizing solution to both of these networks. A system-optimized network never exhibits the Braess paradox.

Interestingly, as reported in the *New York Times* by Kolata [34], the Braess paradox phenomenon has been observed in practice, in the case of New York City, when, in 1990, 42nd Street was closed for Earth Day and the traffic flow actually improved. Just to show that it is not a purely New York or US phenomena concerning drivers and their behavior an analogous situation was observed in Stuttgart where a new road was added to the downtown but the traffic flow worsened and, following complaints, the new road was torn down (see Bass [5]).

This phenomenon is also relevant to telecommunications networks and the Braess paradox has provided one of the main linkages between transportation science and computer science. Cohen and Kelly [14] described a paradox analogous to that of Braess in the case of a queuing network. Korilis, Lazar, and Orda [35], in turn, developed methods to show how resources could be added efficiently to a noncooperative network, including the Internet, so that the Braess paradox would not occur and cited the work of Dafermos and Nagurney [21]. Roughgarden [71] further elaborated on the Braess paradox and focused on the quantification of the worst possible loss in network performance arising from noncooperative behavior (see also Roughgarden and Tardos [72]). He also designed algorithms for the design and management of the networks so that "selfish" (a term also used by Beckmann, McGuire, and Winsten [8]), that is, individually-optimizing, behavior, leads to a "socially desirable" outcome. He noted the importance of the work of the computer scientists, Koutsoupias and Papadimitrou [36], who proposed the idea of bounding the inefficiency of Nash equilibria (see also Dafermos and Sparrow [23]).

Network Interactions

Clearly, one of the principal facets of the Network Economy is the interaction among the networks themselves. For example, the increasing use of electronic commerce especially in business to business transactions is changing not only the utilization and structure of the underlying logistical networks but is also revolutionizing how business itself is transacted and the structure of firms and industries. Cellular phones are being using as vehicles move dynamically over transportation networks resulting in dynamic evolutions of the topologies themselves. The unifying concept of supernetworks with associated methodologies allows one to explore the interactions among such networks as transportation networks, telecommunication networks, as well as financial networks.

1.4 Decision-Making Concepts

As the above discussion has revealed, networks in the Information Age are complex, typically, large-scale systems and the study of their efficient operation, often through some outside intervention, has attracted much interest from economists, computer scientists, engineers, as well as transportation scientists and operations researchers.

In particular, the underlying behavior of the users of the network system is essential in studying its operation. Importantly, Wardrop [85] explicitly recognized alternative possible behaviors of users of transportation networks and stated two principles, which are commonly named after him:

First Principle: The journey times of all routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

Second Principle: The average journey time is minimal.

The first principle corresponds to the behavioral principle in which travelers seek to (unilaterally) determine their minimal costs of travel whereas the second principle corresponds to the behavioral principle in which the total cost in the network is minimal.

Beckmann, McGuire, and Winsten [8] were the first to rigorously formulate these conditions mathematically, as had Samuelson [73] in the framework of spatial price equilibrium problems in which there were, however, no congestion effects. Specifically, Beckmann, McGuire, and Winsten established the equivalence between the *traffic network equilibrium* conditions, which state that all used paths connecting an origin/destination (O/D) pair will have equal and minimal travel times (or costs) (corresponding to Wardrop's first principle), and the Kuhn-Tucker [37] conditions of an appropriately constructed optimization problem (cf. Bazaraa, Sherali, and Shetty [6]), under a symmetry assumption on the underlying functions. Hence, in this case, the equilibrium link and path flows could be obtained as the solution of a mathematical programming problem. Their approach made the formulation, analysis, and subsequent computation of solutions to traffic network problems based on actual networks realizable.

Dafermos and Sparrow [23] coined the terms *user-optimized* (U-O) and *system-optimized* (S-O) transportation networks to distinguish between two distinct situations in which, respectively, users act unilaterally, in their own self-interest, in selecting their routes, and in which users select routes according to what is optimal from a societal point of view, in that the total cost in the system is minimized. In the latter problem, marginal total costs on used paths connecting an O/D pair of nodes, rather than average costs, are equilibrated. The former problem coincides with Wardrop's first principle, and the latter with Wardrop's second principle.

The concept of "system-optimization" is also relevant to other types of "routing models" not only in transportation, but also in communications (cf. Bertsekas and Gallager [9]), including those concerned with the routing of freight and computer messages, respectively. Dafermos and Sparrow also provided explicit computational procedures, that is, *algorithms*, to compute the solutions to such network problems in the case where the user travel cost on a link was an increasing function of the flow on the link (in order to handle congestion) and linear.

1.4.1 System-Optimization Versus User-Optimization

The basic network models are now reviewed, under distinct assumptions as to their operation and the corresponding distinct behavior of the users of the network. The models are classical and are due to Beckmann, McGuire, and Winsten [8] and Dafermos and Sparrow [23]. We later present more general models.

For definiteness, and for easy reference, we present the classical system-optimized network model and then the classical user-optimized network model. Although these models were first developed for transportation networks, here they are presented in the broader setting of network systems, since they are as relevant in other application settings, in particular, in telecommunication networks and, more generally, in supernetworks.

More general models are then outlined, in which the user link cost functions are no longer separable and are also asymmetric. We provide the variational inequality formulations of the governing equilibrium conditions (see Kinderlehrer and Stampacchia [33] and Nagurney [43]), since, in this case, the conditions can no longer be reformulated as the Kuhn-Tucker conditions of a convex optimization problem. Finally, we present the variational inequality formulations in the case of elastic demands.

The System-Optimized Problem

Consider a general network $\mathcal{G} = [\mathcal{N}, \mathcal{L}]$, where $\mathcal{N}$ denotes the set of nodes, and $\mathcal{L}$ the set of directed links. Let a denote a link of the network connecting a pair of nodes, and let p denote a path consisting of a sequence of links connecting an O/D pair. In transportation networks (see also Table 1.1), nodes correspond to origins and destinations, as well as to intersections. Links, on the other hand, correspond to roads/streets in the case of urban transportation networks and to railroad segments in the case of train networks. A path in its most basic setting, thus, is a sequence of "roads" which comprise a route from an origin to a destination. In the telecommunication context, however, nodes can correspond to switches or to computers and links to telephone lines, cables, microwave links, etc. In the supernetwork setting, a path is viewed more broadly and need not be limited to a route-type decision.

Let P_ω denote the set of paths connecting the origin/destination (O/D) pair of nodes ω. Let P denote the set of all paths in the network and assume that there are J origin/destination pairs of nodes in the set Ω. Let x_p represent the flow on path p and let f_a denote the flow on link a. The path flows on the network are grouped into the column vector $x \in R_+^{n_P}$, where n_P denotes the number of paths in the network. The link flows, in turn, are grouped into the column vector $f \in R_+^n$, where n denotes the number of links in the network.

The following conservation of flow equation must hold:

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in \mathcal{L}, \tag{1.1}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. Expression (1.1) states that the flow on a link a is equal to the sum of all the path flows on paths p that contain (traverse) link a.

Moreover, if one lets d_ω denote the demand associated with O/D pair ω, then one must have that

$$d_\omega = \sum_{p \in P_\omega} x_p, \quad \forall \omega \in \Omega, \tag{1.2}$$

where $x_p \geq 0$, $\forall p \in P$; that is, the sum of all the path flows between an origin/destination pair ω must be equal to the given demand d_ω.

Let c_a denote the user link cost associated with traversing link a, and let C_p denote the user cost associated with traversing the path p.

Assume that the user link cost function is given by the *separable* function

$$c_a = c_a(f_a), \quad \forall a \in \mathcal{L}, \tag{1.3}$$

where c_a is assumed to be an increasing function of the link flow f_a in order to model the effect of the link flow on the cost. The link cost functions are also assumed to be continuous and continuously differentiable.

The total cost on link a, denoted by $\hat{c}_a(f_a)$, hence, is given by:

$$\hat{c}_a(f_a) = c_a(f_a) \times f_a, \quad \forall a \in \mathcal{L}, \tag{1.4}$$

that is, the total cost on a link is equal to the user link cost on the link times the flow on the link. Here the cost is interpreted in a general sense. From a transportation engineering perspective, however, the cost on a link is assumed to coincide with the travel time on a link. Later in this chapter, we consider generalized cost functions of the links which are constructed using weights and different criteria.

In the system-optimized problem, there exists a central controller who seeks to minimize the total cost in the network system, where the total cost is expressed as

$$\sum_{a \in \mathcal{L}} \hat{c}_a(f_a), \tag{1.5}$$

where the total cost on a link is given by expression (1.4).

The system-optimization problem is, thus, given by:

$$\text{Minimize} \quad \sum_{a \in \mathcal{L}} \hat{c}_a(f_a) \tag{1.6}$$

subject to:

$$\sum_{p \in P_\omega} x_p = d_\omega, \quad \forall \omega \in \Omega, \tag{1.7}$$

$$f_a = \sum_{p \in P} x_p, \quad \forall a \in \mathcal{L}, \tag{1.8}$$

$$x_p \geq 0, \quad \forall p \in P. \tag{1.9}$$

The constraints (1.7) and (1.8), along with (1.9), are commonly referred to in network terminology as *conservation of flow equations*. In particular, they guarantee that the flow in the network, that is, the users (whether these are travelers or computer messages, for example) do not "get lost."

The total cost on a path, denoted by $\hat{C}_p$, is the user cost on a path times the flow on a path, that is,

$$\hat{C}_p = C_p x_p, \quad \forall p \in P, \tag{1.10}$$

where the user cost on a path, C_p, is given by the sum of the user costs on the links that comprise the path, that is,

$$C_p = \sum_{a \in \mathcal{L}} c_a(f_a)\delta_{ap}, \quad \forall a \in \mathcal{L}. \tag{1.11}$$

In view of (1.8), one may express the cost on a path p as a function of the path flow variables and, hence, an alternative version of the above system-optimization problem can be stated in path flow variables only, where one has now the problem:

$$\text{Minimize} \quad \sum_{p \in P} C_p(x)x_p \tag{1.12}$$

subject to constraints (1.7) and (1.9).

System-Optimality Conditions

Under the above imposed assumptions on the user link cost functions, which recall are assumed to be increasing functions of the flow, the objective function in the S-O problem is convex, and the feasible set consisting of the linear constraints is also convex. Therefore, the optimality conditions, that is, the Kuhn-Tucker conditions are: For each O/D pair $\omega \in \Omega$, and each path $p \in P_\omega$, the flow pattern x (and link flow pattern f), satisfying (1.7)–(1.9) must satisfy:

$$\hat{C}'_p \begin{cases} = \mu_\omega, & \text{if} \quad x_p > 0 \\ \geq \mu_\omega, & \text{if} \quad x_p = 0, \end{cases} \tag{1.13}$$

where $\hat{C}'_p$ denotes the marginal of the total cost on path p, given by:

$$\hat{C}'_p = \sum_{a \in \mathcal{L}} \frac{\partial \hat{c}_a(f_a)}{\partial f_a}\delta_{ap}, \tag{1.14}$$

and in (1.13) it is evaluated at the solution.

Note that in the S-O problem, according to the optimality conditions (1.13), it is the marginal of the total cost on each used path connecting an O/D pair which is equalized and minimal. Indeed, conditions (1.13) state that a system-optimized flow pattern is such that for each origin/destination pair the incurred marginals of the total costs on all used paths are equal and minimal.

We return now to the Braess network(s) in Figure 1.2. The system-optimizing solution to the first network in Figure 1.2 would be:

$$x_{p_1} = x_{p_2} = 3,$$

with marginal total path costs given by:

$$\hat{C}'_{p_1} = \hat{C}'_{p_2} = 116.$$

This would remain the system-optimizing solution, even after the addition of link e, since the marginal total cost of the new path p_3, $\hat{C}'_{p_3}$, at this feasible flow pattern (with the flow on the new path p_3 being zero) is equal to 130. Hence, in the case of a system-optimizing solution, path p_3 would not even be used and we would have that $x_{p_3} = 0$.

The addition of a new link to a network cannot increase the total cost of the network system, in the case of system-optimization, as formulated above.

The User-Optimized Problem

We now describe the user-optimized network problem, also commonly referred to in the literature as the *traffic assignment* problem or the *traffic network equilibrium* problem. Again, as in the system-optimized problem, the network $\mathcal{G} = [\mathcal{N}, \mathcal{L}]$, the demands associated with the origin/destination pairs, as well as the user link cost functions are assumed as given. Recall that user-optimization follows Wardrop's first principle.

Network Equilibrium Conditions

Now, however, one seeks to determine the path flow pattern x^* (and link flow pattern f^*) which satisfies the conservation of flow equations (1.7), (1.8), and the nonnegativity assumption on the path flows (1.9), and which also satisfies the network equilibrium conditions given by the following statement.

For each O/D pair $\omega \in \Omega$ and each path $p \in P_\omega$:

$$C_p \begin{cases} = \lambda_\omega, & \text{if} \quad x_p^* > 0 \\ \geq \lambda_\omega, & \text{if} \quad x_p^* = 0. \end{cases} \tag{1.15}$$

Hence, in the user-optimization problem there is no explicit optimization concept, since now users of the network system act independently, in a non-cooperative manner, until they cannot improve on their situations unilaterally and, thus, an equilibrium is achieved, governed by the above equilibrium conditions. Indeed, conditions (1.15) are simply a restatement of Wardrop's (1952) first principle mathematically and mean that only those paths connecting an O/D pair will be used which have equal and minimal user costs. Otherwise, a user of the network could improve upon his situation by switching to a path with lower cost. User-optimization represents decentralized decision-making, whereas system-optimization represents centralized decision-making.

In order to obtain a solution to the above problem, Beckmann, McGuire, and Winsten [8] established that the solution to the equilibrium problem, in the case of user link cost functions (cf. (1.3)) in which the cost on a link only depends on the flow on that link could be obtained by solving the following optimization problem:

$$\text{Minimize} \quad \sum_{a\in\mathcal{L}} \int_0^{f_a} c_a(y)dy \tag{1.16}$$

subject to:

$$\sum_{p\in P_\omega} x_p = d_\omega, \quad \forall \omega \in \Omega, \tag{1.17}$$

$$f_a = \sum_{p\in P} x_p \delta_{ap}, \quad \forall a \in \mathcal{L}, \tag{1.18}$$

$$x_p \geq 0, \quad \forall p \in P. \tag{1.19}$$

Note that the conservation of flow equations are identical in both the user-optimized network problem (see (1.17)–(1.19)) and the system-optimized problem (see (1.7) – (1.9)). The behavior of the individual decision-makers termed "users," however, is different. Users of the network system, which generate the flow on the network now act independently, and are not controlled by a centralized controller. The relevance of these two distinct behavioral concepts to telecommunications networks is clear.

The objective function given by (1.16) is simply a device constructed to obtain a solution using general purpose convex programming algorithms. It does not possess the economic meaning of the objective function encountered in the system-optimization problem given by (1.6), equivalently, by (1.12). Of course, algorithms that fully exploit the network structure of these problems can be expected to perform more efficiently (cf. Dafermos and Sparrow [23], Nagurney [42, 43]).

1.4.2 Models with Asymmetric Link Costs

There has been much dynamic research activity in the past several decades in both the modeling and the development of methodologies to enable the formulation and computation of more general network equilibrium models, with a focus on traffic networks. Examples of general models include those that allow for multiple modes of transportation or multiple classes of users, who perceive cost on a link in an individual way. We now consider network models in which the user cost on a link is no longer dependent solely on the flow on that link. Other network models, including dynamic traffic models, can be found in Mahmassani et al. [40], and in the books by Ran and Boyce [69], Nagurney and Zhang [64], Nagurney [43], and the references therein.

We now consider user link cost functions which are of a general form, that is, in which the cost on a link may depend not only on the flow on the link but on other link flows on the network, that is,

$$c_a = c_a(f), \quad \forall a \in \mathcal{L}. \tag{1.20}$$

In the case where the symmetry condition holds, that is, $\frac{\partial c_a(f)}{\partial f_b} = \frac{\partial c_b(f)}{\partial f_a}$, for all links $a, b \in \mathcal{L}$, one can still reformulate the solution to the network equilibrium problem satisfying equilibrium conditions (1.15) as the solution to an optimization problem (cf. Dafermos [16] and the references therein), albeit, again, with an objective function that is artificial and simply a mathematical device. However, when the symmetry assumption is no longer satisfied, such an optimization reformulation no longer exists and one must appeal to *variational inequality theory*.

Indeed, it was in the problem domain of traffic network equilibrium problems that the theory of finite-dimensional variational inequalities realized its earliest success, beginning with the contributions of Smith [81] and Dafermos [17]. For an introduction to the subject, as well as applications ranging from traffic network equilibrium problems to financial equilibrium problems, see the book by Nagurney [43]. The methodology of finite-dimensional variational inequalities has also been utilized more recently in order to develop a spectrum of supernetwork models (see Nagurney and Dong [51]).

The system-optimization problem, in turn, in the case of nonseparable (cf. (1.20)) user link cost functions becomes (see also (1.6)–(1.9)):

$$\text{Minimize} \quad \sum_{a \in \mathcal{L}} \hat{c}_a(f), \tag{1.21}$$

subject to (1.7)–(1.9), where $\hat{c}_a(f) = c_a(f) \times f_a$, $\forall a \in \mathcal{L}$.

The system-optimality conditions remain as in (1.13), but now the marginal of the total cost on a path becomes, in this more general case:

$$\hat{C}'_p = \sum_{a,b \in \mathcal{L}} \frac{\partial \hat{c}_b(f)}{\partial f_a} \delta_{ap}, \quad \forall p \in P. \tag{1.22}$$

Variational Inequality Formulations of Fixed Demand Problems

As mentioned earlier, in the case where the user link cost functions are no longer symmetric, one cannot compute the solution to the U-O, that is, to the network equilibrium, problem using standard optimization algorithms. Such cost functions are very important from an application standpoint since they allow for asymmetric interactions on the network. For example, allowing for asymmetric cost functions permits one to handle the situation when the flow on a particular link affects the cost on another link in a different way than the cost on the particular link is affected by the flow on the other link.

Since equilibrium is such a fundamental concept in terms of supernetworks and since variational inequality theory is one of the basic ways in which to study such problems we now, for completeness, also give variational inequality formulations of the network equilibrium conditions (1.15). These formulations are presented without proof (for derivations, see Smith [81] and Dafermos [18], as well as Florian and Hearn [29] and the book by Nagurney [43]).

First, the definition of a variational inequality problem is recalled. We then give both the variational inequality formulation in path flows as well as in link flows of the network equilibrium conditions. Subsequently, in this chapter, these concepts are extended to multicriteria, multiclass network equilibrium problems.

Specifically, the variational inequality problem (finite-dimensional) is defined as follows:

Definition 1.1: Variational Inequality Problem
The finite-dimensional variational inequality problem, $\mathrm{VI}(F, \mathcal{K})$*, is to determine a vector* $X^* \in \mathcal{K}$ *such that*

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{1.23}$$

where F *is a given continuous function from* $\mathcal{K}$ *to* R^N*,* $\mathcal{K}$ *is a given closed convex set, and* $\langle \cdot, \cdot \rangle$ *denotes the inner product in* R^N*.*

Variational inequality (1.23) is referred to as being in *standard form*. Hence, for a given problem, typically an *equilibrium* problem, one must determine the function F that enters the variational inequality problem, the vector of variables X, as well as the feasible set $\mathcal{K}$.

The variational inequality problem contains, as special cases, such well-known problems as systems of equations, optimization problems, and complementarity problems. Thus, it is a powerful unifying methodology for equilibrium analysis and computation.

Theorem 1.1: Variational Inequality Formulation of Network Equilibrium with Fixed Demands – Path Flow Version
A vector $x^* \in K^1$ *is a network equilibrium path flow pattern, that is, it satisfies equilibrium conditions (1.15) if and only if it satisfies the variational inequality problem:*

$$\sum_{\omega \in \Omega} \sum_{p \in P_\omega} C_p(x^*) \times (x - x^*) \geq 0, \quad \forall x \in K^1, \tag{1.24}$$

or, in vector form:

$$\langle C(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K^1, \tag{1.25}$$

where C *is the* n_P*-dimensional column vector of path user costs and* K^1 *is defined as:* $K^1 \equiv \{x \geq 0, \text{ such that } (1.17) \text{ holds}\}$*.*

Theorem 1.2: Variational Inequality Formulation of Network Equilibrium with Fixed Demands – Link Flow Version
A vector $f^* \in K^2$ *is a network equilibrium link flow pattern if and only if it satisfies the variational inequality problem:*

$$\sum_{a \in \mathcal{L}} c_a(f^*) \times (f_a - f_a^*) \geq 0, \quad \forall f \in K^2, \tag{1.26}$$

or, in vector form:

$$\langle c(f^*), f - f^* \rangle \geq 0, \quad \forall f \in K^2, \tag{1.27}$$

where c *is the* n*-dimensional column vector of link user costs and* K^2 *is defined as:* $K^2 \equiv \{f \mid \text{there exists an } x \geq 0 \text{ and satisfying } (1.17) \text{ and } (1.18)\}$*.*

Note that one may put variational inequality (1.25) in standard form (1.23) by letting $F \equiv C$, $X \equiv x$, and $\mathcal{K} \equiv K^1$. Also, one may put variational inequality (1.27) in standard form where now $F \equiv c$, $X \equiv f$, and $\mathcal{K} \equiv K^2$.

Alternative variational inequality formulations of a problem are useful in devising other models, including dynamic versions, as well as for purposes of computation using different algorithms.

Variational Inequality Formulations of Elastic Demand Problems

The general network equilibrium model with elastic demands due to Dafermos [20] is now recalled. Specifically, it is assumed that now one has associated with each O/D pair ω in the network a disutility λ_ω, where here the general case is considered in which the disutility may depend upon the entire vector of demands, which are no longer fixed, but are now variables, that is,

$$\lambda_\omega = \lambda_\omega(d), \quad \forall \omega \in \Omega, \tag{1.28}$$

where d is the J-dimensional column vector of the demands.

The notation, otherwise, is as described earlier, except that here we also consider user link cost functions which are general, that is, of the form (1.20). The conservation of flow equations (see also (1.1) and (1.2)), in turn, are given by

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in \mathcal{L}, \tag{1.29}$$

$$d_\omega = \sum_{p \in P_\omega} x_p, \quad \forall \omega \in \Omega, \tag{1.30}$$

$$x_p \geq 0, \quad \forall p \in P. \tag{1.31}$$

Hence, in the elastic demand case, the demands in expression (1.30) are now variables and no longer given, as was the case for the fixed demand expression in (1.2). Elastic demand models are very useful in that the demands are allowed to adjust. Hence, such models may be viewed as providing a more long-term perspective as to how decision-makers adjust given the network parameters, vis a vis the fixed demand scenario.

Network Equilibrium Conditions in the Case of Elastic Demand

The network equilibrium conditions (see also (1.15)) now take on in the elastic demand case the following form: For every O/D pair $\omega \in \Omega$, and each path $p \in P_\omega$, a vector of path flows and demands (x^*, d^*) satisfying (1.30)–(1.31) (which induces a link flow pattern f^* through (1.29)) is a network equilibrium pattern if it satisfies:

$$C_p(x^*) \begin{cases} = \lambda_\omega(d^*), & \text{if} \quad x_p^* > 0 \\ \geq \lambda_\omega(d^*), & \text{if} \quad x_p^* = 0. \end{cases} \tag{1.32}$$

Equilibrium conditions (1.32) state that the costs on used paths for each O/D pair are equal and minimal and equal to the disutility associated with that O/D pair. Costs on unutilized paths can exceed the disutility.

In the next two theorems, both the path flow version and the link flow version of the variational inequality formulations of the network equilibrium conditions (1.32) are presented. These are analogues of the formulations (1.24) and (1.25), and (1.26) and (1.27), respectively, for the fixed demand model.

Theorem 1.3: Variational Inequality Formulation of Network Equilibrium with Elastic Demands – Path Flow Version
A vector $(x^*, d^*) \in K^3$ is a network equilibrium path flow pattern, that is, it satisfies equilibrium conditions (1.32) if and only if it satisfies the variational inequality problem:

$$\sum_{\omega\in\Omega}\sum_{p\in P_\omega} C_p(x^*)\times(x-x^*)-\sum_{\omega\in\Omega}\lambda_\omega(d^*)\times(d_\omega-d^*_\omega)\geq 0, \quad \forall(x,d)\in K^3, \tag{1.33}$$

or, in vector form:

$$\langle C(x^*), x-x^*\rangle - \langle\lambda(d^*), d-d^*\rangle \geq 0, \quad \forall(x,d)\in K^3, \tag{1.34}$$

where λ is the J-dimensional vector of disutilities and K^3 is defined as: $K^3 \equiv \{x \geq 0,$ such that (1.30) holds$\}$.

Theorem 1.4: Variational Inequality Formulation of Network Equilibrium with Elastic Demands – Link Flow Version
A vector $(f^*, d^*) \in K^4$ is a network equilibrium link flow pattern if and only if it satisfies the variational inequality problem:

$$\sum_{a\in\mathcal{L}} c_a(f^*)\times(f_a-f^*_a)-\sum_{\omega\in\Omega}\lambda_\omega(d^*)\times(d_\omega-d^*_\omega)\geq 0, \quad \forall(f,d)\in K^4, \tag{1.35}$$

or, in vector form:

$$\langle c(f^*), f-f^*\rangle - \langle\lambda(d^*), d-d^*\rangle \geq 0, \quad \forall(f,d)\in K^4, \tag{1.36}$$

where $K^4 \equiv \{(f,d),$ such that there exists an $x \geq 0$ satisfying $(1.29), (1.31)\}$

Note that, under the symmetry assumption on the disutility functions, that is, if $\frac{\partial\lambda_w}{\partial d_\omega} = \frac{\partial\lambda_\omega}{\partial d_w}$, for all w, ω, in addition to such an assumption on the user link cost functions (see following (1.20)), one can obtain (see [8]) an optimization reformulation of the network equilibrium conditions (1.32), which in the case of separable user link cost functions and disutility functions is given by:

$$\text{Minimize} \quad \sum_{a\in\mathcal{L}}\int_0^{f_a} c_a(y)dy - \sum_{\omega\in\Omega}\int_0^{d_\omega}\lambda_\omega(z)dz \tag{1.37}$$

subject to: (1.29)–(1.31).

An example of a simple elastic demand network equilibrium problem is now given.

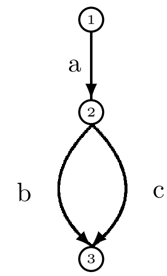

Figure 1.3: An Elastic Demand Example

Example 2: An Elastic Demand Network Equilibrium Problem

Consider the network depicted in Figure 1.3 in which there are three nodes: 1, 2, 3; three links: a, b, c; and a single O/D pair $\omega_1 = (1, 3)$. Let path $p_1 = (a, b)$ and path $p_2 = (a, c)$.

Assume that the user link cost functions are:

$$c_a(f) = 2f_a + 10, \quad c_b(f) = 7f_b + 3f_c + 11, \quad c_c(f) = 6f_c + 4f_b + 14,$$

and the disutility (or inverse demand) function is given by:

$$\lambda_{\omega_1}(d_{\omega_1}) = -2d_{\omega_1} + 104.$$

Observe that in this example, the user link cost functions are non-separable for links b and c and asymmetric and, hence, the equilibrium conditions (cf. (1.32)) cannot be reformulated as the solution to an optimization problem, but, rather, as the solution to the variational inequalities (1.33) (or (1.34)), or (1.35) (or (1.36)).

The U-O flow and demand pattern that satisfies equilibrium conditions (1.32) is: $x^*_{p_1} = 5$, $x^*_{p_2} = 4$, and $d^*_{\omega_1} = 9$, with associated link flow pattern: $f^*_a = 9$, $f^*_b = 5$, $f^*_c = 4$.

The incurred user costs on the paths are: $C_{p_1} = C_{p_2} = 86$, which is precisely the value of the disutility λ_{ω_1}. Hence, this flow and demand pattern satisfies equilibrium conditions (1.32). Indeed, both paths p_1 and p_2 are utilized and their user paths costs are equal to each other. In addition, these costs are equal to the disutility associated with the origin/destination pair that the two paths connect.

1.5 Multiclass, Multicriteria Supernetworks

In this part of the chapter, we describe how the concept of a multicriteria supernetwork can be utilized to address decision-making in the Information Age. We then present a specific applications, in particular, telecommuting versus commuting decision-making. This section is expository. The theoretical foundations can be found in Nagurney and Dong [51].

The term "multicriteria" captures the multiplicity of criteria that decision-makers are often faced with in making their choices, be they regarding consumption, production, transportation, location, or investment. Criteria which are considered as part of the decision-making process may include: cost minimization, time minimization, opportunity cost minimization, profit maximization, as well as risk minimization, among others.

Indeed, the Information Age with the increasing availability of new computer and communication technologies, along with the Internet, have transformed the ways in which many individuals work, travel, and conduct their daily activities today. Moreover, the decision-making process itself has been altered through the addition of alternatives which were not, heretofore, possible or even feasible. As stated in a recent issue of *The Economist* [26], "The boundaries for employees are redrawn... as people work from home and shop from work."

The first publications in the area of multicriteria decision-making on networks focused on transportation networks, and were by Schneider [74] and Quandt [67]. However, they assumed fixed travel times and travel costs. Here, in contrast, these functions (as well as any other appropriate criteria functions) are flow-dependent. The first flow-dependent such model was by Dafermos [19], who considered an infinite number of decision-makers, rather than a finite number as is done here. Furthermore, she assumed two criteria, whereas we consider a finite number, where the number can be as large as necessary. Moreover, the modeling framework set out in this chapter can also handle elastic demands. The first general elastic demand multicriteria network equilibrium model was developed by Nagurney and Dong [50], who considered two criteria and fixed weights but allowed the weights to be class- and link-dependent. The models in this chapter, in contrast, allow the particular application to be handled with as many finite criteria as are relevant and retain the flexible feature of allowing the weights associated with the criteria to be both class- and link-dependent. We refer the reader to Nagurney and Dong [51] for additional references. We now recall the multiclass, multicriteria network equilibrium models with elastic demand and with fixed demand, respectively. Each class of decision-maker is allowed to have weights associated with the criteria which are also permitted to be link-dependent for modeling flexibility purposes. Subsequently, the governing equilibrium conditions along with the variational inequality formulations are presented.

1.5.1 The Multiclass, Multicriteria Network Equilibrium Models

In this section, the multiclass, multicriteria network equilibrium models are described. The elastic demand model is presented first and then the fixed demand model. The equilibrium conditions are, subsequently, shown to satisfy finite-dimensional variational inequality problems.

Consider a general network $G = [\mathcal{N}, \mathcal{L}]$, where $\mathcal{N}$ denotes the set of nodes in the network and $\mathcal{L}$ the set of directed links. Let a denote a link of the network connecting a pair of nodes and let p denote a path, assumed to be acyclic, consisting of a sequence of links connecting an origin/destination (O/D) pair of nodes. There are n links in the network and n_P paths. Let Ω denote the set of J O/D pairs. The set of paths connecting the O/D pair ω is denoted by P_ω and the entire set of paths in the network by P.

Note that in the supernetwork framework a link may correspond to an actual physical link of transportation or to an abstract or virtual link corresponding to telecommunications. Furthermore, the supernetwork representing the problem under study can be as general as necessary and a path may consist also of a set of links corresponding to a combination of physical and virtual choices. A path, hence, in the supernetwork framework, abstracts a decision as a sequence of links or possible choices from an origin node, which represents the beginning of the decision, to the destination node, which represents its completion.

Assume that there are now k classes of decision-makers in the network with a typical class denoted by i. Let f_a^i denote the flow of class i on link a and let x_p^i denote the nonnegative flow of class i on path p. The relationship between the link flows by class and the path flows is:

$$f_a^i = \sum_{p \in P} x_p^i \delta_{ap}, \quad \forall i, \quad \forall a \in \mathcal{L}, \tag{1.38}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. Hence, the flow of a class of decision-maker on a link is equal to the sum of the flows of the class on the paths that contain that link.

In addition, let f_a denote the total flow on link a, where

$$f_a = \sum_{i=1}^{k} f_a^i, \quad \forall a \in \mathcal{L}. \tag{1.39}$$

Thus, the total flow on a link is equal to the sum of the flows of all classes on that link. Group the class link flows into the kn-dimensional column vector $\tilde{f}$ with components: $\{f_1^1, \ldots, f_n^1, \ldots, f_1^k, \ldots, f_n^k\}$, the total link flows: $\{f_1, \ldots, f_n\}$ into the n-dimensional column vector f, and the class path flows into the kn_P-dimensional column vector $\tilde{x}$ with components: $\{x_{p_1}^1, \ldots, x_{p_{n_P}}^k\}$.

The demand associated with origin/destination (O/D) pair ω and class i will be denoted by d_ω^i. Group the demands into a column vector $d \in R^{kJ}$. Clearly, the demands must satisfy the following conservation of flow equations:

$$d_\omega^i = \sum_{p \in P_\omega} x_p^i, \quad \forall i, \forall \omega, \tag{1.40}$$

that is, the demand for an O/D pair for each class is equal to the sum of the path flows of that class on the paths that join the O/D pair.

The functions associated with the links are now described. In particular, assume that there are H criteria which the decision-makers may utilize in their decision-making with a typical criterion denoted by h. Assume that C_{ha} denotes criterion h associated with link a, where

$$C_{ha} = C_{ha}(f), \quad \forall a \in \mathcal{L}, \tag{1.41}$$

where C_{ha} is assumed to be a continuous function.

For example, criterion 1 may be time, in which case we would have

$$C_{1a} = C_{1a}(f) = t_a(f), \quad \forall a \in \mathcal{L}, \tag{1.42}$$

where $t_a(f)$ denotes the time associated with traversing link a. In the case of a transportation link, one would expect the function to be higher than for a telecommunications link. Another relevant criterion may be cost, that is,

$$C_{2a} = C_{2a}(f) = c_a(f), \quad \forall a \in \mathcal{L}, \tag{1.43}$$

which might reflect (depending on the link a) an access cost in the case of a telecommunications link, or a transportation or shipment cost in the case of a transportation link. One can expect both time and cost to be relevant criteria in decision-making in the Information Age especially since telecommunications is at times a substitute for transportation and it is typically associated with higher speed and lower cost (cf. Mokhtarian [41]).

In addition, another relevant criterion in evaluating decision-making in the Information Age is opportunity cost since one may expect that this cost would be high in the case of teleshopping, for example (since one cannot physically experience and evaluate the product), and lower in the case of shopping. Furthermore, in the case of telecommuting, there may be perceived to be a higher associated opportunity cost by some classes of decision-makers who may miss the socialization provided by face-to-face interactions with coworkers and colleagues. Hence, a third possible criterion may be opportunity cost, where

$$C_{3a} = C_{3a}(f) = o_a(f), \quad \forall a \in \mathcal{L}, \tag{1.44}$$

with $o_a(f)$ denoting the opportunity cost associated with link a. Finally, a decision-maker may wish to associate a safety cost in which case the fourth criterion may be

$$C_{4a} = C_{4a}(f) = s_a(f), \quad \forall a \in \mathcal{L}, \tag{1.45}$$

where $s_a(f)$ denotes a security or safety cost measure associated with link a. In the case of teleshopping, for example, decision-makers may be concerned with revealing personal or credit information, whereas in the case of transportation, commuters may view certain neighborhood roads as being dangerous.

We assume that each class of decision-maker has a potentially different perception of the tradeoffs among the criteria, which are represented by the nonnegative weights: $w^i_{1a}, \ldots, w^i_{Ha}$. Hence, w^i_{1a} denotes the weight on link a associated with criterion 1 for class i, w^i_{2a} denotes the weight associated with criterion 2 for class i, and so on. Observe that the weights are link-dependent and can incorporate specific link-dependent factors which could include for a particular class factors such as convenience and sociability. A typical weight associated with class i, link a, and criterion h is denoted by w^i_{ha}.

Nagurney and Dong [50] were the first to model link-dependent weights but only considered two criteria. Nagurney, Dong, and Mokhtarian [53], in turn, used link-dependent weights but assumed only three criteria, in particular, travel time, travel cost, and opportunity cost in their integrated multicriteria network equilibrium models for telecommuting versus commuting.

Here, a generalized cost function is defined as follows.

Definition 1.2: Generalized Link Cost Function
A generalized link cost of class i associated with link a and denoted by C_a^i is given by:

$$C_a^i = \sum_{h=1}^{H} w_{ha}^i C_{ha}, \quad \forall i, \quad \forall a \in \mathcal{L}. \tag{1.46}$$

For example, (1.46) states that each class of decision-maker i when faced by H distinct criteria on each link a assigns his own weights $\{w_{ha}^i\}$ to the links and criteria.

In lieu of (1.39) – (1.46), one can write

$$C_a^i = C_a^i(\tilde{f}), \quad \forall i, \quad \forall a \in \mathcal{L}, \tag{1.47}$$

and group the generalized link costs into the kn-dimensional column vector C with components: $\{C_1^1, \ldots, C_n^1, \ldots, C_1^k, \ldots, C_n^k\}$.

For example, if there are four criteria associated with decision-making and they are given by (1.42) through (1.45), then the generalized cost function on a link a as perceived by class i would have the form:

$$C_a^i = w_{1a}^i C_{1a}(\tilde{f}) + w_{2a}^i C_{2a}(\tilde{f}) + w_{3a}^i C_{3a}(\tilde{f}) + w_{4a}^i C_{4a}(\tilde{f}). \tag{1.48}$$

Let now C_p^i denote the generalized cost of class i associated with path p in the network where

$$C_p^i = \sum_{a \in \mathcal{L}} C_a^i(\tilde{f}) \delta_{ap}, \quad \forall i, \quad \forall p. \tag{1.49}$$

Thus, the generalized cost associated with a class and a path is that class's weighted combination of the various criteria on the links that comprise the path.

Note from the structure of the criteria on the links as expressed by (1.41) and the generalized cost structure assumed for the different classes on the links according to (1.46) and (1.47), that it is explicitly being assumed that the relevant criteria are functions of the total flows on the links, where recall that the total flows (see (1.39)) correspond to the total number of decision-makers of all classes that selects a particular link. This is not unreasonable since one can expect that the greater the number of decision-makers that select a particular link (which comprises a part of a path), the greater the congestion on that link and, hence, one can expect the time of traversing the link as well as the cost to increase.

In the case of the elastic demand model, assume, as given, the inverse demand functions λ_ω^i for all classes i and all O/D pairs ω, where:

$$\lambda_\omega^i = \lambda_\omega^i(d), \quad \forall i, \quad \forall \omega, \tag{1.50}$$

and these functions are assumed to be smooth and continuous. Group the inverse demand functions into a column vector $\lambda \in R^{kJ}$.

The Behavioral Assumption

Assume that the decision-making involved in the particular application is repetitive in nature such as, for example, in the case of commuting versus telecommuting, or shopping versus teleshopping. The behavioral assumption that is proposed, hence, is that decision-makers select their paths so that their generalized costs are minimized.

Specifically, the behavioral assumption utilized is similar to that underlying traffic network assignment models (cf. (1.15) and (1.32)) in that it is assumed that each class of decision-maker in the network selects a path so as to minimize the generalized cost on the path, given that all other decision-makers have made their choices.

In particular, the following are the network equilibrium conditions for the problem outlined above:

Multiclass, Multicriteria Network Equilibrium Conditions for the Elastic Demand Case
For each class i, for all O/D pairs $\omega \in \Omega$, and for all paths $p \in P_\omega$, the flow pattern $\tilde{x}^*$ is said to be *in equilibrium* if the following conditions hold:

$$C_p^i(\tilde{f}^*) \left\{ \begin{array}{lll} = \lambda_\omega^i(d^*), & \text{if} & x_p^{i*} > 0 \\ \geq \lambda_\omega^i(d^*), & \text{if} & x_p^{i*} = 0. \end{array} \right. \tag{1.51}$$

In other words, all utilized paths by a class connecting an O/D pair have equal and minimal generalized costs and the generalized cost on a used path by a class is equal to the inverse demand/ disutility for that class and the O/D pair that the path connects.

In the case of the fixed demand model, in which the demands in (1.40) are now assumed known and fixed, the multicriteria network equilibrium conditions now take the form:

Multiclass, Multicriteria Network Equilibrium Conditions for the Fixed Demand Case
For each class i, for all O/D pairs $\omega \in \Omega$, and for all paths $p \in P_\omega$, the flow pattern $\tilde{x}^*$ is said to be *in equilibrium* if the following conditions hold:

$$C_p^i(\tilde{f}^*) \left\{ \begin{array}{lll} = \lambda_\omega^i, & \text{if} & x_p^{i*} > 0 \\ \geq \lambda_\omega^i, & \text{if} & x_p^{i*} = 0, \end{array} \right. \tag{1.52}$$

where now the λ_ω^i denotes simply an indicator representing the minimal incurred generalized path cost for class i and O/D pair ω. Equilibrium conditions (1.52) state that all used paths by a class connecting an O/D pair have equal and minimal generalized costs.

We now present the variational inequality formulations of the equilibrium conditions governing the elastic demand and the fixed demand problems, respectively, given by (1.51) and (1.52).

Theorem 1.5: Variational Inequality Formulation of the Elastic Demand Model
The variational inequality formulation of the multicriteria network model with elastic demand satisfying equilibrium conditions (1.51) is given by: determine $(\tilde{f}^*, d^*) \in \mathcal{K}^1$*, satisfying*

$$\sum_{i=1}^{k} \sum_{a \in \mathcal{L}} C_a^i(\tilde{f}^*) \times (f_a^i - f_a^{i*}) - \sum_{i=1}^{k} \sum_{\omega \in \Omega} \lambda_\omega^i(d^*) \times (d_\omega^i - d_\omega^{i*}) \geq 0, \quad \forall (\tilde{f}, d) \in \mathcal{K}^1, \tag{1.53}$$

where $\mathcal{K}^1 \equiv \{(\tilde{f}, d) | \tilde{x} \geq 0,$ *and (1.38), (1.39), and (1.40) hold*$\}$*; equivalently, in standard variational inequality form:*

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{1.54}$$

where $F \equiv (C, \lambda)$*,* $X \equiv (\tilde{f}, d)$*, and* $\mathcal{K} \equiv \mathcal{K}^1$*.*

Hence, a flow and demand pattern satisfies equilibrium conditions (1.51) if and only if it also satisfies the variational inequality problem (1.53) or (1.54).

In the case of fixed demands, we have the following:

Theorem 1.6: Variational Inequality Formulation of the Fixed Demand Model
The variational inequality formulation of the fixed demand multiciteria network equilibrium model satisfying equilibrium conditions (1.52) is given by: determine $\tilde{f} \in \mathcal{K}^2$*, satisfying*

$$\sum_{i=1}^{k} \sum_{a \in \mathcal{L}} C_a^i(\tilde{f}^*) \times (f_a^i - f_a^{i*}) \geq 0, \qquad \forall \tilde{f} \in \mathcal{K}^2, \tag{1.55}$$

where $\mathcal{K}^2 \equiv \{\tilde{f} | \exists \tilde{x} \geq 0,$ *and satisfying (1.38), (1.39), and (1.40), with d known*$\}$*; equivalently, in standard variational inequality form:*

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{1.56}$$

where $F \equiv C$*,* $X \equiv \tilde{f}$*, and* $\mathcal{K} \equiv \mathcal{K}^2$*.*

Therefore, a flow pattern satisfies equilibrium conditions (1.52) if and only if it satisfies variational inequality (1.55) or (1.56).

Note that the above are finite-dimensional variational inequality problems. Finite-dimensional variational inequality formulations were also obtained by Nagurney [44] for her bicriteria fixed demand traffic network equilibrium model in which the weights were fixed and only class-dependent. Nagurney and Dong [50], in turn, formulated an elastic demand traffic network problem with two criteria and weights which were fixed but class- and link-dependent as a finite-dimensional variational inequality problem. The first use of a finite-dimensional variational inequality formulation of a multicriteria network equilibrium problem is due to Leurent [39] (see also, e.g., [38]), who, however, only allowed one of the two criteria to be flow-dependent. Moreover, although his model was an elastic demand model, the demand functions were separable and not class-dependent as above.

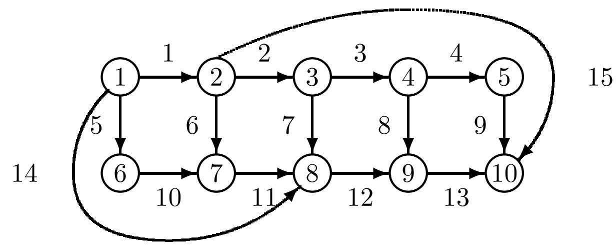

Figure 1.4: Network Topology for Telecommuting versus Commuting Example

1.5.2 An Application - Modeling Telecommuting versus Commuting Decision-Making

In this subsection, an application of the multiclass, multicriteria network equilibrium framework is presented. In particular, the fixed demand multicriteria network equilibrium model is applied to telecommuting versus commuting.

Note that, in the supernetwork framework, a link may correspond to an actual physical link of transportation or an abstract or virtual link corresponding to a telecommuting link. Furthermore, the supernetwork representing the problem under study can be as general as necessary and a path may also consist of a set of links corresponding to physical and virtual transportation choices such as would occur if a worker were to commute to a work center from which she could then telecommute.

Consider the four criteria, given by (1.42) through (1.45), and representing, respectively, travel time, travel cost, the opportunity cost, and safety cost. Consider a generalized link cost for each class given by (1.46). Thus, the generalized cost on a path as perceived by a class of traveler is given by (1.49).

The behavioral assumption is that travelers of a particular class are assumed to choose the paths associated with their origin/destination pair so that the generalized cost on that path is minimal. An equilibrium is assumed to be reached when the multicriteria network equilibrium conditions (1.52) are satisfied. Hence, only those paths connecting an O/D pair are utilized such that the generalized costs on the paths, as perceived by a class, are equal and minimal. The governing variational inequality for this problem is given by (1.55); equivalently, by (1.56).

For illustrative purposes, we now present a numerical example, which is governed by variational inequality (1.55); equivalently, (1.56). In order to compute the equilibrium flow pattern for the problem, the modifed projection method was applied. (See the book by Nagurney and Dong [51] for complete details.)

The numerical example had the topology depicted in Figure 1.4. Links 1 through 13 are transportation links whereas links 14 and 15 are telecommunication links. The network consisted of ten nodes, fifteen links, and two O/D

Table 1.3: The Weights and Travel Time Functions for the Links for the Telecommuting Example

Link a	w^1_{1a}	w^2_{1a}	$t_a(f)$
1	.25	.5	$.00005f_1^4 + 4f_1 + 2f_3 + 2$
2	.25	.5	$.00003f_2^4 + 2f_2 + f_5 + 1$
3	.4	.4	$.00005f_3^4 + f_3 + .5f_2 + 3$
4	.5	.3	$.00003f_4^4 + 7f_4 + 3f_1 + 1$
5	.4	.5	$5f_5 + 2$
6	.5	.7	$.00007f_6^4 + 3f_6 + f_9 + 4$
7	.2	.4	$4f_7 + 6$
8	.3	.3	$.00001f_8^4 + 4f_8 + 2f_{10} + 1$
9	.6	.2	$2f_9 + 8$
10	.3	.1	$.00003f_{10}^4 + 4f_{10} + f_{12} + 7$
11	.2	.4	$.00004f_{11}^4 + 6f_{11} + 2f_{13} + 2$
12	.3	.5	$.00002f_{12}^4 + 4f_{12} + 2f_5 + 1$
13	.2	.4	$.00003f_{13}^4 + 7f_{13} + 4f_{10} + 8$
14	.5	.3	$f_{14} + 2$
15	.5	.2	$f_{15} + 1$

pairs where $\omega_1 = (1, 8)$ and $\omega_2 = (2, 10)$ with travel demands by class given by: $d^1_{\omega_1} = 10$, $d^1_{\omega_2} = 20$, $d^2_{\omega_1} = 10$, and $d^2_{\omega_2} = 30$. The paths connecting the O/D pairs were: for O/D pair ω_1: $p_1 = (1, 2, 7)$, $p_2 = (1, 6, 11)$, $p_3 = (5, 10, 11)$, $p_4 = (14)$, and for O/D pair ω_2: $p_5 = (2, 3, 4, 9)$, $p_6 = (2, 3, 8, 13)$, $p_7 = (2, 7, 12, 13)$, $p_8 = (6, 11, 12, 13)$, and $p_9 = (15)$.

The travel time functions and the travel cost functions, for this example, along with the associated weights for the two classes, are reported, respectively, in Tables 1.3 and 1.4. The opportunity cost functions and the safety cost functions for the links for this example, along with the associated weights for the two classes and these criteria, are reported in Table 1.5.

The generalized link cost functions were constructed according to (1.46).

Note that the opportunity costs associated with links 14 and 15 were high since these are telecommunication links and users by choosing these links forego the opportunities associated with working and associating with colleagues from a face to face perspective. Observe, however, that the weights for class 1 associated with the opportunity costs on the telecommunication links are low (relative to those of class 2). This has the interpretation that class 1 does not weight such opportunity costs highly and may, for example, prefer to be working from the home for a variety, including familial, reasons. Also, note that class 1 weights the travel time on the telecommunication links more highly than class 2 does. Furthermore, observe that class 1 weights the safety or security cost higher than class 2.

Table 1.4: The Weights and Travel Cost Functions for the Links for the Telecommuting Example

Link a	w^1_{2a}	w^2_{2a}	$c_a(f)$
1	.25	.5	$.00005f_1^4 + 5f_1 + 1$
2	.25	.4	$.00003f_2^4 + 4f_2 + 2f_3 + 2$
3	.4	.3	$.00005f_3^4 + 3f_3 + f_1 + 1$
4	.5	.2	$.00003f_4^4 + 6f_4 + 2f_6 + 4$
5	.5	.4	$4f_5 + 8$
6	.3	.6	$.00007f_6^4 + 7f_6 + 2f_2 + 6$
7	.4	.3	$8f_7 + 7$
8	.5	.2	$.00001f_8^4 + 7f_8 + 3f_5 + 6$
9	.2	.3	$8f_9 + 5$
10	.4	.4	$.00003f_{10}^4 + 6f_{10} + 2f_8 + 3$
11	.7	.5	$.00004f_{11}^4 + 4f_{11} + 3f_{10} + 4$
12	.4	.5	$.00002f_{12}^4 + 6f_{12} + 2f_9 + 5$
13	.3	.6	$.00003f_{13}^4 + 9f_{13} + 3f_8 + 3$
14	.2	.4	$.1f_{14} + 1$
15	.3	.2	$.2f_{15} + 1$

Table 1.5: The Weights and the Opportunity Cost and Safety Cost Functions for the Links for the Example

Link a	w^1_{3a}	w^2_{3a}	$o_a(f)$	w_{4a}	w^2_{4a}	$s_a(f)$
1	1.	.5	$2f_1 + 4$	.2	.1	$f_1 + 1$
2	1.	.4	$3f_2 + 2$	.2	.1	$f_2 + 2$
3	1.	.7	$f_3 + 4$	.2	.1	$f_3 + 1$
4	2.	.6	$f_4 + 2$	.2	.1	$f_4 + 2$
5	1.	.5	$2f_5 + 1$	.2	.1	$2f_5 + 2$
6	2.	.7	$f_6 + 2$	.2	.1	$f_6 + 1$
7	1.	.8	$f_7 + 3$	.2	.1	$f_7 + 1$
8	1.	.6	$2f_8 + 1$	.2	.1	$2f_8 + 2$
9	2.	.9	$3f_9 + 2$	.2	.1	$3f_9 + 3$
10	1.	.8	$f_{10} + 1$	.2	.1	$f_{10} + 2$
11	1.	.9	$4f_{11} + 3$	.2	.1	$2f_{11} + 3$
12	1.	.7	$3f_{12} + 2$	.2	.1	$3f_{12} + 3$
13	2.	.9	$f_{13} + 1$	.2	.1	$f_{13} + 2$
14	.1	1.	$6f_{14} + 1$	.2	.1	$.5f_{14} + .1$
15	.1	.2	$7f_{15} + 4$	.2	.1	$.4f_{15} + .1$

Table 1.6: The Equilibrium Link Flows for the Example

Link a	Class 1 - f_a^{1*}	Class 2 - f_a^{2*}	Total flow - f_a^*
1	0.000	0.0000	0.0000
2	0.0000	24.0109	24.0109
3	0.0000	22.7600	22.7600
4	0.0000	17.3356	17.3356
5	0.0000	4.6901	4.6901
6	0.0000	5.9891	5.9891
7	0.0000	1.2509	1.2509
8	0.0000	5.4244	5.4244
9	0.0000	17.3556	17.3556
10	0.0000	4.6901	4.6901
11	0.0000	10.6792	10.6792
12	0.0000	7.2400	7.2400
13	0.0000	12.6644	12.6644
14	10.0000	5.3090	15.3099
15	20.0000	0.0000	20.0000

The equilibrium multiclass link flow and total link flow patterns are reported in Table 1.6, which were induced by the equilibrium multiclass path flow pattern given in Table 1.7.

The generalized path costs were: for Class 1, O/D pair ω_1:

$$C^1_{p_1} = 13478.4365,\ C^1_{p_2} = 11001.0342,\ C^1_{p_3} = 8354.5420,\ C^1_{p_4} = 1025.4167,$$

for Class 1, O/D pair ω_2:

$$C^1_{p_5} = 45099.8047,\ C^1_{p_6} = 27941.5918,\ C^1_{p_7} = 25109.3223,\ C^1_{p_8} = 22631.9199,$$

$$C^1_{p_9} = 2314.7222;$$

for Class 2, O/D pair ω_1:

$$C^2_{p_1} = 15427.5996,\ C^2_{p_2} = 15427.2021,\ C^2_{p_3} = 8721.8945,\ C^2_{p_4} = 8721.3721,$$

and for Class 2, O/D pair ω_2:

$$C^2_{p_5} = 34924.6602,\ C^2_{p_6} = 34924.6094,\ C^2_{p_7} = 34925.3789,\ C^2_{p_8} = 34924.9805,$$

$$C^2_{p_9} = 41574.2617.$$

It is interesting to see the separation by classes in the equilibrium solution. Note that all members of class 1, whether residing at node 1 or node 2, were telecommuters, whereas all members of class 2 chose to commute to work. This outcome is realistic, given the weight assignments of the two classes on the

Table 1.7: The Equilibrium Path Flows for the Example

Path p	Class 1 - x_p^{1*}	Class 2 - x_p^{2*}
p_1	0.0000	0.0000
p_2	0.0000	0.0000
p_3	0.0000	4.6901
p_4	10.0000	5.3099
p_5	0.0000	17.3357
p_6	0.0000	5.4244
p_7	0.0000	1.2509
p_8	0.0000	5.9892
p_9	20.0000	0.0000

opportunity costs associated with the links (as well as the weight assignments associated with the travel times). Of course, different criteria functions, as well as their numerical forms and associated weights, will lead to different equilibrium patterns.

This example demonstrates the flexibility of the modeling approach. Moreover, it allows one to conduct a variety of "what if" simulations in that, one can modify the functions and the associated weights to reflect the particular telecommuting versus commuting scenario. For example, during a downturn in the economy, the opportunity costs associated with the telecommuting links may be high, and, also, different classes may weight this criteria on such links higher, resulting in a new solution. On the other hand, highly skilled employees who are in demand may have lower weights associated with such links in regards to the opportunity costs. This framework is, hence, sufficiently general to capture a variety of realistic situations while, at the same time, allowing decision-makers to identify their specific values and preferences.

1.6 Multitiered and Multilevel Supernetworks

In the preceding section, the focus was on multiclass, multicriteria supernetworks and a specific application to telecommuting versus commuting decision-making was highlighted. Such a framework has also been applied to model teleshopping versus shopping decision-making (see [54] and [55]). Nagurney and Dong [52], on the other hand, proposed a supernetwork model for knowledge production in the case of multiple criteria, assuming system-optimizing behavior.

In this section, we discuss several applications in which the decision-makers are now associated with the nodes of different tiers of a supernetworks. The applications that we discuss (see also Table 1.2) are supply chain networks and financial networks with intermediation and electronic transactions. Appropriate references are noted for complete mathematical formulations and solution procedures.

1.6.1 Supply Chain Networks

The study of supply chain network problems through modeling, analysis, and computation is a challenging topic due to the complexity of the relationships among the various decision-makers, such as suppliers, manufacturers, distributors, and retailers as well as the practical importance of the topic for the efficient movement (and pricing) of products. The topic is multidisciplinary by nature since it involves particulars of manufacturing, transportation and logistics, retailing/marketing, as well as economics. For additional background on supply chains, see the books by Bramel and Simchi-Levi [11] and Pardalos and Tsitsiringos [65], which also discusses financial engineering aspects, and the volume edited by Simchi-Levi, Wu, and Shen [80].

In particular, the introduction of electronic commerce has unveiled new opportunities in terms of research and practice in supply chain analysis and management since electronic commerce (e-commerce) has had an immense effect on the manner in which businesses order goods and have them transported with the major portion of e-commerce transactions being in the form of business-to-business (B2B). Estimates of B2B electronic commerce range from approximately .1 trillion dollars to 1 trillion dollars in 1998 and with forecasts reaching as high as $4.8 trillion dollars in 2003 in the United States (see Federal Highway Administration [28], Southworth [82]). It has been emphasized that the principal effect of business-to-business (B2B) commerce, estimated to be 90% of all e-commerce by value and volume, is in the creation of new and more profitable supply chain networks.

In Figure 1.5, a four-tiered supply chain network is depicted (cf. [51]) in which the top tier consists of suppliers of inputs into the production processes used by the manufacturing firms (the second tier), who, in turn, transform the inputs into products which are then shipped to the third tier of decision-makers, the retailers, from whom the consumers can then obtain the products. In this context, not only are physical transactions allowed but also virtual transactions, in the form of electronic transactions via the Internet to represent electronic commerce. In the supernetwork framework, both B2B and B2C can be considered, modeled, and analyzed. The decision-makers may compete independently across a given tier of nodes of the network and cooperate between tiers of nodes.

In particular, Nagurney et al. [60] have applied the supernetwork framework to supply chain networks with electronic commerce in order to predict product flows between tiers of decision-makers as well as the prices associated with the different tiers. They assumed that the manufacturers as well as the retailers are engaged in profit maximizing behavior whereas the consumers seek to minimize the costs associated with their purchases. The model therein determines the volumes of the products transacted electronically or physically. That work was based on the model of Nagurney, Dong, and Zhang [56], which was the first supply chain network equilibrium model. It assumed decentralized decision-making and competition across a tier of decision-makers but cooperation between tiers.

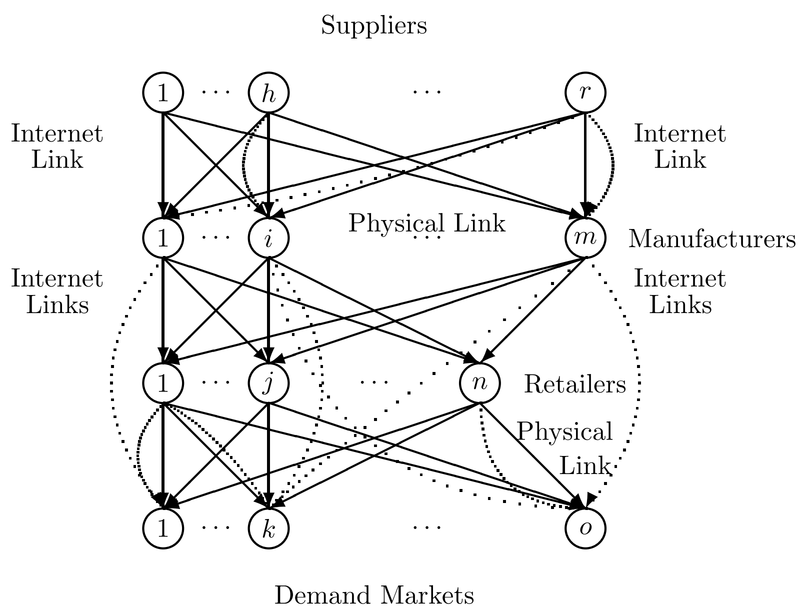

Figure 1.5: The Supernetwork Structure of the Supply Chain Network with Suppliers, Manufacturers, Retailers, and Demand Markets and Electronic Commerce

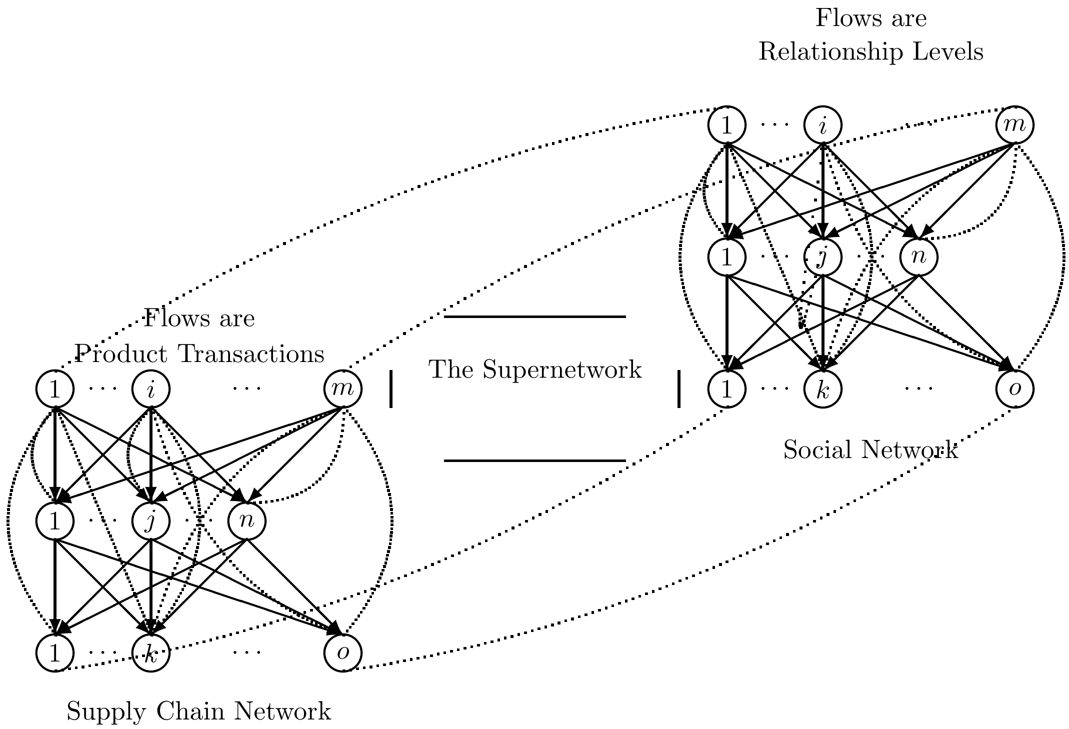

Figure 1.6: The Multilevel Supernetwork Structure of the Integrated Supply Chain / Social Network System

As mentioned earlier, supernetworks may also be multilevel in structure. In particular, Nagurney et al. [59] demonstrated how supply chain networks can be depicted and studied as multilevel networks in order to identify not only the product shipments but also the financial flows as well as the informational ones. In Figure 1.6 we provide a graphic of an integrated social and supply chain networks as a multilevel supernetwork due to Wakolbinger and Nagurney [84]. In their model, they introduced flows into social networks in the form of relationship levels and allowed for transactions costs to be functions of both product transactions as well as relationship levels. The decision-making behavior assumed profit maximization, risk minimization, as well as relationship value maximization with individual associated weights for the manufacturers and the retailers.

Obviously, in the setting of supply chain networks and, in particular, in global supply chains, there may be much risk and uncertainty associated with the underlying functions. Some research along those lines has recently been undertaken (cf. Dong, Zhang, and Nagurney [25], and Nagurney, Cruz, and Matsypura [49]). Continuing efforts to include uncertainty and risk into modeling and computational efforts in a variety of supernetworks and their applications is of paramount importance given the present economic and political climate.

In addition, we emphasize that the inclusion of environmental variables and criteria is also an important topic for research and practice in the context of supply chain networks (cf. Nagurney and Fuminori [61]). Recently, a multitiered supply chain network equilibrium framework has been developed for reverse logistics and the recycling of electronic wastes (see Nagurney and Fuminori [62]).

1.6.2 Financial Networks with Electronic Transactions

As noted earlier, financial networks have been utilized in the study of financial systems since the work of Quesnay [68], who, in 1758, depicted the circular flow of funds in an economy as a network. His conceptualization of the funds as a network, which was abstract, is the first identifiable instance of a supernetwork.

Advances in telecommunications and, in particular, the adoption of the Internet by businesses, consumers, and financial institutions have had an enormous effect on financial services and the options available for financial transactions. Distribution channels have been transformed, new types of services and products introduced, and the role of financial intermediaries altered in the new economic networked landscape. Furthermore, the impact of such advances has not been limited to individual nations but, rather, through new linkages, has crossed national boundaries.

The topic of *electronic* finance has been a growing area of study (cf. Claessens, Glaessner, and Klingebiel [12], Claessens et al. [13], Nagurney [46], and the references therein), due to its increasing impact on financial markets and financial intermediation, as well as related regulatory issues and governance. Of particular emphasis has been the conceptualization of the major issues involved and the role of networks is the transformations (cf. Nagurney and Dong [51] and the references therein).

Nevertheless, the complexity of the interactions among the distinct decision-makers involved, the supply chain aspects of the financial product accessibilities and deliveries, as well as the availability of physical as well as electronic options, and the role of intermediaries, have defied the construction of a unified, quantifiable framework in which one can assess the resulting financial flows and prices.

Here we briefly describe a supernetwork framework for the study of financial decision-making in the presence of intermediation and electronic transactions. Further details can be found in Nagurney and Ke [57] and [58]. The framework is sufficiently general to allow for the modeling, analysis, and computation of solutions to such problems.

The financial network model consists of: agents or decision-makers with sources of funds, financial intermediaries, as well as consumers associated with the demand markets. In the model, the sources of funds can transact directly electronically with the consumers through the Internet and can also conduct their financial transactions with the intermediaries either physically or electronically. The intermediaries, in turn, can transact with the consumers either physically in the standard manner or electronically. The depiction of the network at

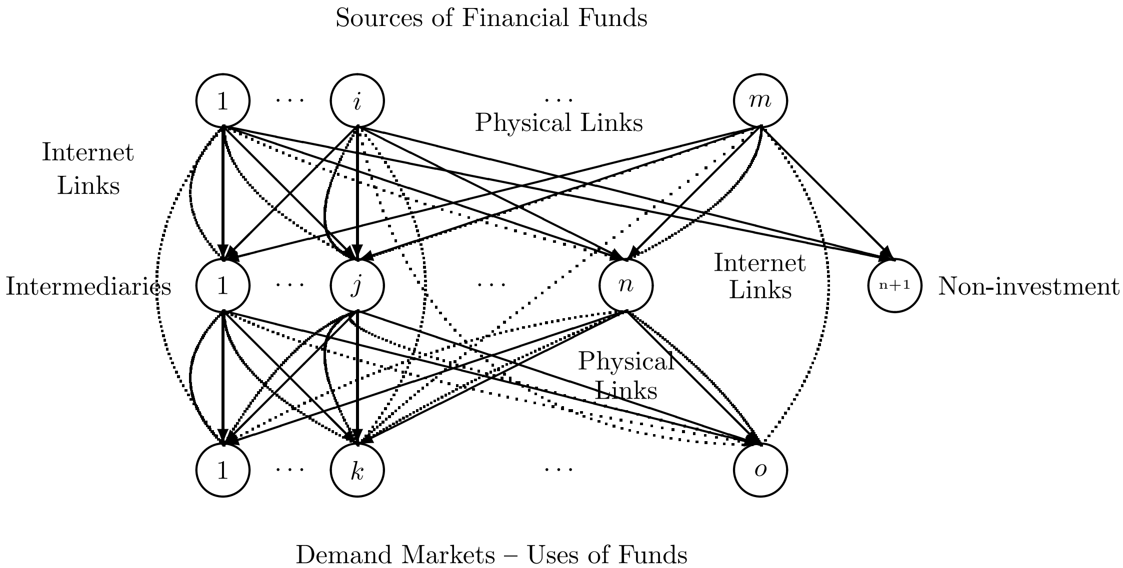

Figure 1.7: The Structure of the Financial Network with Electronic Transactions

equilibrium is given in Figure 1.7.

It is assumed that the agents with sources of funds as well as the financial intermediaries seek to maximize their net revenue (in the presence of transaction costs) while, at the same time, minimizing the risk associated with the financial products. The solution of the model yields the financial flows between the tiers as well as the prices. Here we also allow for the option of having the source agents not invest a part (or all) of their financial holdings. More recently, Nagurney and Cruz [48] have demonstrated that the financial supernetwork framework can also be extended to model international financial networks with intermediation in which there are distinct agents in different countries and the financial products are available in different currencies.

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