A network efficiency measure for congested networks

ANNA NAGURNEY[1] and QIANG QIANG[1]

[1] *University of Massachusetts Amherst, Massachusetts 01003*
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Abstract. - In this paper, we propose a network efficiency measure for congested networks, that captures demands, costs, flows, and behavior. The network efficiency/performance measure can identify which network components, that is, nodes and links, have the greatest impact in terms of their removal, due to, for example, natural disasters, structural failures, terrorist attacks, etc., and, hence, are important from both vulnerability as well as security standpoints. The new measure is applied to the Braess paradox network in which the demands are varied over the horizon and explicit formulae are derived for the importance values of the network nodes and links. This measure is applicable to such congested networks as urban transportation networks and the Internet.

Introduction. – In this paper, we propose a new network efficiency measure that is appropriate for congested networks, that is, networks in which the cost associated with a link is an increasing function of the flow on the link. It is well-known that congestion is a fundamental problem in a variety of modern network systems, including urban transportation networks, electric power generation and distribution networks, as well as the Internet (*cf.* [1], [2], [3], [4]). Hence, an appropriate network efficiency measure for such network systems can have wide application. In addition, it can be used, as we show in this paper, to identify which nodes or links are critical or most important in a congested network in that their removal will result in a large relative efficiency drop. Furthermore, the identified important network components are those, clearly, that should be better protected or secured since their removal has a greater impact on the network system.

This paper is organized as follows. We first briefly review the well-known traffic network equilibrium model, which has been applied to congested urban transportation networks and the Internet, and which is also closely related to electric power generation and distribution networks (see, *e.g.*, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]). We then present the new network efficiency measure for congested networks. Subsequently, we revisit the Braess Paradox [11] (see also [12]), which is as relevant to the Internet as it is to transportation networks and we apply the network measure to the case in which the demand is varied over the nonnegative real line. Explicit formulae are obtained for both the efficiency measure as well as for the importance identification of the nodes and links (along with their rankings). We conclude the paper with a summary of the results and suggestions for future research.

Traffic Network Equilibrium Model. – We now recall the traffic network equilibrium model ([1] - [10]), which is widely used and applied in practice. Consider a network G with the set of directed links L with n_L elements and the set of origin/destination (O/D) pairs W with n_W elements. We denote the set of acyclic paths joining O/D pair w by P_w. The set of (acyclic) paths for all O/D pairs is denoted by P and there are n_P paths in the network. Links are denoted by a, b, etc; paths by p, q, etc., and O/D pairs by w_1, w_2, etc.

We assume that the demand d_w is known for all $w \in W$. We denote the nonnegative flow on path p by x_p and the flow on link a by f_a and we group the path flows into the vector $x \in R^{n_P}_+$ and the link flows into the vector $f \in R^{n_L}_+$.

The following conservation of flow equations must hold:

$$\sum_{p \in P_w} x_p = d_w, \quad \forall w \in W, \tag{1}$$

which means that the sum of path flows on paths connecting each O/D pair must be equal to the demand for that O/D pair.

The link flows are related to the path flows, in turn, through the following conservation of flow equations:

$$f_a = \sum_{p\in P} x_p \delta_{ap}, \quad \forall a \in L, \tag{2}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and $\delta_{ap} = 0$, otherwise. Hence, the flow on a link is equal to the sum of the flows on paths that contain that link.

The user (travel) cost on a path p is denoted by C_p and the user (travel) cost on a link a by c_a. The user costs on paths are related to user costs on links through the following equations:

$$C_p = \sum_{a\in L} c_a \delta_{ap}, \quad \forall p \in P, \tag{3}$$

that is, the user cost on a path is equal to the sum of user costs on links that make up the path. In engineering practice (see [9]), the travel time on a link is used as a proxy for the travel cost.

Since we are concerned with congested networks, we allow the user link cost function on each link to depend, in general, upon the vector of link flows, so that

$$c_a = c_a(f), \quad \forall a \in L. \tag{4}$$

We assume that the link cost functions are continuous and monotonically increasing. In view of (1), (2), and (3), we may write

$$C_p = C_p(x), \quad \forall p \in P. \tag{5}$$

A network equilibrium is defined as follows. A path flow pattern $x^* \in \mathcal{K}^1$, where $\mathcal{K}^1 \equiv \{x|x \in R^{n_P}_+ \text{and}\,(1)\,\text{holds}\}$, is said to be a network equilibrium, if the following conditions hold for each O/D pair $w \in W$ and each path $p \in P_w$:

$$C_p(x^*) \begin{cases} = \lambda_w, & \text{if} \quad x^*_p > 0, \\ \geq \lambda_w, & \text{if} \quad x^*_p = 0. \end{cases} \tag{6}$$

The interpretation of conditions (6) is that all used paths connecting an O/D pair w have equal and minimal costs (with the minimal path costs equal to the equilibrium travel disutility, denoted by λ_w). These conditions are also referred to as the user-optimized conditions (*cf.* [6]). As established in [7] and [8], the equilibrium pattern according to above definition is also the solution to the following variational inequality problem. A path flow pattern $x^* \in \mathcal{K}^1$ is a network equilibrium according to the above definition if and only if it satisfies the variational inequality problem: determine $x^* \in \mathcal{K}^1$ such that

$$\sum_{w\in W} \sum_{p\in P_w} C_p(x^*) \times \left[x_p - x^*_p\right] \geq 0, \quad \forall x \in \mathcal{K}^1. \tag{7}$$

Existence of a solution to variational inequality (7) is guaranteed under the sole assumption that the link cost functions (4) are continuous, and, hence, so are the path cost functions (5), since the feasible set $\mathcal{K}^1$ is compact. Algorithms for the solutions of variational inequality (7) can be found in [10], [16], and the references therein. In the classical traffic network equilibrium problem, in which the cost on each link (*cf.* (4)) depends solely on the flow on that link, the traffic network equilibrium conditions (6) can be reformulated as the solution to an appropriately constructed optimization problem, as established in [5]. Indeed, in this special case. in which the link cost functions are separable, that is, $c_a = c_a(f_a)$, for all links $a \in L$, then the equilibrium link flow (and path flow pattern) can be obtained via the solution of the following optimization problem:

$$\text{Minimize}_{f\in\mathcal{K}^2} \quad \sum_{a\in L} \int_0^{f_a} c_a(y)dy \tag{8}$$

where $\mathcal{K}^2 \equiv \{f \in R^n_+ | x \in R^{n_P}_+ \text{ satisfying}\,(1),(2)\}$. For additional background on this model, along with its impacts, see [13]. In particular, we know that if the user link cost functions are strictly monotone (*cf.* [10]) then the equilibrium link flow pattern is unique.

The New Measure . – We now propose a new network efficiency measure for congested networks. The measure is defined in the context of network equilibrium, and it captures demands and costs, and the underlying behavior of "users" of the network. The formal definition is as follows. The network performance/efficiency measure, $\mathcal{E}(G, d)$, for a given network topology G and demand vector d, is defined as:

$$\mathcal{E} = \mathcal{E}(G, d) = \frac{\sum_{w\in W} \frac{d_w}{\lambda_w}}{n_W}, \tag{9}$$

where recall that n_W is the number of O/D pairs in the network, and d_w and λ_w are, respectively, the demand and the equilibrium disutility for O/D pair w (*cf.* (6)).

Note that this measure has a nice, economic meaning in that it measures the average (O/D pair based) performance versus cost or price, with the performance being measured by the demands and the cost or price by the travel disutility. For example, in the context of transportation networks, the demand d_w is measured over a period of time, typically, an hour, whereas λ_w is the minimum equilibrium travel cost (or time) associated with the O/D pair w. Suppose that we have only a single O/D pair w in a network, and that the $d_w = 100$ vehicles with $\lambda_w = .5$ hour. Then $\mathcal{E} = 200$ (vehicles/hour). Consequently, the network can process, in effect, 200 vehicles in the hour. If λ_w was, instead, 1 hour, then the efficiency $\mathcal{E}$ would be 100 (vehicles/hour), and this network would be half as efficient as the original network. Depending upon the congested network under consideration the unit of measurement would correspond to the type of flow on the network.

The network efficiency measure (9) induces the following definition of the importance of a network component. The importance of a network component $g \in G$, $I(g)$, is measured by the relative network efficiency drop, after g

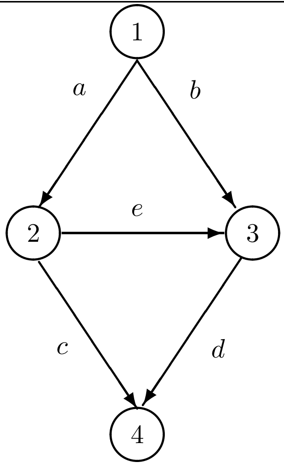

Fig. 1: The Braess Network

is removed from the network:

$$I(g) = \frac{\triangle \mathcal{E}}{\mathcal{E}} = \frac{\mathcal{E}(G,d) - \mathcal{E}(G-g,d)}{\mathcal{E}(G,d)}, \quad (10)$$

where $G-g$ is the resulting network after component g is removed from network G. Obviously, $I(g)$ is unitless and bounded above by 1.

The elimination of a link is treated in the new measure by removing that link while the removal of a node is managed by removing the links entering and exiting that node. In the case that the removal results in no path connecting an O/D pair, we simply assign the demand for that O/D pair to an abstract path with a cost of infinity. Hence, our measure is well-defined even in the case of disconnected networks. Notably, Latora and Marchiori [14] and [15] also mentioned this important characteristic which gives their measure an attractive property over the measure used for the small-world network model (*cf.* [17]). However, their measure considers the inverse of geodesic distances and is not directly applicable to congested networks in which the cost on a link is an increasing function of the flows on the links. Moreover, the Latora and Marchiori measure has no underlying equilibrium or behavioral concept as does our measure.

An Application of the Measure to the Braess Network with Varying Demands. – Consider the Braess Paradox example after the addition of a new link e and as depicted in fig. 1 (see also [11] and [12]). There are four nodes: 1, 2, 3, 4; five links: a, b, c, d, e; and a single O/D pair $w = (1,4)$. There are, hence, three paths connecting the single O/D pair, which are denoted, respectively, by: $p_1 = (a,c)$, $p_2 = (b,d)$ and $p_3 = (a,e,d)$.

The link cost functions are:

$$c_a(f_a) = 10f_a, \quad c_b(f_b) = f_b + 50,$$

$$c_c(f_c) = f_c + 50, \quad c_d(f_d) = 10f_d, \quad c_e(f_e) = f_e + 10.$$

We can also write down the path cost functions (*cf.* (5)) as follows:

$$C_{p_1}(x) = 11x_{p_1} + 10x_{p_3} + 50, \quad C_{p_2}(x) = 11x_{p_2} + 10x_{p_3} + 50,$$

$$C_{p_3}(x) = 10x_{p_1} + 21x_{p_3} + 10x_{p_2} + 10.$$

By referring to [3] and [19], who provided an evolutionary variational inequality formulation of the time-dependent (demand varying) Braess Paradox, which was formulated in a static setting without any qualitative analysis by Pas and Principio [18], we know that different paths are used (that, is, have positive flow in equilibrium) in three different demand ranges. Therefore, the importance and the ranking of individual nodes and links can be expected to be different depending upon which demand range the demand of concern is in.

Moreover, recall that the Braess Paradox [11] demonstrated that for a fixed demand of $d_w = 6$ the addition of link e, which provides the users with the new path p_3, as in the network in fig. 1, actually makes all users worse off since without the link e, the travel disutility and path costs are 83, whereas with the new link/path, the travel disutility and path costs go up for all users to 92!

Furthermore, since the Braess Paradox occurs in a certain part of Demand Range I (as referred to in the above references), in the following analysis, we discuss the importance and the ranking of the network components in four (rather than three) different demand ranges. It is notable that we are able to derive explicit formulae for both the network efficiencies as well as the importance of network components as a function of the demand d_w.

Demand Range I: $d_w \in [0, 2\frac{18}{31})$

Assume that the demand $d_w \in [0, 2\frac{18}{31})$. By referring to [3], [18], and [19], we know that, in this demand range, only path p_3 is used at the equilibrium and the Braess Paradox does not occur. Hence, in this range of demand, we have that the equilibrium path flow pattern, satisfying (6) is: $x^*_{p_1} = x^*_{p_2} = 0$ and $x^*_{p_3} = d_w$. The equilibrium disutility is $\lambda_w = 21d_w + 10$. The efficiency according to (9) for this range of demand is $\mathcal{E} = \frac{d_w}{21d_w+10}$.

The importance and the rankings of the links and the nodes are given, respectively, in tables 1 and 2.

Table 1: Importance and Ranking of Links in Demand Range I: $d_w \in [0, 2\frac{18}{31})$

Link	Importance Value	Importance Ranking
a	$\frac{10(4-d_w)}{11d_w+50}$	1
b	0.00	3
c	0.00	3
d	$\frac{10(4-d_w)}{11d_w+50}$	1
e	$\frac{(80-31d_w)}{(11d_w+100)}$	2

Table 2: Importance and Ranking of Nodes in Demand Range I: $d_w \in [0, 2\frac{18}{31})$

Node	Importance Value	Importance Ranking
1	1.00	1
2	$\frac{10(4-d_w)}{11d_w+50}$	2
3	$\frac{10(4-d_w)}{11d_w+50}$	2
4	1.00	1

Demand Range II: $d_w \in [2\frac{18}{31}, 3\frac{7}{11}]$

Assume now that $d_w \in [2\frac{18}{31}, 3\frac{7}{11}]$. By referring, again, to to [3], [18], and [19], we know that, similar to the results for Demand Range I, only path p_3 is used in equilibrium but now the Braess Paradox occurs. Hence, we have that the equilibrium solution in this demand range is: $x^*_{p_1} = x^*_{p_2} = 0$ and $x^*_{p_3} = d_w$. The equilibrium disutility is $\lambda_w = 21d_w + 10$. The efficiency is now: $\mathcal{E} = \frac{d_w}{21d_w+10}$. The importance ranking of the links and the nodes are given, respectively, in tables 3 and 4.

Table 3: Importance and Ranking of Links in Demand Range II: $d_w \in [2\frac{18}{31}, 3\frac{7}{11}]$

Link	Importance Value	Importance Ranking
a	$\frac{10(4-d_w)}{11d_w+50}$	1
b	0.00	2
c	0.00	2
d	$\frac{10(4-d_w)}{11d_w+50}$	1
e	$\frac{(80-31d_w)}{(11d_w+100)}$	3

Table 4: Importance and Ranking of Nodes in Demand Range II: $d_w \in [2\frac{18}{31}, 3\frac{7}{11}]$

Node	Importance Value	Importance Ranking
1	1.00	1
2	$\frac{10(4-d_w)}{11d_w+50}$	2
3	$\frac{10(4-d_w)}{11d_w+50}$	2
4	1.00	1

Demand Range III: $d_w \in (3\frac{7}{11}, 8\frac{8}{9}]$

Assume now that the demand $d_w \in (3\frac{7}{11}, 8\frac{8}{9}]$. We know that, in this range of demand, all three paths are used in equilibrium and the Braess Paradox still occurs. We now have that: $x^*_{p_1} = x^*_{p_2} = \frac{11}{13}d_w - \frac{40}{13}$ and $x^*_{p_3} = -\frac{9}{13}d_w + \frac{80}{13}$. The equilibrium disutility is now $\lambda_w = \frac{31d_w+1010}{13}$. The efficiency now is $\mathcal{E} = \frac{13d_w}{31d_w+1010}$. The importance rankings of links and nodes are given, respectively, in tables 5 and 6.

Table 5: Importance and Ranking of Links in Demand Range III: $d_w \in (3\frac{7}{11}, 8\frac{8}{9}]$

Link	Importance Value	Importance Ranking
a	$\frac{8(14d_w-45)}{13(11d_w+50)}$	1
b	$\frac{121(11d_w-40)}{13(131d_w+560)}$	2
c	$\frac{121(11d_w-40)}{13(131d_w+560)}$	2
d	$\frac{8(14d_w-45)}{13(11d_w+50)}$	1
e	$\frac{9(9d_w-80)}{13(11d_w+100)}$	3

Table 6: Importance and Ranking of Nodes in Demand Range III: $d_w \in (3\frac{7}{11}, 8\frac{8}{9}]$

Node	Importance Value	Importance Ranking
1	1.00	1
2	$\frac{8(14d_w-45)}{13(11d_w+50)}$	2
3	$\frac{8(14d_w-45)}{13(11d_w+50)}$	2
4	1.00	1

Demand Range IV: $d_w \in (8\frac{8}{9}, \infty)$

Assume now that $d_w \in (8\frac{8}{9}, \infty)$. We know that only paths p_1 and p_2 are now used in equilibrium and the Braess Paradox vanishes. Hence, we have that: $x^*_{p_1} = x^*_{p_2} = \frac{d_w}{2}$ and $x^*_{p_3} = 0$. The equilibrium disutility is now given by the expression: $\lambda_w = \frac{11}{2}d_w + 50$. The efficiency is now $\mathcal{E} = \frac{2d_w}{(11d_w+100)}$. The importance and the rankings of the links and the nodes are given, respectively, in tables 7 and 8.

Table 7: Importance and Ranking of Links in Demand Range IV: $d_w \in (8\frac{8}{9}, \infty)$

Link	Importance Value	Importance Ranking
a	$\frac{11d_w}{2(11d_w+50)}$	1
b	$\frac{5(13d_w-8)}{(131d_w+560)}$	2
c	$\frac{5(13d_w-8)}{(131d_w+560)}$	2
d	$\frac{11d_w}{2(11d_w+50)}$	1
e	0.00	3

Table 8: Importance and Ranking of Nodes in Demand Range IV: $d_w \in (8\frac{8}{9}, \infty)$

Node	Importance Value	Importance Ranking
1	1.00	1
2	$\frac{11d_w}{2(11d_w+50)}$	2
3	$\frac{11d_w}{2(11d_w+50)}$	2
4	1.00	1

Discussion. – Note that, the above example demonstrates the importance ranking of a link may be different in different demand ranges. For example, links b and c are less important in Demand Range I than in Demand Ranges III and IV. This is due to the fact that in Demand Range I, links b and c carry zero flows and, therefore, they are not critical links in evaluating network performance in those ranges. However, in Demand Ranges III and IV, links b and c carry positive flows and, thus, in these ranges of demand, the removal of these links will deteriorate the network performance/efficiency.

The importance rankings of the nodes in this transportation network example remain the same across all the demand ranges.

The different ranking results for links b and c clearly explain why "flow matters" and why an appropriate network performance/efficiency measure for congested networks should capture not only costs/distances but also flows as well as the behavior of the users of the network.

Summary and Conclusions. – In this paper, we have demonstrated how a network efficiency/performance measure given by (9) that captures demands, flows, costs, as well as behavior of users of the network can be applied to assess the efficiency of congested networks as well as the importance and ranking of network components, that is, the nodes and links; see (10). The network measure is well-defined, even in the case of disconnected networks. An application to a transportation network, the well-known Braess network, in which the demands were varied, and with relevance to the Internet, was also given. The results in this paper can assist in the identification of critical network components, whose removal, be it through natural disasters, structural failures, and/or terrorist attacks, etc., has implications for the network system and its vulnerability. Clearly, the identified important network components, through the importance definition that we provide in this paper should be more carefully monitored, in practice. Future research will include the application of the new network measure to a variety of large-scale congested networks.

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