A Network Equilibrium Framework for Internet Advertising: Models, Qualitative Analysis, and Algorithms

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Abstract: In this paper, we develop a network equilibrium framework for the modeling and analysis of competitive firms engaged in Internet advertising among multiple websites. The model allows for the determination of both the equilibrium online advertising budget as well as the advertising expenditures on the different websites. We then specialize the model to the case of fixed online budgets for the firms. The governing equilibrium conditions of both models are shown to satisfy finite-dimensional variational inequalities. We present qualitative properties of the solution patterns as well as computational procedures that exploit the underlying abstract network structure of these problems. The models and algorithms are illustrated with numerical examples. This paper adds to the growing literature of the application of network-based techniques derived from operations research to the advertising/marketing arena.

Key Words: Online advertising; Internet marketing; Optimal budgeting; Resource allocation; Competitive firms, Network equilibrium; Variational inequalities

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1. Introduction

The determination of the size of a firm's marketing budget as well as how its budget should be allocated to online marketplaces isn't only a single firm issue. Indeed, as argued in Zhao and Nagurney (2005) and in Park and Fader (2004), the success of a firm's advertising/marketing efforts is affected by other firms' advertising efforts. In addition, since consumers of a particular product receive advertisements of all the firms in the industry, their responses to the particular product are a function of the aggregate advertising effort of the entire industry, and not simply a function of the advertising of an individual firm. Further, according to the literature in marketing, consumer behavior in an online environment differs from that in traditional shopping environments, with the difference being due, in part, to the inherent nature of the medium, the degree to which consumers have adopted the Web, etc. Indeed, according to Chatterjee et al. (2003), the Internet offers consumers relatively more control of the communication and exchange process than has been the case in such traditional media as print and broadcast media and, hence, those consumers who are most likely to attend and click will do so at the first exposure. Moreover, Danaher et al. (2004) note that better-known brands have greater than expected loyalty when bought online as compared with an offline environment and, conversely, for small share brands. Hence, the allocation of firms' advertising budgets on the Internet medium vs. spending on other mediums should be influenced by this difference.

This paper develops a network equilibrium framework for Internet-based advertising which considers the reality of competition among firms. First, the optimal allocation of a single firm's advertising budget on the Internet and other media is formulated as an optimization problem. It is then argued that a firm's online budget should not be fixed but, rather, should be elastic. A network equilibrium model is then constructed which considers multiple firms competing through Internet advertising. The governing concept is that of Nash equilibrium. We prove that the equilibrium Internet advertising budget size and resource allocation among websites satisfies a variational inequality (cf. Nagurney, 1999). We also provide the corresponding variational inequality in the case of fixed Internet advertising budget sizes. We then conduct qualitative analysis of the equilibrium patterns and establish both existence and uniqueness results, under reasonable conditions. Subsequently, we show how the network structure of the competitive equilibrium problems can be exploited algorithmically and computationally.

The results in this paper generalize and extend those in Zhao and Nagurney (2005) in which a network optimization framework was proposed for the determination of Internet-based advertising strategies and pricing. In addition, in that paper, a quantitative explanation of two paradoxes was given, along with numerical examples. The novelty of this paper is the utilization of networks and variational inequality theory for the formulation, analysis, and solution of competitive equilibrium problems faced by firms engaged in Internet-based advertising. Moreover, we view the online marketing arena as competition among N firms, rather than considering a two-firm duopoly as in Banerjee and Bandyopadhyay (2004).

This paper is structured as follows. Section 1 provides an introduction and outline of the paper. Section 2 describes the first stage of the competition in which firms decide how much should be spent in the Internet medium. Section 3 discusses the second stage of the competition in which firms decide how their budgets should be allocated to the available websites. The network equilibrium model is constructed and the variational inequality formulation of the governing equilibrium conditions derived. For completeness, we also give the variational inequality formulation of the equilibrium conditions in the case of fixed budget sizes. In Section 4, the existence and uniqueness of the equilibrium solutions are investigated under appropriate assumptions. Subsequently, in Section 5, algorithms are proposed for computational purposes and applied to several numerical examples. Finally, in Section 6, we conclude the paper, summarize the results obtained, and lay out some possible directions for future research.

2. Optimization of a Single Firm's Advertising Budget Allocation

In this Section, we formulate the optimization problem faced by a firm in determining its advertising/marketing budget allocation. Through the analysis of the resulting problem formulation, we also establish that a firm's Internet advertising budget is an increasing function of the marginal response.

We assume that a single product may be advertised by each of the N firms in all the mediums. For firm n; $n = 1, 2, ..., N$: let f_{nw} denote the advertising expenditure on the *Internet* and let f_{no} denote the advertising expenditure on the *other* traditional media such as TV, print, radio, etc. For simplicity, we do not distinguish among websites in the model in this Section. However, we do consider multiple websites in the network equilibrium model in Section 3. We group the f_{nw} and the f_{no}; $n = 1, 2, \ldots, N$, respectively, into the vectors f_w and f_o. All vectors in this paper are assumed to be column vectors, except where noted otherwise.

Let $r_{nw}(f_w)$ and $r_{no}(f_o)$ denote the consumers' responses induced by the expenditures f_w and f_o, respectively. Here we assume that the consumers' responses to the advertising expenditures on the Internet depend only on the expenditures made on that medium and the same holds for the other media. This assumption, is less restrictive than that in many papers in marketing science. As noted by Reibstein and Wittink (2005), the marketing literature contains many articles on market response based on both aggregate and disaggregate data. However, there are few papers that deal with competitive reactions. Indeed, as Reibstein and Wittink (2005) further emphasize, "the marketing mix models offered by leading data suppliers often gloss over competitive spending and never include reaction functions." Here, we choose to neglect cross-media effects, but we retain the cross-firm effects, in order to make the paper more focused, while still retaining the basic marketing science mechanism.

Also, we assume that $r_{ni}(f_i)$; $i = w, o$ are increasing, differentiable, and concave functions of f_i (see, also, Zhao and Nagurney, 2005). Each firm n is assumed to have a total advertising budget denoted by C_n.

The optimal budget allocation problem faced by firm n, assuming that it wishes to maximize the consumers' responses over all the media, given its budget, can be expressed as the following optimization problem:

$$\max_{f_{nw}, f_{no}} \{r_{nw}(f_w) + r_{no}(f_o)\} \tag{1}$$

subject to:

$$f_{nw} + f_{no} \leq C_n \tag{2}$$

$$f_{nw} \geq 0, \quad f_{no} \geq 0. \tag{3}$$

Let f_{ns} denote the (possible) slack associated with constraint (2) with this variable denoting the amount not spent from the advertising budget on the advertising mediums. A derivation of the Kuhn-Tucker optimality conditions (see Bazaraa et al., 1993) for the optimization problem (1), subject to (2) – (3), yields that $(f^*_{no}, f^*_{nw}, f^*_{ns})$ is an optimal budget allocation for firm n if and only if it satisfies the system of equalities and inequalities:

For $i = w, o$:

$$\frac{\partial r_{ni}(f_{1i}, ..., f_{(n-1)i}, f^*_{ni}, f_{(n+1)i}, ..., f_{Ni})}{\partial f_{ni}} \begin{cases} = \lambda_n, & \text{if} \quad f^*_{ni} > 0, \\ \leq \lambda_n, & \text{if} \quad f^*_{ni} = 0, \end{cases} \tag{4}$$

$$0 \begin{cases} = \lambda_n, & \text{if} \quad f^*_{ns} > 0, \\ \leq \lambda_n, & \text{if} \quad f^*_{ns} = 0, \end{cases} \tag{5}$$

$$f^*_{nw} + f^*_{no} + f^*_{ns} = C_n, \tag{6}$$

where λ_n is the Lagrange multiplier associated with the budget constraint (2).

If we let $\eta_{nw}(\cdot) = \frac{\partial r_{nw}(\cdot)}{\partial f_{nw}}$, from the above system of equalities and inequalities, we see that the optimal online advertising expenditure f^*_{nw} is the inverse function of λ_n, that is, $f^*_{nw} = \eta^{-1}_{nw}(\lambda_n)$, where λ_n is the marginal response per additional capital spent on the Internet. In other words, if the firm notices that the marginal response to the Internet medium is different from that to the other media, it will adjust its online spending to make the marginal responses in all the media equal. Because $\eta_{nw}(\cdot)$ is a decreasing function of f_{nw}, when the Internet marginal response is higher than in other media, it is more beneficial for the firm to increase f_{nw}, and when the firm does so, it will bring its Internet marginal response down until all the marginal responses are equal. Thus, the firm's online spending is an increasing function of its marginal response to the Internet medium and we denote this function as $b_n(\eta_{nw})$.

Table 1: Adjustments by firm n of f_{nw} given different initial advertising expenditures (Units of \$1000)

Initial online expenditure f_{nw}^0	Initial offline expenditure f_{no}^0	Online margin η_{nw}	Offline margin η_{no}	Adjustment δf_{nw}
\$500	\$400	0.020	-0.008000	\$300
\$600	\$300	0.016	-0.002667	\$200
\$700	\$200	0.012	0.002667	\$100
\$800	\$100	0.008	0.008000	\$0

We now present an example for illustrative purposes.

Example 1

In this example, the response functions (in units of 1000) for online advertising and traditional advertising are given by:

$$r_w = -\frac{2}{100000} f_w^2 + \frac{4}{100} f_w + 2,$$
$$r_o = -\frac{4}{150000} f_o^2 + \frac{2}{150} f_o + 1.$$

These two functions are different because the two media are different and the consumers who use the two media are different. The advertising budget is assumed to be \$900K (that is, in units of 1000 dollars).

Table 1 illustrates the adjustments that firm n should make given different initial investments in online advertising denoted by f_{nw}^0.

We emphasize that $f_{nw} = \$800$ and $f_{no} = \$100$ reflect the optimal allocation of the budget at which the online margin is equal to the offline margin. If the online spending differs from \$800, then the adjustment δf_{nw} needs to be made. Observe that δf_{nw} is increasing with η_{nw} as follows:

$$\delta f_{nw} = 25000(\eta_{nw} - 0.008),$$

and, in order to be optimal, the firm's online advertising budget should be

$$f_{nw} = f_{nw}^0 + 25000(\eta_{nw} - 0.008).$$

In summary, the size of a firm's online advertising budget should not be a pre-fixed number but, rather, the size should be elastically adjusted with the marginal responses, as in Example 1. Moreover, the marginal response is affected by the inherent nature of the Internet medium captured in the function $r_{nw}(f_w)$, as well as by the performance of the online advertising efforts. In Section 3, we further explore these ideas in the context of the network equilibrium model(s) of Internet advertising competition among firms.

3. The Network Equilibrium Models

In this Section, we develop the network equilibrium models of competitive firms engaged in Internet advertising. We first present the elastic demand model and then the fixed demand model. We assume that there are N firms now competing in M websites, each of which seeks to maximize its individual response. Since we focus from this point on exclusively on Internet advertising, we suppress the subscript w used in Section 2 to denote the Internet. In addition, we now make explicit the possibility of each firm advertising on multiple websites. Let f_{mn}; $m = 1, 2, ..., M$ and $n = 1, 2, ...N$ denote the advertising expenditures of firm n on website m, where $f_{mn} \geq 0$. We group the f_{mn} into a nonnegative vector $f \in R_+^{MN}$. For firm n, we use η_n to denote the marginal response to the firm's online advertising efforts. According to Section 2, firm n's online advertising budget b_n is then an increasing function of η_n, and may be expressed as

$$b_n = b_n(\eta_n), \quad n = 1, 2, ..., N. \tag{7}$$

If r_{mn} denotes the consumers' response that firm n receives from website m, then it is reasonable to assume that

$$r_{mn} = r_{mn}(f), \tag{8}$$

which is an increasing and concave function of f, and that

$$r_n = r_n(f) = \sum_{m=1}^{M} r_{mn}(f) \tag{9}$$

is then the firm's total response in the Internet medium. The function r_n must also be an increasing and concave function of f (see Zhao and Nagurney, 2005; Chatterjee et al., 2003). Note that, according to (8), the response from the consumers to a firm's advertising expenditures on a website is a function of, in general, the advertising expenditures of all the firms on all the websites. Recall that, in this paper, we are considering firms competing in a particular industry.

Each of the firms is now assumed to be maximizing its online response subject to its online budget constraint. Hence, firm n; $n = 1, \ldots, N$, in the presence of competition, is now faced with the following optimization problem:

$$\max_{f_{1n}, \ldots, f_{Mn}} r_n(f) \tag{10}$$

subject to:

$$\sum_m f_{mn} \leq \; b_n(\eta_n) \tag{11}$$

$$f_{mn} \geq 0, \quad m = 1, 2, \ldots, M. \tag{12}$$

Here, for a moment, we view the marginal response η_n as extraneously given. When η_n is given, the firm should assign $b_n(\eta_n)$ to online advertising, with problem (10) – (12) then solved to determine the optimal f_{mn}; $m = 1, \ldots, M$. The marketing performance will be improved and η_n will increase, resulting in an increase in the budget $b_n(\eta_n)$. At equilibrium, however, η_n is no longer an extraneous value but, rather, is determined by the allocation f. We formally explain this in the following theorem in which we also assume that the competition among the firms is in the sense of Nash (1950, 1951) yielding a noncooperative game.

Theorem 1: Internet Advertising Nash Equilibrium

The vector $f^* = \{f^*_{mn}; m = 1, 2, ..., M; n = 1, 2, ..., N\}$ *is an equilibrium budget allocation for all firms in all the websites in the sense of Nash if and only if it satisfies the following equalities and inequalities for all firms* n *and for all websites* m*:*

$$\frac{\partial r_n(f^*)}{\partial f_{mn}} \begin{cases} = \lambda^*_n, & \textit{if} \quad f^*_{mn} > 0, \\ \leq \lambda^*_n, & \textit{if} \quad f^*_{mn} = 0, \end{cases} \tag{13}$$

$$0 \begin{cases} = \lambda^*_n, & \textit{if} \quad f^*_{ns} > 0, \\ \leq \lambda^*_n, & \textit{if} \quad f^*_{ns} = 0, \end{cases} \tag{14}$$

$$\sum_{m=1}^{M} f^*_{mn} + f^*_{ns} = \; b_n(\lambda^*_n). \tag{15}$$

Proof: Applying the Kuhn-Tucker optimality conditions to problem (10), subject to (11) and (12), simultaneously for all firms, we obtain that f^* is the equilibrium point of the competition if and only if

$$\frac{\partial r_n(f^*)}{\partial f_{mn}} \begin{cases} = \lambda^*_n, & \text{if} \quad f^*_{mn} > 0, \\ \leq \lambda^*_n, & \text{if} \quad f^*_{mn} = 0, \end{cases} \tag{16}$$

$$0 \begin{cases} = \lambda_n^*, & \text{if} \quad f_{ns}^* > 0, \\ \leq \lambda_n^*, & \text{if} \quad f_{ns}^* = 0, \end{cases} \tag{17}$$

$$\sum_{m=1}^{M} f_{mn}^* + f_{ns}^* = \; b_n(\eta_n). \tag{18}$$

Note that the firm's online spending $f_n = \sum_{m=1}^{M} f_{mn}$ and also that $f_n = f_{nw}$, as in Section 2. We, hence, have that

$$\frac{\partial f_{mn}}{\partial f_n} = \frac{\partial f_{nw}}{\partial f_n} = 1.$$

Therefore,

$$\frac{\partial r_n(f^*)}{\partial f_{mn}} = \lambda_n^* = \frac{\partial r_n(f^*)}{\partial f_n} = \eta_n.$$

Thus, (16), together with the above equation, implies that, at equilibrium, the marginal responses in all of the websites are equal to the marginal response per additional unit of online advertising spending for this firm, if this website is used. Replacing η_n with λ_n^*, (13) – (15) are obtained. $\square$

We now show that the above equilibrium conditions representing the Nash equilibrium for the N firms competing in M websites coincide with the equilibrium conditions of a network equilibrium problem over the network depicted in Figure 1. Indeed, let $\mathcal{N}$ be a network with $N+1$ nodes denoted, respectively, by: $0, 1, 2, \ldots, N$; with $N(M+1)$ links denoted, respectively, by: $11, \ldots, 1M, 1s$; $\ldots$; $1n, \ldots, Mn, ns$; $\ldots$; $1N, \ldots, MN, Ns$, and with N origin/destination (O/D) pairs: denoted, respectively, by $w_1 = (0,1)$, $w_2 = (0,2)$,$\ldots$,$w_N = (0,N)$. Each origin/destination pair is connected by M paths and a "dummy" path. There are, hence, a total of $N(M+1)$ paths in the network. The vector of path flows is then given by $f = (f_{11}, f_{12}, ..., f_{MN})^T$, which is a vector in R_+^{MN}, with f_{mn} representing the flow on path mn. The flows: f_{ns}; $n = 1, 2, ...N$ represent the nonnegative flows on the respective dummy paths and, as we will soon show, correspond to the slacks or unused portion of the budgets. Note that in the network in Figure 1 each path consists of a single link.

Let now $u_{mn}(f)$ denote the "marginal utility" induced by the network flow f on path mn, which is specified as:

$$u_{mn}(f) = \frac{\partial r_n(f)}{\partial f_{mn}}; \quad m = 1, 2, ..., M; n = 1, 2, ..., N, \tag{19}$$

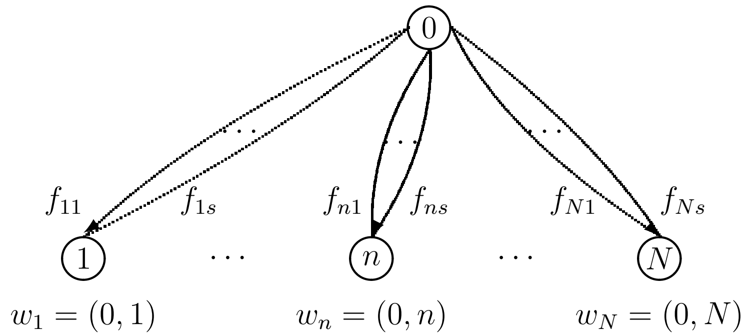

Figure 1: Network Structure of the Competitive Internet Advertising Problem

with the marginal utilities on the dummy paths being all set to zero. The *elastic* demand associated with O/D pair n is given by $b_n(\cdot)$ for $n = 1, \ldots, N$. Then the equilibrium conditions (13) – (15) have the following network equilibrium interpretation: only those paths that provide maximal marginal utilities, that is, maximal marginal responses are used (i.e., have positive flow) in equilibrium. These equilibrium conditions are now contrasted/compared to those governing the network equilibrium problem par excellence – the traffic network equilibrium problem, which, in the case of elastic demand traffic network equilibrium problems, is due to Beckmann et al. (1956) in the classical case and to Dafermos (1982) who used variational inequality theory in the asymmetric case in which the equilibrium conditions could no longer be reformulated as the Kuhn-Tucker conditions of an associated optimization problem (see also, Dafermos and Nagurney, 1984a, b). In traffic network equilibrium problems, at the equilibrium, only those paths connecting an O/D pair are used that have travel costs that are *minimal* and, of course, the demand is equal to the sum of the path flows on paths connecting each O/D pair. Travelers, hence, seek to determine their cost-minimizing routes of travel from their respective origins to their destinations; whereas in the case of the Internet network equilibrium advertising problem the marketers are seeking to maximize their responses in a unilateral fashion, and, hence, it is the marginal responses that are maximized/equalized at the equilibrium across the used websites/paths. The flows on the network model in Figure 1 correspond to financial resource flows and the demands, in turn, are the budgets of the firms.

Additional background on network equilibrium problems including traffic network equilibrium problems and a variety of economic equilibrium problems, can be found in the book by Nagurney (1999). The impact of the book by Beckmann et al. (1956) is recorded in the paper by Boyce et al. (2005). It is also important to note that Gabay and Moulin (1980) proved that game theoretic problems in the sense of Nash and, hence, in the case of many oligopoly problems, admit variational inequality formulations of the equilibrium conditions. The corresponding proof for the above model is given directly in Theorem 2.

Before turning to establishing the variational inequality of the Nash equilibrium conditions (13) – (15), we first introduce the following notation. Since the firm's online budget $b_n(\lambda_n)$ is an increasing function of the marginal response λ_n, then,

$$\lambda_n = \lambda_n(b_n) \tag{20}$$

is the inverse function of $b_n(\cdot)$, and it is also an increasing function. We now define the following vectors: let $u(f) = (u_{mn}(f);\, m = 1, 2, ...M;\, n = 1, 2, ...N)$, $b = (b_n; n = 1, 2, ...N)$; and $\lambda(b) = (\lambda_n(b_n);\, n = 1, 2, ...N)$. Then $u(f) \in R^{MN}$, $b \in R^n_+$, and $\lambda_n(b) \in R^n$, respectively. Equilibrium conditions (13) – (15) can, hence, be, equivalently, written as: for all firms n; $n = 1, \ldots, N$ and for all websites m; $m = 1, \ldots, M$:

$$\frac{\partial r_n(f^*)}{\partial f_{mn}} \begin{cases} = \lambda_n(b_n^*), & \text{if} \quad f^*_{mn} > 0, \\ \leq \lambda_n(b_n^*), & \text{if} \quad f^*_{mn} = 0, \end{cases} \tag{21}$$

$$0 \begin{cases} = \lambda_n(b_n^*), & \text{if} \quad f^*_{ns} > 0, \\ \leq \lambda_n(b_n^*), & \text{if} \quad f^*_{ns} = 0, \end{cases} \tag{22}$$

$$\sum_{m=1}^{M} f^*_{mn} + f^*_{ns} = b^*_n. \tag{23}$$

We are now ready to state the following where $\langle \cdot, \cdot \rangle$ denotes the inner product in the properly dimensioned Euclidean space:

Theorem 2: Variational Inequality Formulation of Internet Advertising Nash Equilibrium

The vector $(f^*, b^*) \in \mathcal{K}^1$ *is an equilibrium according to conditions (21) – (23); equivalently, (13) – (15), if and only if it is a solution of the variational inequality problem:*

$$\langle u(f^*), f - f^* \rangle - \langle \lambda(b^*), b - b^* \rangle \leq 0, \quad \forall (f,b) \in \mathcal{K}^1, \tag{24}$$

$$\mathcal{K}^1 \equiv \{(f,b) | (f,b) \in R_+^{MN+N}, \sum_{m=1}^{M} f_{mn} + f_{ns} = b_n, f_{ns} \geq 0; n = 1, 2, ..., N\}.$$

Proof: If (f^*, b^*) is a solution of (21) – (23), then for $m = 1, \ldots, M$; $n = 1, \ldots, N$:

$$u_{mn}(f^*) - \lambda_n(b^*_n) = 0, \quad \text{if} \quad f^*_{mn} > 0, \tag{25}$$

$$u_{mn}(f^*) - \lambda^*_n(b^*_n) \leq 0, \quad \text{if} \quad f^*_{mn} = 0, \tag{26}$$

$$-\lambda_n(b^*_n) = 0, \quad \text{if} \quad f^*_{ns} > 0, \tag{27}$$

$$-\lambda^*_n(b^*_n) \leq 0, \quad \text{if} \quad f^*_{ns} = 0, \tag{28}$$

$$\sum_{m=1}^{M} f^*_{mn} + f^*_{ns} = \ b^*_n. \tag{29}$$

Inequalities (25) and (26) imply that for $(f,b) \in \mathcal{K}^1$ the following inequalities hold:

$$(u_{mn}(f^*) - \lambda_n(b^*_n)) \times (f_{mn} - f^*_{mn}) \leq 0, \tag{30}$$

whereas (27) and (28) imply that

$$-\lambda_n(b^*_n) \times (f_{ns} - f^*_{ns}) \leq 0. \tag{31}$$

Summing up (30) over all m and n, and (31) over all n, and grouping like terms, we obtain:

$$\sum_{m=1}^{M} \sum_{n=1}^{N} u_{mn}(f^*) \times (f_{mn} - f^*_{mn}) - \sum_{n=1}^{N} \sum_{m=1}^{M} \lambda_n(b^*_n) \times ((f_{mn} - f^*_{mn}) + (f_{ns} - f^*_{ns})) \leq 0,$$

$$\forall (f,b) \in \mathcal{K}^1. \tag{32}$$

Using now (29) and the definition of the feasible set $\mathcal{K}^1$, (32) reduces to:

$$\sum_{m=1}^{M}\sum_{n=1}^{N} u_{mn}(f^*) \times (f_{mn} - f^*_{mn}) - \sum_{n=1}^{N} \lambda_n(b^*_n) \times (b_n - b^*_n) \leq 0, \quad \forall (f,b) \in \mathcal{K}^1. \tag{33}$$

But variational inequality (24) is simply variational inequality (33) in vector notation.

Conversely, if (f^*, b^*) is a solution of (24), for any $f^*_{ij} > 0$, we take $f_{mn} = f^*_{mn}, \forall (m,n) \neq (i,j)$, $f_{ns} = f^*_{ns}$, and $f_{ij} = f^*_{ij} + \delta$. Thus, $b_j = \sum_{m=1, m\neq i}^{M} f^*_{mj} + f^*_{js} + f^*_{ij} + \delta = b^*_j + \delta$, and $b_n = b^*_n, \forall n \neq j$. Inequality (24) is reduced to:

$$u_{ij}(f^*)\delta - \lambda_j(b^*_j)\delta \leq 0. \tag{34}$$

Note that since δ can take a positive or a negative value, (34) is equivalent to (21). On the other hand, for any $f^*_{ij} = 0$, we use the same approach but δ has to be positive. Therefore, (31) is again equivalent to (21).

Further, we see that (24) is equivalent to (32). For any $f^*_{ks} > 0$, we take $f_{ns} = f^*_{ns}, \forall n \neq k$, $f_{mn} = f^*_{mn}, \forall m, n$, and $f_{ks} = f^*_{ks} + \delta$. Then substitution into (32) reduces it to:

$$-\lambda_j(b^*_j)\delta \leq 0.$$

Note that since δ may take on a positive or a negative value, the above inequality implies the first condition of (22). On the other hand, if $f^*_{ks} = 0$, then δ must be positive, and the above inequality implies the second condition of (22). $\square$.

It is important to note that the set $\mathcal{K}^1$ is a convex, unbounded set in the mathematical sense. However, business-wise, we would expect that this set would be a convex, and bounded compact set since a firm's online budget b_n would be less than or equal to a firm's total advertising budget which clearly, in practice, can't be infinite (and, thus, unbounded).

In the case where the budget is fixed for each firm n, b_n is no longer a variable but assumed known and given and, say, equal to $\bar{b}_n$. Then (21) – (23) reduce to: for $m = 1, \ldots, M$; $n = 1, \ldots, N$:

$$\frac{\partial r_n(f^*)}{\partial f_{mn}} \begin{cases} = \lambda_n, & \text{if} \quad f^*_{mn} > 0, \\ \leq \lambda_n & \text{if} \quad f^*_{mn} = 0, \end{cases} \tag{35}$$

$$0 \left\{ \begin{array}{lll} = \lambda_n & \text{if} & f^*_{ns} > 0, \\ \leq \lambda_n, & \text{if} & f^*_{ns} = 0, \end{array} \right. \tag{36}$$

$$\sum_{m=1}^{M} f^*_{mn} + f^*_{ns} = \bar{b}_n. \tag{37}$$

The following result is immediate:

Corollary 1: Variational Inequality Formulation of Internet Advertising Nash Equilibrium with Fixed Budgets

A vector $f^* \in \mathcal{K}^2$ is a solution of equilibrium conditions (35) – (36), subject to (37) if and only if it satisfies the variational inequality problem:

$$\langle u(f^*), f - f^* \rangle \leq 0, \quad \forall f \in \mathcal{K}^2, \tag{38}$$

where $\mathcal{K}^2 \equiv \{f | f \in R_+^{MN}, \sum_{m=1}^{M} f_{mn} + f_{ns} = \bar{b}_n, f_{ns} \geq 0; n = 1, \ldots, N\}$.

Proof: Since $\sum_{m=1}^{M} f_{mn} + f_{ns} = \bar{b}_n$; $n = 1, \ldots, N$ and both f and f^* must satisfy these constraints, the second term in variational inequality (24) collapses to zero and the conclusion follows. □

The λ_n; $n = 1, \ldots, N$ in equations (5), (14), (17), (22), and (36) are the Lagrange multipliers associated with the budget constraints, which are derived from the Kuhn-Tucker optimality conditions. Their business meaning here is that if the budget is non-binding, then the firm should spend at where the marginal responses are all equal to zero, while if the marginal responses are positive, the firm should should spend until the budget becomes binding.

4. Qualitative Properties: Existence and Uniqueness of Equilibria

In this Section, we provide existence and uniqueness results for the solutions of variational inequalities (24) and (38). For simplicity of presentation and easy reference to the literature, we consider the variational inequality problem in standard form (cf. Nagurney, 1999, and also Kinderlehrer and Stampacchia, 1980): determine $X^* \in \mathcal{K}$ satisfying:

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{39}$$

where $\mathcal{K}$ is assumed to be closed and convex and $F(X)$ is a continuous function from $\mathcal{K} \rightarrow R^{\mathcal{M}}$. We assume compactness of $\mathcal{K}$ since, in reality (see discussion at end of preceding section), it is reasonable to assume that, in advertising practice, $\mathcal{K}$ will be bounded.

Note that if we let $X \equiv (f, b)$, $F(X) \equiv (-u(f), \lambda(b))$, and $\mathcal{K} \equiv \mathcal{K}^1$ then variational inequality (24) can be put into form (39). Similarly, if we let $X \equiv f$, $F(X) \equiv -u(f)$, and $\mathcal{K} \equiv \mathcal{K}^2$ then variational inequality (38) can also be put into standard form (39).

We now recall some classical results from variational inequality theory (see the above references for additional details and proofs).

In particular, before establishing the existence and uniqueness of solutions to variational inequalities (24) and (38) we need to recall the following definitions:

Definition 1: Strong Monotonicity

A vector function $F(X)$ is strongly monotone on $\mathcal{K}$ if

$$\langle F(X^1) - F(X^2), X^1 - X^2 \rangle \geq \alpha ||X^1 - X^2||^2, \quad \forall X^1, X^2 \in \mathcal{K}, \tag{40}$$

where α is a positive constant.

Definition 2: Strict Monotonicity

A vector function $F(X)$ is strictly monotone on $\mathcal{K}$ if

$$\langle F(X^1) - F(X^2), X^1 - X^2 \rangle > 0, \quad \forall X^1, X^2 \in \mathcal{K}, \quad X^1 \neq X^2. \tag{41}$$

We now recall some basic existence and uniqueness results from standard variational inequality theory and then apply them to the elastic budget size and the fixed budget size models of Section 3.

Theorem 3: Existence

Existence of a solution X^* *to variational inequality (39) follows under the sole assumption that* F *is continuous on* $\mathcal{K}$*, provided that the feasible set* $\mathcal{K}$ *is compact.*

Theorem 4: Uniqueness under Strict Monotonicity

Uniqueness of a solution X^* *to variational inequality (39) is guaranteed, if the function* F *is strictly monotone, provided that a solution* X^* *exists.*

We, hence, have the following:

Corollary 2

Existence of a solution f^* *to variational inequality (38) is guaranteed since the feasible set* $\mathcal{K}^2$ *is compact and the function* $-u(f)$ *is assumed to be continuous.*

Proof: Follows from Theorem 3. □

In addition, uniqueness of the equilibrium pattern f^* satisfying variational inequality (38) holds under the following:

Corollary 3

The equilibrium pattern f^* *satisfying variational inequality (38) is unique, provided that* $-u(f)$ *is strictly monotone on* $\mathcal{K}^2$*.*

Proof: Follows from Theorem 4.

In the case that the feasible set $\mathcal{K}$ is no longer compact as is the case in the competitive equilibrium Internet advertising model with elastic budgets, the following theorem may be used to establish both existence and uniqueness of a solution X^*.

Theorem 5: Existence and Uniqueness under Strong Monotonicity

If the function F is strongly monotone, then there exists a unique solution X^* to variational inequality (39).

The following results are then immediate:

Corollary 4

There exists a unique solution (f^*, b^*) to variational inequality (24), provided that $(-u(f), \lambda(b))$ is strongly monotone on the set $\mathcal{K}^1$, that is, if

$$\langle(-u(f^1), \lambda(b^1)) - (-u(f^2), \lambda(b^2)), (f^1, b^1) - (f^2, b^2)\rangle \geq \alpha(||f^1 - f^2||^2 + ||b^1 - b^2||^2),$$

$$\forall (f^1, b^1), (f^2, b^2) \in \mathcal{K}^1, \tag{42}$$

where α, β are positive constants.

We now state the following result which allows us to evaluate strong monotonicity of the entire function by establishing strong monotonicity of two functions individually and separately.

Corollary 5

The vector function $(-u(f), \lambda(b))$ is strongly monotone on $\mathcal{K}^1$ if and only if $-u(f)$ and $\lambda(b)$ are strongly monotone with respect to their own vectors f and b, i.e.:

$$\langle u(f^2) - u(f^1), f^1 - f^2\rangle \geq \zeta||f^1 - f^2||^2, \quad \forall f^1, f^2 \in \mathcal{K}^1, \tag{43}$$

and

$$\langle \lambda(b^1) - \lambda(b^2), b^1 - b^2\rangle \geq \tau||b^1 - b^2||^2, \quad \forall b^1, b^2 \geq 0, \tag{44}$$

where ζ, τ are positive constants.

Proof: If the strong monotonicity condition (42) holds true, then by letting $b^1 = b^2$ or $f^1 = f^2$ in (42), respectively, strong monotonicity conditions (43) and (44) for functions $-u(f)$ and $\lambda(b)$ hold true.

Conversely, if monotonicity conditions (43) and (44) hold true for $c(f)$ and $\lambda(b)$, take $\alpha = \min\{\tau, \zeta\}$ and then condition (42) holds true. $\square$.

We now turn to providing the economic interpretation of strong monotonicity of $-u(f)$ and $\lambda(b)$.

From the definition of $u(f)$ we see that the Jacobian matrix

$$[\frac{\partial u}{\partial f}] = [\frac{\partial^2 r_n}{\partial f_m \partial f_n}; m = 1, 2, ..., M; n = 1, 2, ...N].$$

If $-u(f)$ is strongly monotone, then the matrix of the second derivatives of $r_n(f)$ is negative definite on $\|^1$. Negative definiteness of the matrix $[\frac{\partial^2 r_n}{\partial f_m \partial f_n}; m = 1, 2, ..., M; n = 1, 2, ...N]$ implies the concavity of all $r_n(f)$.

From the definition of $\lambda_n(b_n)$ in (16) we see that it is an inverse function of $b_n(\lambda_n)$, where λ_n is the marginal response gained by firm n. $b_n(\lambda_n)$ is an increasing function of λ_n if and only if $\lambda_n(b_n)$ is an increasing function. If $\lambda_n(\cdot)$ is continuously differentiable, strong monotonicity of $\lambda_n(\cdot)$ implies the increase of $\lambda_n(\cdot)$ as well as the increase of $b_n(\cdot)$.

Obviously, an analogous result to Corollary 5 holds true in the case of strict monotonicity of $-u(f), \lambda(b)$ and strict monotonicity of $-u(f)$ and $\lambda(b)$ separately.

5. Algorithms for the Computation of the Equilibrium Advertising Budgets and Budget Allocations

We first introduce an exact algorithm for a variational inequality of special form which allows for the determination of the equilibrium budget and advertising expenditures explicitly in closed form. We then adapt the algorithm for the case of fixed budgets. We subsequently show that the solution of variational inequality (24) (as well as (38)) can be approached by a sequence of solutions to the respective variational inequalities of special structure. The algorithms exploit the underlying network structure of the Internet advertising resource allocation problems. Moreover, they are motivated by the exact equilibration algorithms devised by Dafermos and Sparrow (1969) for the fixed budget size case and by Dafermos and Nagurney (1989) (see also, e.g., Nagurney, 1999) for the case of elastic budgets.

Theorem 6

If the variational inequality (24) is quadratic separable in the sense of: for $m = 1, 2, \ldots, M$ and $n = 1, \ldots, N$:

$$u_{mn}(f) = \frac{\partial r_n(f)}{\partial f_{mn}} = a_{mn} f_{mn} + k_{mn} \tag{45}$$

and

$$\lambda_n(b_n) = \alpha_n b_n + \beta_n, \tag{46}$$

with $a_{mn} < 0$ and $\alpha_n > 0$ in order to guarantee strong monotonicity of $-u(f)$ and $\lambda(b)$; $k_{mn} > 0$ in order to guarantee increase of $r_n(f)$ on feasible set $\mathcal{K}^1$, and $\beta_n \leq 0$ in order to guarantee the nonnegativity of the budget b_n for any nonnegative margin, then the equilibrium Internet budget and allocation for all the firms can be calculated by the following formulae:

For each $n = 1, \ldots, N$:

Step 0: *Sort the $\{k_{mn}\}$ in nonascending order. Without lost of generality, we assume, henceforth, that*

$$k_{1n} \geq k_{2n} \geq, \ldots, \geq k_{Mn}. \tag{47}$$

Step 1: *Let $j = 1$ and calculate*

$$b_n^j = \frac{\beta_n \sum_{m=1}^{j} \frac{1}{a_{mn}} - \sum_{m=1}^{j} \frac{k_{mn}}{a_{mn}}}{1 - \alpha_n \sum_{m=1}^{j} \frac{1}{a_{mn}}} \tag{48}$$

and

$$\lambda_n^j = \alpha_n b_n^j + \beta_n. \tag{49}$$

Compare λ_n^j with k_{jn} in (47):

$$k_{1n} \geq k_{2n} \geq, ..., k_{ln} \geq \lambda_n^j > k_{(l+1)n} \geq, ..., \geq k_{Mn}. \tag{50}$$

If $l = j$, then set

$$b_n^* = b_n^j, \tag{51}$$

$$\lambda_n^j = \alpha_n b_n^* + \beta_n, \tag{52}$$

where b_n^* is the equilibrium budget for firm n and go to Step 2; if $j < l$ then set $j = j + 1$ and go to Step 1; if $j > l$ then set $j = j - 1$ and go to Step 1.

Step 2: *Set*

$$\lambda_n^* = \max\{0, \lambda_n^j\}. \tag{53}$$

Step 3: *Calculate the equilibrium Internet budget allocation: for $n = 1, \ldots, N$:*

$$f_{mn}^* = \frac{\lambda_n^* - k_{mn}}{a_{mn}}, \quad m = 1, 2, ..., j; \tag{54}$$

$$f_{mn}^* = 0, \quad m = j + 1, ..., M. \tag{55}$$

Here we would like to point out that the marginal response in website m for firm n is λ_n^*. If $\lambda_n^j < 0$, the firm reaches the maximum response at an interior point of the feasible set $\mathcal{K}^1$, and the budget surplus of $f_{ns}^* = b_n^* - \sum_{m=1}^{M} f_{mn}^*$ occurs.

Proof: Under the quadratic separability assumption, and using (19), the equilibrium conditions (21) – (22) reduce to:

$$a_{mn} f_{mn}^* + k_{mn} = \alpha_n b_n^* + \beta_n, \quad \text{if} \quad f_{mn}^* > 0, \tag{56}$$

$$a_{mn} f_{mn}^* + k_{mn} \le \alpha_n b_n^* + \beta_n, \quad \text{if} \quad f_{mn}^* = 0. \tag{57}$$

Without loss of generality, we assume that (56) holds true for $m = 1, 2, ..., j$.

Solving (56) for f_{mn}^* yields

$$f_{mn}^* = \frac{\alpha_n b_n^* + \beta_n - k_{mn}}{a_{mn}}, \quad m = 1, \ldots, j. \tag{58}$$

Substituting now (58) into the budget constraint (23) and letting $f_{ns}^* = 0$ yields

$$\sum_{m=1}^{j} \frac{\alpha_n b_n^* + \beta_n - k_{mn}}{a_{mn}} = b_n^*. \tag{59}$$

Solving (59) for b_n^*, we obtain (48), which, together with values obtained by formulae (49), (51), (52), (54), and (55), thereafter, will satisfy the equilibrium conditions (21) – (23). □

In the case when the budget is fixed for each of the firms, then variational inequality (38) governs the equilibrium conditions. Variational inequality (38) is quadratic separable if (45) holds true. The algorithm for (38) under condition (45) is as follows:

Corollary 6

If variational inequality (38) is quadratic separable in the sense that (45) holds true for $m = 1, 2, ..., M$ and $n = 1, ..., N$ with $a_{mn} < 0$ and $k_{mn} > 0$ in order to guarantee strong monotonicity of $-u(f)$ on the feasible set $\mathcal{K}^2$, then the equilibrium Internet budget allocation can be calculated by the following formulae:

For each $n = 1, \ldots, N$:

Step 0: *Sort the $\{k_{mn}\}$ in nonascending order. Without loss of generality, we assume that*

$$k_{1n} \ge k_{2n} \ge, ..., \ge k_{Mn}. \tag{60}$$

Step 1: *Let $j = 1$ and calculate*

$$\lambda_n^j = \frac{\overline{b_n} + \sum_{m=1}^{j} \frac{k_{mn}}{a_{mn}}}{\sum_{m=1}^{j} \frac{1}{a_{mn}}}. \tag{61}$$

Compare λ_n^j with k_{jn} in (60):

$$k_{1n} \geq k_{2n} \geq, ..., k_{ln} \geq \lambda_n^j > k_{(l+1)n} \geq, ..., \geq k_{Mn}. \tag{62}$$

If $l = j$, then set

$$\lambda_n^* = \max\{0, \lambda_n^j\}, \tag{63}$$

where λ_n^* is the marginal response at equilibrium for firm n and go to Step 2; if $j < l$ then set $j = j + 1$ and go to Step 1; if $j > l$ then set $j = j - 1$ and go to Step 1.

Step 2: *Calculate the equilibrium budget allocation:*

$$f_{mn}^* = \frac{\lambda_n^* - k_{mn}}{a_{mn}}, \quad m = 1, 2, ..., j; \tag{64}$$

or

$$f_{mn}^* = 0, \quad m = j + 1, ..., M; \tag{65}$$

$$f_{ns}^* = \overline{b_n} - \sum_{i=1}^{M} f_{mn}^*. \tag{66}$$

Proof: Follows from Theorem 6. See also Zhao and Nagurney (2005). □

Example 2

We now present a small numerical example to illustrate an application of the exact computational procedure stated in Theorem 6. There are two firms competing over three websites in this example. The functions (cf. (45) and (46)) are:

$$u_{11}(f) = -2f_{11} + 100, \quad u_{21}(f) = -4f_{21} + 80, \quad u_{31}(f) = -2f_{31} + 45,$$

$$u_{12}(f) = -1f_{12} + 90, \quad u_{22}(f) = -3f_{22} + 80, \quad u_{32}(f) = -5f_{32} + 90,$$

and

$$\lambda_1(b_1) = 5b_1 - 10, \quad \lambda_2(b_2) = 8b_2 - 20.$$

We applied the exact equilibration algorithm outlined in Theorem 6 to this numerical example and obtained the following solution:

$$f_{11}^* = 14.2105, \quad f_{21}^* = 2.1052, \quad f_{31}^* = 0.0000, \quad b_1^* = 16.3158, \quad \lambda_1^* = 71.5790,$$

$$f_{12}^* = 10.3015, \quad f_{22}^* = 0.1005, \quad f_{32}^* = 2.0603, \quad b_2^* = 12.4623, \quad \lambda_2^* = 79.6985.$$

Clearly, the conditions (21) – (23) (equivalently, (13) – (15)) were satisfied exactly by this solution.

Example 3

For completeness, we now also present an example with fixed Internet budget sizes in order to illustrate the computational procedure in Corollary 6. The data were as in Example 2 except that now we asumed that the budget sizes were fixed and were given by $\bar{b}_1 = 35$ and $\bar{b}_2 = 25$. An application of the algorithm in Corollary 6 yielded the following solution:

$$f_{11}^* = 26.6666, \quad f_{21}^* = 8.3333, \quad f_{31}^* = 0.0000, \quad \lambda_1^* = 46.6666,$$

$$f_{12}^* = 18.3333, \quad f_{22}^* = 2.7777, \quad f_{32}^* = 3.6460, \quad \lambda_2^* = 71.6777.$$

In general, however, the functions $u(f)$ and $\lambda(b)$ in (24) may not be separable and quadratic. Hence, we now propose an algorithm which constructs a sequence of separable quadratic network equilibrium problems, each of which can be solved using the exact procedure in Theorem 6. We then specialize the algorithm to the case of fixed budgets. The algorithm is the general iterative procedure of Dafermos (1983) (see also Nagurney, 1999).

A Variational Inequality Algorithm

We first simplify the notation by introducing a vector function $g(x) = (-u(f), \lambda(b)) : \mathcal{K}^1 \mapsto R^{MN+N}$, where $x \in R^{MN+N}$. Then, we construct a smooth function $G(x, y) : \mathcal{K}^1 \times \mathcal{K}^1 \mapsto R^{MN+N}$ with the following properties:

(i). $G(x, x) = g(x), \quad \forall x \in \mathcal{K}^1$,

(ii). for every $x, y \in \mathcal{K}^1$, the $(MN + N) \times (MN + N)$ matrix $\nabla_x G(x, y)$ is positive definite.

Any smooth function $G(x, y)$ with the above properties generates the following algorithm.

Step 0: Initialization

Initialize with an $x^0 \in \mathcal{K}^1$. Set $\tau := 1$.

Step 1: Construction and Computation

Compute x^τ by solving the variational inequality:

$$\langle G(x^\tau, x^{\tau-1})^T, x - x^\tau \rangle \geq 0, \quad \forall x \in \mathcal{K}^1. \tag{67}$$

Step 2: Convergence Verification

If $|x^\tau - x^{\tau-1}| < \varepsilon$, with $\varepsilon > 0$, a pre-specified tolerance, then stop; otherwise, set $\tau := \tau + 1$, and go to Step 1.

Projection Method for Elastic Internet Advertising Budgets

In particular, if $G(x^\tau, x^{\tau-1})$ is chosen to be

$$G(x^\tau, x^{\tau-1}) = g(x^{\tau-1}) - \frac{1}{\rho} A(x^\tau - x^{\tau-1}) \tag{68}$$

where $(MN + N) \times (MN + N)$ matrix A is a diagonal matrix with diagonal elements:

$$a_{ii} = \frac{\partial g_i(x^0)}{\partial x_i}, \quad \text{for } i = 1, 2, \ldots, MN + N,$$

and ρ is a contraction parameter in the range $0 < \rho \leq 1$ then the variational inequality (67) is quadratic separable and its solution $(f_1^\tau, f_2^\tau, ..., f_{MN}^\tau)$ can be obtained by the exact computational procedure given in Theorem 6. Thus, iteratively using the variational inequality algorithm outlined above and Theorem 6, we obtain a sequence $\{x^\tau\}$. This sequence converges to the solution when $-u(f)$ and $\lambda(b)$ are strongly monotone with respect to their own variables (for detailed proofs, see Dafermos, 1983; Bertsekas and Tsitsiklis, 1989, and Zhao and Dafermos, 1991). The induced method is known as a projection method.

Table 2: Solution $x^\tau = (f^\tau, b^\tau)$

Iteration τ	$f^\tau = (f_{11}^\tau, f_{21}^\tau, f_{31}^\tau, f_{12}^\tau, f_{22}^\tau, f_{32}^\tau)$	$b = (b_1^\tau, b_2^\tau)$
0	$(14.0000, 12.000, 13.000, 12.000, 20.000, 3.0000)$	$(39.0000, 35.0000)$
1	$(10.5489, 6.2120, 0.0000, 4.2539, 7.5951, 1.6782)$	$(16.7608, 13.5272)$
2	$(12.3010, 3.2269, 0.0000, 5.9299, 3.8858, 2.2148)$	$(15.5280, 12.0304)$
...	...	...
23	$(12.3052, 3.0774, 0.0000, 8.4774, 0.5177, 2.9260)$	$(15.3826, 11.9212)$

Example 4

We now demonstrate an application of the projection method outlined above, coupled with the exact equilibration procedure of Theorem 6, to a numerical example. In this example, $M = 3$ and $N = 2$, that is, two firms are advertising on three websites. The functions $u_{mn}(\cdot)$ and $\lambda_n(\cdot)$ are given as follows:

$$u_{11}(f) = -2f_{11} - f_{12} + 100, \quad u_{21}(f) = -4f_{21} - 1.5f_{22} + 80, \quad u_{31}(f) = -2f_{21} + f_{32} + 45,$$

and

$$\lambda_1(b_1) = 5b_1 - 10;$$

$$u_{12}(f) = -1f_{12} - 0.5f_{11} + 90, \quad u_{22}(f) = -3f_{22} - f_{21} + 80, \quad u_{32}(f) = -5f_{21} + 2f_{31} + 90,$$

and

$$\lambda_2(b_2) = 8b_2 - 20.$$

We used the projection formula (68) to construct $G(\cdot,\cdot)$ with $\rho = 0.5$. The sequence $x^\tau = (f^\tau, b^\tau)$ is reported in Table 2.

It is clear that, at iteration $\tau = 23$, the equilibrium conditions (21)–(23) are satisfied (almost exactly), where for the first firm we have that:

$$u_{11} = \frac{\partial r_1}{\partial f_{11}} = 66.9122, \text{ with } f_{11}^* = 12.3052, \quad u_{21} = \frac{\partial r_1}{\partial f_{21}} = 66.9138, \text{ with } f_{21}^* = 3.0774,$$

$$u_{31} = \frac{\partial r_1}{\partial f_{31}} = 47.9260, \text{ with } f_{31}^* = 0;$$

and for the second firm we have that:

$$u_{12} = \frac{\partial r_2}{\partial f_{12}} = 75.36997655, \text{ with } f_{12}^* = 8.4774, \quad u_{22} = \frac{\partial r_2}{\partial f_{22}} = 75.36977639, \text{ with } f_{22}^* = 2.9260,$$

$$u_{32} = \frac{\partial r_2}{\partial f_{32}} = 75.36938516, \text{ with } f_{32}^* = 0.5177;$$

$$f_{11}^* + f_{21}^* + f_{31}^* = b_1^* = 15.3826; \quad f_{12}^* + f_{22}^* + f_{32}^* = b_2^* = 11.9212$$

and

$$\lambda_1 = 66.9130, \quad \lambda_2 = 75.3696.$$

The selection of the contraction parameter ρ is important. According to Dafermos (1983) and Zhao and Dafermos (1991), if ρ is sufficiently small and, of course, always less than or equal to 1, the above variational inequality algorithm with $G(\cdot, \cdot)$ specified by (68) will converge. It is our observation that although small ρ guarantees the convergence, the speed of the convergence is slower than with a larger ρ. On the other hand, if ρ is larger than the criterion required for convergence specified in Dafermos (1983), the algorithm may not converge at all. One should then, in practice, start out with a mid-range value for ρ in the range $0 < \rho \leq 1$ and, if convergence is attained, then stop; otherwise, one can recuce the valur of ρ. Additional computational experience but on entirely different network applications, can be found in Nagurney (1999) and the references therein.

The algorithm for solving variational inequality (38) in the case of fixed Internet advertising budgets, in turn, takes the form:

Construct a smooth function $G(x, y) : \mathcal{K}^2 \times \mathcal{K}^2 \mapsto R^{MN+N}$ with following properties:

(i). $G(x, x) = -u(x), \quad \forall x \in \mathcal{K}^2,$

(ii). for every $x, y \in \mathcal{K}^2$, the $(MN) \times (MN)$ matrix $\nabla_x G(x, y)$ is positive definite.

Any smooth function $G(x, y)$ with the above properties generates the following algorithm:

Step 0: Initialization

Initialize with an $x^0 \in \mathcal{K}^2$. Set $\tau := 1$.

Step 1: Construction and Computation

Compute x^τ by solving the variational inequality:

$$\langle G(x^\tau, x^{\tau-1})^T, x - x^\tau \rangle \geq 0, \quad \forall x \in \mathcal{K}^2. \tag{69}$$

Step 2: Convergence Verification

If $|x^\tau - x^{\tau-1}| < \varepsilon$, with $\varepsilon > 0$, a pre-specified tolerance, then stop; otherwise, set $\tau := \tau + 1$, and go to Step 1.

Projection Method for Fixed Internet Advertising Budgets

In particular, if $G(x^\tau, x^{\tau-1})$ is now chosen to be

$$G(x^\tau, x^{\tau-1}) = -u(x^{\tau-1}) - \frac{1}{\rho} A(x^\tau - x^{\tau-1}), \tag{70}$$

where $(MN) \times (MN)$ matrix A is a diagonal matrix with diagonal elements:

$$a_{ijij} = \frac{\partial u_{ij}(x^0)}{\partial x_{ij}}, \quad \text{for } i = 1, 2, \ldots, MN; j = 1, 2, \ldots, MN,$$

where ρ is in the range $(0, 1]$, then the variational inequality (69) is quadratic separable and its solution $(f_1^\tau, f_2^\tau, ..., f_{MN}^\tau)$ can be obtained by the exact computational procedure given in Corollary 6. Thus, iteratively using the variational inequality algorithm outlined above and Corollary 6, we obtain a sequence $\{x^\tau\}$. This sequence converges to the equilibrium solution when $-u(f)$ is strongly monotone (see also Dafermos, 1983, and Bertsekas and Tsitsiklis, 1989) since the feasible set $\mathcal{K}^2$ is compact. This algorithm is also a projection method.

Example 5

We now present an example to illustrate the solution of variational inequality (38) using the projection method outlined above for the solution of the fixed Internet advertising budget size network equilibrium advertising problem. In this example, $M = 3$ and $N = 2$, that is, two firms are advertising on three websites. Each now has a fixed budget of $\bar{b}_1 = \$20$ and $\bar{b}_2 = \$15$, respectively. The functions $u_{mn}(\cdot)$ and $\lambda_n(\cdot)$ are given as follows:

$$u_{11}(f) = -2f_{11} - 0.5f_{12} + 100, \quad u_{21}(f) = -4f_{21} - f_{22} + 80, \quad u_{31}(f) = -2f_{21} + 0.5f_{32} + 45;$$

Table 3: Solution f^τ

Iteration τ	$f^\tau = (f_{11}^\tau, f_{21}^\tau, f_{31}^\tau, f_{12}^\tau, f_{22}^\tau, f_{32}^\tau)$
0	$(10.000, 5.000, 5.000, 4.000, 7.000, 4.000)$
1	$(14.067, 5.933, 0.000, 5.525, 5.698, 3.777)$
2	$(14.621, 5.379, 0.000, 6.724, 3.600, 4.676)$
...	...
14	$(16.210, 3.790, 0.000, 11.011, 1.054, 2.935)$
15	$(16.216, 3.784, 0.000, 11.057, 1.016, 2.927)$

Table 4: Values of u

Iteration τ	$u(f^\tau) = (u_{11}, u_{21}, u_{31}, u_{12}, u_{22}, u_{32})$
0	$(78.000, 53.000, 37.000, 84.000, 56.500, 75.000)$
1	$(69.104, 50.569, 46.889, 81.662, 59.939, 71.113)$
2	$(67.397, 53.806, 46.800, 80.351, 63.283, 72.001)$
...	...
14	$(62.076, 63.786, 46.467, 75.747, 74.943, 75.326)$
15	$(62.040, 63.847, 46.464, 75.700, 75.364, 75.060)$

$$u_{12}(f) = -1f_{12} - 0.2f_{11} + 90, \quad u_{22}(f) = -3f_{22} - 0.5f_{21} + 80, \quad u_{32}(f) = -5f_{21} + f_{31} + 90.$$

We used the projection formula (70) to construct $G(\cdot, \cdot)$ with $\rho = 0.2$. The sequence f^τ generated is recorded in Table 3.

It is clear that equilibrium conditions (35)–(37) are approached with the progress of τ as shown in Table 4. At $\tau = 15$, the stopping rule $|x^\tau - x^{\tau-1}| < 0.06$ is attained.

6. Summary and Conclusions

In this paper, we proposed a network equilibrium framework for Internet advertising in the case of multiple firms competing on multiple websites. We first argued that such an approach is warranted since online advertising is different from advertising on other media. We identified the network structure of the competitive equilibrium problem whose solution yields the equilibrium online budgets sizes for the competing firms, as well as the equilibrium budget allocations to the various websites in terms of advertising expenditures. Due to the special network structure of the problem, we then proposed first special-purpose algorithms and then showed how these algorithms could be embedded in more general variational inequality algorithms, in particular, projection methods. We illustrated our approach throughout this paper with numerical examples.

This research demonstrated how tools from operations research and, in particular, network-based tools and variational inequalities could be applied to the marketing/advertising arena.

Although this paper discusses the equilibrium (which is also optimal) Internet advertising strategies, the optimal state of non-Internet advertising is reached simultaneously by spending what is left in the budget after allocating the amount calculated by this model for Internet advertising. Indeed, the budget function b_n (or λ_n, equivalently) used in this model is a result of the optimizing firm's total response in both media; when the variational inequality (24) is solved, the optimal conditions of the maximization problem (1), subject to (2) and (3), are satisfied simultaneously.

We note that it has been a great challenge to marketers to measure the effectiveness of advertisement in traditional media, simply, because it is impossible to collect data regarding exposure and responses. Hence, it is difficult to determine in such cases, in a scientific manner, the amount of advertising investment and the payoffs. In the case of the Internet, in contrast, the exposures and responses can be accurately measured. With our model, one can focus on finding the equilibrium (which, as we mentioned earlier, is also a firm's optimum) Internet ad expenditures, and what is left in the budget is the optimal amount to be allocated to the traditinal media.

In addition, we can obtain the response functions based on real data, although the examples presented here are theoretical. We suggest collecting data for the number of clicks and the corresponding ad costs involved, and then to use a quadratic regression model to construct the functions. In practice, collecting one own's company's web-based data is very easy; the difficulty lies in that the data of other companies' may be difficult to obtain since they are usually competitive rivals. Thus, the response functions may have to be built on asymmetric information. The issue of competition with asymmetric information lies in another territory of research (see Grossman and Stiglitz, 1980) which will be discussed in our next paper.

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