A Network Modeling Approach for the Optimization of Internet-Based Advertising Strategies and Pricing with a Quantitative Explanation of Two Paradoxes

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Abstract: This paper addresses the determination and evaluation of optimal Internet marketing strategies when a firm is advertising on multiple websites. An optimization model is constructed for the determination of the optimal amount of click-throughs subject to a budget constraint. The underlying network structure of the problem is then revealed and exploited to obtain both qualitative properties of the solution pattern as well as computational procedures. In addition, three different pricing schemes used in Internet marketing are quantitatively compared and indices that can guide marketers to shift from one scheme to another are proposed. Finally, two numerical examples are constructed that demonstrate two paradoxes: (1). that advertising on more websites may reduce the total responses and (2). that advertising on more websites may reduce the click-through rate. Through the analysis of the network model, such puzzling phenomena are then quantitatively explained.

Key Words: Internet marketing, online advertising, pricing, network optimization, paradoxes.

1. Introduction

The Internet is becoming a more important and recognized advertising medium for firms. According to Jupiter Research (2004), US web-based advertising expenditures are expected to reach $8.4 billion by the year 2004 and, yet, they represent only 3.3% of the $243 billion in advertising expenditures across traditional media outlets such as television, print, and direct mail. This indicates a tremendous potential for Internet-based advertising in the future.

At the same time, Internet advertising offers a powerful vehicle for marketers to be able to measure the effectiveness of their marketing efforts due to a wealth of statistical data that can be captured on a weekly and even on a daily basis on the exposure, the number of clicks, and the number of types of purchases generated by the ads. This timely data availability also provides the opportunity for the construction of rigorous mathematical models for the identification of firms' optimal Internet-based advertising strategies.

Moreover, with the Internet penetrating virtually every aspect of business as well as the lives of consumers and their decision-making, the options available for Internet marketing are becoming increasingly interrelated, more complex, and more competitive. Firms can strategize their advertising efforts through a variety of portals such as AOL, YAHOO, MSN, etc.; through search engines such as GOOGLE, and thousands of affiliated websites. In addition, consumers' click-throughs, the unit of measurement for the success of Internet advertising for a particular Internet ad, is no longer dependent only on a particular advertising effort; but, rather, is a function of collective advertising efforts in all avenues of Internet media (cf. Park and Fader (2004)). Thus, appropriate tools that can enhance and optimize decision-making in this arena are needed for the identification of the global optimal advertising strategies in the highly interrelated Internet marketplace.

Network optimization (and equilibrium) models and associated algorithms have been proven to be powerful tools for the modeling and solution of complex problems, in which there is network structure. Such problems range from transportation and spatial economic problems including oligopolistic market equilibrium problems to financial optimization problems (cf. Ahuja, Magnanti, and Orlin (1993), Nagurney and Siokos (1997), Nagurney (1999, 2003), and the references therein). In many network-based models, the behavior of the decision-makers and whether they compete or cooperate is of paramount importance (see, e.g., Nagurney and Dong (2002)). The understanding of and subsequent modeling of decision-makers' behavior on networks is also critical in capturing (and preventing) such phenomena as the Braess (1968) paradox in which the addition of a link in the network may actually make all the consumers worse-off! McKelvey(1993) pointed out that adding a link to a network may also be viewed as adding another shop or service center because spatial market equilibrium problems can be modeled as network equilibrium problems.

More recently, Nagurney et al. (2002) demonstrated how a supply chain network equilibrium model (see also, e.g., Nagurney, Dong, and Zhang (2002)) could be generalized to include electronic commerce. In their paper, the economy contained manufacturers, distributors, and retailers, each seeking to maximize their profits in the presence of e-commerce channels in addition to the traditional chain of manufacturers - distributors - retailers. The authors provided a variational inequality formulation (cf. Nagurney (1999) and the references therein) of the governing equilibrium conditions and also proposed a computational procedure for the determination of equilibrium product prices and quantities.

This paper builds on the above-cited literature but is the first to demonstrate that the underlying structure of a spectrum of marketing problems in the context of Internet advertising also possesses a network structure. In particular, in this paper, a network model is constructed to allow for the determination of optimal Internet advertising strategies. In addition, qualitative properties of existence and uniqueness of the optimal solution are obtained and algorithms are proposed for the computation of solutions. Different pricing schemes prevalent in the Internet advertising industry, notably, pricing on exposure, pricing on click-throughs, and pricing on the percent of purchase - are, subsequently, quantitatively compared and index criteria for improved pricing schemes are provided.

Finally, two puzzles consisting of:

(1). why has the click-through rate declined (Bicknell (1999)) and

(2). is advertising on more websites really better? (see also Liu, Putter, and Weinberg (2003))

are quantitatively explained. Gaining insights into these conundrums is made possible through the reformulation of the advertising problems as network optimization problems in which the underlying structure of the network and the associated flows is revealed.

This paper is structured as follows. This Section provides an introduction and outline of the paper. Section 2 contains basic assumptions and their real-life rationality. In addition, an optimization model and the equivalent network optimization model are built for the determination of the optimal advertising strategies. In Section 3, the qualitative properties of existence and uniqueness of an optimal solution are investigated under appropriate assumptions. Algorithms that exploit the network structure of the problem are also proposed for computational purposes.

In Section 4, different pricing schemes utilized by the Internet marketing industry are compared and contrasted and index criteria for better pricing are derived. Section 5 then discusses why the click-through rate is declining, and why advertising on more websites is not necessarily better and presents numerical examples that exhibit this seemingly paradoxical behavior. Section 6 summarizes the results in this paper, highlights some of the remaining issues, and provides suggestions for future research.

2. The Network Optimization Model

In this Section, we formulate the basic network optimization model for Internet-based advertising. We first construct the marketing problem and then present its network optimization reformulation. In Section 4, we propose variants of this model to explore different pricing schemes.

We assume that a firm is marketing in n online markets. We let e_i denote the amount of exposures in market i with $i = 1, 2, \ldots, n$. We group the e_is into a vector e, and assume that each term is nonnegative.

Let r_i be the amount of click-throughs induced by the advertising effort on website i. It is easy to see that r_i is a function of e. Moreover, r_i is an increasing function of e_i and a concave function of e, because the amount of click-throughs increases with an increase in exposure e_i and the rate of click-throughs decreases with each additional exposure (see, e.g., DoubleClick (1996), Chatterjee, Hoffman, and Novak (2003)). We assume that $r_i(e)$ for $i = 1, \ldots, n$ is continuously differentiable to any order needed for purposes of later analysis in this paper. We also point out that we assume that each r_i is mainly determined by e_i but also is influenced by e_j; $j \neq i$.

In regards to online marketing strategies, a firm's objective is to achieve the maximum response (that is, the amount of click-throughs), rather than to maximize revenues or profits. This is so because (a). a majority (75%) of marketers use online ads for brand awareness and for acquiring new customers, rather than for driving immediate sales (cf. DoubleClick (2002)) and (b). consumers' data is very valuable to a firm. For example, partial or complete information about a consumer may significantly reduce the costs in terms of customer relationship management. In the case of Internet media, as long as consumers click-through on the advertisement, their information, at least, their partial information, is recorded in Internet logs and the advertising firms can make use of this information.

A firm needs to maximize the click-throughs from its advertisements subject to its marketing budget, that is, a firm is faced with the optimization problem given by:

$$\text{Maximize} \quad \sum_{i=1}^{n} r_i(e) \tag{1}$$

subject to:

$$\sum_{i=1}^{n} c_i e_i \leq C \tag{2}$$

$$e_i \geq 0, \quad i = 1, \ldots, n, \tag{3}$$

where c_i denotes the cost of advertisement per unit of exposures on site i, for $i = 1, \ldots, n$ and C is the firm's budget available for the advertisements.

Expression (1) is the firm's objective function stating that the firm wishes to maximize the amount of click-throughs over all the online markets. Expression (2) is the constraint reflecting that the firm cannot exceed its Internet advertising budget, whereas constraint (3) guarantees that the exposures are nonnegative.

We denote the optimal solution of (1), subject to constraints (2) and (3), yielding the optimal amounts of exposures in each of the markets, by:

$$e_1^*, e_2^*, \ldots, e_n^*. \tag{4}$$

We now construct the network representation of the above problem. For details, see Figure 1. Let $E = (N, L, W)$ denote the network consisting of the set of nodes N, the set of links L, and the set of origin/destination pairs of nodes W. The set of nodes N consists of two nodes: $\{o, d\}$; the set of links L consists of $n + 1$ links: $\{1, 2, \ldots, n, n + 1\}$ and the set of origin/destination pairs of nodes W consists of a single origin/destination pair of nodes: $w = (o, d)$. Let f_i denote the flow on link i; $i = 1, 2, \ldots, n, n + 1$. The first n of the link flows f_i are grouped into a vector f; with b_i denoting the benefit associated with link i generated by having f flows from node o to node d. Then b_i is a function of f. The network optimization problem equivalent to (1) through (3) can then be stated as follows.

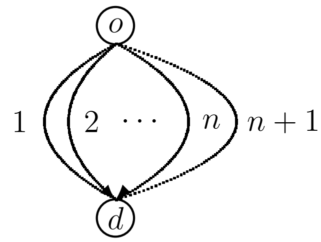

Figure 1: Network structure of the Internet-based advertising problem

Theorem 1

The optimization problem (1) – (3) for optimizing Internet-based advertising decision-making is equivalent to the network optimization problem:

$$\textit{Maximize} \quad B(f) = \sum_{i=1}^{n} b_i(f) \tag{5}$$

subject to:

$$\sum_{i=1}^{n} f_i + f_{n+1} = C \tag{6}$$

$$f_i \geq 0, \quad i = 1, \ldots, n+1. \tag{7}$$

Proof: Denote the optimal solution to network problem (5) subject to constraints (6) and (7) as:

$$f_1^*, f_2^*, \ldots, f_n^*, f_{n+1}^*. \tag{8}$$

The equivalence of (1) – (3) with (5) – (7) is easy to see. Indeed, if we let $f_i = c_i e_i$; $i = 1, \ldots n$, and we let f_{n+1} be the slack variable associated with the budget constraint (2), and $n+1$ is the dummy link that absorbs any unallocated funds of the firm's budget (Hillier and Lieberman (1985)); a simple substitution gives us:

$$b_i(f) = r_i(\frac{f_1}{c_1}, \frac{f_2}{c_2}, \ldots, \frac{f_n}{c_n}), \quad i = 1, \ldots, n. \tag{9}$$

The equivalence of (1) – (3) and (5) – (7) is thus established. □

The above result transforms the problem of seeking the vector of optimal exposures e^* into the problem of determining the vector of optimal advertising expenditures f^* in all the online markets for maximum response. As we will see later, the transformation of the original problem into a network optimization problem greatly facilitates the theory, algorithm development, and interpretation of the problem.

3. Some Theoretical Preliminaries

Before we investigate the existence, uniqueness, and convergence of algorithms for the computation of the optimal advertising expenditures f^*, we need the following:

Lemma 1

If the response $r_i(e)$ is an increasing function of e_i for a given i and a concave function of e, then $b_i(f)$ is also an increasing concave function of f.

Proof: We have that

$$\frac{\partial b_i}{\partial f_i} = \frac{\partial r_i}{\partial e_i} \times \frac{\partial e_i}{\partial f_i} = \frac{1}{c_i}\frac{\partial r_i}{\partial e_i} \tag{10}$$

and

$$\Delta f^T \nabla^2 b_i \Delta f = \{c_1\Delta e_1, c_2\Delta e_2, \ldots, c_n\Delta e_n\} \left\{ \begin{array}{ccc} \frac{\partial^2 b_i}{\partial^2 f_1} & \ldots & \frac{\partial^2 b_i}{\partial f_n \partial f_1} \\ \ldots & \ldots & \ldots \\ \frac{\partial^2 b_i}{\partial f_n \partial f_1} & \ldots & \frac{\partial^2 b_i}{\partial^2 f_n} \end{array} \right\} \left\{ \begin{array}{c} c_1\Delta e_1 \\ \ldots \\ c_n \Delta e_n \end{array} \right\} \tag{11}$$

$$= \{c_1\Delta e_1, c_2\Delta e_2, \ldots, c_n\Delta e_n\} \left\{ \begin{array}{ccc} \frac{1}{c_1^2}\frac{\partial^2 r_i}{\partial^2 e_1} & \ldots & \frac{1}{c_1 c_n}\frac{\partial^2 r_i}{\partial e_n \partial e_1} \\ \ldots & \ldots & \ldots \\ \frac{1}{c_1 c_n}\frac{\partial^2 r_i}{\partial e_n \partial e_1} & \ldots & \frac{1}{c_n^2}\frac{\partial^2 r_i}{\partial^2 e_n} \end{array} \right\} \left\{ \begin{array}{c} c_1\Delta e_1 \\ \ldots \\ c_n \Delta e_n \end{array} \right\}$$

$$= \Delta e^T \nabla^2 r_i \Delta e. \tag{12}$$

Equations (10) through (12) demonstrate the equivalence between $r_i(e)$ and $b_i(f)$ in terms of the fact that they are both increasing functions and concave. □

Now we are ready to state:

Theorem 2

If the response functions $r_i(e)$; $i = 1, \ldots, n$ are increasing and strictly concave, then the network optimization problem (5) – (7) has a unique solution (see Bazaraa, Sherali, and Shetty (1993)).

The optimal solution (8), under the above assumptions, satisfies the following *variational inequality*, which is equivalent to the Kuhn-Tucker optimality conditions for this problem (see, e.g., Bertsekas and Tsitsiklis (1989), Bazaraa, Sherali, and Shetty (1993), and Nagurney (1999)): determine $f^* \in \mathcal{K}$ satisfying

$$\langle \nabla B^T(f^*), f - f^* \rangle \leq 0, \quad \forall f \in \mathcal{K}, \tag{13}$$

where $\nabla B^T(f^*)$ is the gradient of the function $B(f)$ at f^*, that is,

$$\nabla B^T(f^*) = (\frac{\partial B(f^*)}{\partial f_1}, \ldots, \frac{\partial B(f^*)}{\partial f_n}, 0),$$

$\langle \cdot, \cdot \rangle$ is the inner product in N-dimensional Euclidean space where here $N = n + 1$, and $\mathcal{K}$ is the feasible set underlying this problem and defined as:

$$\mathcal{K} \equiv \{f | \, \text{such that} \, (6), (7) \, \text{hold}\}.$$

The economic meaning of (13) is that, at the optimal marketing strategy vector f^*, which provides the optimal advertising expenditures, the click-throughs cannot be increased any more along any feasible direction.

In the special case where the function in (5) is a separable quadratic function of f, the optimal solution (8) can be obtained by explicit formulae. This result is derived by the following theorem.

Theorem 3

If $B(f)$ is a separable quadratic function in the sense that

$$\frac{\partial B(f)}{\partial f_i} = a_i f_i + b_i, \quad \forall i = 1, 2, ..., n, \tag{14}$$

where a_i, b_i are constants, then the optimal solution of (5) – (7) can be obtained in two steps:

An Exact Procedure

(i). calculate λ by formula

$$\lambda = \frac{C + \sum_{i=1}^{n} \frac{b_i}{a_i}}{\sum_{i=1}^{n} \frac{1}{a_i}}; \tag{15}$$

(ii). calculate $f_i^*; i = 1, 2, ..., n, n+1$ by the formula

$$f_i^* = Max\{0, \frac{\lambda - b_i}{a_i}\}, \quad \forall i = 1, 2, ..., n;\ f_{n+1}^* = 0, \quad if \quad \lambda > 0, \tag{16}$$

or,

$$f_i^* = Max\{0, \frac{-b_i}{a_i}\}, \quad \forall i = 1, 2, ..., n;\ f_{n+1}^* = C - \sum_{i=1}^{n} f_i^*, \quad if \quad \lambda \leq 0. \tag{17}$$

We point out that, because of the concavity of $B(f)$: $a_i < 0, \quad \forall i = 1, 2, ..., n$.

Proof: By applying the network optimality conditions (see Dafermos (1972)), we have that $(f_1^*, f_2^*, ..., f_n^*, f_{n+1}^*)$ is a solution to (5) – (7) if and only if

$$a_i f_i^* + b_i \begin{cases} = \lambda, & \text{if} \quad f_i^* > 0 \\ \leq \lambda, & \text{if} \quad f_i^* = 0 \end{cases}, \quad \forall i = 1, 2, ..., n, \tag{18}$$

$$0 \begin{cases} = \lambda, & \text{if} \quad f_{n+1}^* > 0 \\ \leq \lambda, & \text{if} \quad f_{n+1}^* = 0, \end{cases} \tag{19}$$

$$\sum_{i=1}^{n} f_i^* + f_{n+1}^* = C. \tag{20}$$

Thus, the problem of solving (5) – (7) becomes the problem of solving the linear system (18)–(20).

Solving (18) for f_i^*, we obtain

$$f_i^* = \frac{\lambda - b_i}{a_i}, \quad \forall i = 1, 2, ..., n. \tag{21}$$

Substituting then (21) into equation (20) and letting $f_{n+1}^* = 0$, equation (20) then generates:

$$\lambda = \frac{C + \sum_{i=1}^{n} \frac{b_i}{a_i}}{\sum_{i=1}^{n} \frac{1}{a_i}}. \tag{22}$$

If $\lambda > 0$, then $f_i^*;\ i = 1, 2, ..., n+1$ calculated by (16) satisfies the optimality conditions (18) – (20); if $\lambda \leq 0$, then we let $\lambda = 0$, and $f_i^*;\ i = 1, 2, ..., n+1$ calculated by (17) satisfies the optimality conditions (18) – (20). □

Dafermos and Sparrow (1969) considered a network of the structure in Figure 1 in the case of traffic network equilibrium problems in which the underlying functions were also quadratic and separable and proposed an exact computational procedure similar to the one above. However, in that paper, the objective function was a minimization one and all the links had costs associated with them.

In general, the function $B(f)$ in (5) may not be separable and quadratic. Recall that solving (5) for the optimal network flow f^* is equivalent to solving the *variational inequality* problem (13). We are now ready to propose an algorithm which constructs a sequence of separable quadratic network optimization problems, each of which can be solve using the exact procedure above. The algorithm is based on the general iterative scheme of Dafermos (1983).

A Variational Inequality Algorithm

Construct a smooth function $g(f, y) : \mathcal{K} \times \mathcal{K} \mapsto R^{n+1}$ with the following properties:

(i). $g(f, f) = -\nabla B(f), \quad \forall f \in \mathcal{K}$,

(ii). for every $f, y \in \mathcal{K}$, the $(n+1) \times (n+1)$ matrix $\nabla_f g(f, y)$ is positive definite.

Any smooth function $g(f, y)$ with the above properties generates the following algorithm.

Step 0: Initialization

Start with some $f^0 \in \mathcal{K}$. Set $k := 1$.

Step 1: Construction and Computation

Compute f^k by solving the *Variational Inequality*

$$\langle g(f^k, f^{k-1})^T, f - f^k \rangle \geq 0, \quad \forall f \in \mathcal{K}. \tag{23}$$

Step 2: Convergence Verification

If $|f^k - f^{k-1}| < \varepsilon$, with $\varepsilon > 0$, a pre-specified tolerance, then stop; otherwise, set $k := k + 1$, and go to step 1.

In particular, if $g(f^k, f^{k-1})$ is chosen to be

$$g(f^k, f^{k-1}) = -\nabla B(f^{k-1}) - A(f^k - f^{k-1})$$

where $n \times n$ matrix A is a diagonal matrix with diagonal elements:

$$a_{ii} = \frac{\partial^2 B(f^{k-1})}{\partial f_i^2}, \quad \text{for } i = 1, 2, \ldots, n,$$

then the variational inequality (14) is equivalent to a *quadratic programming problem* given by:

$$\mathrm{M}ax_{f \in \mathcal{K}} \left[\nabla B(f^{k-1}) \cdot (f - f^{k-1}) + \frac{1}{2}(f - f^{k-1})^T \cdot A \cdot (f - f^{k-1}) \right]. \tag{24}$$

Problem (24) is a separable quadratic network optimization problem with $a_i = \frac{\partial^2 B(f^{k-1})}{\partial f_i^2}$, and $b_i = \frac{\partial B(f^{k-1})}{\partial f_i} - \frac{\partial^2 B(f^{k-1})}{\partial f_i^2} f_i^{k-1}$, and its solution $(f_1^k, f_2^k, ...f_n^k, f_{n+1}^k)$ can be obtained by the exact computational procedure given in Theorem 3. Thus, iteratively using the Variational Inequality algorithm outlined above and Theorem 3, we obtain a sequence $\{f^k\}$. This sequence converges to the solution of (5)–(7) when $B(f)$ is strictly concave on $\mathcal{K}$ (for a proof, see Zhao and Dafermos (1991)).

We note here that, ultimately, we would like to also construct a framework for the modeling of multiple firms competing with their advertising decision-making across multiple websites. The variational inequality framework introduced above, along with the computational algorithm(s), will help to support that initiative as well. In particular, we expect that the competition will take place in a noncooperative setting and will lead to Nash equilibria (cf. Nagurney (1999) and the references therein).

4. Evaluation of Alternative Internet-Based Advertising Pricing Schemes

In this Section, we compare different pricing schemes for Internet advertising. There are basically three pricing schemes in the industry:

(1). pricing on exposure (Scheme 1),

(2). pricing on response (Scheme 2), and

(3). pricing based on the percent of revenue generated by the advertisement (Scheme 3).

We let c_i; $i = 1, 2 \ldots, n$ denote the price per unit of exposure and we let c_i^r; $i = 1, 2 \ldots, n$ denote the price per unit of response. Also, we let p_i denote the percent of revenue generated by the responders. Then, the mathematical formulations of the optimization problems represented by the three schemes can be written, respectively, as:

Scheme 1:

$$\text{Maximize} \quad B(e) = \sum_{i=1}^{n} r_i(e) \tag{25}$$

subject to:

$$\sum_{i=1}^{n} c_i e_i \leq C \tag{26}$$

$$e_i \geq 0, \quad i = 1, \ldots, n. \tag{27}$$

Scheme 2:

$$\text{Maximize} \quad B(e) = \sum_{i=1}^{n} r_i(e) \tag{28}$$

subject to:

$$\sum_{i=1}^{n} c_i^r r_i(e) \leq C \tag{29}$$

$$e_i \geq 0, \quad i = 1, \ldots, n. \tag{30}$$

Scheme 3:

$$\text{Maximize} \quad B(e) = \sum_{i=1}^{n} r_i(e) \tag{31}$$

subject to:

$$\sum_{i=1}^{n} p_i v_i r_i(e) \leq C \tag{32}$$

$$e_i \geq 0, \quad i = 1, \ldots, n, \tag{33}$$

where v_i denotes the revenue resulting from a click-through in online market i, for $i = 1, \ldots, n$.

Note that each of the above three schemes is characterized by the same objective function (that is, (25), (28), and (31) are identical); the same nonnegativity assumption on the variables (cf. (27), (30), and (33)) but are distinct in the constraints: (26), (29), and (32).

Theorem 4

Let $r_i^* = r_i(e^*)$, where e^* is a solution to Scheme 1. Then, if the c_i^r; $i = 1, 2, \ldots, n$, are chosen so that $c_i^r \leq c_i e_i^* / r_i^*$; $i = 1, 2, \ldots, n$, for Scheme 2, then e^* is a feasible solution for Scheme 2.

Proof: Substituting e^* into the constraint (29), we see that

$$\sum_{i=1}^{n} c_i^r r_i(e^*) \leq \sum_{i=1}^{n} \frac{c_i e_i^*}{r_i^*} r_i(e^*) = \sum_{i=1}^{n} c_i e_i^* \leq C. \qquad \square$$

Thus, the optimal solution to Scheme 1 satisfies constraint (29) of Scheme 2, which implies that if a firm placing the Internet-based ad wants to switch the payment scheme from pricing on exposure to pricing on response (the amount of click-throughs), the firm needs to negotiate for the price (per unit of response) to be less than or equal to $c_i e_i^* / r_i^*$, in order to guarantee that the firm is not worse-off.

Example 1

In this example, we consider a simple case where a firm advertises on two websites with a budget of $9000 and the response functions are, respectively:

$$r_1 = -0.0002 e_1^2 + 0.4 e_1 + 2$$

$$r_2 = -0.0004 e_2^2 + 0.2 e_2 + 1,$$

where e_i; $i = 1, 2$ are the exposures in units of 1000 and $r_i; i = 1, 2$ are the click-throughs in units of 100. If the firm uses Scheme 1 with unit prices of c_1 =\$10 and c_2=\$15, per thousand of click-throughs, respectively, then $f_1 = 10e_1$, $f_2 = 15e_2$ are the capital flows to each of the websites and f_3 is the slack (unused amount of the budget) that flows on the dummy link (website). The firm is faced with the problem:

$$\text{Maximize} \quad B(f) = r_1(f_1/10) + r_2(f_2/15)$$

subject to:

$$f_1 + f_2 + f_3 = 9000$$

$$f_i \geq 0, \quad i = 1, 2, 3.$$

It is a separable quadratic network optimization problem. By Theorem 3, the solution to this problem is: f_1^* =\$7764.70588, f_2^* =\$1235.294115, and $r_1^* = 192.0069572$, $r_2^* = 14.75778544$. The response rate in each of the websites is 2.473% and 1.792%, respectively.

If the firm uses Scheme 2 to model the optimal advertising strategy problem, according to our previous discussion, it will tolerate prices up to $c_1^r = 40.44$ and $c_2^r = 83.70$ per hundreds of click-throughs. In this case, the firm is faced with the problem:

$$\text{Maximize} \quad B(e) = r_1(e) + r_2(e)$$

subject to:

$$40.44 r_i(e) + 83.70 r_2(e) \leq 9000$$

$$e_i \geq 0, \quad i = 1, 2.$$

The solution to this problem: is $e_1^* = 999.999522$, $e_2^* = 450.818924$ and $r_1^* = 202.0000$, $r_2^* = 9.868703905$. Note that the firm still spends \$9000 but the responses are higher. Scheme 2 will, hence, benefit the firm. Comparing Scheme 1 with Scheme 2, we see that the lower response rate website loses capital and the higher rate website will gain capital. The optimal results are summarized in Tables 1 and 2 below.

Pricing Scheme 3 is often associated with affiliate programs where a percent of the revenues is paid to the affiliate websites. If v denotes the value per response, which is usually the lifetime value of a customer times the purchase/click-through ratio, if we set $p_i = c_i^r/v$, then Scheme 2 and Scheme 3 are equivalent.

Table 1: Solution Results for Example 1: Scheme 1

	Exposures(in 1000s)	Click-throughs (in 100s)	Rate	Expenditures
Site 1	776.470588	192.0069572	2.473%	$7764.705880
Site 2	82.35294	14.75778544	1.792%	$1235.294115

Table 2: Solution Results for Example 1: Scheme 2

	Exposures (in 1000s)	Click-throughs(in 100s)	Rate	Expenditures
Site 1	999.999522	202.0000000	2.020%	$8168.825977
Site 2	450.818924	9.868703905	2.19%	$826.0556371

5. Analysis and Explanation of Paradoxes

It has been widely accepted that the click-through rate has been declining since the late 1990s (Double Click (2004)). Many marketers argue that this is due to the pervasiveness and increasing popularity of the Internet (see, e.g., Shen (2000)) and that online advertising is growing faster than the number of Internet users (cf. Baker (2000)). However, mathematically, given a firm's budget, if a firm puts ads in more diverse markets, the exposures on a single market would decrease due to wider spreading. As a result, the click-through rate should increase due to the concavity of the response function.

Equivalently, more links (or channels) in the isomorphic networks should theoretically benefit the amount of click-throughs or the click-through rate. Why then, in reality, is a decline of the click-through rate observed? Similar paradoxes have been observed in other contexts (cf. Braess (1968), Dafermos and Nagurney (1984), McKelvey (1992), Korilis, Lazar, and Orda (1999), Nagurney (2000), Liu, Putler, and Weinberg (2003)). In this Section, we will illuminate this seemingly paradoxical behavior and will explain the reason by revealing the inherent quantitative aspects.

Let's keep in mind that an additional website used for advertising is equivalent to saying that an additional link *new* of the isomorphic network is introduced. We also assume that the advertising budget C is completely spent under optimization with or without the additional link new. Hence, we can eliminate link $n+1$ from the network in Figure 1 since it has zero flow both before and after the addition due to our assumption that the budget is exhausted and, consequently, there is no slack. Let then

$$(\hat{f}, \hat{f}_{new}) = (\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n, \hat{f}_{new}), \text{ with } \hat{f} = (\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n)$$

be the optimal flow on the enlarged network. We assume that $\hat{f}_{new} > 0$ since $\hat{f}_{new} = 0$ is a trivial case.

Explanation of Conundrum #2: Is Advertising on More Websites Really Better?

We now consider question #2 posed in the Introduction and we investigate the conditions under which advertising in an additional website causes the total response to decrease. We construct the change in response after the new website is used, with the change expressed as:

$$\sum_{i=1}^{n} b_i(\hat{f}, \hat{f}_{new}) + b_{new}(\hat{f}, \hat{f}_{new}) - \sum_{i=1}^{n} b_i(f^*, 0). \tag{34}$$

We note that (24) is equal to:

$$= \sum_{i=1}^{n} b_i(\hat{f}, \hat{f}_{new}) - \sum_{i=1}^{n} b_i(\hat{f}, 0) + \sum_{i=1}^{n} b_i(\hat{f}, 0) - \sum_{i=1}^{n} b_i(f^*, 0) + b_{new}(\hat{f}, \hat{f}_{new}). \tag{35}$$

Expression (35) represents that the change in response (34) can be decomposed into three parts:

(i). $\sum_{i=1}^{n} b_i(\hat{f}, \hat{f}_{new}) - \sum_{i=1}^{n} b_i(\hat{f}, 0)$, the change in the response in the 'old' websites impacted by advertisements in the new website, whose value may or may not be negative;

(ii). $\sum_{i=1}^{n} b_i(\hat{f}, 0) - \sum_{i=1}^{n} b_i(f^*, 0)$, the change in the response in the 'old' websites caused by the decrease in associated advertising capital. Since $\sum_{i=1}^{n} \hat{f} < \sum_{i=1}^{n} f^*$, this change is negative since f^* is an optimal solution in the case of the 'old' websites; and

(iii). $b_{new}(\hat{f}, \hat{f}_{new})$, the amount of click-throughs in the new website, which is nonnegative. If item (iii) cannot prevail over the sum of (i) and (ii), then the use of the new website for advertising, makes the firm worse-off!

We now present an example to show that using a new website can actually reduce the response.

Table 3: Solution Results for Example 2

	Exposures(in 1000s)	Click-throughs (in 100s)	Rate	Expenditures
Site 1	771.16	191.5253189	2.484%	$7711.60
Site 2	82.12	14.66992	1.786%	$1231.80
Site 3	5.66	0.029	0.050%	$56.60

Example 2

Example 2 is constructed from Example 1 by adding a new website for the advertising. Let e_3 be the exposure (in units of 1000) in the new website, $c_3 = \$10$ per 1000 of exposures, and let

$$r_3(e) = -0.0002e_3^2 + 0.1e_3 + 2.8 - 0.004e_1 - 0.003e_2 \tag{36}$$

be the amount of click-throughs. Please note that the click-throughs in the new website are impacted, although not significantly, by exposures in websites 1 and 2. Similarly, the click-throughs in the two 'old' websites are also impacted by the exposures in the new website, so we now have that

$$r_1 = -0.0002e_1^2 + 0.4e_1 + 2 - 0.0002e_3$$

and

$$r_2 = -0.0004e_2^2 + 0.2e_2 + 1 - 0.01e_3.$$

The corresponding network optimization problem

Maximize $\quad B(f) = r_1(f_1/10, f_2/15, f_3/10) + r_2(f_1/10, f_2/15, f_3/10) + r_3(f_1/10, f_2/15, f_3/10)$

subject to:

$$f_1 + f_2 + f_3 = 9000$$

$$f_i \geq 0, \quad i = 1, 2, 3, \tag{37}$$

is also a separable quadratic network problem in the sense that the partial derivatives have the form of $\frac{\partial B}{\partial f_i} = a_i f_i + b_i$, given in Theorem 3. Hence, we obtain the solution summarized in Table 3.

The result shows that the total amount of click-throughs is less by 54 than the number without the new website (obtained by summing $b_1 + b_2 + b_3$ in Table 3 minus $b_1 + b_2$ in Table 1 and multiplying this difference by 100). From the structure of the click-through functions $r_i(f)$, we see that the three websites are interrelated; the click-throughs generated by the new website do not make up for the reduction in the click-throughs in the 'old' websites due to the reduction of advertising capital of \$56.60.

We would like to point out that all solutions of the optimization problems have been also cross-checked by using the SAS 8.02/OR software.

Explanation of Conundrum #1: Why has the Click-Through Rate Declined?

We now explain why the click-through rate declines. For simplicity, we introduce the notation

$$\overline{f} = \sum_{i=1}^{n} \hat{f}_i = C - \hat{f}_{new}.$$

As before, $f_1^*, f_2^*, \ldots, f_n^*$ denotes the optimal flow on the 'old' network (before the addition of the new website) and, as noted, we assume that

$$\sum_{i=1}^{n} f_i^* = C.$$

We consider now the difference between the click-throughs per unit of advertising capital, which is positively correlated with the click-through rate, for the participating markets at the two optimal solutions. In particular, we assume that the budget is fully spent, that is,

$$\frac{\sum_{i=1}^{n} b_i(\hat{f}, \hat{f}_{new})}{\overline{f}} - \frac{\sum_{i=1}^{n} b_i(f_i^*)}{C}$$

$$= \frac{C \sum_{i=1}^{n} b_i(\hat{f}, \hat{f}_{new}) - \overline{f} \sum_{i=1}^{n} b_i(f^*)}{C\overline{f}}. \tag{38}$$

Expanding upon the right-hand side expression in (28), we obtain:

$$(38) = \frac{C \sum_{i=1}^{n} b_i(\hat{f}, \hat{f}_{new}) - C \sum_{i=1}^{n} b_i(\hat{f}, 0) + C \sum_{i=1}^{n} b_i(\hat{f}, 0) - \overline{f} \sum_{i=1}^{n} b_i(f^*)}{C\overline{f}} \tag{39}$$

$$= \frac{C\left[\sum_{i=1}^{n} b_i(\hat{f}, \hat{f}_{new}) - \sum_{i=1}^{n} b_i(\hat{f}, 0) + \sum_{i=1}^{n} b_i(\hat{f}, 0) - \sum_{i=1}^{n} b_i(f^*, 0) + \sum_{i=1}^{n} b_i(f^*, 0)\right] - \overline{f}\sum_{i=1}^{n} b_i(f^*)}{C\overline{f}}$$

$$= \frac{C\left[\sum_{i=1}^{n} b_i(\hat{f}, \hat{f}_{new}) - \sum_{i=1}^{n} b_i(\hat{f}, 0) + \sum_{i=1}^{n} b_i(\hat{f}, 0) - \sum_{i=1}^{n} b_i(f^*, 0)\right] + \hat{f}_{new}\sum_{i=1}^{n} b_i(f^*)}{C\overline{f}} \tag{40}$$

$$= \frac{C\hat{f}_{new}\left[\frac{\sum_{i=1}^{n} b_i(\hat{f}, \hat{f}_{new}) - \sum_{i=1}^{n} b_i(\hat{f}, 0)}{\hat{f}_{new}} + \frac{\sum_{i=1}^{n} b_i(\hat{f}, 0) - \sum_{i=1}^{n} b_i(f^*, 0)}{\hat{f}_{new}} + \frac{\sum_{i=1}^{n} b_i(f^*)}{C}\right]}{C\overline{f}.} \tag{41}$$

The first term in (41) is the change in the click-throughs in the ‘old’ websites per unit of capital spent in the new website caused by the advertising effort in the new website. It may or may not be negative. The second term is the change in the click-throughs in the ‘old’ websites per unit of capital spent in the new website, caused by the reduction of capital in the ‘old’ websites. Its value is negative because f^* is the optimal solution of the original network. The third term is the amount of click-throughs per unit of capital in the ‘old’ markets. Expression (41) indicates that if a decrease in click-throughs caused by shifting a unit of advertising expenditure from ‘old’ markets to the new one prevails over the click-throughs per unit of advertising expenditures in the ‘old’ markets, then a decrease in the click-through rate occurs.

6. Summary and Conclusions

We conclude that if the click-through function $r(\cdot)$ is an increasing concave function of the exposure e, an optimal advertising strategy exists. Moreover, if the function $r(\cdot)$ is strictly concave, then the optimal strategy is unique and can be found by network equilibrium/optimization algorithms. We also conclude that the pricing scheme based on the click-throughs could be better (and certainly not worse) than the pricing scheme based on exposure, if the price per click-through is set to be $c_i e_i^*/r_i^*$, and that the click-through rate in participating markets may decline if a firm advertises in more markets, due to the negative impact of exposures in other markets relative to the participating markets.

We realize that the reality may be more complicated than that supported by our assumptions in this paper in which we focus on the behavior of a single firm. In the Internet marketplace, there are, in fact, m firms advertising in n online markets, competing for the market shares and the limited resources of consumers. Moreover, the consumers' response to the explosion in Internet advertising may not necessarily be an increasing function of the firms' advertising efforts. Nevertheless, we would expect that equilibrium would be attained during the course of the interactions. The next step in this research will be to develop a network equilibrium model to represent competition among firms in the Internet marketplace.

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