Financial Networks with Intermediation: Risk Management with Variable Weights

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Abstract:

In this paper, we develop a framework for the modeling, analysis, and computation of solutions to multitiered financial network problems with intermediaries in which both the sources of financial funds as well as the intermediaries are multicriteria decision-makers. In particular, we assume that these decision-makers seek not only to maximize their net revenues but also to minimize risk with the risk being penalized by a variable weight. We make explicit the behavior of the various decision-makers, including the consumers at the demand markets for the financial products. We derive the optimality conditions, and demonstrate that the governing equilibrium conditions of the financial network economy can be formulated as a finite-dimensional variational inequality problem. Qualitative properties of the equilibrium financial flow and price pattern are provided. A computational procedure that exploits the network structure of the problem is proposed and then applied to several numerical examples.

Key Words: Finance; Networks; Risk management; Financial intermediation; Multicriteria decision-making; Equilibrium; Variational inequalities

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1. Introduction

Many decision-making applications today involve not only a single decision-maker but several (or many) interacting in some fashion on what may be viewed as *multitiered* networks. In such networks, there may be competition within a tier but in order for, say, the products to be ultimately delivered to the consumers at the demand markets there must also be some degree of cooperation between the tiers. Examples par excellence include supply chain networks (cf. Nagurney and Dong (2002), Nagurney, Dong, and Zhang (2002), Dong, Zhang, and Nagurney (2004) and the references therein) as well as financial networks with intermediation (see Nagurney and Ke (2001, 2003), Nagurney and Cruz (2003a, b)). For an annotated bibliography on network optimization for supply chain and financial engineering problems, see Guenes and Pardalos (2003). For additional models and analyses, see the edited volume by Pardalos and Tsitsiringos (2002).

We note that the origin of the use of networks for the representation of *financial systems* with many interacting decision-makers lies in the work of Quesnay (1758), who depicted the circular flow of funds in an economy as a network. His basic idea was subsequently applied in the construction of flow of funds accounts, which are a statistical description of the flows of money and credit in an economy (cf. Board of Governors (1980), Cohen (1987), Nagurney and Hughes (1992)). Thore (1969) had earlier introduced networks, along with the mathematics, for the study of systems of linked portfolios (see also Charnes and Cooper (1967)) in the context of credit networks and made use of linear programming. Storoy, Thore, and Boyer (1975), in turn, presented a network model of the interconnection of capital markets and demonstrated how decomposition theory of mathematical programming could be exploited for the computation of equilibrium. The utility functions facing a sector/agent were no longer restricted to being linear functions. Thore (1980) recognized some of the shortcomings of financial flow of funds accounts and developed, instead, network models of linked portfolios with financial intermediation, using the decentralization/decomposition theory.

Nagurney, Dong, and Hughes (1992) presented a multi-sector, multi-instrument financial equilibrium model and recognized the network structure underlying the subproblems encountered in their proposed decomposition scheme, which was based on finite-dimensional variational inequality theory. The book by Nagurney and Siokos (1997) presents a plethora of static and dynamic (single country as well as international) financial network models developed to that date. Nagurney and Ke (2001), in turn, focused on modeling the behavior of not only the sources of funds and the consumers of the financial products but also on modeling the behavior of the intermediaries. They developed a multitiered network framework for the modeling, analysis and computation of solutions of such problems and also, more recently, considered the incorporation of electronic transactions into that framework (see Nagurney and Ke (2003)).

In this paper, we will advance the work of Nagurney and Ke (2001) by introducing a class of objective functions with variable weights for bicriteria decision-making and the resulting network model will allow the decision-makers to optimize their objective functions according to their risk attitudes. Moreover, we will consider general risk functions rather than risk functions of a particular form. Our work will also be based on the theoretical framework proposed by Dong and Nagurney (2001) who introduced state-dependent weights for the modeling of a sector's bicriteria decision-making problem in the context of a financial network but without any intermediation. Moreover, no numerical results were provided in that paper. Nagurney, Dong, and Mokhtarian (2002) also considered variable weights in the context of a multicriteria network equilibrium model but the model was single-tiered and not financial.

In particular, in this paper, we assume that the agents with sources of funds as well as the financial intermediaries are faced with two objectives or criteria, i.e., net revenue maximization and risk minimization with the weight associated with the latter criterion being distinct and variable for each such decision-maker. Our approach is in concert with risk-return analysis widely used in the financial arena which dates to Markowitz (1952) who introduced the concept of portfolio selection based on the mean and variance. Note that mean-variance procedures based on the seminal work of Markowitz (see also, various extensions due to Sharpe (1971), Stone (1973), Kroll, Levy, and Markowitz (1984), and Spronk and Hallerbach (1997)) for portfolio selection are known to be consistent with maximization of expected utility if either the predictive distribution of returns is jointly normal and the objective function is concave or if the objective function is quadratic. We will extend the above work by introducing a general function of risk to avoid the restriction of the use of a probability distribution of returns.

Moreover, we note that although there have been many extensions to Markowitz's model most of the model extensions imply an average weight assumption and that there is an equal trade-off between the two basic criteria. However, a decision-maker's attitude towards risk may have a major impact on his decisions and, hence, also on the monetary payoffs. For example, a risk-averse decision-maker may select a more conservative portfolio instead of a riskier one offering the potential of a higher payoff (cf. Freund (1956), Arrow (1965), Pratt (1964), Eliashberg and Winkler (1978)). More recently, various researchers have argued that equally weighted objective functions might not adequately reveal an agent's preference. Chow (1995) argued that the reverse of the risk tolerance could be regarded as a weight for the risk criterion. Ballestero and Romero (1996), in turn, proposed a surrogate of Lagrangian optimization to approximate the utility function associated with the weighted return and risk for an "average" investor within the context of incomplete information. Zouponidis, Doumpos, and Zanakis (1999) presented an application of a multicriteria decision analysis sorting methodology for portfolio selection (see also Hurson and Zopounidis (1995) and Zopunidis (1995)).

Choo and Wedley (1985) surveyed procedures for estimating implied criterion weights. See also Ballestero and Romero (1991), Weber and Borcherding (1993), Yu (1997), Brugha (1998), and Ma, Fan, and Huang (1999). For additional background on decision-making in general and on multicriteria decision-making, in particular, see Karwan, Spronk, and Wallenius (1997). Subsequently, Choo, Schoner, and Wedley (1999) provided interpretations of criteria weights and their appropriate roles in distinct multicriteria decision-making models. The paper by Steuer and Na (2003) contains a categorized bibliography on the techniques of multiple criteria decision-making applied to problems and issues in finance. The recent book by Rustem and Howe (2002) develops a variety of models, along with computational procedures, for risk management in both finance and engineering with a focus on design issues.

The proposed framework that we develop is sufficiently general to allow for the modeling, analysis, and computation of solutions to financial network problems with intermediation and with multicriteria decision-makers. Moreover, since here we consider not only the individual decision-maker's behavior but the complete financial system through the medium of networks and equilibrium analysis, our equilibrium perspective provides a valuable benchmark against which existing prices and financial flows can be compared. We emphasize that risk management in the context of multitiered supply chain networks has also been the subject of recent research activity (see Nagurney et al. (2003) and the references therein).

This paper is organized as follows: In Section 2, we present the financial network model with intermediaries and with risk management through the use of variable weights. We first derive the optimality conditions for each set of decision-makers and identify the underlying network structure of these subproblems. We construct the network for the entire financial system and obtain the governing equilibrium conditions. We derive the finite-dimensional variational inequality formulation of the equilibrium conditions which is used in subsequent sections for qualitative analysis as well as computation purposes. This formulation allows for the determination of the equilibrium financial flows between the tiers of the network as well as the equilibrium prices for the various financial instruments/products.

Section 3 provides qualitative properties of the equilibrium financial flow and price pattern, notably, existence and uniqueness results. Section 4 then describes an algorithm that resolves the variational inequality problem into simpler network subproblems, each of which can be solved exactly and in closed form. In Section 5 several numerical examples are presented to illustrate both the model and the computational procedure. We conclude the paper with a summary of results and suggestions for future research.

2. The Financial Network Model with Intermediation and Variable Weights

In this section, we develop the financial network model consisting of: decision-makers or agents with sources of funds; financial intermediaries; as well as consumers for the financial products associated with the demand markets. Specifically, we consider m sources of financial funds, such as households and businesses, involved in the allocation of their financial resources among a portfolio of financial instruments which can be obtained by transacting with distinct n financial intermediaries, such as banks, insurance and investment companies, etc. The financial intermediaries, in turn, in addition to transacting with the source agents, also determine how to allocate the incoming financial resources among distinct uses, as represented by o demand markets with a demand market corresponding to, for example, the market for real estate loans, household loans, business loans, etc.

The financial network for the entire financial system is now described and depicted graphically in Figure 1. The top tier of nodes consists of the agents with sources of funds. A typical source agent is denoted by the symbol i and associated with the node i. The middle or second tier of nodes consists of the intermediaries as well as node $n+1$ which corresponds to the non-investment option. A typical intermediary is denoted by the symbol j and associated with the node j in the second tier of nodes in the network. The bottom tier of nodes consists of the demand markets, with a typical demand market denoted by the symbol k and corresponding to the node k in the bottom tier.

For simplicity of notation, we assume that there are L instruments associated with each intermediary. Hence, from each source of funds node, there are L links connecting such a node with an intermediary node with the l-th such link corresponding to the l-th financial instrument available from the intermediary. In addition, we allow the option of non-investment and to denote this option, we then also construct an additional link from each source node to the middle tier node $n+1$, which recall represents non-investment. From each intermediary node j; $j = 1, \dots, n$, we subsequently construct o links, one to each "use" node or demand market in the bottom tier of nodes in the network.

Let q_{ijl} denote the nonnegative amount of the funds that source i "invests" in financial instrument l obtained from intermediary j. We group the financial flows q_{ijl} into the column vector $q_i \in R_+^{nL}$ for each source i. We then group the vectors q_i for all the sources $i = 1, \dots, m$

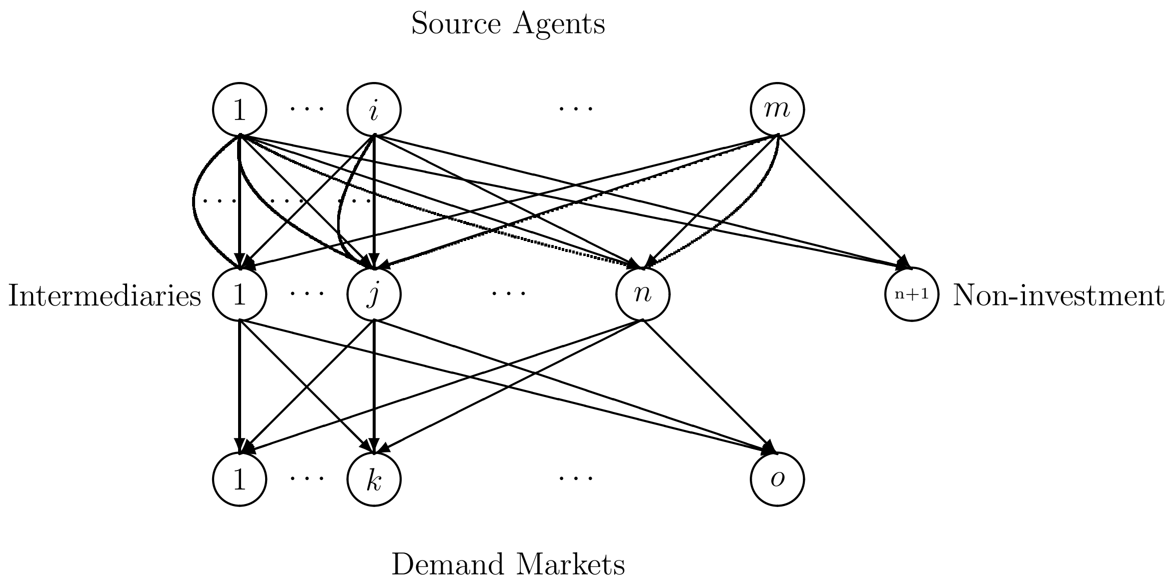


Figure 1: The Network Structure of the Financial Economy with Intermediation and Non-investment Allowed

into the column vector $Q^1 \in R_+^{mnL}$. We assume that each source i has, at his disposal, an amount of funds S_i and we denote the unallocated portion of this amount (and flowing on the link joining node i with node $n+1$) by s_i.

We associate a distinct financial product k with each demand market, bottom-tiered node k, and we let q_{jk} denote the amount of the financial product obtained by consumers at demand market k from intermediary j. We group these "consumption" quantities into the column vector $Q^2 \in R_+^{no}$. The intermediaries, hence, convert the incoming financial flows Q^1 into the outgoing financial flows Q^2.

The notation for the prices is now given. Note that there will be prices associated with each of the tiers of nodes in the financial network. Let ρ_{ijl}^1 denote the price associated with instrument l as quoted by intermediary j to source agent i. We group the first tier prices into the column vector $\rho^1 \in R_+^{mnL}$. Also, let ρ_{jk}^2 denote the price charged by intermediary j for the product at demand market k and group all such prices into the column vector $\rho^2 \in R_+^{no}$. Finally, let ρ_k^3 denote the price of the financial product at the third or bottom-tiered node k and group all such prices into the column vector $\rho^3 \in R_+^o$.

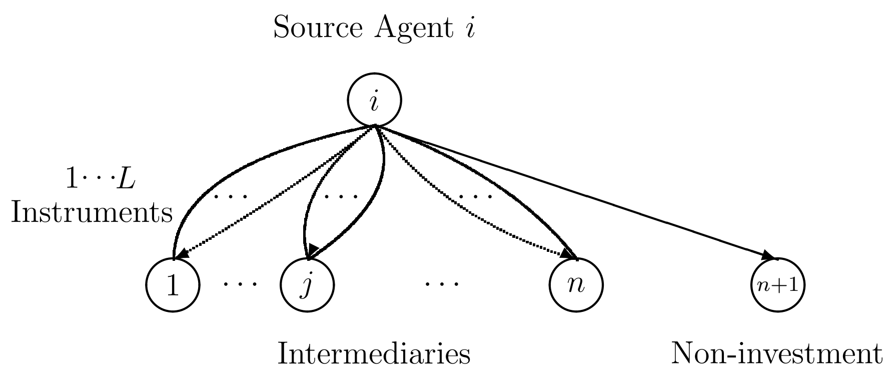


Figure 2: The Network Structure of Source i's Transactions

We now turn to describing the behavior of the various economic agents/decision-makers represented by the three tiers of nodes in Figure 1. We first focus on the top tier agents. We then turn to the intermediaries and, finally, to the demand markets.

2.1 The Behavior of the Agents with Sources of Funds and their Optimality Conditions

In order to describe the allowable transactions of a typical source agent i with the financial intermediaries, we provide a graphical network depiction in Figure 2.

For each source agent i the following conservation of flow equation must hold:

$$\sum_{j=1}^{n} \sum_{l=1}^{L} q_{ijl} \leq S^i, \quad \forall i, \tag{1}$$

that is, the amount of funds allocated by a source agent cannot exceed his financial holdings. In Figure 2, we represent the "slack" associated with constraint (1) for source agent i as the flow on the link joining node i with the non-investment node $n+1$.

Let c_{ijl} denote the transaction cost associated with source i "investing" in instrument l obtained from intermediary j. We consider the situation in which the transaction cost between a source agent and intermediary pair depends upon the volume of financial instrument transactions between the particular pair, that is:

$$c_{ijl} = c_{ijl}(q_{ijl}), \quad \forall i, j, l. \tag{2}$$

We assume that the transaction cost functions given by (2) are continuously differentiable and convex.

Each source agent's uncertainty, or assessment of risk, depends on the volume of financial transactions associated with the source agent. In particular, we assume, as given, a risk function for source agent i is denoted by r^i, such that

$$r^i = r^i(q_i), \quad \forall i, \tag{3}$$

where r^i is assumed to be strictly convex and continuously differentiable. A possible risk function could be constructed as:

$$r^i(q_i) = {q_i}^T V^i q_i, \quad \forall i, \tag{4}$$

where V^i is the variance-covariance matrix associated with agent i, and of dimension $Ln \times Ln$, and is assumed to be symmetric and positive definite. In such a case, the source agent's uncertainty is based on a variance-covariance matrix representing the source agent's assessment of the standard deviation of the prices of the financial instruments (see also Markowitz (1959)).

As noted in the Introduction, we assume that each source agent faces a bicriteria decision-making problem, with the first objective reflecting net revenue maximization and expressed as:

$$\text{Maximize } z^1_{1i} = \sum_{j=1}^{n}\sum_{l=1}^{L} \rho^1_{ijl} q_{ijl} - \sum_{j=1}^{n}\sum_{l=1}^{L} c_{ijl}(q_{ijl}), \tag{5}$$

and the second objective denoting risk minimization and expressed as:

$$\text{Minimize } z^1_{2i} = r^i(q_i), \tag{6}$$

where the superscript "1" refers to a decision-maker in the first (or top) tier of the financial network (cf. Figure 1). Indeed, in (5) the first term in the objective function denotes the revenue whereas the second denotes the total cost associated with transacting for the various instruments with the intermediaries.

Furthermore, a source agent is faced with trading off the gain of one objective against the other objective. The essence of the issue is, "How much achievement on objective z^1_{1i} is the decision-maker willing to give up in order to improve achievement on objective z_{2i}^1 by some amount?" It is rational to assume that financial investors may be risk-averse and for them, the weights of the two objectives may not be equal. For example, a risk-averse investor may be willing to accept a portfolio with a lower mean return if the portfolio has a lower associated risk. In other words, the risk-averse investor may be willing to take on certain risks only if the return is much higher. As discussed in Dong and Nagurney (2001), more attention will generally be given to reduce the risk when the risk is high and this kind of decision rationality argues that the objective function should penalize the states with high risk by imposing a greater weight to z_{2i}^1 associated with high risks than to those z_{2i}^1 with low risks. Now, for definiteness, we introduce several terms. We first derive the general expressions since we will use an analogous weight also for the intermediaries' bicriteria decision-making problems.

Definition 1: Criterion-Dependent Weight

A weight $w_{h\mathcal{I}}^t = w_{h\mathcal{I}}^t(z_{h\mathcal{I}}^t)$ is called a criterion-dependent weight for criterion h and decision-maker $\mathcal{I}$ associated with tier t of the financial network if it is strictly increasing, convex, smooth, and nonnegative.

In this paper, the weighted criterion of concern will be that of risk minimization so we will have that $h = 2$ in the case of the source agents and the intermediaries. Also, since we have that both the source agents and the intermediaries are faced with variables weights, in the case of the source agents: $\mathcal{I} = i$; $i = 1, \ldots, m$, and $t = 1$. In the case of the intermediaries, we will have: $\mathcal{I} = j$; $j = 1, \ldots, n$, and $t = 2$.

Hence, $w_{2i}^1(z_{2i}^1)$ denotes the risk-penalizing weight associated with the value of the risk objective of source agent i. Furthermore, according to Definition 1, w_{2i}^1 is a strictly increasing, convex, smooth, and nonnegative function. We now state the following definition.

Definition 2: Risk-Penalizing Value Function of Source Agent i

A value function U_i for source agent i is called a risk-penalizing value function if

$$U_i = z_{1i}^1 - w_{2i}^1(z_{2i}^1)z_{2i}^1, \tag{7}$$

where $w_{2i}^1(z_{2i}^1)$ is indicated as in Definition 1.

Thus, we can express the optimization problem facing source agent i as:

$$\text{Maximize } U_i(q_i) = \sum_{j=1}^{n} \sum_{l=1}^{L} (\rho_{ijl}^1 q_{ijl} - c_{ijl}(q_{ijl})) - w_{2i}^1(r^i(q_i))r^i(q_i), \tag{8}$$

subject to $q_{ijl} \geq 0$, $\forall j, l$; and the constraint (1) for source agent i.

The expression consisting of the first two terms to the right-hand side of the equal sign in (8) represents the net revenue (which is to be maximized), whereas the last term in (8) represents the weighted dollar value of risk (which is to be minimized) by source agent i. Observe that such an objective function is in concert with those used in classical portfolio optimization (see Markowitz (1952, 1959)) and such a construction has been used in other financial network problems as well (cf. Nagurney and Siokos (1997) and Nagurney and Ke (2003) and the references therein).

We note that value functions have been studied extensively and used for decision problems with multiple criteria (cf. Fishburn (1970), Zeleny (1982), Chankong and Haimes (1983), Yu (1985), Keeney and Raiffa (1993)). Of course, a special example of a constant weight value function is the one with equal weights (see, e.g., Nagurney and Ke (2001)). For some references to the use of value functions with equal weights that have been used in financial applications, see Dong and Nagurney (2001).

We now prove a theorem and then derive the optimality conditions of the source agents. The proof is similar to that found in Dong and Nagurney (2001).

Theorem 1: Concavity

The value function U_i defined in (8) is strictly concave with respect to $q_i \in K_i$, for all i, where $K_i \equiv \{q_i | q_i \in R_+^{nL}$ and satisfies (1) for that $i\}$.

Proof: Let $g_i(z_{2i}^1) = w_{2i}^1(z_{2i}^1)z_{2i}^1$.

Since $w_{2i}^1(z_{2i}^1)$ is assumed to be convex, strictly increasing, and nonnegative, and $z_{2i}^1 > 0$, we have that

$$\frac{dg_i(z_{2i}^1)}{dz_{2i}^1} = \frac{dw_{2i}^1(z_{2i}^1)}{dz_{2i}^1} z_{2i}^1 + w_{2i}^1(z_{2i}^1) > 0 \tag{9}$$

$$\frac{d^2 g_i(z_{2i}^1)}{d{z_{2i}^1}^2} = \frac{d^2 w_{2i}^1(z_{2i}^1)}{d{z_{2i}^1}^2} z_{2i}^1 + 2\frac{dw_{2i}^1(z_{2i}^1)}{dz_{2i}^1} > 0. \tag{10}$$

Combining (9) and (10), we know that g_i is nondecreasing and strictly convex.

z_{2i}^1 is convex with respect to q_i according to the assumptions following (3).

Hence, the composition $h_i \equiv -g_i \circ z_{2i}^1$ is strictly concave with respect to q_i.

Since z_{1i}^1 is linear with respect to q_i, the proof is complete. □

Since U_i is concave with respect to $q_i \in K_i$, the necessary and sufficient conditions (cf. Bazaraa, Sherali, and Shetty (1993)) for $q_i^* \in K_i$ to be optimal for source agent i for problem (8) are that the following inequality is satisfied:

$$-\sum_{j=1}^{n}\sum_{l=1}^{L} \frac{\partial U_i(q_i^*)}{\partial q_{ijl}} \times (q_{ijl} - q_{ijl}^*) \geq 0, \quad \forall q_i \in K_i. \tag{11}$$

We assume now that the top-tiered prices are at the equilibrium values denoted by * (we show how these equilibrium prices are recovered after we construct the complete financial network equilibrium model). Therefore, the optimality conditions for all source agents *simultaneously* can be expressed as the following inequality (see also Nagurney and Ke (2001)): determine $Q^{1*} \in \prod_{i=1}^m K_i$, such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{L}\left[w_{2i}^1(r^i(q_i^*))\frac{\partial r^i(q_i^*)}{\partial q_{ijl}} + \frac{\partial w_{2i}^1(r^i(q_i^*))}{\partial q_{ijl}} r^i(q_i^*) + \frac{\partial c_{ijl}(q_{ijl}^*)}{\partial q_{ijl}} - \rho_{ijl}^{1*}\right] \times \left[q_{ijl} - q_{ijl}^*\right] \geq 0,$$

$$\forall Q^1 \in \prod_{i=1}^{m} K_i. \tag{12}$$

2.2 The Behavior of the Intermediaries and their Optimality Conditions

We now describe the behavior of the financial intermediaries. For a graphical depiction of the transactions associated with intermediary j, see Figure 3. Note that the intermediaries are involved in transactions both with the source agents, as well as with the ultimate consumers associated with the markets for the distinct types of loans/products at the bottom

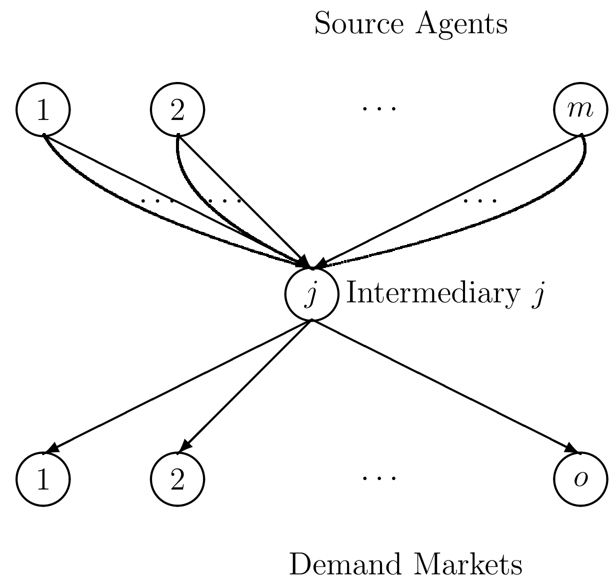


Figure 3: The Network Structure of Intermediary j's Transactions

tier of the network. We assume that an intermediary j is faced with what we term a *handling/conversion* cost, which may include, for example, the cost of converting the incoming financial flows into the financial products at the demand markets. We denote this cost by c_j. In general, we would have that c_j is a function of $\sum_{i=1}^m \sum_{l=1}^L q_{ijl}$, that is, the conversion cost of an intermediary is a function of how much he has obtained from the various source agents and the amounts held by other intermediaries. We may write:

$$c_j = c_j(Q^1), \quad \forall j. \tag{13}$$

We assume that the handling cost functions are continuously differentiable and convex.

The intermediaries also have associated transaction costs in regards to transacting with the source agents. We denote the transaction cost associated with intermediary j transacting with source agent i investing on instrument l by $\hat{c}_{ijl}$ and we assume that the function can depend on the the financial flow q_{ijl}, that is,

$$\hat{c}_{ijl} = \hat{c}_{ijl}(q_{ijl}), \quad \forall i, j, l. \tag{14}$$

Recall that q_{jk} denotes the quantity of the financial product transacted by demand market

k from intermediary j. We assume that an intermediary j incurs a transaction cost c_{jk} associated with transacting with consumers at demand market k, where

$$c_{jk} = c_{jk}(q_{jk}), \quad \forall j, k. \tag{15}$$

We assume that the above transaction cost functions (14) and (15) are also continuously differentiable and convex.

The intermediaries may have risk associated with transacting with the various source agents and with the demand markets. Let r^j denote the risk function associated with intermediary j's transactions. We assume that it is strictly convex and continuously differentiable function and that it depends on the financial transactions q_j, where q_j, without loss of generality, denotes the $(Lm + o)$-dimensional column vector with components: q_{ijl}; $i = 1, \ldots, m$; $l = 1, \ldots, L$; q_{jk}; $k = 1, \ldots, o$. The risk function associated with intermediary j can, hence, be written as:

$$r^j = r^j(q_j), \quad \forall j. \tag{16}$$

A possible risk function for intermediary j could be represented by a variance-covariance matrix denoted by V^j with this matrix being positive definite and of dimensions $(Lm+o) \times (Lm+o)$. Such a matrix reflects the risk associated with transacting for the various financial instruments/products with the various source agents and demand markets. In this case, we may write

$$r^j(q_j) = {q_j}^T V^j q_j, \quad \forall j. \tag{17}$$

Therefore, each intermediary faces a bicriteria decision-making problem, with the first objective expressed as:

$$\text{Maximize } z^2_{1j} = \sum_{k=1}^{o} (\rho^2_{jk} q_{jk} - c_{jk}(q_{jk})) - c_j(Q^1) - \sum_{i=1}^{m} \sum_{l=1}^{L} (\hat{c}_{ijl}(q_{ijl}) + \rho^1_{ijl} q_{ijl}) \tag{18}$$

and denoting the net revenue to be maximized and the second objective expressed as:

$$\text{Minimize } z^2_{2j} = r^j(q_j) \tag{19}$$

and denoting the risk to be minimized. Here the superscript "2" denotes the second tier (the intermediary level of nodes) of the network (cf. Figure 1).

Each intermediary, as was the case for each source agent above, will be faced with a value trade-off problem. Depending upon the risk attitude of the particular intermediary, a variable weight associated with his risk objective can be constructed in a manner similar to that done above for the source agents. We assume that the risk-penalizing weight of intermediary j is denoted by $w^2_{2j}(z^2_{2j})$ and that it is strictly increasing, convex, smooth, and nonnegative for each j.

We now, for completeness, provide the following definition (akin to Definition 1).

Definition 3: Risk-Penalizing Value Function of Intermediary j

A value function U_j for intermediary j is called a risk-penalizing value function if

$$U_j = z^2_{2j} - w^2_{2j}(z^2_{2j})z^2_{2j}, \tag{20}$$

where $w^2_{2j}(z^2_{2j})$ is as defined in Definition 1.

The optimization problem of intermediary j can, thus, be expressed as:

$$\text{Maximize } U_j(q_j) = \sum_{k=1}^{o}(\rho^2_{jk}q_{jk} - c_{jk}(q_{jk})) - c_j(Q^1) - \sum_{i=1}^{m}\sum_{l=1}^{L}(\hat{c}_{ijl}(q_{ijl}) + \rho^1_{ijl}q_{ijl})$$
$$-w^2_{2j}(r^j(q_j))r^j(q_j), \tag{21}$$

subject to:

$$\sum_{k=1}^{o} q_{jk} \leq \sum_{i=1}^{m}\sum_{l=1}^{L} q_{ijl}, \tag{22}$$

and the non-negativity assumption on all the q_{ijl}s and q_{jk}s.

The expression consisting of the first five terms to the right-hand side of the equal sign in (21) represents the net revenue (to be maximized), whereas the last term in (21) represents the weighted dollar value of risk (to be minimized) associated with the intermediary j's risk attitude. Constraint (22) reflects that the financial intermediary cannot allocate more financial flows than it has as financial holdings (obtained from the various source agents).

We can apply an analogous proof to that of Theorem 1 to establish that U_j is strictly concave with respect to $q_j \in R^{Lm+o}_+$ under the above stated assumptions on the transaction cost functions and the risk function for intermediary j.

We assume now that the financial intermediaries can compete, with the governing optimality/equilibrium concept underlying noncooperative behavior being that of Nash (1950, 1951), which states that each decision-maker or agent will determine his optimal strategies, given the optimal ones of his competitors. The optimality conditions for all financial intermediaries simultaneously, under the above stated assumptions, can be succinctly expressed as (see also Bertsekas and Tsitsiklis (1989), Bazaraa, Sherali, and Shetty (1993), Gabay and Moulin (1980), Dafermos and Nagurney (1987), and Nagurney and Ke (2001)): determine $(Q^{1*}, Q^{2*}, \gamma^*) \in R_+^{Lmn+no+n}$, such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{L}\left[w_{2j}^2(r^j(q_j^*))\frac{\partial r^j(q_j^*)}{\partial q_{ijl}} + \frac{\partial w_{2j}^2(r^j(q_j^*))}{\partial q_{ijl}}r^j(q_j^*) + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \rho_{ijl}^{1*} + \frac{\partial \hat{c}_{ijl}(q_{ijl}^*)}{\partial q_{ijl}} - \gamma_j^*\right] \times \left[q_{ijl} - q_{ijl}^*\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[w_{2j}^2(r^j(q_j^*))\frac{\partial r^j(q_j^*)}{\partial q_{jk}} + \frac{\partial w_{2j}^2(r^j(q_j^*))}{\partial q_{jk}}r^j(q_j^*) + \frac{\partial c_{jk}(q_{jk}^*)}{\partial q_{jk}} - \rho_{jk}^{2*} + \gamma_j^*\right] \times \left[q_{jk} - q_{jk}^*\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{L} q_{ijl}^* - \sum_{k=1}^{o} q_{jk}^*\right] \times \left[\gamma_j - \gamma_j^*\right] \geq 0, \quad \forall (Q^1, Q^2, \gamma) \in R_+^{mnL+no+n}, \tag{23}$$

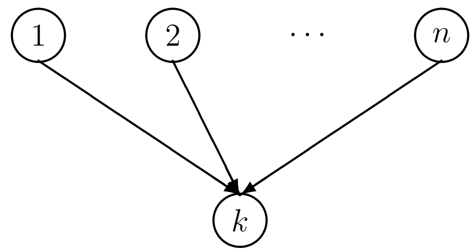
where γ_j is the Lagrange multiplier associated with constraint (22) (see Bazaraa, Sherali, and Shetty (1993)), γ is the n-dimensional column vector of Lagrange multipliers of all the intermediaries, and the top and middle tier prices (without loss of generality) are at their equilibrium values (more discussion on the pricing mechanism follows at the end of this section).

2.3 The Consumers at the Demand Markets and the Equilibrium Conditions

In terms of the financial flows between the intermediaries and the demand markets, we know that, ultimately, the flows accepted by the consumers at the demand markets must coincide with those "shipped out" by the former decision-makers. The consumers at the demand markets take into account in making their consumption decisions not only the price charged for the financial products but also their transaction costs associated with obtaining the financial products. See Figure 4 for the transactions associated with demand market k.

Let $\hat{c}_{jk}$ denote the transaction cost associated with obtaining the financial product at demand market k from intermediary j. We assume that the transaction cost is continuous

Intermediaries



Demand Market

Figure 4: The Network Structure of Transactions at Demand Market k

and is of the general form:

$$\hat{c}_{jk} = \hat{c}_{jk}(Q^2), \quad \forall j, k, \tag{24}$$

that is, the cost of transacting, as perceived by consumers at a given demand market, can depend upon the volume of financial flows between all the intermediary/demand market pairs. The generality of this cost structure enables the modeling of competition on the demand side.

Let ρ_k^3 denote the price of the financial product at demand market k and group the demand market prices into the column vector $\rho^3 \in R_+^o$. Further, let d_k denote the demand for the product at demand market k. We assume continuous demand functions of the general form:

$$d_k = d_k(\rho^3), \quad \forall k. \tag{25}$$

The equilibrium conditions (with the middle tier prices set at their equilibrium values) for demand market k (cf. Nagurney and Ke (2001, 2003), thus, take the form: for all intermediaries: j; $j = 1, ..., n$:

$$\rho_{jk}^{2*} + \hat{c}_{jk}(Q^{2*}) \begin{cases} = \rho_k^{3*}, & \text{if} \quad q_{jk}^* > 0 \\ \geq \rho_k^{3*}, & \text{if} \quad q_{jk}^* = 0. \end{cases} \tag{26}$$

In addition, we must have that

$$d_k(\rho^{3*}) \begin{cases} = \sum_{j=1}^{n} q^*_{jk}, & \text{if} \quad \rho^{3*}_k > 0 \\ \leq \sum_{j=1}^{n} q^*_{jk}, & \text{if} \quad \rho^{3*}_k = 0. \end{cases} \tag{27}$$

Conditions (26) state that the consumers at demand market k will purchase the financial product from intermediary j if the price charged by the intermediary plus the transaction cost (from the perspective of the consumers) does not exceed the price that the consumers are willing to pay for the product. Condition (27), on the other hand, states that, if the price that the consumers are willing to pay for the financial product is positive, then the quantity of the product at the demand market is precisely equal to the demand.

In equilibrium, conditions (26) and (27) will have to hold for all demand markets and these, in turn, can be expressed also as an inequality akin to (12) and (23) given by: determine $(Q^{2^*}, \rho^{3*}) \in R^{no+o}$, such that

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\rho^{2*}_{jk} + \hat{c}_{jk}(Q^{2^*}) - \rho^{3*}_k\right] \times \left[q_{jk} - q^*_{jk}\right] + \sum_{k=1}^{o}\left[\sum_{j=1}^{n} q^*_{jk} - d_k(\rho^{3*})\right] \times \left[\rho^3_k - \rho^{3*}_k\right] \geq 0,$$

$$\forall (Q^2, \rho^3) \in R^{no+o}_+. \tag{28}$$

2.4 The Equilibrium Conditions of the Financial Network with Variable Weights

In equilibrium, the financial flows that the source agents transact with the intermediaries must be equal to those that the intermediaries accept from the source agents. In addition, the amounts that are obtained by the consumers at the demand markets must be equal to the volume that the intermediaries transact with the demand markets. Hence, the equilibrium financial flow and price pattern must satisfy the sum of the optimality conditions (12) and (23), and the equilibrium conditions (28), in order to formalize the agreements between tiers of the financial network.

We now state this formally:

Definition 4: Financial Network Equilibrium with Intermediation and Variable Weights

The equilibrium state of the financial network with intermediation and variable weights is one where the financial flows between tiers coincide and the financial flows and prices satisfy the sum of conditions (12), (23), and (28).

We now establish the following:

Theorem 2: Variational Inequality Formulation

The equilibrium state governing the financial network with intermediation and variable weights is equivalent to the solution of the variational inequality given by:
determine $(Q^{1*}, Q^{2*}, \gamma^*, \rho^{3*}) \in \mathcal{K}$, *satisfying*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{L}\Bigg[w_{2i}^1(r^i(q_i^*))\frac{\partial r^i(q_i^*)}{\partial q_{ijl}} + \frac{\partial w_{2i}^1(r^i(q_i^*))}{\partial q_{ijl}}r^i(q_i^*) + w_{2j}^2(r^j(q_j^*))\frac{\partial r^j(q_j^*)}{\partial q_{ijl}} + \frac{\partial w_{2j}^2(r^j(q_j^*))}{\partial q_{ijl}}r^j(q_j^*)$$

$$+\frac{\partial c_{ijl}(q_{ijl}^*)}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q_{ijl}^*)}{\partial q_{ijl}} - \gamma_j^*\Bigg] \times \left[q_{ijl} - q_{ijl}^*\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\Bigg[w_{2j}^2(r^j(q_j^*))\frac{\partial r^j(q_j^*)}{\partial q_{jk}} + \frac{\partial w_{2j}^2(r^j(q_j^*))}{\partial q_{jk}}r^j(q_j^*) + \frac{\partial c_{jk}(q_{jk}^*)}{\partial q_{jk}} + \hat{c}_{jk}(Q^{2*}) + \gamma_j^* - \rho_k^{3*}\Bigg]\times\left[q_{jk} - q_{jk}^*\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{L} q_{ijl}^* - \sum_{k=1}^{o} q_{jk}^*\right] \times \left[\gamma_j - \gamma_j^*\right] + \sum_{k=1}^{o}\left[\sum_{j=1}^{n} q_{jk}^* - d_k(\rho^{3*})\right] \times \left[\rho_k^3 - \rho_k^{3*}\right] \geq 0,$$

$$\forall (Q^1, Q^2, \gamma, \rho^3) \in \mathcal{K}, \qquad (29)$$

where $\mathcal{K} \equiv \{\prod_{i=1}^m K_i \times R_+^{no+n+o}\}$.

Proof: We first establish that the equilibrium state implies the satisfaction of variational inequality (29). Indeed, the summation of (12), (23), and (28) yields, after algebraic simplification, the variational inequality (29).

We now establish the converse, that is, that a solution to variational inequality (29) satisfies the sum of conditions (12), (23), and (28), and is, hence, an equilibrium according to Definition 4.

To inequality (29), add the term: $-\rho_{ijl}^{1*}+\rho_{ijl}^{1*}$ to the term in the first set of brackets, preceding the multiplication sign. Similarly, add the term: $-\rho_{jk}^{2*}+\rho_{jk}^{2*}$ to the term preceding the second multiplication sign. Such "terms" do not change the inequality since they are identically equal to zero, with the resulting inequality of the form:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{L}\Bigg[w_{2i}^{1}(r^i(q_i^*))\frac{\partial r^i(q_i^*)}{\partial q_{ijl}}+\frac{\partial w_{2i}^{1}(r^i(q_i^*))}{\partial q_{ijl}}r^i(q_i^*)+w_{2j}^{2}(r^j(q_j^*))\frac{\partial r^j(q_j^*)}{\partial q_{ijl}}+\frac{\partial w_{2j}^{2}(r^j(q_j^*))}{\partial q_{ijl}}r^j(q_j^*)$$
$$+\frac{\partial c_{ijl}(q_{ijl}^*)}{\partial q_{ijl}}+\frac{\partial c_j(Q^{1*})}{\partial q_{ijl}}+\frac{\partial \hat{c}_{ijl}(q_{ijl}^*)}{\partial q_{ijl}}-\gamma_j^*-\rho_{ijl}^{1*}+\rho_{ijl}^{1*}\Bigg]\times\Big[q_{ijl}-q_{ijl}^*\Big]$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\Bigg[w_{2j}^{2}(r^j(q_j^*))\frac{\partial r^j(q_j^*)}{\partial q_{jk}}+\frac{\partial w_{2j}^{2}(r^j(q_j^*))}{\partial q_{jk}}r^j(q_j^*)+\frac{\partial c_{jk}(q_{jk}^*)}{\partial q_{jk}}+\hat{c}_{jk}(Q^{2*})+\gamma_j^*-\rho_k^{3*}-\rho_{jk}^{2*}+\rho_{jk}^{2*}\Bigg]$$
$$\times\Big[q_{jk}-q_{jk}^*\Big]+\sum_{j=1}^{n}\Bigg[\sum_{i=1}^{m}\sum_{l=1}^{L}q_{ijl}^*-\sum_{k=1}^{o}q_{jk}^*\Bigg]\times\Big[\gamma_j-\gamma_j^*\Big]$$
$$+\sum_{k=1}^{o}\Bigg[\sum_{j=1}^{n}q_{jk}^*-d_k(\rho^{3*})\Bigg]\times\Big[\rho_k^3-\rho_k^{3*}\Big]\geq 0,\quad \forall(Q^1,Q^2,\gamma,\rho_3)\in\mathcal{K}, \tag{30}$$

which, in turn, can be rewritten as:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{L}\Bigg[w_{2i}^{1}(r^i(q_i^*))\frac{\partial r^i(q_i^*)}{\partial q_{ijl}}+\frac{\partial w_{2i}^{1}(r^i(q_i^*))}{\partial q_{ijl}}r^i(q_i^*)+\frac{\partial c_{ijl}(q_{ijl}^*)}{\partial q_{ijl}}-\rho_{ijl}^{1*}\Bigg]\times\Big[q_{ijl}-q_{ijl}^*\Big]$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{L}\Bigg[w_{2j}^{2}(r^j(q_j^*))\frac{\partial r^j(q_j^*)}{\partial q_{ijl}}+\frac{\partial w_{2j}^{2}(r^j(q_j^*))}{\partial q_{ijl}}r^j(q_j^*)+\frac{\partial c_j(Q^{1*})}{\partial q_{ijl}}+\rho_{ijl}^{1*}+\frac{\partial \hat{c}_{ijl}(q_{ijl}^*)}{\partial q_{ijl}}-\gamma_j^*\Bigg]\times\Big[q_{ijl}-q_{ijl}^*\Big]$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\Bigg[w_{2j}^{2}(r^j(q_j^*))\frac{\partial r^j(q_j^*)}{\partial q_{jk}}+\frac{\partial w_{2j}^{2}(r^j(q_j^*))}{\partial q_{jk}}r^j(q_j^*)+\frac{\partial c_{jk}(q_{jk}^*)}{\partial q_{jk}}-\rho_{jk}^{2*}+\gamma_j^*\Bigg]\times\Big[q_{jk}-q_{jk}^*\Big]$$
$$+\sum_{j=1}^{n}\Bigg[\sum_{i=1}^{m}\sum_{l=1}^{L}q_{ijl}^*-\sum_{k=1}^{o}q_{jk}^*\Bigg]\times\Big[\gamma_j-\gamma_j^*\Big]+\sum_{j=1}^{n}\sum_{k=1}^{o}\Big[\rho_{jk}^{2*}+\hat{c}_{jk}(Q^{2*})-\rho_k^{3*}\Big]\times\Big[q_{jk}-q_{jk}^*\Big]$$
$$+\sum_{k=1}^{o}\Bigg[\sum_{j=1}^{n}q_{jk}^*-d_k(\rho^{3*})\Bigg]\times\Big[\rho_k^3-\rho_k^{3*}\Big]\geq 0,\quad \forall(Q^1,Q^2,\gamma,\rho^3)\in\mathcal{K}. \tag{31}$$

But inequality (31) is equivalent to the price and financial flow pattern satisfying the sum of conditions (12), (23), and (28). The proof is complete. □

For easy reference in the subsequent sections, variational inequality problem (29) can be rewritten in standard variational inequality form (cf. Nagurney (1999)) as follows:

$$\langle F(X^*)^T, X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{32}$$

where $X \equiv (Q^1, Q^2, \gamma, \rho^3)$, and $F(X) \equiv (F_{ijl}, F_{jk}, F_j, F_k)_{i=1,\ldots,m; j=1,\ldots,n; l=1,\ldots,L; k=1,\ldots,o}$, and the specific components of F given by the functional terms preceding the multiplication signs in (29), respectively. The term $\langle \cdot, \cdot \rangle$ denotes the inner product in N-dimensional Euclidean space.

We now consider a special case of variational inequality (29) that has appeared in the literature. The proof of the corollary below is immediate.

Corollary 1

Assume that the risk functions associated with the source agents are of the form (4) for all source agents i; the risk functions associated with the intermediaries are of the form (17) for all intermediaries j but depend only upon Q^2 (and not upon Q^1). Also, assume that $w^t_{h\mathcal{I}} = 1$ for $t = 1, 2$; $h = 2$, and $\mathcal{I}$=i, j for $i = 1, \ldots, m$; $j = 1, \ldots, n$. The variational inequality (29) then collapses to the variational inequality obtained in Nagurney and Ke (2001) governing the financial network equilibrum problem with intermediation with equal unit weights and risk functions of the special variance-covariance form above.

We now discuss how to recover the prices ρ^{1*}_{ijl}, for all i, j, l, and ρ^{2*}_{jk}, for all j, k, from the solution of variational inequality (29). Observe that these prices do not appear in variational inequality (29). However, they do play an important role in terms of *pricing* of the various financial instruments/products. Note that from (23), if $q^*_{jk} > 0$, for some j, and k, then ρ^{2*}_{jk} is precisely equal to $\left[w^2_{2j}(r^j(q^*_j))\frac{\partial r^j(q^*_j)}{\partial q_{jk}} + \frac{\partial w^2_{2j}(r^j(q^*_j))}{q_{jk}} r^j(q^*_j) + \frac{\partial c_{jk}(q^*_{jk})}{\partial q_{jk}} + \gamma^*_j\right]$ or, equivalently, to (cf. (26)) $[\rho^{3*}_k - \hat{c}_{jk}(Q^{2*})]$. The prices ρ^{1*}_{ijl}, in turn (cf. also (12)), can be obtained by finding a $q^*_{ijl} > 0$, and then setting $\rho^{1*}_{ijl} = \left[w^1_{2i}(r^i(q^*_i))\frac{\partial r^i(q^*_i)}{\partial q_{ijl}} + \frac{\partial w^1_{2i}(r^i(q^*_i))}{\partial q_{ijl}} r^i(q^*_i) + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}}\right]$, or, equivalently (see (23)), to $\left[\gamma^*_j - w^2_{2j}(r^j(q^*_j))\frac{\partial r^j(q^*_j)}{\partial q_{ijl}} - \frac{\partial w^2_{2j}(r^j(q^*_j))}{\partial q_{ijl}} r^j(q^*_j) - \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} - \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}}\right]$, for all such i, j, l.

Moreover, it is easy to establish that if the top and middle tier prices are set as above, then the optimality conditions (12) and (23) and the equilibrium conditions (28) each hold (separately).

Hence, using the variational inequality formulation one cannot only (as we shall further demonstrate in Sections 4 and 5) determine the equilibrium financial flows between the tiers of the financial network but, in addition, the equilibrium prices associated with the financial products at the demand markets, ρ^{3*}, the equilibrium prices at the source agents, ρ^{1*}, and at the intermediaries, ρ^{2*}.

We note that the optimality conditions (12) for the top tier of decision-makers are expressed as a variational inequality since both the optimality/equilibrium conditions for all the intermediaries (which compete in the sense of Nash), as well as the equilibrium conditions (28) governing the demand markets take on variational inequality formulations. Indeed, since we have no symmetry assumptions on the handling cost functions, the transaction cost functions between intermediaries and the demand markets, as well as the demand functions at the demand markets, we need to appeal to a variational inequality formulation for the equilibrium associated with both the intermediaries as well as the demand markets(see also, e.g., Gabay and Moulin (1980) and Nagurney (1999)). By formulating the optimality conditions associated with the top tier of decision-makers also as a variational inequality, the equilibrium conditions of the entire financial network with intermediation (in which there can be competition between intermediaries and also between demand markets) take on a variational inequality form through the summation of the optimality/equilibrium conditions associated with each of the tiers of the financial network.

3. Qualitative Properties

In this section, we provide some qualitative properties of the solution to the variational inequality (29). In particular, we derive existence and uniqueness results. We also investigate properties of the function F (cf. (32)) that enters the variational inequality of interest here.

Since the feasible set is not compact we cannot derive existence simply from the assumption of continuity of the functions. Nevertheless, we can impose a rather weak condition to guarantee existence of a solution pattern. Let

$$\mathcal{K}_b = \{(Q^1, Q^2, \gamma, \rho^3) | 0 \leq Q^1 \leq b_1;\ 0 \leq Q^2 \leq b_2;\ 0 \leq \gamma \leq b_3;\ 0 \leq \rho^3 \leq b_4\},$$

where $b = (b_1, b_2, b_3, b_4) \geq 0$ and $Q^1 \leq b_1; Q^2 \leq b_2; \gamma \leq b_3; \rho^3 \leq b_4$ means that $q_{ijl} \leq b_1$; $q_{jk} \leq b_2$; $\gamma_j \leq b_3$; and $\rho^3_k \leq b_4$ for all i, j, l, k. Then $\mathcal{K}_b$ is a bounded closed convex subset of $R^{mnL+no+n+o}$. Thus, the following variational inequality

$$\langle F(X^b)^T, X - X^b \rangle \geq 0, \quad \forall\, X^b \in \mathcal{K}_b, \tag{33}$$

admits at least one solution $X^b \in \mathcal{K}_b$, from the standard theory of variational inequalities, since $\mathcal{K}_b$ is compact and F is continuous. Following Kinderlehrer and Stampacchia (1980) (see also Theorem 1.5 in Nagurney (1999)), we then have:

Theorem 3

Variational inequality (29) admits a solution if and only if there exists a $b > 0$, such that variational inequality (33) admits a solution in $\mathcal{K}_b$ with

$$Q^{1b} < b_1, \quad Q^{2b} < b_2, \quad \gamma^b < b_3, \quad \rho^{3b} < b_4. \tag{34}$$

Theorem 4: Existence

Suppose that there exist positive constants M, N, R with $R > 0$, such that:

$$w^1_{2i}(r^i(q_i))\frac{\partial r^i(q_i)}{\partial q_{ijl}} + \frac{\partial w^1_{2i}(r^i(q_i))}{\partial q_{ijl}} r^i(q_i) + w^2_{2j}(r^j(q_j))\frac{\partial r^j(q_j)}{\partial q_{ijl}} + \frac{\partial w^2_{2j}(r^j(q_j))}{\partial q_{ijl}} r^j(q_j)$$

$$+\frac{\partial c_{ijl}(q_{ijl})}{\partial q_{ijl}} + \frac{\partial c_j(Q^1)}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q_{ijl})}{\partial q_{ijl}} \geq M, \quad \forall Q^1 \text{ with } q_{ijl} \geq N, \quad \forall i, j, l, \tag{35}$$

$$w_{2j}^2(r^j(q_j))\frac{\partial r^j(q_j)}{\partial q_{jk}} + \frac{\partial w_{2j}^2(r^j(q_j))}{\partial q_{jk}} r^j(q_j) + \frac{\partial c_{jk}(q_{jk})}{\partial q_{jk}} + \hat{c}_{jk}(Q^2) \geq M,$$

$$\forall Q^2 \ \textit{with} \ q_{jk} \geq N, \quad \forall j, k, \tag{36}$$

$$d_k(\rho^3) \leq N, \quad \forall \rho^3 \ \textit{with} \ \rho_k^3 > R, \quad \forall k. \tag{37}$$

Then variational inequality (29); equivalently, variational inequality (32), admits at least one solution.

Proof: Follows using analogous arguments as the proof of existence for Proposition 1 in Nagurney and Zhao (1993). □

Assumptions (35) to (37) are economically reasonable, since when the financial flow between a source agent and intermediary is large, we can expect the corresponding sum of the associated marginal costs of handling and transaction from either the source agent's or the intermediary's perspectives to exceed a positive lower bound. Moreover, in the case where the demand price of the financial product as perceived by consumers at a demand market is high, we can expect that the demand for the financial product at the demand market to not exceed a positive bound.

We now establish additional qualitative properties both of the function F that enters the variational inequality problem (cf. (32) and (29)), as well as uniqueness of the equilibrium pattern. Monotonicity and Lipschitz continuity of the function F (under assumptions given below) will be utilized in Section 4 for proving convergence of the algorithmic scheme.

Theorem 5: Monotonicity

Assume that the risk functions r^i; $i = 1, \ldots, m$; and r^j; $j = 1, \ldots, n$, are strictly convex and that the c_{ijl}, c_j, $\hat{c}_{ijl}$, and c_{jk} functions are convex; the $\hat{c}_{jk}$ functions are monotone increasing, and the d_k functions are monotone decreasing functions, for all i, j, k, l. Assume also that the variable weights are all positive. Then the vector function F that enters the variational inequality (32) is monotone, that is,

$$\langle (F(X') - F(X''))^T, X' - X'' \rangle \geq 0, \quad \forall X', X'' \in \mathcal{K}. \tag{38}$$

Proof: From the definition of $F(X)$, the left-hand side of inequality (38) is:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{L}\Bigg[w^1_{2i}(r^i(q^{'}_i))\frac{\partial r^i(q^{'}_i)}{\partial q_{ijl}}+\frac{\partial w^1_{2i}(r^i(q^{'}_i))}{\partial q_{ijl}}r^i(q^{'}_i)+w^2_{2j}(r^j(q^{'}_j))\frac{\partial r^j(q^{'}_j)}{\partial q_{ijl}}+\frac{\partial w^2_{2j}(r^j(q^{'}_j))}{\partial q_{ijl}}r^j(q^{'}_j)$$

$$+\frac{\partial c_{ijl}(q^{'}_{ijl})}{\partial q_{ijl}}+\frac{\partial c_j(Q^{1'})}{\partial q_{ijl}}+\frac{\partial \hat{c}_{ijl}(q^{'}_{ijl})}{\partial q_{ijl}}-\gamma^{'}_j$$

$$-(w^1_{2i}(r^i(q^{''}_i))\frac{\partial r^i(q^{''}_i)}{\partial q_{ijl}}+\frac{\partial w^1_{2i}(r^i(q^{''}_i))}{\partial q_{ijl}}r^i(q^{''}_i)+w^2_{2j}(r^j(q^{''}_j))\frac{\partial r^j(q^{''}_j)}{\partial q_{ijl}}+\frac{\partial w^2_{2j}(r^j(q^{''}_j))}{q_{ijl}}r^j(q^{''}_j)$$

$$+\frac{\partial c_{ijl}(q^{''}_{ijl})}{\partial q_{ijl}}+\frac{\partial c_j(Q^{1''})}{\partial q_{ijl}}+\frac{\partial \hat{c}_{ijl}(q^{''}_{ijl})}{\partial q_{ijl}}-\gamma^{''}_j)\Bigg]\times\left[q^{'}_{ijl}-q^{''}_{ijl}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\Bigg[w^2_{2j}(r^j(q^{'}_j))\frac{\partial r^j(q^{'}_j)}{\partial q_{jk}}+\frac{\partial w^2_{2j}(r^j(q^{'}_j))}{\partial q_{jk}}r^j(q^{'}_j)+\frac{\partial c_{jk}(q^{'}_{jk})}{\partial q_{jk}}+\hat{c}_{jk}(Q^{2'})+\gamma^{'}_j-\rho^{3'}_k$$

$$-(w^2_{2j}(r^j(q^{''}_j))\frac{\partial r^j(q^{''}_j)}{\partial q_{jk}}+\frac{\partial w^2_{2j}(r^j(q^{''}_j))}{\partial q_{jk}}r^j(q^{''}_j)+\frac{\partial c_{jk}(q^{''}_{jk})}{\partial q_{jk}}+\hat{c}_{jk}(Q^{2''})+\gamma^{''}_j-\rho^{3''}_k)\Bigg]\times\left[q^{'}_{jk}-q^{''}_{jk}\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{L}q^{'}_{ijl}-\sum_{k=1}^{o}q^{'}_{jk}-(\sum_{i=1}^{m}\sum_{l=1}^{L}q^{''}_{ijl}-\sum_{k=1}^{o}q^{''}_{jk})\right]\times\left[\gamma^{'}_j-\gamma^{''}_j\right]$$

$$+\sum_{k=1}^{o}\left[\sum_{j=1}^{n}q^{'}_{jk}-d_k(\rho^{3'})-(\sum_{j=1}^{n}q^{''}_{jk}-d_k(\rho^{3''}))\right]\times\left[\rho^{3'}_k-\rho^{3''}_k\right]. \qquad (39)$$

After simplifying (39), we obtain

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{L}\Bigg[w^1_{2i}(r^i(q^{'}_i))\frac{\partial r^i(q^{'}_i)}{\partial q_{ijl}}+\frac{\partial w^1_{2i}(r^i(q^{'}_i))}{\partial q_{ijl}}r^i(q^{'}_i)+w^2_{2j}(r^j(q^{'}_j))\frac{\partial r^j(q^{'}_j)}{\partial q_{ijl}}+\frac{\partial w^2_{2j}(r^j(q^{'}_j))}{\partial q_{ijl}}r^j(q^{'}_j)$$

$$+\frac{\partial c_{ijl}(q^{'}_{ijl})}{\partial q_{ijl}}+\frac{\partial c_j(Q^{1'})}{\partial q_{ijl}}+\frac{\partial \hat{c}_{ijl}(q^{'}_{ijl})}{\partial q_{ijl}}$$

$$-(w^1_{2i}(r^i(q^{''}_i))\frac{\partial r^i(q^{''}_i)}{\partial q_{ijl}}+\frac{\partial w^1_{2i}(r^i(q^{''}_i))}{\partial q_{ijl}}r^i(q^{''}_i)+w^2_{2j}(r^j(q^{''}_j))\frac{\partial r^j(q^{''}_j)}{\partial q_{ijl}}+\frac{\partial w^2_{2j}(r^j(q^{''}_j))}{q_{ijl}}r^j(q^{''}_j)$$

$$+\frac{\partial c_{ijl}(q^{''}_{ijl})}{\partial q_{ijl}}+\frac{\partial c_j(Q^{1''})}{\partial q_{ijl}}+\frac{\partial \hat{c}_{ijl}(q^{''}_{ijl})}{\partial q_{ijl}})\Bigg]\times\left[q^{'}_{ijl}-q^{''}_{ijl}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[w_{2j}^2(r^j(q_j^{'}))\frac{\partial r^j(q_j^{'})}{\partial q_{jk}}+\frac{\partial w_{2j}^2(r^j(q_j^{'}))}{\partial q_{jk}}r^j(q_j^{'})+\frac{\partial c_{jk}(q_{jk}^{'})}{\partial q_{jk}}+\hat{c}_{jk}(Q^{2'})\right.$$

$$\left.-(w_{2j}^2(r^j(q_j^{''}))\frac{\partial r^j(q_j^{''})}{\partial q_{jk}}+\frac{\partial w_{2j}^2(r^j(q_j^{''}))}{\partial q_{jk}}r^j(q_j^{''})+\frac{\partial c_{jk}(q_{jk}^{''})}{\partial q_{jk}}+\hat{c}_{jk}(Q^{2''}))\right]\times\left[q_{jk}^{'}-q_{jk}^{''}\right]$$

$$+\sum_{k=1}^{o}\left[d_k(\rho^{3''})-d_k(\rho^{3'})\right]\times\left[\rho_k^{3'}-\rho_k^{3''}\right]. \quad (40)$$

It is easy to verify that under the above imposed assumptions the term in (40) is greater than or equal to zero. □

Monotonicity plays a role in the qualitative analysis of variational inequality problems similar to that played by convexity in the context of optimization problems.

Since the proof of Theorems 6 below is similar to that of Theorem 5, it is omitted here.

Theorem 6: Strict Monotonicity

Assume all the conditions of Theorem 5. In addition, suppose that one of the families of convex functions $c_{ijl}; i=1,...,m;\ j=1,...,n;\ l=1,...,L;\ c_j;\ j=1,...,n;\ \hat{c}_{ijl};\ i=1,\dots,m;$ $j=1,\dots,n;\ l=1,\dots,L;$ and $c_{jk};\ j=1,\dots,n;\ k=1,\dots,o,$ is a family of strictly convex functions. Suppose also that $\hat{c}_{jk}; j=1,...,n;\ k=1,...,o,$ and -d_k; $k=1,...,o,$ are strictly monotone. Then, the vector function F that enters the variational inequality (32) is strictly monotone, with respect to (Q^1,Q^2,ρ^3), that is, for any two X',X'' with $(Q^{1'},Q^{2'},\rho^{3'})\neq(Q^{1''},Q^{2''},\rho^{3''})$

$$\langle (F(X')-F(X''))^T, X'-X''\rangle > 0. \quad (41)$$

Theorem 7: Uniqueness

Assuming the conditions of Theorem 6, there must be a unique financial flow pattern (Q^{1*},Q^{2*}), and a unique demand price vector ρ^{3*} satisfying the equilibrium conditions of the financial network with variable weights. In other words, if the variational inequality (29) admits a solution, then that is the only solution in (Q^1,Q^2,ρ^3).

Proof: Under the strict monotonicity result of Theorem 6, uniqueness follows from the standard variational inequality theory (cf. Kinderlehrer and Stampacchia (1980)). $\square$

Theorem 8: Lipschitz Continuity

The function that enters the variational inequality problem (32) is Lipschitz continuous, that is,

$$\|F(X') - F(X'')\| \leq L\|X' - X''\|, \quad \forall X', X'' \in \mathcal{K}, \text{ where } L > 0, \tag{42}$$

under the following condition:

the F_{ijl}, F_{jk}, F_j, and F_k terms comprising the function F in (32) have bounded first order derivatives.

Proof: The result is direct by applying a mid-value theorem from calculus to the vector function F that enters the variational inequality problem (32). $\square$

In the next section, we utilize the monotonicity and Lipschitz continuity properties in order to establish the convergence of the algorithm for the solution of the equilibrium financial flows and prices satisfying variational inequality (29).

4. The Algorithm

In this section, we consider the computation of solutions to variational inequality (29). We will apply the modified projection method of Korpelevich (1977), which is guaranteed to solve any variational inequality problem in standard form. The realization of the modified projection method for the variational inequality (29) (for further details in the context of a more specialized model, see Nagurney and Ke (2001)) is as follows.

Step 0: Initialization Step

Set $(Q^{10}, Q^{20}, \gamma^0, \rho^{3^0}) \in \mathcal{K}$. Let $\tau = 1$, where τ is the iteration counter. Set α so that $0 < \alpha \leq \frac{1}{L}$, where L is the Lipschitz constant (cf. (42)) for the problem.

Step 1: Computation Step

Compute $(\bar{Q}^{1\tau}, \bar{Q}^{2\tau}, \bar{\gamma}^\tau, \bar{\rho}^{3\tau}) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{L}\Bigg[\bar{q}_{ijl}^{\tau} + \alpha(w_{2i}^1(r^i(q_i^{\tau-1}))\frac{\partial r^i(q_i^{\tau-1})}{\partial q_{ijl}} + \frac{\partial w_{2i}^1(r^i(q_i^{\tau-1}))}{\partial q_{ijl}} r^i(q_i^{\tau-1}) + w_{2j}^2(r^j(q_j^{\tau-1}))\frac{\partial r^j(q_j^{\tau-1})}{\partial q_{ijl}}$$

$$+\frac{\partial w_j^2(r^j(q_j^{\tau-1}))}{\partial q_{ijl}} r^j(q_j^{\tau-1}) + \frac{\partial c_{ijl}(q_{ijl}^{\tau-1})}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1\tau-1})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q_{ijl}^{\tau-1})}{\partial q_{ijl}} - \gamma_j^{\tau-1}) - q_{ijl}^{\tau-1}\Bigg] \times \left[q_{ijl} - \bar{q}_{ijl}^{\tau}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\Bigg[\bar{q}_{jk}^{\tau} + \alpha(w_{2j}^2(r^j(q_j^{\tau-1}))\frac{\partial r^j(q_j^{\tau-1})}{\partial q_{jk}} + \frac{\partial w_{2j}^2(r^j(q_j^{\tau-1}))}{\partial q_{jk}} r^j(q_j^{\tau-1}) + \frac{\partial c_{jk}(q_{jk}^{\tau-1})}{\partial q_{jk}} + \hat{c}_{jk}(Q^{2\tau-1})$$

$$+\gamma_j^{\tau-1} - \rho_k^{3^{\tau-1}}) - q_{jk}^{\tau-1}\Bigg] \times \left[q_{jk} - \bar{q}_{jk}^{\tau}\right]$$

$$+\sum_{j=1}^{n}\left[\bar{\gamma}_j^{\tau} + \alpha(\sum_{i=1}^{m}\sum_{l=1}^{L} q_{ijl}^{\tau-1} - \sum_{k=1}^{o} q_{jk}^{\tau-1}) - \gamma_j^{\tau-1}\right] \times \left[\gamma_j - \bar{\gamma}_j^{\tau}\right]$$

$$+\sum_{k=1}^{o}\left[\bar{\rho}_k^{3\tau} + \alpha(\sum_{j=1}^{n} q_{jk}^{\tau-1} - d_k(\rho^{3^{\tau-1}})) - \rho_k^{3^{\tau-1}}\right] \times \left[\rho_k^3 - \bar{\rho}_k^{3\tau}\right] \geq 0,$$

$$\forall (Q^1, Q^2, \gamma, \rho^3) \in \mathcal{K}. \tag{43}$$

Step 2: Adaptation

Compute $(Q^{1\tau}, Q^{2\tau}, \gamma^{\tau}, \rho^{3\tau}) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{L}\Bigg[q_{ijl}^{\tau} + \alpha(w_{2i}^{1}(r^{i}(\bar{q}_i^{\tau}))\frac{\partial r^{i}(\bar{q}_i^{\tau})}{\partial q_{ijl}} + \frac{\partial w_{2i}^{1}(r^{i}(\bar{q}_i^{\tau}))}{\partial q_{ijl}}r^{i}(\bar{q}_i^{\tau}) + w_{2j}^{2}(r^{j}(\bar{q}_j^{\tau}))\frac{\partial r^{j}(\bar{q}_j^{\tau})}{\partial q_{ijl}}$$

$$+\frac{\partial w_{2j}^{2}(r^{j}(\bar{q}_j^{\tau}))}{\partial q_{ijl}}r^{j}(\bar{q}_j^{\tau}) + \frac{\partial c_{ijl}(\bar{q}_{ijl}^{\tau})}{\partial q_{ijl}} + \frac{\partial c_j(\bar{Q}^{1\tau})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(\bar{q}_{ijl}^{\tau})}{\partial q_{ijl}} - \bar{\gamma}_j^{\tau}) - q_{ijl}^{\tau-1}\Bigg] \times \Big[q_{ijl} - q_{ijl}^{\tau}\Big]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\Bigg[q_{jk}^{\tau} + \alpha(w_{2j}^{2}(r^{j}(\bar{q}_j^{\tau}))\frac{\partial r^{j}(\bar{q}_j^{\tau})}{\partial q_{jk}} + \frac{\partial w_{2j}^{2}(r^{j}(\bar{q}_j^{\tau}))}{\partial q_{jk}}r^{j}(\bar{q}_j^{\tau})$$

$$+\frac{\partial c_{jk}(\bar{q}_{jk}^{\tau})}{\partial q_{jk}} + \hat{c}_{jk}(\bar{Q}^{2\tau}) + \bar{\gamma}_j^{\tau} - \bar{\rho}_k^{3\tau}) - q_{jk}^{\tau-1}\Bigg] \times \Big[q_{jk} - q_{jk}^{\tau}\Big]$$

$$+\sum_{j=1}^{n}\Bigg[\gamma_j^{\tau} + \alpha(\sum_{i=1}^{m}\sum_{l=1}^{L}\bar{q}_{ijl}^{\tau} - \sum_{k=1}^{o}\bar{q}_{jk}^{\tau}) - \gamma_j^{\tau-1}\Bigg] \times \Big[\gamma_j - \gamma_j^{\tau}\Big]$$

$$+\sum_{k=1}^{o}\Bigg[\rho_k^{3\tau} + \alpha(\sum_{j=1}^{n}\bar{q}_{jk}^{\tau} - d_k(\bar{\rho}^{3\tau})) - \rho_k^{3\tau-1}\Bigg] \times \Big[\rho_k^{3} - \rho_k^{3\tau}\Big] \geq 0,$$

$$\forall (Q^1, Q^2, \gamma, \rho^3) \in \mathcal{K}. \tag{44}$$

Step 3: Convergence Verification

If $|q_{ijl}^{\tau} - q_{ijl}^{\tau-1}| \leq \epsilon$, $|q_{jk}^{\tau} - q_{jk}^{\tau-1}| \leq \epsilon$, $|\gamma_j^{\tau} - \gamma_j^{\tau-1}| \leq \epsilon$, $|\rho_k^{3\tau} - \rho_k^{3\tau-1}| \leq \epsilon$, for all $i = 1, \cdots, m$; $j = 1, \cdots, n$; $k = 1, \ldots, o$; $l = 1, \cdots, L$, with $\epsilon > 0$, a pre-specified tolerance, then stop; otherwise, set $\tau := \tau + 1$, and go to Step 1.

Note that the variational inequality subproblems (43) and (44) encountered at each iteration of the algorithm can be solved explicitly and in closed form since they are actually quadratic programming problems over simple feasible sets with network structure. Indeed, the feasible set $\mathcal{K}$ is a Cartesian product consisting of the product of the individual sectors' feasible sets, and the nonnegative orthants, R_+^{no}, R_+^{n}, and R_+^{o}, corresponding to the variables Q^1, Q^2, γ, and ρ^3, respectively. In fact, the subproblems in (43) and (44) corresponding to the q_{ijl}s can be solved using exact equilibration (cf. Dafermos and Sparrow (1969) and

Nagurney (1999)), whereas the remainder of the variables in (43) and (44) can be obtained by explicit formulae. We now, for completeness, and also to illustrate the simplicity of the proposed computational procedure in the context of the financial network model, state the explicit formulae for the computation of the q^{τ}_{jk}, the γ^{τ}_j, and the $\rho^{3^{\tau}}_k$ in the Adaptation Step (44). The solution of the corresponding variables in (43) can be obtained analogously.

Computation of Financial Flows and Products

In particular, compute, at iteration τ, the q^{τ}_{jk}s, according to:

$$q^{\tau}_{jk} = \max\{0, q^{\tau-1}_{jk} - \alpha(w^2_{2j}(r^j(\bar{q}^{\tau}_j))\frac{\partial r^j(\bar{q}^{\tau}_j)}{\partial q_{jk}} + \frac{\partial w^2_{2j}(r^j(\bar{q}^{\tau}_j))}{\partial q_{jk}} r^j(\bar{q}^{\tau}_j) + \frac{\partial c_{jk}(\bar{q}^{\tau}_{jk})}{\partial q_{jk}} + \hat{c}_{jk}(\bar{Q}^{2\tau}) + \bar{\gamma}^{\tau}_j - \bar{\rho}^{3\tau}_k)\},$$

$$\forall j, k. \tag{45}$$

where

$$\bar{q}^{\tau}_{jk} = \max\{0, q^{\tau-1}_{jk} - \alpha(w^2_{2j}(r^j(q^{\tau-1}_j))\frac{\partial r^j(q^{\tau-1}_j)}{\partial q_{jk}} + \frac{\partial w^2_{2j}(r^j(q^{\tau-1}_j))}{\partial q_{jk}} r^j(q^{\tau-1}_j) + \frac{\partial c_{jk}(q^{\tau-1}_{jk})}{\partial q_{jk}}$$

$$+\hat{c}_{jk}(Q^{2\tau-1}) + \gamma^{\tau-1}_j - \rho^{3^{\tau-1}}_k)\}, \quad \forall j, k. \tag{46}$$

$$\bar{\gamma}^{\tau}_j = \max\{0, \gamma^{\tau-1}_j - \alpha(\sum_{i=1}^m \sum_{l=1}^L q^{\tau-1}_{ijl} - \sum_{k=1}^o q^{\tau-1}_{jk})\}, \quad \forall j, \tag{47}$$

and

$$\bar{\rho}^{3\tau}_k = \max\{0, \rho^{3^{\tau-1}}_k - \alpha(\sum_{j=1}^n q^{\tau-1}_{jk} - d_k(\rho^{3^{\tau-1}}))\}, \quad \forall k. \tag{48}$$

Computation of the Prices

At iteration τ, compute the γ^{τ}_js according to:

$$\gamma^{\tau}_j = \max\{0, \gamma^{\tau-1}_j - \alpha(\sum_{i=1}^m \sum_{l=1}^L \bar{q}^{\tau}_{ijl} - \sum_{k=1}^o \bar{q}^{\tau}_{jk})\}, \quad \forall j, \tag{49}$$

whereas the $\rho^{3^{\tau}}_k$s are computed explicitly and in closed form according to:

$$\rho^{3^{\tau}}_k = \max\{0, \rho^{3^{\tau-1}}_k - \alpha(\sum_{j=1}^n \bar{q}^{\tau}_{jk} - d_k(\bar{\rho}^{3\tau}))\}, \quad \forall k. \tag{50}$$

where $\bar{Q}^{2\tau}$ and $\bar{\rho}_k^{3\tau}$ can be achieved from the equation (46) and (48) respectively, whereas $\bar{Q}^{1\tau}$ can be obtained through exact equilibration.

Note that in the computation process, the financial flows and the prices can be updated simultaneously at each iteration.

We now state the convergence result for the modified projection method for this model.

Theorem 9: Convergence

Assume that the function that enters the variational inequality (29) (or (32)) has at least one solution and satisfies the conditions in Theorem 5 and in Theorem 8. Then the modified projection method described above converges to the solution of the variational inequality (29) or (32).

Proof: According to Korpelevich (1977), the modified projection method converges to the solution of the variational inequality problem of the form (29), provided that the function F that enters the variational inequality is monotone and Lipschitz continuous and that a solution exists. Existence of a solution follows from Theorem 4, monotonicity follows Theorem 5, and Lipschitz continuity, in turn, follows from Theorem 8. □

Of course, the algorithm may converge even if the conditions in Theorem 5 and 8 do not hold in which case the algorithm, nevertheless, converges to the equilibrium solution.

5. Numerical Examples

In this section, we apply the modified projection method to several numerical financial network examples. The algorithm was implemented in FORTRAN and the computer system used was a DEC Alpha system located at the University of Massachusetts at Amherst. For the solution of the induced network subproblems in the Q^1 variables we utilized the exact equilibration algorithm (see Dafermos and Sparrow (1969), Nagurney (1999), and Nagurney and Ke (2001)). The other subproblems in the Q^2, γ, and the ρ_3 variables were solved exactly and in closed form as described in Section 4 (cf. (45) – (50)).

The convergence criterion used was that the absolute value of the financial flows and prices between two successive iterations differed by no more than 10^{-4}.

The algorithm was initialized as follows: we set $q_{ij1} = \frac{S^i}{n}$ for each source agent i and all intermediaries j. All the other variables were initialized to zero.

Example 1

The first example consisted of two source agents, two intermediaries, a single financial instrument between each source agent and intermediary pair, and two demand markets, as depicted in Figure 5. This example serves as a baseline.

The data for the first example were constructed for easy interpretation purposes. The financial holdings of the two source agents were: $S^1 = 20$ and $S^2 = 20$. We assumed risk functions of the form (4) and (14) for the source agents and the intermediaries, respectively. The variance-covariance matrices V^i and V^j were set equal to the identity matrices for all source agents i and all intermediaries j. All the weights were fixed and set equal to 1. We set $\alpha = .2$ in the modified projection method.

The transaction cost functions faced by the source agents associated with transacting with the intermediaries (cf. (2)) were given by:

$$c_{ijl}(q_{ijl}) = .5q_{ijl}^2 + 3.5q_{ijl}, \quad \text{for } i = 1, 2; j = 1, 2; l = 1.$$

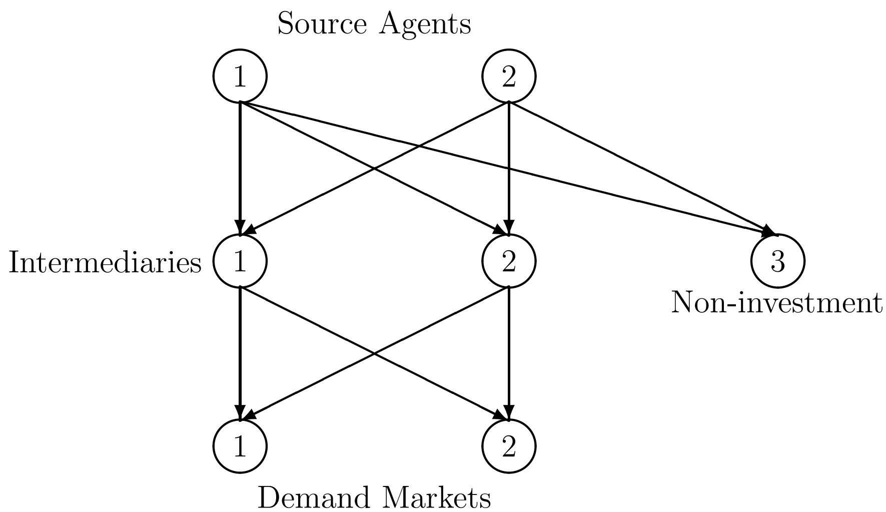


Figure 5: The Financial Network Structure of Example 1

The handling costs of the intermediaries, in turn (see (13)), were given by:

$$c_j(Q^1) = .5(\sum_{i=1}^{2} q_{ij1})^2, \quad \text{for } j = 1, 2.$$

The transaction costs of the intermediaries associated with transacting with the source agents were (cf. (14)) given by:

$$\hat{c}_{ijl}(q_{ijl}) = 1.5q_{ijl}^2 + 3q_{ijl}, \quad \text{for } i = 1, 2; j = 1, 2; l = 1.$$

The demand functions at the demand markets (see (25)) were:

$$d_1(\rho^3) = -2\rho_1^3 - 1.5\rho_2^3 + 1000, \quad d_2(\rho^3) = -2\rho_2^3 - 1.5\rho_1^3 + 1000,$$

and the transaction costs between the intermediaries and the consumers at the demand markets (see (24)) were given by:

$$\hat{c}_{jk}(q_{jk}) = q_{jk} + 5, \quad \text{for } j = 1, 2; k = 1, 2.$$

We assumed for this and the subsequent examples that the transaction costs as perceived by the intermediaries and associated with transacting with the demand markets were all zero (cf. (15)), that is, $c_{jk}(q_{jk}) = 0$, for all j, k.

The modified projection method converged in 106 iterations and yielded the following equilibrium financial flow pattern:

$$Q^{1*} := q^*_{111} = q^*_{121} = q^*_{211} = q^*_{221} = 10.00,$$

$$Q^{2*} := q^*_{11} = q^*_{12} = q^*_{21} = q^*_{22} = 10.00.$$

The vector γ^* had components: $\gamma^*_1 = \gamma^*_2 = 245.00$, and the computed demand prices at the demand markets were: $\rho^{3*}_1 = \rho^{3*}_2 = 280.00$.

Also, for completeness (see discussion following Corollary 1), we computed the top-tiered and the middle tiered equilibrium prices. The top-tiered prices associated with the source agents were: $\rho^{1*}_{ijl} = 152.00$ for all i, j, l, and the middle-tiered prices associated with the financial intermediaries were: $\rho^{2*}_{jk} = 265.00$, for all j, k.

It is easy to verify that the optimality/equilibrium conditions were satisfied with good accuracy. Note that in this example, constraint (1) was tight for both source agents, that is, there was zero flow on the links connecting node 3 with top tier nodes 1 and 2. Thus, it was optimal for both source agents to invest their entire financial holdings in each instrument made available by each of the two intermediaries.

Example 2

Example 2 was constructed from the first example as follows. We kept the data as in Example 1 except that we doubled the weights associated with the source agents transacting with the first financial intermediary. We set $\alpha = .1$ in the modified projection method (since we did not observe convergence with $\alpha = .2$). The modified projection method required 190 iterations for convergence and yielded the following new equilibrium financial flow and price pattern:

$$Q^{1*} := q^*_{111} = 9.29;\ q^*_{121} = 10.71;\ q^*_{211} = 9.29;\ q^*_{221} = 10.71;$$

$$Q^{2*} := q^*_{11} = 9.29;\ q^*_{12} = 9.29;\ q^*_{21} = 10.71;\ q^*_{22} = 10.71.$$

The vector γ^* now had components: $\gamma_1^* = 247.14$; $\gamma_2^* = 242.86$. The computed equilibrium demand prices at the demand markets remained: $\rho_1^{3*} = 280.00 = \rho_2^{3*}$. This is, as expected, since the demand at each demand market was still 20.

The top tier prices were now: $\rho_{111}^{1*} = 142.00$; $\rho_{121}^{1*} = 143.43$; $\rho_{211}^{1*} = 142.00$; and $\rho_{221}^{1*} = 143.43$. The middle tier prices, in turn were: $\rho_{11}^{2*} = 265.71$; $\rho_{12}^{2*} = 261.71$; $\rho_{21}^{2*} = 264.73$: and $\rho_{22}^{2*} = 264.73$.

Observe that since there was more risk associated with transacting with the first financial intermediary both source agents reduced the volume transacted with that intermediary. Also, since the intermediaries now handled a different amount of financial funds invested in the instrument (than in Example 1) the value of γ_j^* changed accordingly.

Example 3

The final numerical example had the identical data as Example 1 but now we assumed that the source agents had variable weights associated with their risk criteria. In particular, we assumed that each source agent had $w_{2i}^1 = c^i z_{2i}^1$ for $i = 1, 2$. We set $c^i = 1$ for both source agents. We set $\alpha = .01$ for which the modified projection method required 1475 iterations for convergence and yielded the following equilibrium financial flow and price pattern:

$$Q^{1*} := q_{111}^* = 3.10;\ q_{121}^* = 3.10;\ q_{211}^* = 3.10;\ q_{221}^* = 3.10;$$

$$Q^{2*} := q_{11}^* = 3.10;\ q_{12}^* = 3.10;\ q_{21}^* = 3.10;\ q_{22}^* = 3.10.$$

Now, however, since the source agents were more risk-averse, we had that $s_1^* = s_2^* = 13.8$. In other words, the source agents opted to not to invest a sizeable portion of their financial holdings.

The γ^* values were: $\gamma_1^* = \gamma_2^* = 269.63$. The computed equilibrium demand prices at the demand markets were: $\rho_1^{3*} = 283.94$; $\rho_2^{3*} = 283.94$.

The above numerical results illustrate both the financial network equilibrium model as well as the algorithm. Obviously, the examples are stylized but they reflect the power of the methodological framework and, moreover, allow for a plethora of simulations to be conducted.

6. Summary and Conclusions

In this paper, we developed a framework for the formulation, qualitative analysis, and computation of solutions to financial network equilibrium problems with intermediation and variable weights. The financial network consisted of a multi-tiered network in which non-investment is also permitted.

We described the behavior of the decision-makers consisting of the source agents, the financial intermediaries, and the consumers associated with the financial products at the demand markets. Each decision-maker in the first two tiers of the financial network faced a bicriteria decision-making problem consisting of net revenue maximization and risk minimization. Unlike in the earlier literature on financial network equilibrium problems with intermediation (cf. Nagurney and Ke (2001), Nagurney and Cruz (2003a, b) and the references therein), the weights associated with the objectives were no longer assumed to be equal. In particular, we applied risk-penalizing weights, which were variable and dependent on the value of the risk objective in the value function associated with each source agent as well as with each financial intermediary. Moreover, we proposed risk functions of a general form rather than a specialized one as done in Nagurney and Ke (2001). We derived the optimality conditions for the sources of financial funds as well as the financial intermediaries, under suitable assumptions on the underlying functions, along with the equilibrium conditions. We identified the network structure associated with each decision-maker's problem and the network structure of the financial network economy.

We derived the variational inequality formulation of the governing equilibrium conditions. The methodology of variational inequalities was then utilized for the qualitative analysis of the equilibrium financial flow and price pattern, as well as for its computation.

This paper demonstrated that financial network problems with different tiers of decision-makers in the presence of risk attitudes associated with the source agents and the intermediaries can be formulated and studied in a rigorous fashion.

Future research will include the extension of this framework to the international arena, the incorporation of other criteria, the introduction of dynamics, as well as empirical applications.

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