**Financial Networks with Electronic Transactions:
Modeling, Analysis, and Computations**

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April 2002

Appears in *Quantitative Finance* **3** (2003), pp. 71-87.

Abstract: Advances in telecommunication networks, and, in particular, the Internet have transformed the economic landscape for financial decision-making. In this paper, we focus on financial networks with electronic transactions and with different tiers of decision-makers and we develop an integrated framework for the modeling, analysis, and computation of solutions to such problems. Specifically, we consider an economy consisting of three types of decision-makers: those with sources of funds; intermediary ones, and consumers associated with the financial products at the demand markets. Those with sources of funds can transact with the intermediaries either physically or electronically as well as directly in an electronic manner with the consumers. The intermediaries, in turn, can also transact with the consumers either in a physical or an electronic fashion. We address the behavior of the decision-makers, identify the network structure of the problem, derive the equilibrium conditions, and establish the variational inequality formulation. In addition, we propose a continuous time adjustment process for the study of the disequilibrium dynamics and prove that the set of stationary points of the resulting *projected* dynamical system coincides with the set of solutions of the variational inequality. We then utilize variational inequality theory to derive qualitative properties of the equilibrium price and financial flow pattern. Finally, we apply an algorithm for the determination of equilibrium prices and financial flows in several examples.

1. Introduction

Advances in telecommunications and, in particular, the adoption of the Internet by businesses, consumers, and financial institutions have had an enormous effect on financial services and the options available for financial transactions. Distribution channels have been transformed, new types of services and products introduced, and the role of financial intermediaries altered in the new economic networked landscape. Furthermore, the impact of such advances has not been limited to individual nations but, rather, through new linkages, has crossed national boundaries.

The topic of *electronic* finance has been a growing area of study (cf. Claessens, Glaessner, and Klingebiel (2000, 2001), Long (2000), Sato and Hawkins (2001), Banks (2001), and Allen, Hawkins, and Sato (2001), and the references therein), due to its increasing impact on financial markets and financial intermediation, as well as related regulatory issues and governance (see also Turner (2001)). Of particular emphasis has been the conceptualization of the major issues involved and the role of networks is the transformations (see McAndrews and Stefanidis (2000), Allen, Hawkins, and Seto (2001), Economides (2001), and Nagurney and Dong (2002)).

Nevertheless, the complexity of the interactions among the distinct decision-makers involved, the supply chain aspects of the financial product accessibilities and deliveries, as well as the availability of physical as well as electronic options, and the role of intermediaries, have defied the construction of a unified, quantifiable framework in which one can assess the resulting financial flows and prices.

In this paper, we propose a theoretical framework for the study of financial decision-making in the presence of intermediation and electronic transactions. The framework is sufficiently general to allow for the modeling, analysis, and computation of solutions to such problems. Our perspective is based on the recent work of Nagurney and Ke (2001) on financial networks with intermediation and that of Nagurney and Dong (2002) on supernetworks which allows for the abstraction of decision-making on interrelated network systems, notably, those that involve telecommunication networks such as the Internet.

In this paper, however, we extend the above work in several directions, principally, by, first, explicitly including electronic transactions in financial networks with intermediation and identifying the resulting network structure and, second, by introducing a continuous time adjustment process for the study of the disequilibrium dynamics. The equilibrium perspective provides a valuable benchmark against which existing prices and financial flows can be compared against. The framework that we propose, hence, allows for the synthesis of financial decision-making surrounding electronic options as well as more standard, which we refer to as, physical, ones.

The paper is organized as follows. In Section 2, we present the financial network model with electronic transactions, derive the optimality conditions for each set of network agents or decision-makers, and then present the governing equilibrium conditions. We also derive the finite-dimensional variational inequality formulation of the problem. For an introduction to variational inequalities in the context of network economics, see the book by Nagurney (1999). For background on financial networks, see the book by Nagurney and Siokos (1997).

In Section 3, we provide qualitative properties of the equilibrium pattern, notably, existence and uniqueness results. In Section 4, we propose the projected dynamical system (cf. Nagurney and Zhang (1996) and the references therein) which describes the dynamic adjustment process associated with the various decision-makers and prove that its set of stationary points coincides with the set of solutions to the derived variational inequality problem. We also establish a stability analysis result for the financial network system.

In Section 5, we outline the computational procedure, which has an interpretation of a discrete-time adjustment process, and can track the dynamic trajectories to the stationary or equilibrium point. Moreover, the algorithm resolves the network problem into subproblems, each of which can be solved exactly and in closed form. In Section 6, we apply the algorithm to numerical financial examples in order to determine the equilibrium financial flows and prices. We conclude the paper with a summary and suggestions for future research in Section 7.

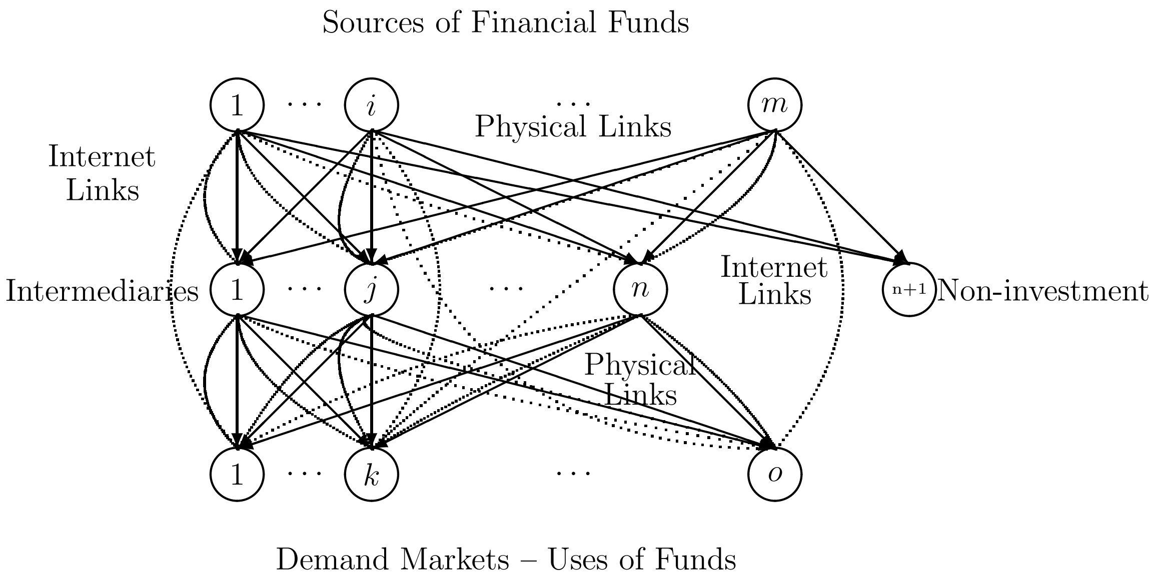

Figure 1: The Structure of the Financial Network with Electronic Transactions

2. The Financial Network Model with Electronic Transactions

In this Section, we develop the financial network model consisting of: agents or decision-makers with sources of funds, financial intermediaries, as well as consumers associated with the demand markets. In this model, the sources of funds can transact directly electronically with the consumers through the Internet and can also conduct their financial transactions with the intermediaries either physically or electronically. The intermediaries, in turn, can transact with the consumers either physically in the standard manner or electronically. The depiction of the network at equilibrium is given in Figure 1.

Specifically, we consider m agents with sources of financial funds, such as households and businesses, involved in the allocation of their financial resources among a portfolio of financial instruments which can be obtained by transacting with distinct n financial intermediaries, such as banks, insurance and investment companies, etc., and/or directly with the consumers associated with the o demand markets. The financial intermediaries, in turn, in addition to transacting with the source agents, determine how to allocate the incoming financial resources among distinct uses or financial products at the demand markets, such as, for example, the market for real estate loans, household loans, or business loans, etc.

The financial network is now described and depicted graphically in Figure 1. The top tier of nodes consists of the decision-makers with sources of funds, with a typical source agent denoted by i and associated with node i. The middle tier of nodes consists of the intermediaries, with a typical intermediary denoted by j and associated with node j in the network. In addition, in contrast to the model of Nagurney and Ke (2001), we now allow for the possibility of source agents not investing their funds (or a portion thereof), which we represent by the node $n+1$ at the middle tier of nodes. The bottom tier of nodes consists of the demand markets, with a typical demand market denoted by k and corresponding to node k (and associated with a particular financial product).

The links in the network in Figure 1 include classical physical links as well as Internet links to allow for electronic financial transactions. Note that the introduction of such transactions allows for "connections" that were, heretofore, not possible, such as allowing, for example, consumers to borrow the money directly from the source agents. In order to conceptualize this type of transaction, we construct a direct link from each top tier node to each bottom tier node. In addition, we consider the situation in which the source agents can now transact not only with the consumers directly but also with the financial intermediaries through the Internet (e.g., online banking). Thus, we add an additional link between each top tier node and each middle tier node to reflect the possibility of Internet transactions between source agents and intermediaries. Hence, a source agent may transact with an intermediary through either a physical link or through the Internet link, or both. For simplicity of notation and presentation, we assume here that each intermediary has, hence, in effect, a financial instrument which can be transacted electronically and/or physically. In Nagurney and Ke (2001), in contrast, it was assumed that each intermediary had available alternative instruments but these could only be transacted physically.

A similar situation can be handled within our framework as regards the transactions between intermediaries and the consumers. The consumers can transact with an intermediary through either a physical link (traditional transaction) or through an Internet link (e.g. electronic brokerage), or both. Hence, there are two links connecting each middle tier node with each bottom tier node.

Of course, depending on the particular application, the network topology depicted in Figure 1 can take on a different form. For example, in the case of electronic brokers who transact only electronically, one would have an intermediate (second tier node) being connected with the source agents and with the demand markets only through Internet links. Moreover, some intermediaries may not allow for online transactions with consumers, in which case, those links would be severed for the application-dependent network representation. On the other hand, in the absence of electronic transactions and non-investment, the financial network in Figure 1 collapses to a special case of the financial network with intermediation studied in Nagurney and Ke (2001).

We now describe the behavior of the various economic decision-makers represented by the three tiers of nodes in Figure 1. We first focus on the source agents. We then turn to the intermediaries and, subsequently, to the demand markets.

The Behavior of the Decision-Makers with Sources of Funds and their Optimality Conditions

In order to depict the allowable transactions of a typical source agent i with the consumers at the demand markets and with the financial intermediaries, we provide a graphical depiction in Figure 2. Note that a source agent may transact with an intermediary via a physical link, and/or electronically via an Internet link. Let S^i denote the nonnegative amount of funds that source i holds and let c_{ijl} denote the transaction cost associated with source i transacting with intermediary j via mode l, where l=1 refers to a physical transaction and l=2 refers to an electronic transaction via the Internet. The quantity of financial funds associated with source agent i, intermediary j, and mode of transaction l, in turn, is denoted by q_{ijl} and is associated with link l joining node i to node j. We group the financial flows for all source agents/intermediaries/modes into the column vector $Q^1 \in R^{2mn}_+$.

In addition, a source agent i may transact directly with consumers located at a demand market k with the transaction cost associated with the electronic transaction denoted by c_{ik} and the associated funds flow from source agent i to demand market k by q_{ik}. Note that q_{ik} is, hence, the financial flow from node i at the top tier of nodes to node k at the bottom tier of nodes. We group such financial flows, in turn, into the column vector $Q^2 \in R^{mo}_+$.

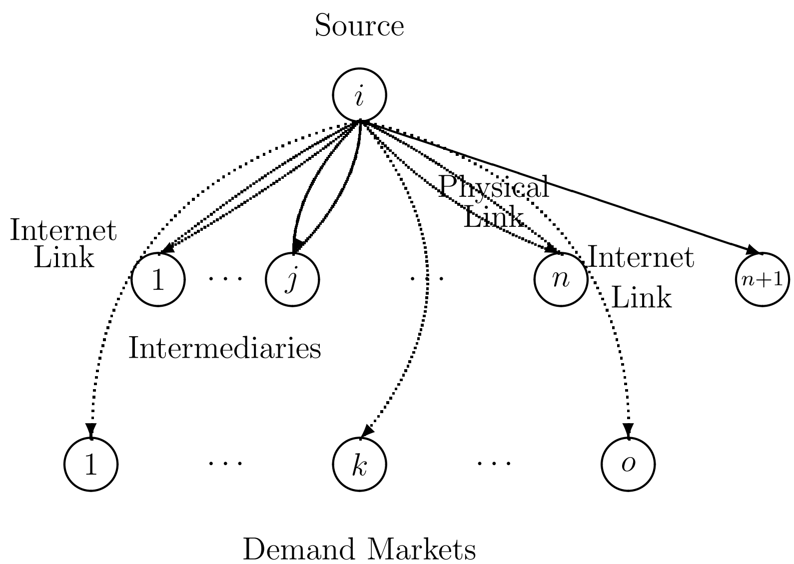

Figure 2: Network Structure of Source i's Transactions

Also, we let q_i denote the $(2n+o)$-dimensional column vector associated with source agent i with components: $\{q_{ijl}; j = 1, \ldots, n; l = 1, 2; q_{ik}; k = 1, \ldots, o\}$. This vector, thus, contains the financial allocations of source agent i. For each source agent i the following conservation of flow equation must hold:

$$\sum_{j=1}^{n} \sum_{l=1}^{2} q_{ijl} + \sum_{k=1}^{o} q_{ik} \leq S^i, \quad \forall i, \tag{1}$$

that is, the amount of funds allocated either electronically and/or physically by a source agent cannot exceed his financial holdings. In Figure 2, we represent the "slack" associated with constraint (1) for source agent i as the flow on the link joining node i with the non-investment node $n + 1$.

We consider the situation in which the transaction cost between a source agent and intermediary pair, as well as, the transaction cost between a source agent and a demand market depend upon the volume of financial transactions between the particular pair (via the particular mode), that is:

$$c_{ijl} = c_{ijl}(q_{ijl}), \quad \forall i, j, l, \tag{2a}$$

and

$$c_{ik} = c_{ik}(q_{ik}), \quad \forall i, k. \tag{2b}$$

Hence, we explicitly allow for the transaction cost to differ depending upon whether the transaction was conducted physically or electronically.

We now construct the optimization problem facing a source agent. We assume that each such agent seeks to maximize his net return while, simultaneously, minimizing his risk, where source agent i's utility function is denoted by U^i. As discussed in Nagurney and Ke (2001), but now in the more general setting of electronic transactions, we assume that each source agent's utility can be defined as a function of the expected future portfolio value where the expected value of the future portfolio is described by two characteristics: the expected mean value and the uncertainty surrounding the expected mean. Here, the expected mean portfolio value is assumed to be equal to the market value of the current portfolio. Each source agent's uncertainty, or assessment of risk, is based on a variance-covariance matrix representing the source agent's assessment of the standard deviation of the prices of the financial instruments. Moreover, we assume that the variance-covariance matrix V^i associated with source agent i, which is of dimension $(2n + o) \times (2n + o)$, is symmetric and positive definite. In addition, we assume that the transaction cost functions given by (2a) and (2b) are continuously differentiable and convex.

Note that, in our framework, we allow for the variance-covariance matrix for any source agent to represent both the risk associated with physical as well as with electronic transactions. Indeed, certain source agents may have a higher perception of risk associated with electronic transactions vis a vis physical transactions due to security concerns, for example.

Let ρ^*_{1ijl} denote the price obtained by source agent i from intermediary j by transacting via mode l and let ρ^*_{1ik}, in turn, denote the price associated with source agent i transacting electronically with demand market k. Later, we discuss how these prices are arrived at.

We can express the optimization problem facing source agent i as:

$$\text{Maximize } U^i(q_i) = \sum_{j=1}^{n} \sum_{l=1}^{2} \rho^*_{1ijl} q_{ijl} + \sum_{k=1}^{o} \rho^*_{1ik} q_{ik} - \sum_{j=1}^{n} \sum_{l=1}^{2} c_{ijl}(q_{ijl}) - \sum_{k=1}^{o} c_{ik}(q_{ik}) - {q_i}^T V^i q_i, \tag{3}$$

subject to $q_{ijl} \geq 0$, $\forall j, l$; $q_{ik} \geq 0$, $\forall k$, and the constraint (1) for this agent i. Thus, the expression consisting of the first four terms to the right-hand side of the equal sign in (3) represents the net revenue (which is to be maximized), whereas the last term in (3) represents the risk (which is to be minimized). Observe that such an objective function is in concert with those used in classical portfolio optimization (see Markowitz (1952, 1959)).

Under the above stated assumptions on the transaction cost functions, and assuming that the variance-covariance matrices are positive definite, the optimality conditions for all source agents *simultaneously* can be expressed as the following inequality (see also Nagurney and Ke (2001)): determine $(Q^{1*}, Q^{2*}) \in K$, such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^i_{z_{jl}} \cdot q_i^* + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \rho^*_{1ijl}\right] \times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^i_{z_{2n+k}} \cdot q_i^* + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}} - \rho^*_{1ik}\right] \times [q_{ik} - q^*_{ik}] \geq 0, \quad \forall (Q^1, Q^2) \in K, \tag{4}$$

where V^i_{jl} denotes the z_{jl}-th row of V^i, z_{jl} is defined as the indicator: $z_{jl} = (l-1)n + j$, with z_{2n+k} defined as the z_{2n+k}-th row, and the feasible set $K \equiv \{(Q^1, Q^2) | q_{ijl} \geq 0, \forall i, j, l; q_{ik} \geq 0, \forall i, k, \text{ and (1) holds}\}$.

The Behavior of the Intermediaries and their Optimality Conditions

We now describe the behavior of the financial intermediaries. For a graphical depiction of the transactions associated with intermediary j, see Figure 3. We assume that an intermediary j is faced with what we term a *handling/conversion* cost, which may include, for example, the cost of converting the incoming financial flows into the financial products at the demand markets. We denote this cost by c_j. In general, we would have that c_j is a function of $\sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}$, that is, the conversion cost of an intermediary is a function of how much he has obtained from the various source agents and the amounts held by other intermediaries. We may write:

$$c_j = c_j(Q^1), \quad \forall j. \tag{5}$$

The intermediaries also have associated transaction costs in regards to transacting with the source agents via the two modal alternatives. We denote the transaction cost associated with intermediary j transacting with source agent i using mode l by $\hat{c}_{ijl}$ and we assume that

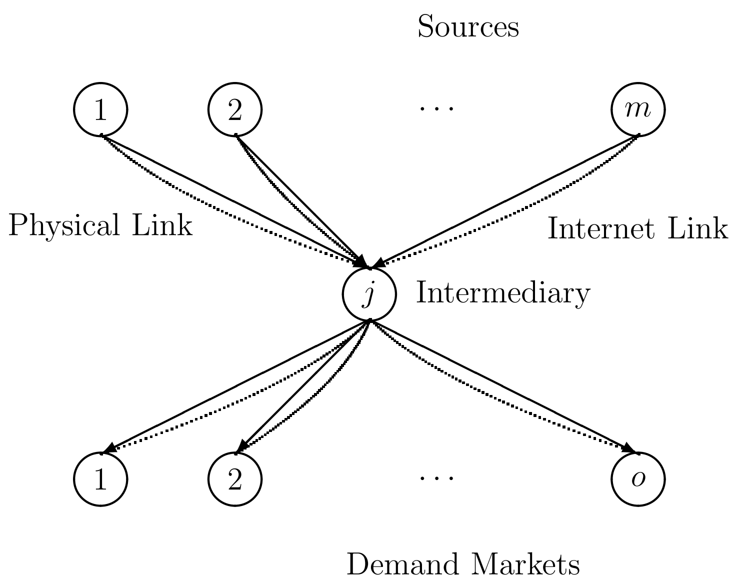

Figure 3: The Network Structure of Intermediary j's Transactions

the function can depend on the the financial flow q_{ijl}, that is,

$$\hat{c}_{ijl} = \hat{c}_{ijl}(q_{ijl}), \quad \forall i, j, l. \tag{6}$$

Let q_{jkl} denote the quantity of the financial product transacted via mode l by demand market k from intermediary j, where recall that $l = 1$ denotes a physical transaction, whereas $l = 2$ denotes an electronic transaction. Such a financial flow is associated with link l joining node j in the middle tier of nodes with node k in the bottom tier. We group these financial quantities into the column vector $Q^3 \in R^{2no}_+$. Hence, the intermediaries convert the incoming financial flows Q^1 into the outgoing financial products Q^3. We assume that an intermediary j incurs a transaction cost c_{jkl} associated with transacting with consumers at demand market k via mode l, where

$$c_{jkl} = c_{jkl}(q_{jkl}), \quad \forall j, k, l. \tag{7}$$

Let ρ^*_{2jkl} denote the price associated with intermediary j, demand market k, and mode l. How these prices are determined is discussed subsequently, after the complete model development. Also, assume that intermediaries may have risk associated with transacting with the various source agents and demand markets with the variance-covariance matrix associated with intermediary j denoted by V^j. This matrix is of dimension $(2m+2o)\times(2m+2o)$ and reflects the risk associated with transacting with the various source agents and demand markets electronically as well as physically. We assume that each such variance-covariance matrix is symmetric and positive definite. Note that in Nagurney and Ke (2001) we assumed that there was risk only associated with transacting with the demand markets. We assume that the underlying transaction cost functions (6) and (7) as well as the handling cost functions (5) are convex and continuously differentiable. Let q_j, without loss of generality, denote the $(2m+2o)$-dimensional column vector with components: q_{ijl}; $i=1,\ldots,m$; $l=1,2$; q_{jkl}; $k=1,\ldots,o$; $l=1,2$.

Let U^j denote the utility function associated with intermediary j and assume that each intermediary seeks to maximize his net return, while his minimizing risk. The optimization problem of intermediary j can now be expressed and is given by:

$$\text{Maximize}\, U^j(q_j)=\sum_{k=1}^{o}\sum_{l=1}^{2}\rho^*_{2jkl}q_{jkl}-c_j(Q^1)-\sum_{i=1}^{m}\sum_{l=1}^{2}\hat{c}_{ijl}(q_{ijl})-\sum_{k=1}^{o}\sum_{l=1}^{2}c_{jkl}(q_{jkl})-\sum_{i=1}^{m}\sum_{l=1}^{2}\rho^*_{1ijl}q_{ijl}$$
$$-q_j^TV^jq_j, \tag{8}$$

subject to:

$$\sum_{k=1}^{o}\sum_{l=1}^{2}q_{jkl}\leq\sum_{i=1}^{m}\sum_{l=1}^{2}q_{ijl}, \tag{9}$$

and the nonnegativity assumption on all the q_{ijl}s and q_{jkl}s.

The expression consisting of the first five terms to the right-hand side of the equal sign in (8) represents the net return (to be maximized), whereas the last term in (8) represents the risk (to be minimized). The constraint (9) reflects that the financial intermediary cannot produce more financial products than it has as financial holdings obtained from the various source agents.

Here we assume that the financial intermediaries can compete, with the governing optimality/equilibrium concept underlying noncooperative behavior being that of Nash (1950, 1951), which states that each decision-maker or agent will determine his optimal strategies, given the optimal ones of his competitors. The optimality conditions for all financial intermediaries simultaneously, under the above stated assumptions, can be succinctly expressed as (see also Bertsekas and Tsitsiklis (1989), Bazaraa, Sherali, and Shetty (1993), Gabay and Moulin (1980), Dafermos and Nagurney (1987), and Nagurney and Ke (2001)): determine $(Q^{1*}, Q^{3*}, \gamma^*) \in R_+^{2mn+2no+n}$, such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V_{z_{il}}^{j}\cdot q_j^* + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \rho_{1ijl}^* + \frac{\partial \hat{c}_{ijl}(q_{ijl}^*)}{\partial q_{ijl}} - \gamma_j^*\right] \times \left[q_{ijl} - q_{ijl}^*\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V_{z_{kl}}^{j}\cdot q_j^* + \frac{\partial c_{jkl}(q_{jkl}^*)}{\partial q_{jkl}} - \rho_{2jkl}^* + \gamma_j^*\right] \times \left[q_{jkl} - q_{jkl}^*\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}^* - \sum_{k=1}^{o}\sum_{l=1}^{2} q_{jkl}^*\right] \times \left[\gamma_j - \gamma_j^*\right] \geq 0, \quad \forall (Q^1, Q^3, \gamma) \in R_+^{2mn+2no+n}, \tag{10}$$

where γ_j is the Lagrange multiplier associatd with constraint (9) (see Bazaraa, Sherali, and Shetty (1993)), and γ is the n-dimensional column vector of Lagrange multiplers of all the intermediaries. For an analogous derivation (but without the electronic transactions), see Nagurney and Ke (2001).

The Consumers at the Demand Markets and the Equilibrium Conditions

In terms of the financial flows between the source agents and the demand markets and those of the intermediaries and the demand markets, we know that, ultimately, the flows accepted by the consumers at the demand markets must coincide with those "shipped out" by the former decision-makers. The consumers at the demand markets take into account in making their consumption decisions not only the price charged but also their transaction costs associated with obtaining the financial products. Note that we allow for the transaction costs to differ depending upon whether the financial product has been obtained in a physical or in an electronic manner and also whether it has been obtained from a source agent or a financial intermediary. See Figure 4 for the transactions associated with demand market k,

Let $\hat{c}_{jkl}$ denote the transaction cost associated with obtaining the financial product at demand market k from intermediary j via mode l. We assume that the transaction cost is continuous and is of the general form:

$$\hat{c}_{jkl} = \hat{c}_{jkl}(Q^2, Q^3), \quad \forall j, k, l, \tag{11}$$

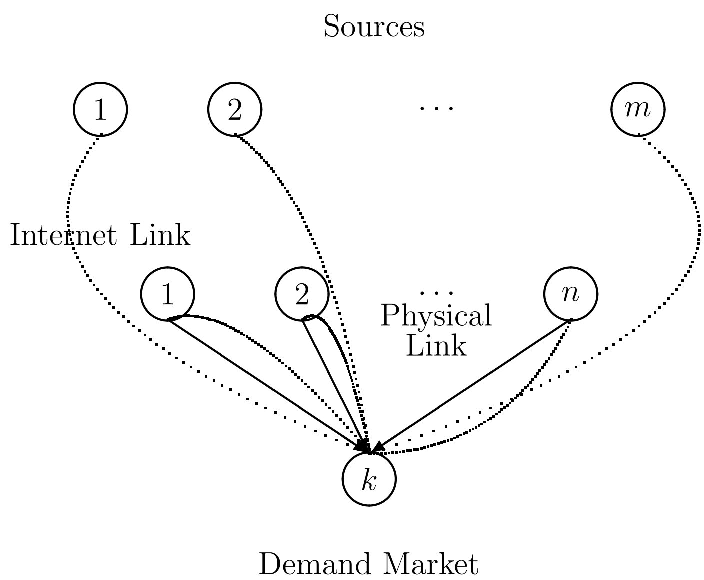

Figure 4: The Network Structure of Transactions at Demand Market k

that is, the cost of transacting, as perceived by consumers at a given demand market, can depend upon the volume of financial flows transacted either physically and/or electronically from intermediaries as well as from source agents. The generality of this cost function structure enables the modeling of competition on the demand side.

Furthermore, let $\hat{c}_{ik}$ denote the transaction cost associated with obtaining the financial product at demand market k from source agent i, where we assume that this transaction cost function is of the general form:

$$\hat{c}_{ik} = \hat{c}_{ik}(Q^2, Q^3), \quad \forall i, k. \tag{12}$$

Hence, the transaction cost associated with transacting directly with the source agents is of a form of the same level of generality as the transaction costs associated with transacting with the financial intermediaries (from the perspective of the consumers). Of course, depending upon the particular application, these functions may be simpler than the general form given above.

Let ρ^*_{3k} denote the price of the financial product at demand market k and group the demand market prices into the column vector $\rho_3 \in R^o_+$. Further, let d_k denote the demand for the product at demand market k and assume continuous demand functions of the general form:

$$d_k = d_k(\rho_3), \quad \forall k. \tag{13}$$

The equilibrium conditions for demand market k, thus, take the form: for all intermediaries: j; $j = 1, ..., n$ and all modes l; $l = 1, 2$:

$$\rho^*_{2jkl} + \hat{c}_{jkl}(Q^{2*}, Q^{3*}) \begin{cases} = \rho^*_{3k}, & \text{if} \quad q^*_{jkl} > 0 \\ \geq \rho^*_{3k}, & \text{if} \quad q^*_{jkl} = 0, \end{cases} \tag{14}$$

and for all source agents i; $i = 1, ..., m$:

$$\rho^*_{1ik} + \hat{c}_{ik}(Q^{2*}, Q^{3*}) \begin{cases} = \rho^*_{3k}, & \text{if} \quad q^*_{ik} > 0 \\ \geq \rho^*_{3k}, & \text{if} \quad q^*_{ik} = 0. \end{cases} \tag{15}$$

In addition, we must have that

$$d_k(\rho^*_3) \begin{cases} = \sum_{j=1}^{n} \sum_{l=1}^{2} q^*_{jkl} + \sum_{i=1}^{m} q^*_{ik}, & \text{if} \quad \rho^*_{3k} > 0 \\ \leq \sum_{j=1}^{n} \sum_{l=1}^{2} q^*_{jkl} + \sum_{i=1}^{m} q^*_{ik}, & \text{if} \quad \rho^*_{3k} = 0. \end{cases} \tag{16}$$

Conditions (14) state that the consumers at demand market k will purchase the financial product from intermediary j, transacted via mode l, if the price charged by the intermediary plus the transaction cost (from the perspective of the consumers) does not exceed the price that the consumers are willing to pay for the product. Conditions (15) state the analogue, but for the case of electronic transactions with the source agents. Condition (16), on the other hand, states that, if the price that the consumers are willing to pay for a financial product is positive, then the quantity of the product at the demand market is precisely equal to the demand.

In equilibrium, conditions (14), (15), and (16) will have to hold for all demand markets and these, in turn, can be expressed also as an inequality akin to (4) and (10) and given by: determine $(Q^{2^*}, Q^{3^*}, \rho^*_3) \in R^{2no+mo+n}$, such that

$$\sum_{j=1}^{n} \sum_{k=1}^{o} \sum_{l=1}^{2} \left[\rho^*_{2jkl} + \hat{c}_{jkl}(Q^{2^*}, Q^{3^*}) - \rho^*_{3k}\right] \times \left[q_{jkl} - q^*_{jkl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\rho_{1ik}^{*}+\hat{c}_{ik}(Q^{2^{*}},Q^{3^{*}})-\rho_{3k}^{*}\right]\times[q_{ik}-q_{ik}^{*}]$$

$$+\sum_{k=1}^{o}\left[\sum_{l=1}^{2}\sum_{j=1}^{n}q_{jkl}^{*}+\sum_{i=1}^{m}q_{ik}^{*}-d_k(\rho_3^{*})\right]\times[\rho_{3k}-\rho_{3k}^{*}]\geq 0, \quad \forall(Q^2,Q^3,\rho_3)\in R_{+}^{mo+2no+n}. \quad (17)$$

For further background on such a result, see Nagurney and Ke (2001).

The Equilibrium Conditions of the Financial Network with Electronic Transactions

In equilibrium, the financial flows that the source agents transact with the intermediaries must be equal to those that the intermediaries accept from the source agents. In addition, the amounts that are obtained by the consumers at the demand markets must be equal to the volume that both the source agents and the intermediaries transact with the demand markets. Hence, the equilibrium financial flow and price pattern must satisfy the sum of the optimality conditions (4), (10), and (17), in order to formalize the agreements between tiers of the financial network.

We now state this formally:

Definition 1: Financial Network Equilibrium with Electronic Transactions

The equilibrium state of the financial network with electronic transactions is one where the financial flows between tiers coincide and the financial flows and prices satisfy the sum of conditions (4), (10), and (17).

We now establish the following:

Theorem 1: Variational Inequality Formulation

The equilibrium conditions governing the financial network with electronic transactions are equivalent to the solution of the variational inequality given by:
determine $(Q^{1*},Q^{2*},Q^{3*},\gamma^*,\rho_3^*)\in\mathcal{K}$, *satisfying:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V_{z_{jl}}^{i}\cdot q_i^{*}+2V_{z_{il}}^{j}\cdot q_j^{*}+\frac{\partial c_{ijl}(q_{ijl}^{*})}{\partial q_{ijl}}+\frac{\partial c_j(Q^{1*})}{\partial q_{ijl}}+\frac{\partial \hat{c}_{ijl}(q_{ijl}^{*})}{\partial q_{ijl}}-\gamma_j^{*}\right]\times\left[q_{ijl}-q_{ijl}^{*}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^{i}_{z_{2n+k}}\cdot q_i^*+\frac{\partial c_{ik}(q_{ik}^*)}{\partial q_{ik}}+\hat{c}_{ik}(Q^{2*},Q^{3*})-\rho_{3k}^*\right]\times[q_{ik}-q_{ik}^*]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V^{j}_{z_{kl}}\cdot q_j^*+\frac{\partial c_{jkl}(q_{jkl}^*)}{\partial q_{jkl}}+\hat{c}_{jkl}(Q^{2*},Q^{3*})+\gamma_j^*-\rho_{3k}^*\right]\times\left[q_{jkl}-q_{jkl}^*\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2}q_{ijl}^*-\sum_{k=1}^{o}\sum_{l=1}^{2}q_{jkl}^*\right]\times\left[\gamma_j-\gamma_j^*\right]$$

$$+\sum_{k=1}^{o}\left[\sum_{l=1}^{2}\sum_{j=1}^{n}q_{jkl}^*+\sum_{i=1}^{m}q_{ik}^*-d_k(\rho_3^*)\right]\times[\rho_{3k}-\rho_{3k}^*]\geq 0,\quad \forall(Q^1,Q^2,Q^3,\gamma,\rho_3)\in\mathcal{K},\qquad(18)$$

where $\mathcal{K}\equiv\{K\times K^1\}$, *where* $K^1\equiv\{(Q^3,\gamma,\rho_3)|(Q^3,\gamma,\rho_3)\in R_+^{2no+n+o}\}$.

Proof: We first establish that the equilibrium conditions imply variational inequality (18). Indeed, the summation of (4), (10), and (17) yields, after algebraic simplification, the variational inequality (18).

We now establish the converse, that is, that a solution to variational inequality (18) satisfies the sum of conditions (4), (10), and (17), and is, hence, an equilibrium.

To inequality (18), add the term: $-\rho_{1ijl}^*+\rho_{1ijl}^*$ to the term in the first set of brackets, preceding the multiplication sign. Similarly, add the term: $-\rho_{1ik}^*+\rho_{1ik}^*$ to the term preceding the second multiplication sign, and, finally, add the term: $-\rho_{2jkl}^*+\rho_{2jkl}^*$ to the term preceding the third multiplication sign. Such "terms" do not change the inequality since they are identically equal to zero, with the resulting inequality of the form:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^{i}_{z_{jl}}\cdot q_i^*+2V^{j}_{z_{il}}\cdot q_j^*+\frac{\partial c_{ijl}(q_{ijl}^*)}{\partial q_{ijl}}+\frac{\partial c_j(Q^{1*})}{\partial q_{ijl}}+\frac{\partial \hat{c}_{ijl}(q_{ijl}^*)}{\partial q_{ijl}}-\gamma_j^*-\rho_{1ijl}^*+\rho_{1ijl}^*\right]\times\left[q_{ijl}-q_{ijl}^*\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^{i}_{z_{2n+k}}\cdot q_i^*+\frac{\partial c_{ik}(q_{ik}^*)}{\partial q_{ik}}+\hat{c}_{ik}(Q^{2*},Q^{3*})-\rho_{3k}^*-\rho_{1ik}^*+\rho_{1ik}^*\right]\times[q_{ik}-q_{ik}^*]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V^{j}_{z_{kl}}\cdot q_j^*+\frac{\partial c_{jkl}(q_{jkl}^*)}{\partial q_{jkl}}+\hat{c}_{jkl}(Q^{2*},Q^{3*})+\gamma_j^*-\rho_{3k}^*-\rho_{2jkl}^*+\rho_{2jkl}^*\right]\times\left[q_{jkl}-q_{jkl}^*\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2}q_{ijl}^*-\sum_{k=1}^{o}\sum_{l=1}^{2}q_{jkl}^*\right]\times\left[\gamma_j-\gamma_j^*\right]$$

$$+\sum_{k=1}^{o}\left[\sum_{l=1}^{2}\sum_{j=1}^{n}q_{jkl}^*+\sum_{i=1}^{m}q_{ik}^*-d_k(\rho_3^*)\right]\times\left[\rho_{3k}-\rho_{3k}^*\right]\geq 0,\quad \forall(Q^1,Q^2,Q^3,\gamma,\rho_3)\in\mathcal{K},\tag{19}$$

which, in turn, can be rewritten as:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V_{z_{jl}}^{i}\cdot q_i^*+\frac{\partial c_{ijl}(q_{ijl}^*)}{\partial q_{ijl}}-\rho_{1ijl}^*\right]\times\left[q_{ijl}-q_{ijl}^*\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V_{z_{2n+k}}^{i}\cdot q_i^*+\frac{\partial c_{ik}(q_{ik}^*)}{\partial q_{ik}}-\rho_{1ik}^*\right]\times\left[q_{ik}-q_{ik}^*\right]$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V_{z_{il}}^{j}\cdot q_j^*+\frac{\partial c_j(Q^{1*})}{\partial q_{ijl}}+\rho_{1ijl}^*+\frac{\partial\hat{c}_{ijl}(q_{ijl}^*)}{\partial q_{ijl}}-\gamma_j^*\right]\times\left[q_{ijl}-q_{ijl}^*\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V_{z_{kl}}^{j}\cdot q_j^*+\frac{\partial c_{jkl}(q_{jkl}^*)}{\partial q_{jkl}}-\rho_{2jkl}^*+\gamma_j^*\right]\times\left[q_{jkl}-q_{jkl}^*\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2}q_{ijl}^*-\sum_{k=1}^{o}\sum_{l=1}^{2}q_{jkl}^*\right]\times\left[\gamma_j-\gamma_j^*\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[\rho_{2jkl}^*+\hat{c}_{jkl}(Q^{2^*},Q^{3^*})-\rho_{3k}^*\right]\times\left[q_{jkl}-q_{jkl}^*\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\rho_{1ik}^*+\hat{c}_{ik}(Q^{2^*},Q^{3^*})-\rho_{3k}^*\right]\times\left[q_{ik}-q_{ik}^*\right]$$

$$+\sum_{k=1}^{o}\left[\sum_{l=1}^{2}\sum_{j=1}^{n}q_{jkl}^*+\sum_{i=1}^{m}q_{ik}^*-d_k(\rho_3^*)\right]\times\left[\rho_{3k}-\rho_{3k}^*\right]\geq 0,\quad \forall(Q^1,Q^2,Q^3,\gamma,\rho_3)\in\mathcal{K}.\tag{20}$$

But inequality (20) is equivalent to the price and financial flow pattern satisfying the sum of conditions (4), (10), and (17). The proof is complete. □

For easy reference in the subsequent sections, variational inequality problem (18) can be rewritten in standard variational inequality form (cf. Nagurney (1999)) as follows:

$$\langle F(X^*)^T, X-X^*\rangle\geq 0,\quad \forall X\in\mathcal{K},\tag{21}$$

where $X\equiv(Q^1,Q^2,Q^3,\gamma,\rho_3)$, and $F(X)\equiv(F_{ijl},F_{ik},F_{jkl},F_j,F_k)_{i=1,\ldots,m;j=1,\ldots,n;l=1,2;k=1,\ldots,o}$, and the specific components of F given by the functional terms preceding the multiplication signs in (18), respectively. The term $\langle\cdot,\cdot\rangle$ denotes the inner product in N-dimensional Euclidean space.

We now discuss how to recover the prices ρ^*_{1ijl}, for all i, j, l, and ρ^*_{2jkl}, for all j, k, from the solution of variational inequality (18). (In Section 5 we describe an algorithm for computing the solution.) Note from (10), that if $q^*_{jkl} > 0$, for some j, k, and l, then ρ^*_{2jkl} is precisely equal to $2V^i_{z_{kl}} \cdot q^*_i + \frac{\partial c_{jkl}(q^*_{jkl})}{\partial q_{jkl}} + \gamma^*_j$ or, equivalently, to (cf. (17)) $\rho^*_{3k} - \hat{c}_{jkl}(Q^{2*}, Q^{3*})$. The prices ρ^*_{1ijl}, in turn (cf. also (4)), can be obtained by finding a $q^*_{ijl} > 0$, and then setting $\rho^*_{1ijl} = \left[2V_{z_{jl}} \cdot q^*_i + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}}\right]$, or, equivalently (see (10)), to $\left[\gamma^*_j - 2V^j_{z_{il}} \cdot q_j - \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} - \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}}\right]$, for all such i, j, l. The prices ρ^*_{1ik}, on the other hand (cf. (4)), can be obtained by finding a $q^*_{ik} > 0$ and setting $\rho^*_{1ik} = \left[2V^i_{z_{2n+k}} \cdot q^*_i + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}}\right]$, or, equivalently (see (17)), to $\left[\rho^*_{3k} - \hat{c}_{ik}(Q^{2^*}, Q^{3^*})\right]$, for all such i, k.

We now construct the financial network in equilibrium (cf. Figure 1), using, as building blocks, the previously drawn networks in Figures 2 through 4 corresponding, respectively, to the transactions of the source agents, the intermediaries, and the demand markets. First, however, we need to establish the result that, in equilibrium, the sum of the financial flows to each intermediary is equal to the sum of the financial products out. This means that each intermediary, assuming profit-maximization, only obtains from the source agents the amount of financial flows that is actually consumed by the consumers. In order to establish this result, we utilize variational inequality (18). Clearly, we know that, if $\gamma^*_j > 0$, then the "market clears" for that intermediary, that is, $\sum_{i=1}^m \sum_{l=1}^2 q^*_{ijl} = \sum_{k=1}^o \sum_{l=1}^2 q^*_{jkl}$. Let us now consider the case where $\gamma^*_j = 0$ for some intermediary j. From the first term in inequality (18), since the transaction cost functions and handling cost functions have been assumed to be convex, and assuming further, which is not unreasonable, that either the marginal transaction costs or the marginal holding cost for each source agent/intermediary/mode combination is strictly positive at equilibrium, then we know that $2V^i_{z_{jl}} \cdot q^*_i + 2V^j_{z_{il}} \cdot q^*_j + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}} > 0$, which implies that $q^*_{ijl} = 0$, and this holds for all i, j, l. It follows then from the third term in (18), that $\sum_{k=1}^o \sum_{l=1}^2 q^*_{jkl} = 0$, and, hence, the market clears also in this case since the flow into an intermediary is equal to the flow out and equal to zero. We have thus, established the following:

Corollary 1

The market for the financial flows clears for each intermediary in the financial network with electronic transactions at equilibrium.

In Figure 1, we depict the structure of the financial network in equilibrium, consisting of all the source agents, all the intermediaries, and all the demand markets. Hence, we replicate Figure 2 for all source agents, Figure 3, for all intermediaries, and Figure 4 for all demand markets. These resulting networks represent the possible transactions of all the economic decision-makers. In addition, since there must be agreement between/among the transactors at equilibrium, the analogous links (and equilibrium flows on them) must coincide, yielding the network structure given in Figure 1.

In this Section, we have proposed an equilibrium framework for the formulation of financial problems with electronic transactions since we believe that the concept of equilibrium provides a valuable benchmark against which existing financial flows between tiers and prices at different tiers of the financial network can be compared. In Section 4, we propose a dynamic adjustment process, which is then formulated as a projected dynamical system, whose set of stationary points coincides with the set of solutions to the variational inequality problem (18). The dynamical system provides a means of addressing the disequilibrium dynamics associated with the financial network with multiple tiers.

Remark

It is worth comparing the financial network in Figure 1 with the supply chain network with electronic commerce constructed in Nagurney and Dong (2002) (see also Nagurney, Loo, Dong, and Zhang (2002)). In particular, we note that, in the case of the supply chain network, the shipments between tiers are that of a commodity. Moreover, there is no node to correspond to a non-investment node in the case of supply chains. Furthermore, in the supply chain context, the manufacturers, which are located at the top tier of the network seek to maximize profits, with risk not entering into their utility functions. In addition, in that supply chain model, only one commodity is considered, whereas in the financial model, distinct products associated with the distinct demand markets are treated. Also, in the supply chain context, the second tiered nodes correspond to retailers, which, in contrast to the financial intermediaries, do not "convert" incoming shipments into distinct products. Nevertheless, the supply chain framework with electronic commerce as discussed in the above citations, does capture both B2B (business to business) and B2C (business to consumer) transactions. In the financial setting, in contrast, the source agents at the top tier can be households and do not necessarily need to be businesses. Furthermore, the financial network in Figure 1 can be interpreted as a supply chain-type of network with distinct tiers of decision-makers and with electronic transactions in the former corresponding to electronic commerce in the latter.

3. Qualitative Properties

In this Section, we provide some qualitative properties of the solution to variational inequality (18). In particular, we derive existence and uniqueness results. We also investigate properties of the function F (cf. (21)) that enters the variational inequality of interest here.

Since the feasible set is not compact we cannot derive existence simply from the assumption of continuity of the functions. Nevertheless, we can impose a rather weak condition to guarantee existence of a solution pattern. Let

$$\mathcal{K}_b = \{(Q^1, Q^2, Q^3, \gamma, \rho_3) | 0 \leq Q^1 \leq b_1;\ 0 \leq Q^2 \leq b_2; 0 \leq Q^3 \leq b_3;\ 0 \leq \gamma \leq b_4;\ 0 \leq \rho_3 \leq b_5\}, \tag{22}$$

where $b = (b_1, b_2, b_3, b_4, b_5) \geq 0$ and $Q^1 \leq b_1; Q^2 \leq b_2; Q^3 \leq b_3; \gamma \leq b_4; \rho_3 \leq b_5$ means that $q_{ijl} \leq b_1$; $q_{ik} \leq b_2$; $q_{jkl} \leq b_3$; $\gamma_j \leq b_4$; and $\rho_{3k} \leq b_5$ for all i, j, l, k. Then $\mathcal{K}_b$ is a bounded closed convex subset of $R^{2mn+mo+2no+n+o}$. Thus, the following variational inequality

$$\langle F(X^b)^T, X - X^b \rangle \geq 0, \quad \forall X^b \in \mathcal{K}_b, \tag{23}$$

admits at least one solution $X^b \in \mathcal{K}_b$, from the standard theory of variational inequalities, since $\mathcal{K}_b$ is compact and F is continuous. Following Kinderlehrer and Stampacchia (1980) (see also Theorem 1.5 in Nagurney (1999)), we then have:

Theorem 2

Variational inequality (18) admits a solution if and only if there exists a $b > 0$, such that variational inequality (23) admits a solution in $\mathcal{K}_b$ with

$$Q^{1b} < b_1, \quad Q^{2b} < b_2, \quad Q^{3b} < b_3, \quad \gamma^b < b_4, \quad \rho_3^b < b_5. \tag{24}$$

Theorem 3: Existence

Suppose that there exist positive constants M, N, R with $R > 0$, such that:

$$2V^i_{z_{jl}} \cdot q_i + 2V^j_{z_{il}} \cdot q_j + \frac{\partial c_{ijl}(q_{ijl})}{\partial q_{ijl}} + \frac{\partial c_j(Q^1)}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q_{ijl})}{\partial q_{ijl}} \geq M,\ \forall Q^1 \ \textit{with}\ q_{ijl} \geq N,\ \forall i, j, l, \tag{25}$$

$$2V^i_{z_{2n+k}} \cdot q_i + \frac{\partial c_{ik}(q_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(Q^2, Q^3) \geq M, \quad \forall Q^2 \textit{ with } q_{ik} \geq N, \quad \forall i, k,$$

$$2V^j_{z_{kl}} \cdot q_j + \frac{\partial c_{jkl}(q_{jkl})}{\partial q_{jkl}} + \hat{c}_{jkl}(Q^2, Q^3) \geq M, \quad \forall Q^3 \textit{ with } q_{jk} \geq N, \quad \forall j, k,$$

$$d_k(\rho_3^*) \leq N, \quad \forall \rho_3 \textit{ with } \rho_{3k} > R, \quad \forall k. \tag{26}$$

Then variational inequality (18); equivalently, variational inequality (21), admits at least one solution.

Proof: Follows using analogous arguments as the proof of existence for Proposition 1 in Nagurney and Zhao (1993) (see also the existence proof in Nagurney and Ke (2001)). □

Assumptions (25) and (26) are reasonable from an economics perspective, since when the financial flow between a source agent and demand market pair or a source agent and intermediary is large, we can expect the corresponding sum of the associated marginal costs of handling and transaction from either the source agent's or the intermediary's perspectives as well as the transaction cost associated with the consumers, to exceed a positive lower bound. Moreover, in the case where the demand price of the financial product as perceived by consumers at a demand market is high, we can expect that the demand for the financial product at the demand market to not exceed a positive bound.

We now establish additional qualitative properties both of the function F that enters the variational inequality problem (cf. (21) and (18)), as well as uniqueness of the equilibrium pattern. Monotonicity and Lipschitz continuity of F will be utilized in Section 4 to establish qualitative properties of the projected dynamical system, whose set of stationary points coincides with the set of solutions of variational inequality (18). Since the proofs of Theorems 4 and 5 below are similar to the analogous proofs in Nagurney and Ke (2001) they are omitted here.

Theorem 4: Monotonicity

Suppose that the variance-covariance matrices V^i; $i = 1, \ldots, m$; and V^j; $j = 1, \ldots, n$, are positive definite and that the c_{ijl}, c_j, $\hat{c}_{ijl}$, c_{ik}, c_{ik}, and c_{jkl} functions are convex; the $\hat{c}_{jkl}$ and *the $\hat{c}_{ik}$ functions are monotone increasing, and the d_k functions are monotone decreasing functions, for all i, j, k, l. Then the vector function F that enters the variational inequality (21) is monotone, that is,*

$$\langle (F(X') - F(X''))^T, X' - X'' \rangle \geq 0, \quad \forall X', X'' \in \mathcal{K}. \tag{27}$$

Theorem 5: Strict Monotonicity

Assume all the conditions of Theorem 4. In addition, suppose that one of the families of convex functions $c_{ijl}; i = 1, ..., m; j = 1, ..., n; l = 1, 2; c_j; j = 1, ..., n; \hat{c}_{ijl}; i = 1, \ldots, m; j = 1, \ldots, n; l = 1, 2;$ and $c_{ik}; i = 1, \ldots, m; k = 1, \ldots, o,$ and $c_{jkl}; j = 1, \ldots, n; k = 1, \ldots, o;$ and $l = 1, 2,$ is a family of strictly convex functions. Suppose also that $\hat{c}_{ik}; i = 1, \ldots, m; k = 1, \ldots, o; \hat{c}_{jkl}; j = 1, ..., n; k = 1, ..., o; l = 1, 2,$ and -d_k; $k = 1, ..., o,$ are strictly monotone. Then, the vector function F that enters the variational inequality (21) is strictly monotone, with respect to (Q^1, Q^2, Q^3, ρ_3), that is, for any two X', X'' with $(Q^{1'}, Q^{2'}, Q^{3'}, \rho_3') \neq (Q^{1''}, Q^{2''}, Q^{3''}, \rho_3'')$

$$\langle (F(X') - F(X''))^T, X' - X'' \rangle > 0. \tag{28}$$

Theorem 6: Uniqueness

Assuming the conditions of Theorem 5, there must be a unique financial flow pattern (Q^{1*}, Q^{2*}, Q^{3*}), and a unique demand price price vector ρ_3^* satisfying the equilibrium conditions of the financial network with electronic transactions. In other words, if the variational inequality (21) admits a solution, then that is the only solution in (Q^1, Q^2, Q^3, ρ_3).

Proof: Under the strict monotonicity result of Theorem 5, uniqueness follows from the standard variational inequality theory (cf. Kinderlehrer and Stampacchia (1980)) $\square$

Theorem 7: Lipschitz Continuity

The function that enters the variational inequality problem (21) is Lipschitz continuous, that is,

$$\|F(X') - F(X'')\| \leq L\|X' - X''\|, \quad \forall X', X'' \in \mathcal{K}, \textit{ where } L > 0, \tag{29}$$

under the following conditions:

(i). c_{ijl}, c_j, $\hat{c}_{ijl}$, c_{ik}, c_{jkl} *have bounded second-order derivatives, for all* i, j, l, k*;*

(ii). $\hat{c}_{ik}$, $\hat{c}_{jkl}$, *and* d_k *have bounded first-order derivatives for all* i, j, l, k.

Proof: The result is direct by applying a mid-value theorem from calculus to the vector function F that enters the variational inequality problem (21). □

In the next Section, we utilize the Lipschitz continuity property in order to guarantee that the dynamic trajectories associated with the proposed continuous time adjustment process are well-defined. Monotonicity, on the other hand, is utilized to establish stability results for the financial network system.

4. The Dynamics

In this Section, we propose a dynamic adjustment process, formulated as a projected dynamical system. We then establish that the set of stationary points of the projected dynamical system coincides with the set of solutions of variational inequality (21), equivalently, variational inequality (18).

In particular, we now turn to describing the dynamics by which the source agents adjust their financial allocations over time, the consumers at the demand markets adjust their consumption amounts of the financial products based on the prices of the products at the demand markets, and the financial intermediaries operate between the two, except in the case of electronic transactions when the consumers at the demand markets can deal with the source agents directly. We also describe the dynamics by which the prices adjust over time.

We first describe the dynamics of the financial flows and then those of the prices. We begin with establishing some precursors to the derivation of the financial flow dynamics.

Precursors to the Derivation of the Dynamics of the Financial Flows

Recall that the source agent i's utility function is given by (3) and represents his net revenue to be maximized and the risk to be minimized. Clearly, in the case of *unconstrained* utility maximization, the gradient of source agent i's utility function with respect to the vector of variables q_i, and denoted by $\nabla_{q_i} U^i$, represents agent i's idealized direction, where

$$\nabla_{q_i} U^i = (\frac{\partial U^i}{\partial q_{i11}}, \ldots, \frac{\partial U^i}{\partial q_{in2}}, \frac{\partial U^i}{\partial q_{i1}}, \ldots, \frac{\partial U^i}{\partial q_{io}}),$$

with the jl-th component of $\nabla_{q_i} U^i$ given by:

$$\left[\rho_{1ijl} - 2V^i_{z_{jl}} \cdot q_i - \frac{\partial c_{ijl}(q_{ijl})}{\partial q_{ijl}} \right], \quad \text{for } j = 1, \ldots, n; l = 1, 2, \tag{30}$$

and the $2n + k$-th component of $\nabla_{q_i} U^i$ given by:

$$\left[\rho_{1ik} - 2V^i_{z_{2n+k}} \cdot q_i - \frac{\partial c_{ik}(q_{ik})}{\partial q_{ik}} \right], \quad \text{for } k = 1, \ldots, o. \tag{31}$$

On the other hand, an intermediary j also seeks to maximize his utility U^j as represented by the function (8), which denotes his net revenue to be maximized and his risk to be minimized. Ignoring for the moment (as we did in the case of a typical source agent) the underlying constraints, we note that intermediary j's idealized direction can be represented by the gradient of his utility function U^j, denoted by $\nabla_{q_j} U^j$, where

$$\nabla_{q_j} U^j = (\frac{\partial U^j}{\partial q_{1j1}}, \ldots, \frac{\partial U^j}{\partial q_{mj2}}, \frac{\partial U^j}{\partial q_{j11}}, \ldots, \frac{\partial U^j}{\partial q_{jo2}}),$$

with component il given by:

$$\left[-2V^j_{z_{il}} \cdot q_j - \frac{\partial c_j(Q^1)}{\partial q_{ijl}} - \frac{\partial \hat{c}_{ijl}(q_{ijl})}{\partial q_{ijl}} - \rho_{1ijl}\right], \quad \text{for } i = 1, \ldots, m; l = 1, 2, \tag{32}$$

and component $2m + kl$ given by:

$$\left[\rho_{2jkl} - 2V^j_{z_{kl}} \cdot q_j - \frac{\partial c_{jkl}(q_{jkl})}{\partial q_{jkl}}\right], \quad \text{for } k = 1, \ldots, 0; l = 1, 2. \tag{33}$$

However, since source agent i must agree with intermediary j as to the shipment q_{ijl} and must respond to the price signal γ_j associated with intermediary j, the addition of expression (30) and (32), with the response to the price signal, yields a "combined force" (see also Nagurney and Dong (2002)), which, after algebraic simplification, gives us:

$$\left[\gamma_j - 2V^i_{z_{jl}} \cdot q_i - 2V^j_{z_{il}} \cdot q_j - \frac{\partial c_{ijl}(q_{ijl})}{\partial q_{ijl}} - \frac{\partial c_j(Q^1)}{\partial q_{ijl}} - \frac{\partial \hat{c}_{ijl}(q_{ijl})}{\partial q_{ijl}}\right]. \tag{34}$$

Expression (34) reflects that the financial flow q_{ijl} will be responsive to the difference between the price signal at intermediary j and the marginal costs and marginal risks.

In addition, from the consumers' perspective, we have that an idealized direction is one where the flow of the financial product between an intermediary/demand market pair (j, k) transacted via mode l is given by:

$$\rho_{3k} - \hat{c}_{jkl}(Q^2, Q^3) - \rho_{2jkl}, \quad \text{for } l = 1, 2, \tag{35}$$

whereas that between a source agent i and demand market k would be given by:

$$\rho_{3k} - \hat{c}_{ik}(Q^2, Q^3) - \rho_{1ik}, \tag{36}$$

where we have ignored, for the time being, the constraints.

Furthermore, since the consumers at demand market k must agree with the intermediary j as to the financial flow q_{jkl}, adding (33) plus the term (35), and noting that they, in turn, must also be responsive to the price signals at the intermediaries, yields a "combined force" of

$$\rho_{3k} - 2V^{j}_{z_{kl}} \cdot q_j - \frac{\partial c_{jkl}(q_{jkl})}{\partial q_{jkl}} - \hat{c}_{jkl}(Q^2, Q^3) - \gamma_j. \tag{37}$$

Expression (37) states that the flow of the financial product between an intermediary/demand market pair transacted via mode l will be responsive to the difference between the price for the product at the demand market subtracted by the various "costs" and marginal risk and the price at the particular intermediary.

Agreement between the source agent i and demand market k, in turn, is given by the addition of the terms (31) and (36), yielding the combined force:

$$\rho_{3k} - 2V^{i}_{z_{2n+k}} \cdot q_i - \frac{\partial c_{ik}(q_{ik})}{\partial q_{ik}} - \hat{c}_{ik}(Q^2, Q^3). \tag{38}$$

The Dynamics of the Financial Flows between the Source Agents and the Intermediaries and Demand Markets

We are now ready to express the dynamics of the financial flows between the source agents and the intermediaries and demand markets. In particular, let the feasible set $K_i \equiv \{q_i | q_{ijl} \geq 0, \forall i, j, l, ; q_{ik} \geq 0, \forall k, \text{and (1) holds}\}$ (see also following (4)). Then K is the Cartesian product given by $K \equiv \Pi_{i=1}^{m} K_i$. Define F_{ijl} as minus the term in (34) and F_{ik} as minus the term in (38) with $F_i = (F_{i11}, \ldots, F_{in2}, F_{i1}, \ldots, F_{io})$.

Then the *best realizable* direction for the vector of financial flows q_i can be mathematically expressed as:

$$\dot{q}_i = \Pi_{K_i}(q_i, -F_i), \tag{39}$$

where $\Pi_\kappa(X, v)$ is defined as (see also Nagurney and Zhang (1996)):

$$\Pi_\kappa(X, v) = \lim_{\delta \to 0} \frac{P_\kappa(X + \delta v) - X}{\delta}, \tag{40}$$

and P_κ is the norm projection defined by

$$P_\kappa(X) = \text{argmin}_{X' \in \kappa} \|X' - X\|. \tag{41}$$

Indeed, expression (39) states that the financial flows associated with the source agents will evolve according to (34) and (38), while, at the same time guaranteeing that the financial flows do not become negative and that the constraint (1) is not violated for the source agent.

The Dynamics of the Financial Products between the Financial Intermediaries and the Demand Markets

The dynamics of the financial products between the financial intermediaries to the demand markets are now described.

The rate of change of the financial product q_{jkl} transacted via mode l is assumed to evolve according to (37), where, of course, one also must guarantee that these financial flows do not become negative. Hence, one may write:

$$\dot{q}_{jkl} = \begin{cases} \rho_{3k} - 2V^j_{z_{kl}} \cdot q_j - \frac{\partial c_{jkl}(q_{jkl})}{\partial q_{jkl}} - \hat{c}_{jkl}(Q^2, Q^3) - \gamma_j, & \text{if} \quad q_{jkl} > 0 \\ \max\{0, \rho_{3k} - 2V^i_{z_{kl}} \cdot q_j - \frac{\partial c_{jkl}(q_{jkl})}{\partial q_{jkl}} - \hat{c}_{jkl}(Q^2, Q^3) - \gamma_j\}, & \text{if} \quad q_{jkl} = 0, \end{cases} \tag{42}$$

where $\dot{q}_{jkl}$ denotes the rate of change of the financial product flow q_{jkl}.

Thus, according to (42), if the price the consumers are willing to pay for the financial product at a demand market exceeds the price the financial intermediaries charge plus the marginal transaction cost (from the perspective of the intermediaries) and the unit transaction cost (at an instant in time) plus the marginal risk associated with the intermediary/market/mode combination, then the volume of the financial product between that financial intermediary and demand market pair will increase; if the price charged by the financial intermediary plus the marginal transaction cost and the unit transaction cost plus the marginal risk, as described above, exceeds the price the consumers are willing to pay, then the volume of flow of the financial product between that pair will decrease.

The Demand Market Price Dynamics

We now turn to describing the dynamics underlying the prices of the financial products associated with the demand markets. Assume that the rate of change of the price ρ_{3k}, denoted by $\dot{\rho}_{3k}$, is equal to the difference between the demand at the demand market k, as a function of the demand market prices, and the amount available from the financial intermediaries and the source agents at the demand market. Hence, if the demand for the financial product at the demand market (at an instant in time) exceeds the amount available, the price at that demand market will increase; if the amount available exceeds the demand at the price, then the price at the demand market will decrease. Furthermore, it is guaranteed that the prices do not become negative. Consequently, the dynamics of the price ρ_{3k} associated with the financial product at demand market k can be expressed as:

$$\dot{\rho}_{3k} = \begin{cases} d_k(\rho_3) - \sum_{j=1}^{n}\sum_{l=1}^{2} q_{jkl} - \sum_{i=1}^{m} q_{ik}, & \text{if} \quad \rho_{3k} > 0 \\ \max\{0, d_k(\rho_3) - \sum_{j=1}^{n}\sum_{l=1}^{2} q_{jkl} - \sum_{i=1}^{m} q_{ik}\}, & \text{if} \quad \rho_{3k} = 0. \end{cases} \tag{43}$$

The Dynamics of the Prices at the Financial Intermediaries

The prices at the financial intermediaries, in turn, must reflect supply and demand conditions as well. In particular, assume that the price associated with financial intermediary j, γ_j, evolves over time according to:

$$\dot{\gamma}_j = \begin{cases} \sum_{k=1}^{o}\sum_{l=1}^{2} q_{jkl} - \sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}, & \text{if} \quad \gamma_j > 0 \\ \max\{0, \sum_{k=1}^{o}\sum_{l=1}^{2} q_{jkl} - \sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}\}, & \text{if} \quad \gamma_j = 0, \end{cases} \tag{44}$$

where $\dot{\gamma}_j$ denotes the rate of change of the price γ_j. Hence, if the amount of the product desired to be transacted by the consumers (at an instant in time) exceeds that available at the financial intermediary, then the price at the financial intermediary will increase; if the amount available is greater than that desired by the consumers, then the price at the financial intermediary will decrease.

The Projected Dynamical System

Consider now the dynamic model in which the financial flows from the source agents evolve according to (39) for all source agents i; $i = 1, \ldots, m$; the financial flows from the intermediaries to the demand markets evolve according to (42) for all intermediaries j; $j = 1, \ldots, n$, and demand markets k; $k = 1, \ldots, n$, and modes l; $l = 1, 2$, the prices associated with the intermediaries evolve according to (44) for all intermediaries j; $j = 1, \ldots, n$, and the demand market prices evolve according to (43) for all k; $k = 1, \ldots, o$.

Let X and $F(X)$ be defined as following (21). Then the dynamic model described by (39), (42), (44), and (43) for all i, j, k, l can be rewritten as the projected dynamical system (PDS) (cf. Nagurney and Zhang (1996)) defined by the following initial value problem:

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X_0, \tag{45}$$

where $\Pi_{\mathcal{K}}$ is the projection operator of $-F(X)$ onto $\mathcal{K}$ at X (cf. (40) and (41)) and $X_0 = (Q^{1^0}, Q^{2^0}, Q^{3^0}, \gamma^0, \rho_3^0)$ is the initial point corresponding to the initial financial flows between the sources and the intermediaries and the demand markets; the initial financial flows between the intermediaries and the demand markets; and the initial intermediaries' prices and the demand prices.

The dynamical system (45) is non-classical in that the right-hand side is discontinuous in order to guarantee that the constraints, which in the context of the above model are nonnegativity constraints on the variables, as well as the constraints (1) are not violated. Such dynamical systems were introduced by Dupuis and Nagurney (1993) and to-date have been used to model a variety of applications ranging from dynamic traffic network problems to dynamic oligopoly problems (cf. Nagurney and Zhang (1996)) and dynamic financial problems (see Nagurney and Siokos (1997) and the references therein).

Stationary/Equilibrium Points

The following theorem states that the projected dynamical system evolves until it reaches a stationary point, that is, $\dot{X} = 0$, in which there is no change in the financial flows and prices, and that the stationary point coincides with the equilibrium point of the financial network model according to Definition 1. The notation "*" is utilized here to denote an equilibrium point, as was also done in Section 2, as well as a stationary point, since these are shown to be equivalent in Theorem 8 below.

Theorem 8: The Set of Stationary Points Coincides with the Set of Equilibrium Points

The set of stationary points of the projected dynamical system (45) coincides with the set of equilibrium points as defined by Definition 1.

Proof: According to Dupuis and Nagurney (1993), the necessary and sufficient condition for X^* to be a stationary point of the PDS (45), that is, to satisfy:

$$\dot{X} = 0 = \Pi_{\mathcal{K}}(X^*, -F(X^*)), \tag{46}$$

is that $X^* \in \mathcal{K}$ solves the variational inequality problem:

$$\langle F(X^*)^T, X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{47}$$

where, in our problem, $F(X)$, X, and $\mathcal{K}$ are as defined following (21). Writing out (47) explicitly, we have that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^i_{z_{jl}} \cdot q^*_i + 2V^j_{z_{il}} \cdot q^*_j + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \gamma^*_j\right] \times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^i_{z_{2n+k}} \cdot q^*_i + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(Q^{2*}, Q^{3*}) - \rho^*_{3k}\right] \times \left[q_{ik} - q^*_{ik}\right],$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V^j_{z_{kl}} \cdot q^*_j + \frac{\partial c_{jkl}(q^*_{jkl})}{\partial q_{jkl}} + \hat{c}_{jkl}(Q^{2*}, Q^{3*}) + \gamma^*_j - \rho^*_{3k}\right] \times \left[q_{jkl} - q^*_{jkl}\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2} q^*_{ijl} - \sum_{k=1}^{o}\sum_{l=1}^{2} q^*_{jkl}\right] \times \left[\gamma_j - \gamma^*_j\right]$$

$$+\sum_{k=1}^{o}\left[\sum_{l=1}^{2}\sum_{j=1}^{n} q^*_{jkl} + \sum_{i=1}^{m} q^*_{ik} - d_k(\rho^*_3)\right] \times \left[\rho_{3k} - \rho^*_{3k}\right] \geq 0, \quad \forall (Q^1, Q^2, Q^3, \gamma, \rho_3) \in \mathcal{K}, \tag{48}$$

where $\mathcal{K} \equiv \{K \times K^1\}$, where $K^1 \equiv \{(Q^3, \gamma, \rho_3) | (Q^3, \gamma, \rho_3) \in R^{2no+n+o}_+\}$.

But variational inequality (48) is precisely the variational inequality (18) (and their corresponding $F(\cdot)$s, Xs, and $\mathcal{K}$s are one and the same), which, in turn, according to Theorem 1 coincides with $(Q^{1^*}, Q^{2^*}, Q^{3^*}, \gamma^*, \rho^*_3)$ being an equilibrium pattern according to Definition 1. The proof is complete. □

Hence, Theorem 8 establishes the linkage between the solution to the variational inequality problem (18) governing the static financial network model with electronic transactions described in Section 2, and the stationary points of the dynamic financial model described by the projected dynamical system (45). Indeed, it shows that they are one and the same. Thus, once a stationary point of the dynamic financial network model has been achieved, that point satisfies the equilibrium conditions, at which the source agents, the financial intermediaries, and the consumers associated with the demand markets for the financial products have formalized their agreements and the financial flows between the tiers coincide.

We now state the following theorem.

Theorem 9: Existence and Uniqueness of a Solution to the Initial Value Problem

Assume the conditions of Theorem 7. Then, for any $X_0 \in \mathcal{K}$, there exists a unique solution $X_0(t)$ to the initial value problem (45).

Proof: Lipschitz continuity of the function F is sufficient for the conclusion based on Theorem 2.5 in Nagurney and Zhang (1996).

Theorem 9 guarantees that, if the Lipschitz property is satisfied, then the disequilibrium dynamics associated with the proposed projected dynamical system model of the financial network are well-defined. In other words, given an initial financial flow and price pattern, there exists a unique trajectory associated with (45). Note that this existence and uniqueness result is not the same as those given in Theorems 3 and 6, respectively, since the latter results are for the equilibrium or stationary point, rather than for the dynamic trajectories.

We now provide a stability result. We first state the following:

Definition 2: Stability of the System

The system as defined by (45) is stable if for every X_0 and every equilibrium point X^*, the Euclidean distance $\|X^* - X_0(t)\|$ is a monotone decreasing function of t.

We now state a global stability result in the next Theorem.

Theorem 10: Stability of the Financial Network

Assume the conditions of Theorem 4. Then the dynamical system (45) underlying the financial network with intermediation and electronic transactions is stable.

Proof: Under the assumptions of Theorem 4, $F(X)$ is monotone and, hence, the conclusion follows directly from Theorem 4.1 of Zhang and Nagurney (1995). $\square$

From the preceding result, one sees that the financial network model developed in this paper (the dynamic version) is well-defined and, moreover, the financial network system is stable according to Theorem 10.

5. The Algorithm

In this Section, we consider the computation of solutions to variational inequality (18); equivalently, the stationary points of the projected dynamical system (45). The algorithm that we propose is the Euler method, which is a special case of the general iterative scheme proposed by Dupuis and Nagurney (1993) for the solution of projected dynamical systems. The algorithm not only yields a solution to variational inequality problem (18) but also provides a time discretization of the continuous time adjustment process (45). Conditions for convergence of this algorithm can be found in Dupuis and Nagurney (1993) and in Nagurney and Zhang (1996).

The Euler Method

Step 0: Initialization Step

Set $(Q^{10}, Q^{20}, Q^{30}, \gamma^0, \rho_3^0) \in \mathcal{K}$. Let $\tau = 1$, where τ is the iteration counter, and set the sequence $\{\alpha_\tau\}$ so that $\sum_{\tau=1}^{\infty} \alpha_\tau = \infty$, $\alpha_\tau > 0$, $\alpha_\tau \to 0$, as $\tau \to \infty$. (Such a sequence is required for convergence of the algorithm.)

Step 1: Computation Step

Compute $(Q^{1\tau}, Q^{2\tau}, Q^{3\tau}, \gamma^\tau, \rho_3^\tau) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[q_{ijl}^{\tau} + \alpha_\tau(2V_{z_{jl}}^{i} \cdot q_i^{\tau-1} + 2V_{z_{ik}}^{j} \cdot q_j + + \frac{\partial c_{ijl}(q_{ijl}^{\tau-1})}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1\tau-1})}{\partial q_{ijl}}\right.$$

$$\left. + \frac{\partial \hat{c}_{ijl}(q_{ijl}^{\tau-1})}{\partial q_{ijl}} - \gamma_j^{\tau-1}) - q_{ijl}^{\tau-1}\right] \times \left[q_{ijl} - q_{ijl}^{\tau}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[q_{ik}^{\tau} + \alpha_\tau(2V_{z_{2n+k}}^{i} \cdot q_i^{\tau-1} + \frac{\partial c_{ik}(q_{ik}^{\tau-1})}{\partial q_{ik}} + \hat{c}_{ik}(Q^{2\tau}, Q^{3\tau}) - \rho_{3k}^{\tau}) - q_{ik}^{\tau-1}\right] \times [q_{ik} - q_{ik}^{\tau}]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[q_{jkl}^{\tau} + \alpha_\tau(2V_{z_{kl}}^{j} \cdot q_j^{\tau-1} + \frac{\partial c_{jkl}(q_{jkl}^{\tau-1})}{\partial q_{jkl}} + \hat{c}_{jkl}(Q^{2\tau-1}, Q^{3\tau-1})\right.$$

$$\left. +\gamma_j^{\tau-1} - \rho_{3k}^{\tau-1}) - q_{jkl}^{\tau-1}\right] \times \left[q_{jkl} - q_{jkl}^{\tau}\right]$$

$$+\sum_{j=1}^{n}\left[\gamma_j^{\tau} + \alpha_\tau(\sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}^{\tau-1} - \sum_{k=1}^{o}\sum_{l=1}^{2} q_{jkl}^{\tau-1}) - \gamma_j^{\tau-1}\right] \times \left[\gamma_j - \gamma_j^{\tau}\right]$$

$$+\sum_{k=1}^{o}\left[\rho_{3k}^{\tau}+\alpha_{\tau}(\sum_{j=1}^{n}\sum_{l=1}^{2}q_{jkl}^{\tau-1}+\sum_{i=1}^{m}q_{ik}^{\tau-1}-d_k(\rho_3^{\tau-1}))-\rho_{3k}^{\tau-1}\right]\times[\rho_{3k}-\rho_{3k}^{\tau}]\geq 0, \tag{49}$$

$$\forall(Q^{1\tau},Q^{2\tau},Q^{3\tau},\gamma,\rho_3)\in\mathcal{K}.$$

Step 2: Convergence Verification

If $|q_{ijl}^{\tau}-q_{ijl}^{\tau-1}|\leq\epsilon$, $|q_{ik}^{\tau}-q_{ik}^{\tau-1}|$, $|q_{jkl}^{\tau}-q_{jkl}^{\tau-1}|\leq\epsilon$, $|\gamma_j^{\tau}-\gamma_j^{\tau-1}|\leq\epsilon$, $|\rho_{3k}^{\tau}-\rho_{3k}^{\tau-1}|\leq\epsilon$, for all $i=1,\cdots,m$; $j=1,\cdots,n$; $k=1,\ldots,o$; $l=1,2$, with $\epsilon>0$, a pre-specified tolerance, then stop; otherwise, set $\tau:=\tau+1$, and go to Step 1.

Note that the variational inequality subproblem (49) encountered at each iteration of the discrete-time algorithm can be solved explicitly and in closed form since it is actually a quadratic programming problem and the feasible set is a Cartesian product consisting of the product of K, which has a simple network structure, and the nonnegative orthants, R_+^{no}, R_+^n, and R_+^o, corresponding to the variables (Q^1,Q^2), Q^3, γ, and ρ_3, respectively.

Computation of Financial Flows and Products

In fact, the subproblem in (49) in the (Q^1,Q^2) variables can be solved using exact equilibration (cf. Dafermos and Sparrow (1969), Nagurney (1999)), whereas the remainder of the variables in (49) can be obtained by explicit formulae, which are provided below for convenience.

In particular, compute, at iteration τ, the q_{jkl}^{τ}s, according to:

$$q_{jkl}^{\tau}=\max\{0,q_{jkl}^{\tau-1}-\alpha_{\tau}(2V_{z_{kl}}^{j}\cdot q_j^{\tau-1}+\frac{\partial c_{jkl}(q_{jkl}^{\tau-1})}{\partial q_{jkl}}+\hat{c}_{jkl}(Q^{2\tau-1},Q^{3\tau-1})+\gamma_j^{\tau-1}-\rho_{3k}^{\tau-1})\},$$

$$\forall j,k,l. \tag{50}$$

Computation of the Prices

At iteration τ, compute the γ_j^τs according to:

$$\gamma_j^\tau = \max\{0, \gamma_j^{\tau-1} - \alpha_\tau(\sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}^{\tau-1} - \sum_{k=1}^{o}\sum_{l=1}^{2} q_{jkl}^{\tau-1})\}, \quad \forall j, \tag{51}$$

whereas the ρ_{3k}^τs are computed explicitly and in closed form according to:

$$\rho_{3k}^\tau = \max\{0, \rho_{3k}^{\tau-1} - \alpha_\tau(\sum_{j=1}^{n}\sum_{l=1}^{2} q_{jkl}^{\tau-1} + \sum_{i=1}^{m} q_{ik}^{\tau-1} - d_k(\rho_3^{\tau-1}))\}, \quad \forall k. \tag{52}$$

Note that in the discrete-time adjustment process, the financial flows and the prices can be updated simultaneously at each iteration.

This algorithm, hence, tracks the dynamic trajectory of the financial flows and prices until the stationary point; equivalently, the equilibrium point is reached.

6. Numerical Examples

In this Section, we apply the discrete-time algorithm (the Euler method) to several numerical financial examples of increasing complexity. The algorithm was implemented in FORTRAN and the computer system used was a DEC Alpha system located at the University of Massachusetts at Amherst. For the solution of the induced network subproblems in (Q^1, Q^2) we utilized the exact equilibration algorithm (see Dafermos and Sparrow (1969), Nagurney (1999), and Nagurney and Ke (2001)). The other subproblems in the Q^3, γ, and the ρ_3 variables were solved exactly and in closed form as described in Section 5 (cf. (50), (51), and (52)).

The convergence criterion used was that the absolute value of the financial flows and prices between two successive iterations differed by no more than 10^{-4}. For the examples, the sequence $\{\alpha_\tau\}$ was set to $\{\alpha_\tau\} = \{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\}$, which is of the form required by the algorithm for convergence.

We initialized the algorithm as follows: we set $q_{ij1} = \frac{S^i}{n}$ for each source agent i and all intermediaries j. All the other variables were initialized to zero.

Example 1

The first example consisted of two source agents, two intermediaries, and two demand markets, as depicted in Figure 5. In this numerical example, no electronic transactions were allowed. In subsequent examples, we include additional links to correspond to electronic transactions. Hence, Example 1 serves as a baseline.

The data for the first example were constructed for easy interpretation purposes. The financial holdings of the two source agents were: $S^1 = 20$ and $S^2 = 20$. The variance-covariance matrices V^i and V^j were equal to the identity matrices for all source agents i and all intermediaries j.

The transaction cost functions faced by the source agents associated with transacting with the intermediaries (cf. (2a)) were given by:

$$c_{ijl}(q_{ijl}) = .5q_{ijl}^2 + 3.5q_{ijl}, \quad \text{for } i = 1, 2; j = 1, 2; l = 1.$$

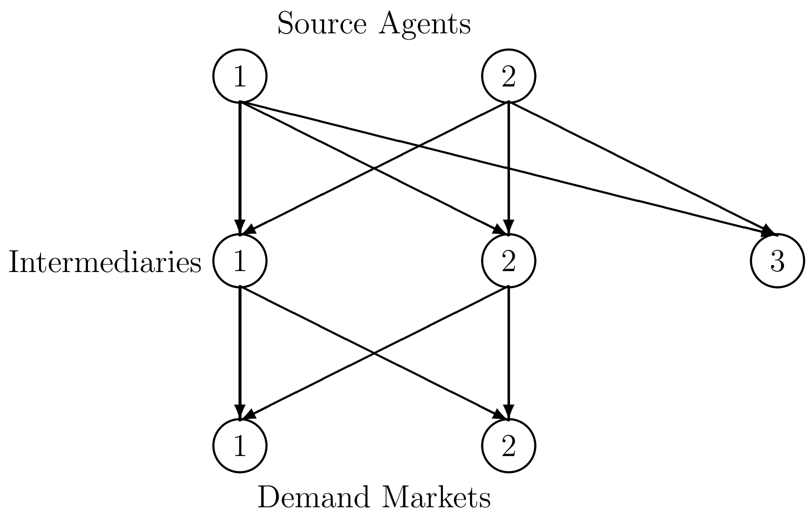

Figure 5: The Financial Network Structure of Example 1

The handling costs of the intermediaries, in turn (see (5)), were given by:

$$c_j(Q^1) = .5(\sum_{i=1}^{2} q_{ij1})^2, \quad \text{for } j = 1, 2.$$

The transaction costs of the intermediaries associated with transacting with the source agents were (cf. (6)) given by:

$$\hat{c}_{ijl}(q_{ijl}) = 1.5q_{ijl}^2 + 3q_{ijl}, \quad \text{for } i = 1, 2; j = 1, 2; l = 1.$$

The demand functions at the demand markets (refer to (13)) were:

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1000, \quad d_2(\rho_3) = -2\rho_{32} - 1.5\rho_{31} + 1000,$$

and the transaction costs between the intermediaries and the consumers at the demand markets (see (11)) were given by:

$$\hat{c}_{jkl}(q_{jkl}) = q_{jkl} + 5, \quad \text{for } j = 1, 2; k = 1, 2; l = 1.$$

We assumed for this and the subsequent examples that the transaction costs as perceived by the intermediaries and associated with transacting with the demand markets were all zero, that is, $c_{jkl}(q_{jkl}) = 0$, for all j, k, l.

The discrete-time algorithm converged and yielded the following equilibrium financial flow pattern:

$$Q^{1*} := q^*_{111} = q^*_{121} = q^*_{211} = q^*_{221} = 10.00,$$

$$Q^{3*} := q^*_{111} = q^*_{121} = q^*_{211} = q^*_{221} = 10.00.$$

Note that since there were no electronic transactions and, hence, only physical ones, the above vectors are only associated with the transaction costs on the physical links, as are the resulting equilibrium financial flows.

The vector γ^* had components: $\gamma^*_1 = \gamma^*_2 = 245.02$, and the computed demand prices at the demand markets were: $\rho^*_{31} = \rho^*_{32} = 280.00$.

It is easy to verify that the optimality/equilibrium conditions were satisfied with good accuracy. Note that in this example, constraint (1) was tight for both source agents, that is, there was zero flow on the links connecting node 3 with top tier nodes 1 and 2. Thus, it was optimal for both source agents to invest their entire financial holdings in each instrument made available by each of the two intermediaries.

Example 2

We then constructed the following variant of Example 1. We kept the data identical to that in Example 1 except that now we added links from the top tier nodes to the bottom tier nodes to represent electronic transactions between source agents and the consumers at the demand markets. The structure of the financial network for Example 2 was, hence, as depicted in Figure 6.

The transaction costs (cf. (2b) and (12)) associated with the new links were, respectively:

$$c_{ik}(q_{ik}) = \frac{1}{2}q^2_{ik} + q_{ik}. \quad \forall i, k,$$

$$\hat{c}_{ik}(Q^2, Q^3) = q_{ik} + 1. \quad \forall i, k.$$

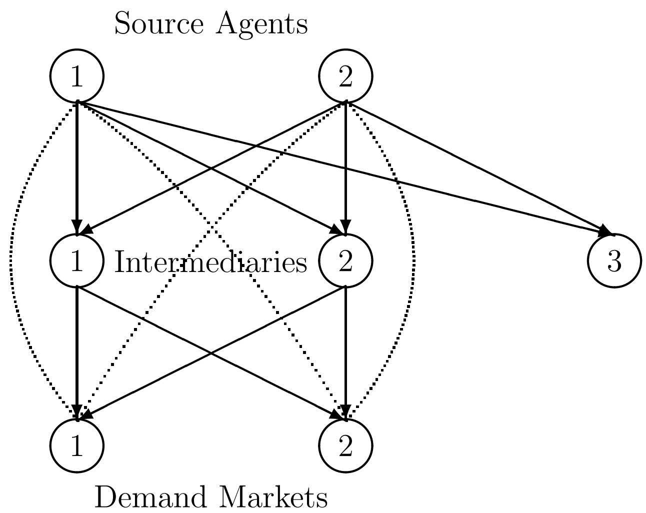

Figure 6: The Financial Network Structure of Example 2

The variance-covariance matrices were now expanded accordingly but still set to the identity matrices.

The discrete-time algorithm converged and yielded the following new equilibrium pattern:

$$Q^{1*} := q^*_{111} = q^*_{121} = q^*_{211} = q^*_{221} = 4.80,$$

$$Q^{2*} := q^*_{11} = q^*_{12} = q^*_{21} = q^*_{22} = 5.20,$$

$$Q^{3*} := q^*_{11} = q^*_{12} = q^*_{21} = q^*_{22} = 4.80.$$

The vector γ^* had components: $\gamma^*_1 = \gamma^*_2 = 260.59$, and the demand prices at the demand markets were: $\rho^*_{31} = \rho^*_{32} = 280.00$.

It is easy to verify that the optimality/equilibrium conditions, again, were satisfied with good accuracy.

Note that the introduction of the option of electronic transactions between source agents and the demand markets (with relatively low associated transaction costs), resulted in more than half of the financial holdings of each source agents being now transacted electronically directly with the consumers at the demand markets. Indeed, whereas in Example 1, each source agent allocated its full financial holdings evenly between each intermediary, now in Example 2, each intermediary obtained a total of 9.6 financial flows from the source agents (which were then evenly distributed between the two products at the demand markets), whereas a total of 10.4 (5.2 from each source agent) financial flows were obtained at each demand market via electronic transactions with the source agents.

Also, it is worth noting, that, as predicted by the theory (see Corollary 1), the sum of the flows to each intermediary (given by 4.8 plus 4.8) was precisely equal to the sum of the flows out (also given by 4.8 plus 4.8). Finally, note that the demand prices at the demand markets did not change whereas those associated with the financial intermediaries did, since now each handled a different volume of financial transactions than it had in Example 1.

Example 3

We then modified Example 2 as follows: The data were identical to that in Example 2 except that now we added links to represent electronic transactions between source agents and the intermediaries. Hence, the financial network was as depicted in Figure 7.

The transaction costs between the source agents and the financial intermediaries associated with the electronic transactions were given by:

$$c_{ij2}(q_{ij2}) = .5q_{ij2}^2 + .5q_{ij2}, \quad \text{for } i = 1,2; j = 1,2,$$

and

$$\hat{c}_{ij2}(q_{ij2}) = .5q_{ij2}^2 + .5q_{ij2}, \quad \text{for } i = 1,2; j = 1,2.$$

We, again, expanded the variance-covariance matrices but set them equal to the identity matrices. The discrete-time algorithm converged, yielding the following new equilibrium pattern:

$$Q^{1*} := q_{111}^* = q_{121}^* = q_{211}^* = q_{221}^* = 3.09; q_{112}^* = q_{122}^* = q_{212}^* = q_{222}^* = 3.33,$$

$$Q^{2*} := q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 3.57,$$

$$Q^{3*} := q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 6.42.$$

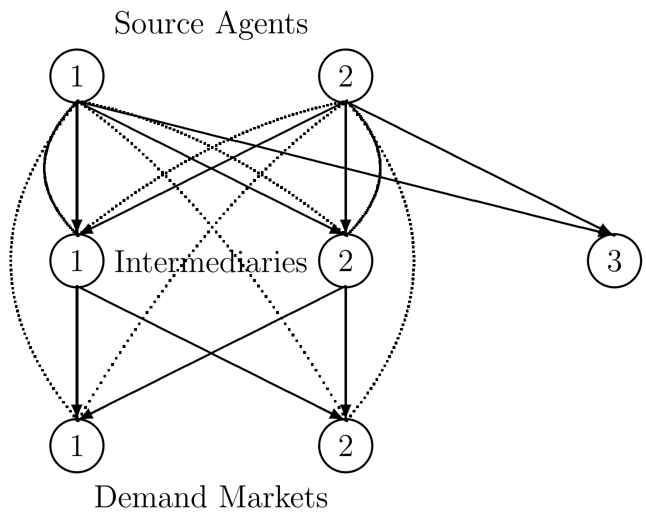

Figure 7: The Financial Network Structure of Example 3

The vector γ^* had components: $\gamma_1^* = \gamma_2^* = 255.72$, and the demand prices at the demand markets were: $\rho_{31}^* = \rho_{32}^* = 280.00$.

Note that in this example the vector Q^{1*} now includes financial volumes transacted electronically with the financial intermediaries. Interestingly, the availability of electronic transactions between source agents and financial intermediaries (at low transaction costs) now resulted in a portion of the volume of financial flows transacted electronically with the demand markets to be transacted with the financial intermediaries. Indeed, whereas in Example 2, the financial flows on the direct links between source agents and demand markets were all equal to 5.2, now, in Example 3, this volume was reduced to 3.57, with a portion of this difference going to the electronic transactions between the two top tiers of nodes. In addition, a portion of the financial volumes between the source agents and intermediaries that were allocated physically in Example 2 were now reallocated to electronic transactions since this option was now allowed. The demand prices associated with the financial products at the demand markets did not change, whereas those associated with the financial intermediaries did since the financial volumes that the intermediaries handled now increased.

It is worth noting that in this, as in the preceding examples, the contraint (1) held tightly for each source agent, that is, all the financial holdings were allocated.

7. Summary and Conclusions

In this paper, we developed a framework for the formulation, qualitative analysis, and computation of solutions to financial network problems with intermediation in the presence of electronic transactions. The financial network consists of a multi-tiered network in which non-investment is also permitted.

We described the behavior of the decision-makers consisting of the source agents, the financial intermediaries, and the consumers associated with the financial products at the demand markets and established the variational inequality formulation of the governing equilibrium conditions. We then established qualitative properties of the equilibrium financial flow and price pattern.

Subsequently, we proposed a dynamic adjustment process, formulated it as a projected dynamical system, and showed that its set of stationary points coincides with the set of solutions of the variational inequality problem. We established that the trajectory, under reasonable conditions, is well-defined and that the financial network system is stable. We then turned to the computation of solutions and proposed a discrete-time algorithm for the time discretization of the continuous time adjustment process. Finally, we applied the algorithm to several numerical financial examples.

This paper demonstrates that financial network problems with different tiers of decision-makers, notably, with the inclusion of financial intermediaries, and in the presence of electronic transactions can be formulated and studied in a rigorous fashion.

Future research will include the extension of this framework to the international arena.

Acknowledgments

This research was supported, in part, by NSF Grant No.: IIS-0002647 and, in part, by NSF Grant No.: CMS-0085720. This support is gratefully acknowledged.

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