A Retrospective on Beckmann, McGuire and Winsten's
Studies in the Economics of Transportation

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Abstract

This paper describes the impact and influence of the book, *Studies in the Economics of Transportation*, by M. Beckmann, C. B. McGuire and C. B. Winsten, published in 1956 by Yale University Press. The focus of this paper is on the book's impacts on innovations in modeling, methodological developments, and applications in transportation, regional science and other disciplines which continue to this day.

1. Introduction

"Demand refers to trips and capacity refers to flows on roads. The connecting link is found in the distribution of trips over the network according to the principle that traffic follows shortest routes in terms of average cost. The idea of equilibrium in a network can then be described as follows: ... the existing traffic conditions are such as to call forth the demand that will sustain the flows that create these conditions." (Beckmann, McGuire and Winsten 1955, p. 3.2; 1956, p. 59). In 1951 when Martin Beckmann, Bartlett McGuire and Christopher Winsten (hereafter BMW) began the research that resulted in *Part I* of *Studies in the Economics of Transportation*, their concept of traffic network equilibrium was novel. That they succeeded to formulate and extensively analyze a nonlinear optimization problem whose optimality conditions correspond to this statement (and related behavioral assumptions) was an enormous advance in the rigorous modeling of network traffic, a result never before achieved for urban traffic, and most unlikely for any complex system involving interactions of human behavior with technology. Moreover, they provided a parallel model and analysis for the case of cost-minimizing (now called *system optimum*) flows in a congested traffic network.

In 1956, their monograph, which included the study of railroad as well as highway transportation, was published by Yale University Press; it was issued earlier as Rand Corporation report, RM-1488-PR (hereafter Rand), on May 12, 1955. The research was conducted at the Cowles Commission for Research in Economics, then located at the University of Chicago, and led by T. C. Koopmans, who in his *Introduction* described the report as "exploratory studies ... addressed to ... various professions, including economists, traffic and railroad engineers, management scientists, operations researchers and mathematicians" (p. xi; Rand, p. 1). Beckmann, a mathematical economist, was especially interested in linear programming and economic activity analysis. Winsten, a mathematician and economist, had an interest in applying probability concepts to industrial issues, whereas McGuire, an economist, provided a pragmatic and realistic check on the model development. Special thanks in Koopman's *Introduction* were extended to George Dantzig, Harry Markowitz and William Vickrey, the last having recommended the manuscript for publication. Decades later, Koopmans, Markowitz and Vickrey were awarded Nobel Memorial Prizes in Economic Science, and Dantzig received the (U.S.) National Medal of Science.

Altogether, the book had three printings as well as a Spanish edition by 1959, and received eight favorable reviews. Smeed (1957) in *The Economic Journal* noted "it is refreshing to read a book that attempts to tackle the subject of road transport in a comprehensive and fundamental way." Thrall (1958) in *Econometrica* found it "useful" as an example of operations research; "good illustrative material; a welcome addition to the literature." Camp (1959) in *Operations Research* wrote that Chapters 1 to 4 "furnish a clear picture of a highway system as a servomechanism with complex feedback interactions among its parts, activated by the behavior of many drivers ... a 'must' in the library of any operations researcher interested in highway or railroad transportation." Peterson in the *Wall Street Journal* on January 2, 1957, stated "this is a 'heavy' theoretical work by a group of economists searching for the optimum efficiency of railroad and highway systems." However, no reviewer recognized the true significance of the network equilibrium formulation achieved in *Part I* of the book, and no one linked that formulation to the need to forecast travel for the urban transportation planning studies that began in the mid-1950s in the US and subsequently spread around the world, studies which sought to solve computationally the same problem formulated by the authors.

Moreover, it took over a decade before the book's contributions gained the attention of researchers in the US and France. At that time no journals and few conferences were devoted to basic research on transportation; information technology was in its infancy and the Internet for the timely dissemination of results was effectively unimaginable. Nevertheless, the impact of the book has been seminal, far-reaching, and continues to this day on disciplines that did not even exist half a century ago, such as computer science.

Some Background

Students of regional science and transportation know that the topic of transportation had been studied prior to the 1950s in the context of optimal allocation of resources through linear programming by Hitchcock (1941) and Kantorovich (1942), who later shared the Nobel Memorial Prize with Koopmans, as well as by Koopmans (1947) and Dantzig (1951). However, there is no congestion in these classical transportation-type linear programming models. The problem of users of a congested transportation network seeking to determine their routes of travel from their origins to respective destinations, interestingly, was considered by Kohl (1841) and by Pigou (1920) for a two-node, two-link (or route) network, with further developments by Knight (1924). Both Pigou and Knight are cited in BMW. Fascinatingly, Koopmans in his introduction also acknowledged the work of Enke (1951) and Samuelson (1952) in terms of commodity transportation and the determination of interregional price differentials, a topic now known as spatial price equilibrium.

Wardrop (1952, p. 345), concurrently with the research of BMW, stated two widely-quoted criteria of traffic network utilization, which have come to be termed, respectively, user-optimization and system-optimization (Dafermos and Sparrow 1969). Wardrop's first criterion expresses that if travelers select their routes of travel from origins to destinations independently, at equilibrium "the journey times of all routes actually used are equal and less than those which would be experienced by a single vehicle on any unused route." The user-optimized solution is also referred to in the literature as a *traffic network equilibrium* or a *traffic assignment*. His second criterion, in contrast, reflects the situation of a central controller who routes flows from origins to destinations so that "the average journey time is a minimum." BMW showed that the minimum is reached when the marginal costs on used routes connecting each origin-destination pair are equal and minimal.

BMW were the first to provide a rigorous mathematical formulation of the conditions described by Wardrop's first criterion that allowed for the ultimate solution of the traffic network equilibrium problem in the context of certain link cost functions which are increasing functions of the flows on the links. In particular, they demonstrated that the optimality conditions in the form of Kuhn-Tucker (1951) conditions of an appropriately constructed optimization problem coincided with Wardrop's first principle. Hence, no traveler, acting unilaterally will have any incentive to alter his route (assuming rational cost (time)-minimizng behavior) since his travel cost (travel time) is minimal. Independently, Charnes and Cooper (1958, 1961) in papers on traffic network equilibrium with fixed origin-destination flows noted the relation to Nash (1951). Earlier, Prager (1954) had informally described a formulation related to Wardrop's principles, and emphasized that the travel time on a link may depend upon the flows on other links.

Thus, as shown by BMW, a traffic network problem in which there are numerous decision-makers acting independently and, as later also noted by Dafermos and Sparrow (1969), competing in the sense of Nash, could be reformulated (under appropriate assumptions) as a convex optimization problem with a single objective function subject to linear constraints and nonnegativity assumptions on the flows on the network. Jorgensen (1963) in his Master's thesis did not cite BMW, but noted the work of Wardrop (1952) and Charnes and Cooper (1961), and independently proposed an optimization formulation of the traffic network equilibrium conditions in the case of fixed travel demands and separable link cost functions. His thesis and Pigou (1920) later influenced the Ph.D. thesis of Dafermos (1968), the basis of their now classical paper, Dafermos and Sparrow (1969).

In this article, we discuss some of the impacts of BMW. We first identify, in Section 2, some of the major transportation-based innovations in modeling and methodological developments, since these two have often been intimately linked. We then, in Section 3, focus on innovations in applications to other fields which continue to this day. Throughout we include direct quotes from BMW to emphasize its timeliness and enduring qualities.

2. Transportation-Based Innovations Stemming from BMW

We now describe some of the major innovations in modeling and methodology that were motivated by BMW. Our emphasis in this section is on transportation and the contributions are presented more or less in chronological order to provide a progression of the intellectual developments and a timeline. How the book has contributed to, directly or indirectly, and anticipated innovations in several additional areas is subsequently highlighted.

Algorithms for Solving the Standard Model of BMW

Despite the extent of their accomplishments, BMW did not propose an algorithm for solving their formulations computationally. As revealed by interviews in 1998 and 1999, the authors were aware of this need, but determined that computers were at such an early stage of development that such efforts were impractical. However, Carroll (1959) reported that first generation mainframe computers were used to solve what amounts to a crude analog of the model of BMW using shortest routes methods (then) recently devised by Moore (1957).

The first major methodological innovation in a paper that cites BMW is Dafermos and Sparrow (1969) who coined the terms "user-optimized" and "system-optimizing" to describe Wardrop's first and second principles and to clarify the underlying behavior of travelers; they also proposed algorithms for solving problems with nonlinear cost (congestion) functions that explicitly exploited the network structure of these two problems and established convergence results for the schemes. Moreover, their paper not only provided solution algorithms for networks of any topology, but also special-purpose ones in the case of topologies for which the flows could be computed exactly and in closed form. Tomlin (1966) had earlier considered linear cost functions, and exploited that feature in the development of his algorithm. Almond (1967) had constructed an algorithm for the determination of the user-optimized solution but only in the case of very simple networks. She cited BMW whereas Tomlin did not, although he did refer to Jorgensen (1963). Subsequently, Leventhal et al. (1973) proposed embedding a column generation procedure in the general equilibration procedures of Dafermos and Sparrow to allow for route generation as needed, rather than a priori which could require large computer memory resources.

The first innovations in algorithms for traffic network equilibrium problems with exogenous origin-destination flows focused on the standard model of BMW, where *standard* means the link cost functions are separable, that is the cost on a link depends upon only the flow on that link. Bruynooghe et al. (1969) investigated two algorithms for this case. Netter (1972) described the properties of system-optimized versus user-optimized solutions. LeBlanc et al. (1975) proposed a linearization algorithm based on Frank and Wolfe (1956), which was later widely implemented in practitioner software. Nguyen (1974, 1976) devised, implemented and compared three algorithms. The contributions of BMW were recognized in 1974 at the International Symposium on Traffic Equilibrium Methods in Montreal organized by Florian (1976).

Additional algorithms are described by Sheffi (1985), Nagurney (1993) and Patriksson (1994), which contains an extensive list of references, and in the survey by Florian and Hearn (1995). Bar-Gera (1999, 2002) recently proposed an origin-based assignment algorithm, which is especially effective for solving large-scale network problems precisely.

Initial attempts in the late 1960s to solve the problem posed by BMW focused on the case of fixed demand; although BMW did not formulate this case mathematically, it is, nevertheless, discussed. In Section 4.2.1 (pp. 83-86; Rand, pp. 3.23-3.30), the authors treat graphically cost minimization on a network with two roads and fixed demand. Section 4.3.2 (pp. 91-92; Rand, pp. 4.19-4.21) provides a mathematical analysis of the minimization of transportation costs on a network with fixed travel demands. Finally, Section 4.5.1 (pp. 99-101; Rand, pp. 4.31-4.34) presents an analysis of the tolling of two roads with a fixed number of users who differ in their values of time.

Efforts to construct algorithms for the more general, and significant, variable demand problem were actually stimulated by the four-step or sequential procedure (generation, distribution, mode split, and assignment) devised by practitioners in the late 1950s for forecasting travel. Two researchers began exploring how to "combine" the distribution and assignment steps into an integrated model, only to rediscover BMW's original and more general formulation. Murchland (1970b) made an initial attempt, but it was Evans (1973, 1976) who proposed a partial linearization scheme which elegantly provided an efficient, convergent algorithm. Florian and Nguyen (1978) also studied this problem and proposed a full linearization scheme. Bar-Gera and Boyce (2003) incorporated Bar-Gera's origin-based algorithm into the variable demand problem using fixed point concepts, and achieved very rapid convergence, as compared with Evans's algorithm and the traditional sequential procedure with feedback.

Toll Policies

BMW proposed “efficiency tolls,” such that by “charging everyone a toll equal to his contribution to the total cost of others, road users can be induced to make an efficient use of the available capacity” and considered how tolls could be constructed in the case of a simple network (cf. p. 94; Rand, pp. 4.23-4.24). Earlier formulations had included those of Pigou (1920) and Vickrey (1952) whose “contributions to the economic theory of incentives under asymmetric information,” which included the efficient pricing of public services, later earned him a Nobel Prize. Walters (1961) utilized the network model of BMW for toll determination. Beckmann (1967a) further described optimal tolls for highways, tunnels, and bridges. Dafermos and Sparrow (1971) and Dafermos (1973) described optimal resource allocation and tolls that guarantee that, once assigned, the user-optimized solution coincides with the system-optimized solution, so that individual travelers behave in a manner that is also optimal from a system or societal point of view.

Congestion pricing through tolls continues today as an area of active research. Bergendorff et al. (1997) suggested a new approach which spawned a number of theses and papers. Verhoef (2002) examined another approach to tolling links in a congested network in which not all links can be tolled. Arnott (2001) proposed a new research agenda to move the economic theory of traffic congestion initiated by BMW and others to the microscopic level, and Small and Chu (2003) considered a model with more realistic cost functions that has its origins in BMW. Time-of-day tolling schemes are being applied with some success in Singapore and London.

Extended Traffic Network Models Including Models of Urban Location

BMW focused on transportation networks in which the cost (or travel time) on a link, that is, a road segment, depended solely upon the flow on that link. Under such an assumption (i.e., separable functions and necessarily symmetric) they could then prove their fundamental result. They noted, however, in their final chapter that included a discussion of “unsolved problems” that (cf. p. 109; Rand, p. 5.13) “there are differences between classes of road users in their contribution to delays and to the deterioration of the road” thus also anticipating the need for multiclass/multimodal traffic network modeling.

Dafermos in a series of papers in the early 1970s developed “extended” traffic network models, whose need was noted by Prager (1954), and also formulated tolls in the case of multiclass networks. In particular, Dafermos (1971, 1972) demonstrated that an analogous reformulation of the traffic network equilibrium conditions as a convex optimization problem could be identified in the case of more general user link cost functions in which the cost on a link could depend on the flows on all links in the network, provided that a symmetry condition held either in the single-class user case or the multi-class user case, which allowed for different classes of travelers who perceive the travel cost on a link in an individual manner. In addition, extensions of the general equilibration algorithms contained in Dafermos and Sparrow (1969), along with convergence results, were obtained.

BMW clearly delineated that one should distinguish between short-run and long-run decision-making regarding transportation networks. Dafermos (1976) demonstrated, through the use of abstract networks, that one could capture long-run decision-making according to BMW in the form not only of route choice but also of origin and/or destination choice within a network equilibrium context. For additional abstract network constructs, see Nagurney and Dong (2002).

Boyce (1980) proposed a framework for constructing network equilibrium models of urban location, which allowed for the incorporation of the trip distribution problem. Boyce et al. (1983), motivated by BMW and Evans (1976), presented a unified approach to deriving models of urban location, destination, mode, and route choice; see also Erlander and Stewart (1990). Calibration of model parameters and estimation of coefficients of generalized link cost functions were also described. Boyce and Bar-Gera (2003) implemented selected parts of the framework for the multi-class case.

Variational Inequality Formulations and Algorithms

Smith (1979) provided an alternative formulation of traffic network equilibrium, which Dafermos (1980) identified as a variational inequality problem. These fundamental papers based on BMW enabled the modeling, analysis, and computation of solutions to traffic network equilibrium problems in which the assumption of a symmetric Jacobian is no longer needed. Simply stated, this condition means that the cost on one link depends on the flow on another link in the same fashion that the cost on the other link depends on the first link's flow. In this case, important from the application standpoint, one could no longer reformulate the network equilibrium conditions as a solution to an optimization problem. This recognition, along with rigorous computational schemes (cf. Dafermos 1980, 1982, 1983) allowed for the solution of more general traffic network problems than had been possible. Moreover, since variational inequality problems are typically solved as series of optimization problems, advances in the development of the solution of symmetric traffic network problems could be applied to more general problem settings. The variational inequality framework revolutionized the formulation, analysis, and solution of network equilibrium problems, generally, as well as other equilibrium problems.

Interestingly, variational inequalities were originally developed for the study of partial differential equations for problems drawn from mechanics (cf. Kinderlehrer and Stampacchia, 1980 and the references therein). In Chapter 3 (p. 63; Rand, p. 3.8) BMW stated that: "A well known characteristic of the equilibrium encountered in theoretical mechanics is that they may be regarded as solutions to certain extremum problems, a fact which has occasionally given rise to some speculation about nature's grand design."

BMW specifically emphasized elastic demand traffic network problems and developed a model which allowed for the prediction not only of the traffic volumes on the links or roads of the network but also the travel demand associated with the origin-destination pairs. Hence, there may be times when travelers opt not to travel at all due to the cost associated with congestion. As noted in Chapter 2 (p. 51; Rand p. 2.19), "In general the demand for transportation between a given pair of locations would depend also on the costs or travel times to and from other locations." Aashtiani and Magnanti (1981) considered a similar model, treating it as a nonlinear complementarity problem. Dafermos (1982), recognizing the generality of the elastic demand traffic network model, proposed a multiclass, asymmetric model and formulated and solved it as a variational inequality problem.

Dynamic Transportation Networks

In the final chapter of *Part I* entitled, "Conclusion; Some Unsolved Problems," BMW stated (p. 107; Rand, p. 5.9) that "The notion of a static equilibrium of flow in a network may be thought of as somewhat limited..." They proceeded to say that "While the equilibrium mechanism is operative during the relatively short periods of a constant load one would like to see a more comprehensive model which contributes to our understanding of the time pattern itself." Moreover "While it is not difficult, by attaching time subscripts to the flow variables, to write down formally the equilibrium conditions ... for a dynamic model this merely makes the analysis more complicated without explaining much that is new. An understanding of dynamic aspects of the traffic really depends on an understanding of demand substitution over time."

Hence, although BMW did not explicitly formulate dynamic traffic network models, the recognition of the importance of such models was explicit in the book. Yagar (1971), Hurdle (1974), and Merchant and Nemhauser (1978a, b) were some of the first contributors to the development of dynamic models with explicit flows, and the work of Merchant and Nemhauser (1978a, b) is often credited with being the first to consider dynamic route choices over general networks. In particular, they studied dynamic system-optimized networks in the case of single destination networks and athough they did not cite BMW, they did reference Dafermos and Sparrow (1969). Mahmassani and Herman (1984), in turn, citing BMW, built upon the work of Hendrickson and Kocur (1981), and generalized it to the situation where a user can adapt to congestion by not only changing his departure time but also by changing routes. Carey (1987), referencing BMW, provided a convex programming formulation of a dynamic system-optimized traffic network which could handle multiple destinations and multiple commodities.

Today, variational inequality theory has since become the theoretical basis for the analysis and computation of Wardrop equilibria in a within-day static traffic network. Motivated by concerns for realism, the within-day dynamic traffic assignment problem is receiving much attention (cf. among others, Janson 1991, Smith 1993, Friesz et al. 1993, Ran and Boyce 1996, and Wu et al. 1998). Underlying a dynamic user equilibrium is a "doubly" dynamic system which is comprised of a day-to-day adjustment process and a within-day realization process. The day-to-day adjustment process addresses user behavior in acquiring information and in adjusting their departure time and route choices; see, e. g., Smith (1984), Mahmassani et al. (1986), Mahmassani (1990), Friesz et al. (1994), Zhang and Nagurney (1996), Nagurney and Zhang (1997). The within-day realization process addresses the real time dynamic traffic flow as the realization of the users' route choices on the particular day, which, in turn, results in updated information feedback to the day-to-day process. A dynamic loading operation (Wu et al. 1998) is involved in this realization process that loads the dynamic route departure rates into dynamic link volumes which determine the dynamic link travel times as the feedback to the travelers. For some additional insights, see Cantarella and Cascetta (1995) and Zhang et al. (2001).

Dupuis and Nagurney (1993), motivated in great part by the need to introduce dynamics into the formal modeling and analysis of network systems, including transportation networks, that had been studied primarily at an equilibrium state, developed the basic theory of existence and uniqueness as well as computational procedures for what are now termed "projected dynamical systems" (cf. Nagurney and Zhang 1996). Importantly, the set of equilibrium states and, hence, solutions to a variational inequality problem coincides with the set of stationary points of a particular non-classical dynamical system. Hence, we see that both the methodologies of (finite-dimensional) variational inequality theory and projected dynamical systems theory can trace the origins of their ultimate development, evolution, and, finally, application, back to the traffic network equilibrium ideas of BMW.

Stability Analysis and Sensitivity Analysis

The importance of stability analysis was also recognized in BMW who stated (p. 70; Rand, p. 3.19): "An equilibrium would be just an extreme state of rare occurrence if it were not stable" "Besides the stability 'in the small' one may consider 'stability in the large' ... This latter type of stability is interesting ... because one may want to use an analog of the adjustment process as a method of computing an equilibrium solution by successive approximations." Furthermore, BMW (p. 103; Rand, p. 103) emphasized the relevance of the equilibrium concept and stated that "the attainment of ... an equilibrium does not exclude altogether the occurrence of fluctuations around the equilibrium points ... All that equilibrium means is that there is some constant time average of delays, and that this mean value is all one can predict about traffic conditions in the future."

Dafermos and Sparrow (1969) obtained stability analysis results in the context of user-optimized models in the static setting. Stability analysis using Lyapunov functions was addressed by Smith (1979, 1984). More recently, Nagurney and Zhang (1996), motivated by the connection between finite-dimensional variational inequality problems and dynamical systems, as discussed above, obtained local and global stability analysis results for dynamic traffic network problems modeled as projected dynamical systems.

Sensitivity analysis is also central to decision-making and, in particular, to the planning of transportation networks. Interestingly, Braess (1968), whose well-known paradox motivated much of the subsequent research in sensitivity analysis and networks, cited neither Wardrop (1952) nor BMW and constructed his paradox without familiarity with these publications (cf. Braess 2003). That paper was followed by the contribution of Murchland (1970a), who elaborated upon the Braess paradox and reflected upon it in the context of BMW and Beckmann (1967b). Fisk (1979) also cited BMW and identified additional paradoxical phenomena in traffic networks. Stewart (1980) and Steinberg and Zangwill (1983) further spurred the investigation of sensitivity analysis in network equilibrium problems.

Dafermos and Nagurney (1984 a, b, c)) obtained sensitivity analysis and stability results for a variety of network equilibrium problems in a variational inequality framework. Dafermos and Nagurney (1984a) demonstrated how, in terms of traffic networks with general (asymmetric) user link cost functions, the addition of a route connecting an origin-destination pair that shared no links with any other route in the network, could never result in the Braess paradox. Moreover, they provided explicit formulae for the effects of cost function and demand changes on the incurred flows and route travel costs. Dafermos and Nagurney (1984b), in turn, used the variational inequality formulation of traffic network equilibrium with fixed demands to provide directional effects of link cost function changes and to demonstrate that small changes in the data yielded small changes in the resulting equilibrium link flows. Dafermos and Nagurney (1984c) provided similar results but in the context of a general spatial price equilibrium problem.

Braess and Koch (1979) established existence results in the case of multiclass user-optimized networks in which the symmetry assumption does not hold using fixed point arguments; they referenced Dafermos (1971, 1972). In addition, they highlighted the importance of stability in the case of multiple equilibria. For additional references see Nagurney (1993).

Today, paradoxes on networks, due to alternative behaviors of decision-makers, notably that of user-optimization, which reflects non-cooperation, are garnering increasing attention in other scientific domains, including that of computer science, which we return to below.

3. Novel Network Applications

Several applications whose further development has benefited from BMW are now briefly described. In his survey article Beckmann (1967b) noted that there are analogues of the elastic demand network equilibrium model for problems other than road traffic and included examples of electricity, steam, water, and natural gas distribution, as well as the routing of messages in a communication network.

Spatial Price Equilibrium Networks and Other Economic Equilibrium Problems

In summarizing the railroad transportation contributions of BMW, Koopmans noted the works of Enke (1951) and Samuelson (1952) in the development of frameworks, the former using an analogue to electronic circuits and the latter a linear programming problem, for the determination of interregional commodity flows and prices in the case of separated markets. BMW stated (p. 105; Rand, p. 5.5): “The principal difference between passenger transportation as discussed in this study and commodity transportation lies in the substitutability of commodity shipments from different origins to a given destination.” Subsequently, Takayama and Judge (1964) in the case of linear regional supply and demand functions and fixed interregional transportation costs, demonstrated how the governing spatial price equilibrium conditions could be reformulated as the Kuhn-Tucker conditions of a quadratic programming problem. In the paper, the authors thank Martin Beckmann for helpful comments.

Recall in spatial price equilibrium problems, the governing equilibrium conditions state that a commodity will be produced at a supply market, and shipped to a demand market, where it is consumed, if the price at the supply market plus the unit transportation cost is equal to the price at the demand market; if the price at the supply market plus the transportation cost exceeds the price that the consumers are willing to pay for the commodity at the demand market, then there will be no trade of the commodity between the pair of markets.

Florian and Los (1982) provided a synthesis of the Samuelson (1952) model and the BMW network equilibrium model with elastic demand to construct a spatial price equilibrium model on a general network. They also considered multicommodity models and demonstrated that the governing equilibrium conditions satisfy a variational inequality problem akin to those arising in traffic network equilibrium models. Others had also been developing and extending the basic spatial price equilibrium models of Samuelson (1952) and Takayama and Judge (1964, 1971) (for a list of references, see Nagurney 1993). However, it was researchers in transportation and regional science that truly exploited the connections between these two subjects, which had been identified concurrently with BMW.

Dafermos and Nagurney (1985) established an isomorphism between spatial price and traffic network equilibrium problems which was further elaborated upon by Dafermos (1986) in the context of multicommodity/multiclass networks. Interestingly, BMW in Chapter 5 (p. 106; Rand 5.7) anticipated the existence of such a relationship in stating that "Passenger transportation may in fact be included as a special case of commodity transportation." Friesz et al. (1983, 1984), citing BMW, provided additional contributions to the modeling, analysis and solution of spatial price network equilibrium problems and forged the topic of freight network equilibrium.

Spatial price equilibrium models, in contrast to general economic equilibrium models, are necessarily partial equilibrium models. The network structure of spatial price equilibrium problems considered today often corresponds to the physical transportation network. The general economic equilibrium problem due to Walras (1874) can also be cast into a network equilibrium form on a simple single origin-destination pair network with disjoint routes as shown in Zhao and Nagurney (1993), who recognized the work of BMW (see also Nagurney 1993). It is fascinating to note that the classical portfolio optimization problem of Markowitz (1959) can be transformed into a system-optimized traffic network problem with fixed demand on a network with special structure (see also, e.g., Nagurney and Siokos 1997).

Supply Chain Networks

BMW explicitly recognized the generality of networks as a means of conceptualizing even decision-making of a firm, with routes corresponding to production processes and the links corresponding to transformations as the material moves down the route from the origin to the destination. The routes abstracted the choices or production possibilities available to a firm. For example, BMW (p. 88; Rand p. 4.13) provided an analogy of transportation networks to the theory of a firm and "consider a chemical or metallurgical material which is capable of various stages or modifications, and a firm which undertakes to transform it from certain stages to certain other stages .... Here the stages of the material correspond to locations, the transitions correspond to roads, and sequences of transformation processes ... of the material – that is, the production methods – correspond to routes."

Another application in which the concept of network equilibrium is garnering interest is *supply chain networks*, which is interdisciplinary by nature since it contains aspects of manufacturing, retailing, transportation, economics, as well as operations research and management science. Zhang et al. (2003) have recently generalized Wardrop's second principle to consider not only routes but *chains* in the network to identify the "winning" supply chains. In that context, routes correspond to production processes and links can be either operation or interface links. Their framework allows for the modeling of competition between supply chains, which may entail several firms (producing, transporting, retailing, etc.).

Nagurney, Dong and Zhang (2001) appear to be the first to utilize network equilibrium concepts in the context of supply chain applications. In this model decision-makers, now located at the nodes of the network, are faced with their individual objective functions, which can include profit-maximization, and seek to determine not only the optimal/equilibrium flows between tiers of nodes but also the prices of the product at the various tiers. The model therein was subsequently generalized to include electronic commerce by Nagurney, Loo, Dong and Zhang (2002).

Supernetworks

The growing impact of the Information Age, coupled with similarities between traffic networks and communications networks in terms of the relevance of such concepts as system-optimization and user-optimization, along with issues of centralized versus decentralized control, have provided a setting in which the relationships between decision-making on such networks and associated trade-offs could be explored. Nagurney, Dong and Mokhtarian (2001, 2002) developed multicriteria network equilibrium models which allowed for distinct classes of decision-makers who individually weigh their criteria associated with utilized transportation versus telecommunications networks in a variety of activities (such as teleshopping and telecommuting). In these and related models, criteria such as time, cost, risk, as well as opportunity cost (all criteria noted in BMW) play a prominent role. Nagurney and Dong (2002) described the governing equilibrium conditions in the cases of fixed and elastic demand and provided computational procedures and numerical examples demonstrating that the user-optimizing principle was relevant in the context of these new types of networks termed *supernetworks*. They also traced the origins of the term back to the transportation and computer science literatures.

Computer Scientists “Discover” Beckmann, McGuire, and Winsten

As mentioned earlier, Beckmann (1967b) noted the relevance of network equilibrium concepts to communication networks. Bertsekas and Gallager (1987) realized the similarities between communication and transportation networks as well and were familiar with the algorithms of Dafermos and Sparrow (1969).

It was, however, the Braess paradox that provided one of the main linkages between transportation science and computer science. Cohen and Kelly (1990) described a paradox analogous to that of Braess in the case of a queuing network. Later, Cohen and Horwitz (1991) investigated paradoxical behavior in electrical and mechanical networks. Korilis et al. (1999), in turn, developed methods to show how resources could be added efficiently to a noncooperative network, including the Internet, so that the Braess paradox would not occur and cited the work of Dafermos and Nagurney (1984a). Roughgarden (2002a) further elaborated upon the Braess paradox and focused on the quantification of the worst possible loss in network performance arising from noncooperative behavior. He also designed algorithms for the design and management of the networks so that “selfish” (a term also used by BMW), that is, individual optimizing, behavior, leads to a “socially desirable” outcome. He recognized the importance of the work of computer scientists, Koutsoupias and Papadimitrou (1999), who proposed the idea of bounding of the inefficiency of Nash equilibria, and that of BMW and Dafermos and Sparrow (1969). The motivations for his thesis, as well as its foundations, are drawn heavily from the transportation science literature in the form of the traffic network equilibrium problem and its game theoretic aspects. The work is generating much interest among computer scientists and is also often referred to as “selfish routing” (cf. Roughgarden and Tardos 2002; Roughgarden 2001; 2002b).

4. Conclusions

Hence, almost 50 years after its publication, BMW is finding applications in disciplines that did not even exist when the book was published! We expect that there will be continuing cross-fertilization between many fields in which networks play a prominent role, with BMW serving as one of the fundamental references. Indeed, BMW laid the intellectual and economic science foundation for transportation systems analysis, planning, and evaluation for the rest of the 20th century and beyond. Moreover, as the above discussion reveals, the work provided linkages to other application areas and fields. The ideas and concepts are fundamental and are unlikely to change in the foreseeable future.

While considerable progress has been made on many of the problems insightfully identified in this seminal work, many remain active areas of investigation. Only recently have observational methods become practical to provide empirical support for the theories and methods addressed in the book. Potentially significant departures from the principles and methods of BMW, however, may occur in the contribution of technology to our ability to manage traffic systems, and, more fundamentally, in the kinds of socio-technical changes that pervasive real-time information and ubiquitous access to the Internet might bring about.

In conclusion, we ask the following question: Have we left the next generation a contribution that is as far-reaching and thoughtful as what Beckmann, McGuire and Winsten have given us?

Notes

In this article, we have attempted to trace some of the major impacts of Beckmann, McGuire and Winsten (1956) in terms of innovations in modeling, methodological developments, and applications. This task has been challenging, not only because the book appeared a half century ago but due to its depth and breadth of influence. This article is an outgrowth of the panel: A Retrospective of Beckmann, McGuire, and Winsten's *Studies in the Economics of Transportation* held at the 50th North American Meetings of the Regional Science Association International in Philadephia, PA, November 20-22, 2003. The authors of this article served as panelists, and Dr. Suzanne Evans of Birkbeck College, London, United Kingdom, was chair and discussant.

Through the efforts of David Boyce, Rand Report, Rand-RM-1488-PR (BMW 1955), is now available on the Internet. For the link to the report, along with the panel PPT presentations of Boyce, Nagurney (with accompanying paper and many additional citations), and Mahmassani, see: http://supernet.som.umass.edu/classics.htm

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