The Co-Evolution and Emergence of Integrated International Financial Networks and Social Networks: Theory, Analysis, and Computations

Anna Nagurney
Department of Finance and Operations Management
Isenberg School of Management
University of Massachusetts
Amherst, Massachusetts 01003

Jose M. Cruz
Department of Operations and Information Management
School of Business
University of Connecticut
Waterbury, Connecticut 06702

and

Tina Wakolbinger
Department of Finance and Operations Management
Isenberg School of Management
University of Massachusetts
Amherst, Massachusetts 01003

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Abstract: Globalization and technological advances, notably, in telecommunications networks and in the development of new financial instruments, have transformed the financial services landscape and their impacts have been studied empirically. In this paper, in contrast, we focus on the theoretical foundations of such transformations and we develop a rigorous dynamic supernetwork theory for the integration of social networks with international financial networks with intermediation in the presence of electronic transactions. We consider decision-makers with sources of funds, financial intermediaries, as well as demand markets for the various financial products who can be located in the same or in different countries.

Through a multilevel supernetwork framework consisting of the international financial network with intermediation and the social network we model the multicriteria decision-making behavior of the various decision-makers, which includes the maximization of net return, the maximization of relationship values, and the minimization of risk. Increasing relationship levels in our framework are assumed to reduce transaction costs as well as risk and to have some additional value for the decision-makers. We explore the dynamic co-evolution of the financial flows, the associated financial product prices, as well as the relationship levels on the supernetwork until an equilibrium pattern is achieved. We provide some qualitative properties of the dynamic trajectories, under suitable assumptions, and propose a discrete-time algorithm which is then applied to track the co-evolution of the relationship levels over time as well as the financial flows and prices. The equilibrium pattern yields, as a byproduct, the emergent structure of the social and international financial networks since it identifies not only which pairs of nodes will have flows but also the size of the flows, i.e., the relationship levels and the financial transactions.

1. Introduction

Globalization and technological advances have made major impacts on financial services in recent years and have allowed for the emergence of electronic finance. Indeed, the financial landscape has been transformed through increased financial integration, increased cross border mergers, and lower barriers between markets. In addition, boundaries between different financial intermediaries have become less clear (cf. Claessens, Glaessner, and Klingebiel (2000, 2001), Claessens (2003), Claessens et al. (2003), G-10 (2001)).

During the period 1980-1990, global capital transactions tripled with telecommunication networks and financial instrument innovation being two of the empirically identified major causes of globalization with regards to international financial markets (Kim (1999)). The growing importance of networks in financial services and their effects on competition have been also addressed by Claessens et al. (2003). Kim (1999), in particular, has argued for the necessity of integrating various theories, including portfolio theory with risk management, and flow theory in order to capture the underlying complexity of the financial flows over space and time.

At the same time that globalization and technological advances have transformed financial services, researchers have identified the importance of social networks in a plethora of financial transactions (cf. Sharpe (1990), Uzzi (1997, 1999), Anthony (1997), Arrow (1998), DiMaggio and Louch (1998), Ghatak (2002)), notably, in the context of personal relationships. Nevertheless, the relevance of social networks within an international financial context has yet to be examined theoretically (or empirically). Clearly, the existence of appropriate social networks can affect not only the risk associated with financial transactions but also transaction costs.

Given the prevalence of networks (be they in the form of telecommunication networks, social networks, as well as the foundational financial networks; cf. Nagurney and Siokos (1997), Nagurney (2003), and the references therein) in the discussions of globalization and international financial flows, it seems natural that any theory for the illumination of the behavior of the decision-makers involved in this context as well as the impacts of their decisions on the financial product flows, prices, appreciation rates, etc., should be network-based. In this paper, hence, we take on a network perspective for the theoretical modeling, analysis, and computation of solutions to international financial networks with intermediation in which we explicitly integrate the social network component. We also capture electronic transactions within our framework since this aspect is critical in the modeling of international financial flows today.

In particular, in this paper, we focus on the development of a *supernetwork* framework for the integration of social networks with international financial networks with intermediation and electronic transactions. Supernetwork theory (cf. Nagurney and Dong (2002) and the references therein) has been used, to-date, to study a variety of network-based applications in which humans interact on two or more networks (very often transportation and telecommunication networks). Applications that have been formulated and solved using this approach include: telecommuting versus commuting decision-making, supply chains with electronic commerce, power/energy networks, as well as knowledge networks.

In addition, in this paper, we build upon the recent work of Nagurney and Cruz (2003, 2004) in the development of international financial network models (static and dynamic) and that of Nagurney, Wakolbinger, and Zhao (2004) and Wakolbinger and Nagurney (2004) in the integration of social networks with other economic networks.

This paper is organized as follows. In Section 2, we develop the multilevel supernetwork model consisting of multiple tiers of decision-makers acting on the international financial network with intermediation and the social network. We describe the decision-makers' optimizing behavior, and establish the governing equilibrium conditions along with the corresponding variational inequality formulation.

In Section 3, we describe the disequilibrium dynamics of the international financial flows, the prices, and the relationship levels as they co-evolve over time and formulate the dynamics as a projected dynamical system (cf. Nagurney and Zhang (1996a, b), Nagurney and Ke (2003), Nagurney and Cruz (2004), and Nagurney, Wakolbinger, and Zhao (2004)). We establish that the set of stationary points of the projected dynamical system coincides with the set of solutions to the derived variational inequality problem.

In Section 4, we present a discrete-time algorithm to approximate (and track) the international financial flow, price, and relationship level trajectories over time until the equilibrium values are reached. We then apply the discrete-time algorithm in Section 5 to several numerical examples to further illustrate the supernetwork model. We conclude with Section 6, in which we summarize our results and suggest possibilities for future research.

2. The Supernetwork Model Integrating International Financial Networks with Intermediation and Social Networks

In this Section, we develop the supernetwork model consisting of the integration of the international financial network with intermediation and the social network in which the decision-makers are those with sources of funds, the financial intermediaries, as well as the consumers associated with the demand markets. Here we describe the model in an equilibrium context, whereas in Section 3, we provide the disequilibrium dynamics and the co-evolution of the international financial flows, the prices, as well as the relationship levels between tiers of decision-makers over time. This model generalizes the model of Nagurney and Cruz (2003) to explicitly include social networks. In addition, it broadens the framework proposed in Nagurney, Wakolbinger, and Zhao (2004) to the international dimension.

As in the model of Nagurney and Cruz (2003), the model consists of L countries, with a typical country denoted by l or $\hat{l}$; I "source" agents in each country with sources of funds, with a typical source agent denoted by i, and J financial intermediaries with a typical financial intermediary denoted by j. Examples of source agents are households and businesses, whereas examples of financial intermediaries include banks, insurance companies, investment companies, brokers, including electronic brokers, etc. Intermediaries in our framework need not be country-specific but, rather, may be virtual.

We assume that each source agent can transact directly electronically with the consumers through the Internet and can also conduct his financial transactions with the intermediaries either physically or electronically in different currencies. There are H currencies in the international economy, with a typical currency being denoted by h. Also, we assume that there are K financial products which can be in distinct currencies and in different countries with a typical financial product (and associated with a demand market) being denoted by k. Hence, the financial intermediaries in the model, in addition to transacting with the source agents, also determine how to allocate the incoming financial resources among distinct uses, which are represented by the demand markets with a demand market corresponding to, for example, the market for real estate loans, household loans, or business loans, etc., which, as mentioned, can be associated with a distinct country and a distinct currency combination. We let m refer to a mode of transaction with $m = 1$ denoting a physical transaction and

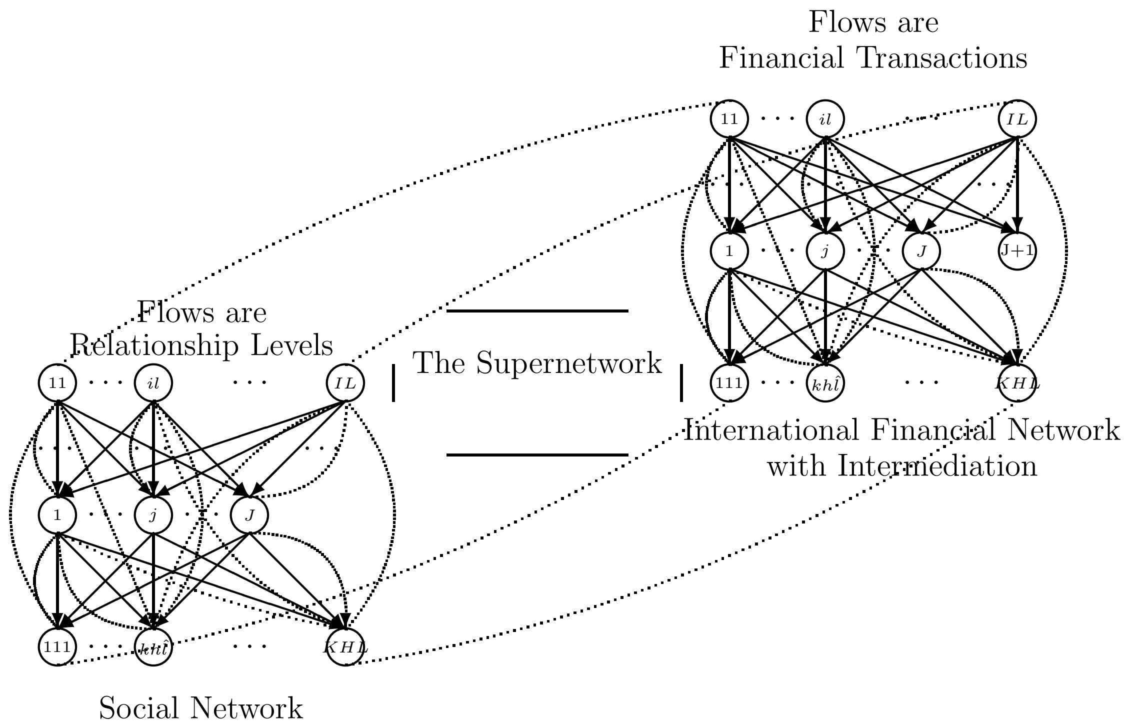

Figure 1: The Multilevel Supernetwork Structure of the Integrated International Financial Network / Social Network System

$m = 2$ denoting an electronic transaction via the Internet.

The depiction of the supernetwork is given in Figure 1. As this figure illustrates, the supernetwork is comprised of the social network, which is the bottom level network, and the international financial network, which is the top level network. Internet links to denote the possibility of electronic financial transactions are denoted in the figure by dotted arcs. In addition, dotted arcs/links are used to depict the integration of the two networks into a supernetwork. Examples of other supernetworks can be found in Nagurney and Dong (2002). Subsequently, we describe the interrelationships between the financial network and the social network through the functional forms and the flows on the links.

The supernetwork in Figure 1 consists of a social and an international financial network with intermediation. Both networks consist of three tiers of decision-makers. The top tier of nodes consists of the agents in the different countries with sources of funds, with agent i in country l being referred to as agent il and associated with node il. There are, hence, IL top-tiered nodes in the network. The middle tier of nodes in each of the two networks consists of the intermediaries (which need not be country-specific), with a typical intermediary j associated with node j in this (second) tier of nodes in the networks. The bottom tier of nodes in both the social network and in the financial network consists of the demand markets, with a typical demand market for product k in currency h and country $\hat{l}$ associated with node $kh\hat{l}$. There are, as depicted in Figure 1, J middle (or second) tiered nodes corresponding to the intermediaries and KHL bottom (or third) tiered nodes in the international financial network. In addition, we add a node $J+1$ to the middle tier of nodes in the financial network only in order to represent the possible non-investment (of a portion or all of the funds) by one or more of the source agents (see also Nagurney and Ke (2003)).

The supernetwork in Figure 1 includes classical physical links as well as Internet links to allow for electronic financial transactions. Electronic transactions are possible between the source agents and the intermediaries, the source agents and the demand markets as well as the intermediaries and the demand markets. Physical transactions can occur between the source agents and the intermediaries and between the intermediaries and the demand markets.

We now turn to the description of the behavior of the various economic decision-makers, i.e., the source agents, the financial intermediaries, and the demand markets.

The Behavior of the Agents with Sources of Funds and their Optimality Conditions

As described in Nagurney and Cruz (2003), we assume that each agent i in country l has an amount of funds S^{il} available in the base currency. Since there are assumed to be H currencies and 2 modes of transaction (physical or electronic), there are $2H$ links joining each top tier node il with each middle tier node j; $j = 1, \ldots, J$, with the first H links representing physical transactions between a source and intermediary, and with the corresponding flow on such a link given, respectively, by x^{il}_{jh1}, and the subsequent H links representing electronic transactions with the corresponding flow given, respectively, by x^{il}_{jh2}. Hence, x^{il}_{jh1} denotes the nonnegative amount invested (across all financial instruments) by source agent i in country

l in currency h transacted through intermediary j using the physical mode whereas x^{il}_{jh2} denotes the analogue but for an electronic transaction. We group the financial flows for all source agents/countries/intermediaries/currencies/modes into the column vector $x^1 \in R^{2ILJH}_+$. In addition, a source agent i in country l may transact directly with the consumers at demand market k in currency h and country $\hat{l}$ via an Internet link. The nonnegative flow on such a link joining node il with node $kh\hat{l}$ is denoted by $x^{il}_{kh\hat{l}}$. We group all such financial flows, in turn, into the column vector $x^2 \in R^{ILKHL}_+$. Also, we let x^{il} denote the $(2JH+KHL)$-dimensional column vector associated with source agent il with components:$\{x^{il}_{jhm}, x^{il}_{kh\hat{l}}; j = 1,\ldots,J; h = 1,\ldots,H; m = 1,2; k = 1,\ldots,K; \hat{l} = 1,\ldots,L\}$. Furthermore, we construct a link from each top tiered node to the second tiered node $J+1$ and associate a flow s^{il} on such a link emanating from node il to represent the possible nonnegative amount not invested by agent i in country l.

For each source agent il the amount of funds transacted either electronically and/or physically cannot exceed his financial holdings. Hence, the following conservation of flow equation must hold:

$$\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2} x^{il}_{jhm} + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} x^{il}_{kh\hat{l}} \leq S^{il}, \quad \forall i, l. \tag{1}$$

In Figure 1 the slack associated with constraint (1) for source agent i in country l is represented as the flow on the link joining node il with the non-investment node $J+1$.

Furthermore, let η^{il}_{jhm} denote the nonnegative level of the relationship between source agent il and intermediary j associated with currency h and mode of transaction m and let $\eta^{il}_{kh\hat{l}}$ denote the nonnegative relationship level associated with the virtual mode of transaction between source agent il and "demand market" $kh\hat{l}$. Each source agent il may actively try to achieve a certain relationship level with an intermediary and/or a demand market. We group the $\eta^{il}_{jhm}s$ for all source agent/country/intermediary/currency/mode combinations into the column vector $\eta^1 \in R^{2ILJH}_+$ and the $\eta^{il}_{kh\hat{l}}s$ for all the source agent/country/demand market/currency/country combinations into the column vector $\eta^2 \in R^{ILKHL}_+$. Moreover, we assume that these relationship levels take on a value that lies in the range [0, 1]. No relationship is indicated by a relationship level of zero and the strongest possible relationship is indicated by a relationship level of one. In the supernetwork depicted in Figure 1 the relationship flows are associated with the links in the social network component of the supernetwork. Specifically, the vector of flows η^1 corresponds to flows on the links between the source agents in the countries and the intermediaries, whereas the vector of flows η^2 corresponds to the flows on the links between the source agents and the demand markets in the various currencies and countries on the social network. The relationship levels, along with the financial flows, are endogenously determined in the model.

The source agent may spend money, for example, in the form of gifts and/or additional time/service in order to achieve a particular relationship level. The production cost functions for relationship levels are denoted by b^{il}_{jhm} and $b^{il}_{kh\hat{l}}$ and represent, respectively, how much money a source agent il has to spend in order to achieve a certain relationship level with intermediary j transacting through mode m and currency h or in order to achieve a certain relationship level with demand market $kh\hat{l}$. These relationship production cost functions are distinct for each such combination. Their specific functional forms may be influenced by such factors as the willingness of intermediaries or demand markets to establish/maintain a relationship and the level of previous business relationships and private relationships that exist. In an international setting, in particular, cultural differences, difficulties with languages, and distances, may also play a role in making it more costly to establish (and maintain) a relationship level.

The relationship production cost function is assumed, hence, to be a function of the relationship level between the source agent il and intermediary j transacting via mode m and currency h or with the consumers at demand market $kh\hat{l}$, that is,

$$b^{il}_{jhm} = b^{il}_{jhm}(\eta^{il}_{jhm}), \quad \forall i, l, j, h, m, \tag{2}$$

$$b^{il}_{kh\hat{l}} = b^{il}_{kh\hat{l}}(\eta^{il}_{kh\hat{l}}), \quad \forall i, l, k, h, \hat{l}. \tag{3}$$

We assume that these functions are convex and continuously differentiable.

We denote the transaction cost associated with source agent il transacting with intermediary j in currency h via mode m by c^{il}_{jhm} (and measured in the base currency) and assume that:

$$c^{il}_{jhm} = c^{il}_{jhm}(x^{il}_{jhm}, \eta^{il}_{jhm}), \quad \forall i, l, j, h, m, \tag{4}$$

that is, the cost associated with source agent i in country l transacting with intermediary j in currency h depends on the volume of financial transactions between the particular pair via the particular mode and on the relationship level between them. If the relationship level increases, the transaction cost may be expected to decrease since higher relationship levels may lead to higher levels of trust which can affect transactions. This is especially important in international exchanges in which transaction costs may be significant.

We denote the transaction cost associated with source agent il transacting with demand market k in country $\hat{l}$ in currency h via the Internet link by $c^{il}_{kh\hat{l}}$ (and also measured in the base currency) and assume that:

$$c^{il}_{kh\hat{l}} = c^{il}_{kh\hat{l}}(x^{il}_{kh\hat{l}}, \eta^{il}_{kh\hat{l}}), \quad \forall i, l, k, h, \hat{l}, \tag{5}$$

that is, the cost associated with source agent i in country l transacting with the consumers for financial product k in currency h and country $\hat{l}$. The transaction cost functions are assumed to be convex and continuously differentiable and depend on the volume of flow of the transaction and on the relationship level.

The source agent il faces total costs that equal the sum of the total transaction costs plus the costs that he incurs for establishing relationship levels. His revenue, in turn, is equal to the sum of the price (rate of return plus the rate of appreciation) that the agent can obtain times the total quantity obtained/purchased. Let now ρ^{il*}_{1jhm} denote the actual price charged agent il for the instrument in currency h by intermediary j transacting via mode m and let $\rho^{il*}_{1kh\hat{l}}$, in turn, denote the actual price associated with source agent il transacting electronically with demand market $kh\hat{l}$. Let e^*_h denote the actual rate of appreciation of currency h against the base currency, which can be interpreted as the rate of return earned due to exchange rate fluctuations (see Nagurney and Siokos (1997)). These "exchange" rates are grouped into the column vector $e^* \in R^H_+$. We later discuss how such prices are recovered.

We assume that each source agent in each country seeks to maximize his net return with the net revenue maximization problem for source agent il being given by:

$$\text{Maximize} \quad \sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}(\rho^{il*}_{1jhm}+e^*_h)x^{il}_{jhm}+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}(\rho^{il*}_{1khl}+e^*_h)x^{il}_{kh\hat{l}}-\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}c^{il}_{jhm}(x^{il}_{jhm}, \eta^{il}_{jhm})$$

$$-\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} c^{il}_{kh\hat{l}}(x^{il}_{kh\hat{l}}, \eta^{il}_{kh\hat{l}}) - \sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2} b^{il}_{jhm}(\eta^{il}_{jhm}) - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} b^{il}_{kh\hat{l}}(\eta^{il}_{kh\hat{l}}) \tag{6}$$

subject to:

$$x^{il}_{jhm} \geq 0, \quad x^{il}_{kh\hat{l}} \geq 0, \quad \forall j, h, m, k, \hat{l}, \tag{7}$$

$$0 \leq \eta^{il}_{jhm} \leq 1, \quad 0 \leq \eta^{il}_{kh\hat{l}} \leq 1, \quad \forall j, h, m, k, \hat{l}, \tag{8}$$

and the constraint (1) for source agent il.

The first two terms in (6) represent the revenue. The next four terms represent the various costs. The constraints are comprised of the conservation of flow equation (1), the nonnegativity assumptions on the financial flows (cf. (7)) and on the relationship levels with the latter being bounded from above by the value one (cf. (8)).

Furthermore, it is reasonable to assume that each source agent tries to minimize risk and has been noted in empirical studies by Kim (1999). Here, for the sake of generality, and as in the papers by Nagurney and Cruz (2003, 2004) and Nagurney, Wakolbinger, and Zhao (2004), we assume, as given, a risk function for source agent il, dealing with intermediary j via mode m and currency h, denoted by r^{il}_{jhm}, and a risk function for source agent il dealing with demand market $kh\hat{l}$ denoted by $r^{il}_{kh\hat{l}}$. These functions depend not only on the quantity of the financial flow transacted between the pair of nodes (and via a particular currency and mode) but also on the corresponding relationship level. If the relationship level increases, the risk is likely to decrease because trust reduces transaction uncertainty. Since international financial transactions are potentially riskier, high relationship levels can be of utmost significance and can, hence, create competitive advantages.

These risk functions are assumed to be as follows:

$$r^{il}_{jhm} = r^{il}_{jhm}(x^{il}_{jhm}, \eta^{il}_{jhm}), \quad \forall i, l, j, h, m, \tag{9}$$

$$r^{il}_{kh\hat{l}} = r^{il}_{kh\hat{l}}(x^{il}_{kh\hat{l}}, \eta^{il}_{kh\hat{l}}), \quad \forall i, l, k, h, \hat{l}, \tag{10}$$

where r^{il}_{jhm} and $r^{il}_{kh\hat{l}}$ are assumed to be convex and continuously differentiable.

Hence, source agent il also faces an optimization problem associated with his desire to minimize the total risk and corresponding to:

$$\text{Minimize} \quad \sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2} r^{il}_{jhm}(x^{il}_{jhm}, \eta^{il}_{jhm}) + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} r^{il}_{kh\hat{l}}(x^{il}_{kh\hat{l}}, \eta^{il}_{kh\hat{l}}) \tag{11}$$

subject to:

$$x^{il}_{jhm} \geq 0, \quad x^{il}_{kh\hat{l}} \geq 0, \quad \forall j, h, m, k, \hat{l}, \tag{12}$$

$$0 \leq \eta^{il}_{jhm} \leq 1, \quad 0 \leq \eta^{il}_{kh\hat{l}} \leq 1, \quad \forall j, h, m, k, \hat{l}. \tag{13}$$

In addition, the source agent also tries to maximize the relationship value generated by interacting with other decision-makers in the network. Here, v^{il}_{jhm} denotes the relationship value function for source agent il, intermediary j, mode m and currency h, and v^{il}_{jhm} is assumed to be a function of the relationship level of il with intermediary j transacting via mode m and currency h . Similarly, $v^{il}_{kh\hat{l}}$ denotes the relationship value function for source agent il and demand market $kh\hat{l}$. It is assumed to be a function of the relationship level with the particular demand market $kh\hat{l}$ such that

$$v^{il}_{jhm} = v^{il}_{jhm}(\eta^{il}_{jhm}), \quad \forall i, l, j, h, m, \tag{14}$$

$$v^{il}_{kh\hat{l}} = v^{il}_{kh\hat{l}}(\eta^{il}_{kh\hat{l}}), \quad \forall i, l, k, h, \hat{l}. \tag{15}$$

We assume that the value functions are continuously differentiable and concave.

Hence, source agent il is also faced with the optimization problem representing the maximization of the total value of his relationships expressed mathematically as:

$$\text{Maximize} \quad \sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2} v^{il}_{jhm}(\eta^{il}_{jhm}) + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} v^{il}_{kh\hat{l}}(\eta^{il}_{kh\hat{l}}) \tag{16}$$

subject to:

$$0 \leq \eta^{il}_{jhm} \leq 1, \quad 0 \leq \eta^{il}_{kh\hat{l}} \leq 1, \quad \forall j, h, m, k, \hat{l}. \tag{17}$$

The Multicriteria Decision-Making Problem Faced by a Source Agent

We can now construct the multicriteria decision-making problem facing a source agent which allows him to weight the criteria of net revenue maximization (cf. (6)), risk minimization (cf. (11)), and total relationship value maximization (see (16)) in an individual manner. Source agent ils multicriteria decision-making objective function is denoted by U^{il}. Assume that source agent il assigns a nonnegative weight α^{il} to the risk generated and a nonnegative weight β^{il} to the relationship value. The weight associated with net revenue maximization serves as the numeraire and is set equal to 1. The nonnegative weights measure the importance of risk and the total relationship value and, in addition, transform these values into monetary units. We can now construct a value function for each source agent (cf. Keeney and Raiffa (1993), Dong, Zhang, and Nagurney (2002), Nagurney, Wakolbinger, and Zhao (2004), and the references therein) using a constant additive weight value function. Therefore, the multicriteria decision-making problem of source agent il can be expressed as:

$$\text{Maximize} \quad U^{il} = \sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}(\rho^{il*}_{1jhm} + e^*_h)x^{il}_{jhm} + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}(\rho^{il*}_{1khl} + e^*_h)x^{il}_{kh\hat{l}}$$

$$-\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2} c^{il}_{jhm}(x^{il}_{jhm}, \eta^{il}_{jhm}) - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} c^{il}_{kh\hat{l}}(x^{il}_{kh\hat{l}}, \eta^{il}_{kh\hat{l}}) - \sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2} b^{il}_{jhm}(\eta^{il}_{jhm})$$

$$-\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} b^{il}_{kh\hat{l}}(\eta^{il}_{kh\hat{l}}) - \alpha^{il}(\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2} r^{il}_{jhm}(x^{il}_{jhm}, \eta^{il}_{jhm}) + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} r^{il}_{kh\hat{l}}(x^{il}_{kh\hat{l}}, \eta^{il}_{kh\hat{l}}))$$

$$+\beta^{il}(\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2} v^{il}_{jhm}(\eta^{il}_{jhm}) + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} v^{il}_{kh\hat{l}}(\eta^{il}_{kh\hat{l}})) \tag{18}$$

subject to:

$$x^{il}_{jhm} \geq 0, \quad x^{il}_{kh\hat{l}} \geq 0, \quad \forall j, h, m, k, \hat{l}, \tag{19}$$

$$0 \leq \eta^{il}_{jhm} \leq 1, \quad 0 \leq \eta^{il}_{kh\hat{l}} \leq 1, \quad \forall j, h, m, k, \hat{l}, \tag{20}$$

and the constraint (1) for source agent il.

The first six terms on the right-hand side of the equal sign in (18) represent the net revenue which is to be maximized, the next two terms represent the weighted total risk which is to be minimized and the last two terms represent the weighted total relationship value, which is to be maximized. We can observe that such an objective function is in concert with those used in classical portfolio optimization (see Markowitz (1952, 1959)) but substantially more general to reflect specifically the additional criteria, notably, that of total relationship value maximization.

Under the above assumed and imposed assumptions on the underlying functions, the optimality conditions for all source agents simultaneously can be expressed as the following inequality (cf. Bazaraa, Sherali, and Shetty (1993), Gabay and Moulin (1980); see also Nagurney (1999)): determine $(x^{1*}, x^{2*}, \eta^{1*}, \eta^{2*}) \in \mathcal{K}^1$, satisfying

$$\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}\left[\alpha^{il}\frac{\partial r^{il}_{jhm}(x^{il*}_{jhm},\eta^{il*}_{jhm})}{\partial x^{il}_{jhm}}+\frac{\partial c^{il}_{jhm}(x^{il*}_{jhm},\eta^{il*}_{jhm})}{\partial x^{il}_{jhm}}-\rho^{il*}_{1jhm}\ -e^*_h\right]\times\left[x^{il}_{jhm}-x^{il*}_{jhm}\right]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\alpha^{il}\frac{\partial r^{il}_{kh\hat{l}}(x^{il*}_{kh\hat{l}},\eta^{il*}_{kh\hat{l}})}{\partial x^{il}_{kh\hat{l}}}+\frac{\partial c^{il}_{kh\hat{l}}(x^{il*}_{kh\hat{l}},\eta^{il*}_{kh\hat{l}})}{\partial x^{il}_{kh\hat{l}}}-\rho^{il*}_{1kh\hat{l}}\ -e^*_h\right]\times\left[x^{il}_{kh\hat{l}}-x^{il*}_{kh\hat{l}}\right]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}\left[\frac{\partial c^{il}_{jhm}(x^{il*}_{jhm},\eta^{il*}_{jhm})}{\partial \eta^{il}_{jhm}}+\frac{\partial b^{il}_{jhm}(\eta^{il*}_{jhm})}{\partial \eta^{il}_{jhm}}-\beta^{il}\frac{\partial v^{il}_{jhm}(\eta^{il*}_{jhm})}{\partial \eta^{il}_{jhm}}\right.$$

$$\left.+\alpha^{il}\frac{\partial r^{il}_{jhm}(x^{il*}_{jhm},\eta^{il*}_{jhm})}{\partial \eta^{il}_{jhm}}\right]\times\left[\eta^{il}_{jhm}-\eta^{il*}_{jhm}\right]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\frac{\partial c^{il}_{kh\hat{l}}(x^{il*}_{kh\hat{l}},\eta^{il*}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}}+\frac{\partial b^{il}_{kh\hat{l}}(\eta^{il*}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}}-\beta^{il}\frac{\partial v^{il}_{kh\hat{l}}(\eta^{il*}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}}+\alpha^{il}\frac{\partial r^{il}_{kh\hat{l}}(x^{il*}_{kh\hat{l}},\eta^{il*}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}}\right]$$

$$\times\left[\eta^{il}_{kh\hat{l}}-\eta^{il*}_{kh\hat{l}}\right]\geq 0,\quad \forall(x^1,x^2,\eta^1,\eta^2)\in\mathcal{K}^1, \tag{21}$$

where

$$\mathcal{K}^1\equiv\Big[(x^1,x^2,\eta^1,\eta^2)|x^{il}_{jhm}\geq 0,\ x^{il}_{kh\hat{l}}\geq 0,\ 0\leq\eta^{il}_{jhm}\leq 1,\ 0\leq\eta^{il}_{kh\hat{l}}\leq 1,$$

$$\forall i,l,j,h,m,k,\hat{l},\ and\,(1)\ holds\Big] \tag{22}$$

Inequality (21) is actually a variational inequality (cf. Nagurney (1999) and the references therein).

The Behavior of the Intermediaries and their Optimality Conditions

The intermediaries (cf. Figure 1), in turn, are involved in transactions both with the source agents in the different countries, as well as with the users of the funds, that is, with the ultimate consumers associated with the markets for the distinct types of loans/products in different currencies and countries and represented by the bottom tier of nodes of the network. Each intermediary node j; $j=1,\ldots,J$, may transact with a demand market via a physical link, and/or electronically via an Internet link. Hence, from each intermediary node j, we construct two links to each node $kh\hat{l}$, with the first such link denoting a physical transaction and the second such link – an electronic transaction. The corresponding flow, in turn, which is nonnegative, is denoted by $y^j_{kh\hat{l}m}$; $m = 1, 2$, and corresponds to the amount of the financial product k in currency h and country $\hat{l}$ transacted from intermediary j via mode m. We group the financial flows between node j and the bottom tier nodes into the column vector $y^j \in R^{2KHL}_+$. All such financial flows for all the intermediaries are then further grouped into the column vector $y \in R^{2JKHL}_+$.

As in the case of source agents, the intermediaries have to bear some costs to establish and maintain relationship levels with source agents and with the consumers. We denote the relationship level between intermediary j and demand market $kh\hat{l}$ transacting through mode m by $\eta^j_{kh\hat{l}m}$. We group the relationship levels for all intermediary/demand market pairs into the column vector $\eta^3 \in R^{2JKHL}_+$. We assume that the relationship levels are nonnegative and that they may assume a value from 0 through 1. These relationship levels represent the flows between the intermediaries and the demand market nodes in the social network level of the supernetwork in Figure 1.

Let $\hat{b}^{il}_{jhm}$ denote the cost function associated with the relationship between intermediary j and source agent il transacting in currency h and via mode m and let $b^j_{kh\hat{l}m}$ denote the analogous cost function but associated with intermediary j, demand market $kh\hat{l}$, and mode m. Note that these functions are from the perspective of the intermediary (whereas (2) and (3) are from the perspective of the source agents). These cost functions are a function of the relationship levels (as in the case of the source agents) and are given by:

$$\hat{b}^{il}_{jhm} = \hat{b}^{il}_{jhm}(\eta^{il}_{jhm}), \quad \forall i, l, j, h, m, \tag{23}$$

$$b^j_{kh\hat{l}m} = b^j_{kh\hat{l}m}(\eta^j_{kh\hat{l}m}), \quad \forall i, l, k, h, \hat{l}, m. \tag{24}$$

The intermediaries also have associated transaction costs in regards to transacting with the source agents, which can depend on the type of currency as well as the source agent. We denote the transaction cost associated with intermediary j transacting with source agent il associated with currency h via mode m by $\hat{c}^{il}_{jhm}$ and we assume that it is of the form

$$\hat{c}^{il}_{jhm} = \hat{c}^{il}_{jhm}(x^{il}_{jhm}, \eta^{il}_{jhm}), \quad \forall i, l, j, h, m, \tag{25}$$

that is, such a transaction cost is allowed to depend on the amount allocated by the particular agent in a currency and transacted with the particular intermediary via the particular mode as well as the relationship level between them. In addition, we assume that an intermediary j also incurs a transaction cost $c^j_{kh\hat{l}m}$ associated with transacting with demand market $kh\hat{l}$ through mode m, where

$$c^j_{kh\hat{l}m} = c^j_{kh\hat{l}m}(y^j_{kh\hat{l}m}, \eta^j_{kh\hat{l}m}), \quad \forall j, k, h, \hat{l}, m. \tag{26}$$

Hence, the transaction costs given in (26) can vary according to the intermediary/product/currency/country/mode combination and are a function of the volume of the product transacted and the relationship level.

In addition, an intermediary j is faced with what we term a *handling/conversion* cost, which may include, for example, the cost of converting the incoming financial flows into the financial loans/products associated with the demand markets. We denote such a cost faced by intermediary j by c_j and, in the simplest case, c_j would be a function of $\sum_{i=1}^I \sum_{l=1}^L \sum_{h=1}^H \sum_{m=1}^2 x^{il}_{jhm}$, that is, the holding/conversion cost of an intermediary is a function of how much he has obtained in the different currencies from the various source agents in the different countries. For the sake of generality, however, we allow the function to depend also on the amounts held by other intermediaries and, therefore, we may write:

$$c_j = c_j(x^1), \quad \forall j. \tag{27}$$

We assume that the cost functions (23) – (27) are convex and continuously differentiable and that the costs are measured in the base currency.

The actual price charged for the financial product k associated with intermediary j transacting with the consumers in currency h via mode m and country $\hat{l}$ is denoted by $\rho^{j*}_{2kh\hat{l}m}$, for intermediary j. Similarly, as in the case of source agents, e^*_h denote the actual rate of appreciation in currency h. Later, we discuss how such prices are arrived at.

We assume that each intermediary seeks to maximize his net revenue with the net revenue criterion for intermediary j being given by:

$$\text{Maximize} \quad \sum_{k=1}^K \sum_{h=1}^H \sum_{\hat{l}=1}^L \sum_{m=1}^2 (\rho^{j*}_{2kh\hat{l}m} + e^*_h) y^j_{kh\hat{l}m} - c_j(x^1) - \sum_{i=1}^I \sum_{l=1}^L \sum_{h=1}^H \sum_{m=1}^2 \hat{c}^{il}_{jhm}(x^{il}_{jhm}, \eta^{il}_{jhm})$$

$$-\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2} c^j_{kh\hat{l}m}(y^j_{kh\hat{l}m},\eta^j_{kh\hat{l}m}) - \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2} \hat{b}^{il}_{jhm}(\eta^{il}_{jhm})$$

$$-\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2} b^j_{kh\hat{l}m}(\eta^j_{kh\hat{l}m}) - \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2} (\rho^{il*}_{1jhm}+e^*_h)x^{il}_{jhm} \tag{28}$$

subject to:

$$\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2} y^j_{kh\hat{l}m} \leq \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2} x^{il}_{jhm} \tag{29}$$

$$x^{il}_{jhm} \geq 0, \quad y^j_{kh\hat{l}m} \geq 0, \quad \forall i,l,h,\hat{l},m. \tag{30}$$

$$0 \leq \eta^{il}_{jhm} \leq 1, \quad 0 \leq \eta^j_{kh\hat{l}m} \leq 1, \quad \forall i,l,h,m,k,\hat{l}. \tag{31}$$

Constraint (29) guarantees that each intermediary does not reallocate more financial flows than he has available. Constraints (30) and (31) guarantee that the financial flows and relationship levels are nonnegative (from the perspective of the intermediary) and that the levels of the relationships do not exceed one.

In addition, we assume that each intermediary is also concerned with risk minimization. For the sake of generality, we assume, as given, a risk function $\hat{r}^{il}_{jhm}$, for intermediary j in transacting with source agent il in currency h through mode m and a risk function $r^j_{kh\hat{l}m}$ for intermediary j associated with his transacting with consumers at demand market $kh\hat{l}$ through mode m. The risk functions are assumed to be continuous and convex and a function of the amount transacted with the particular source agent or demand market and the relationship level with this source agent or demand market. A higher relationship level can be expected to reduce risk since trust reduces transactional uncertainty. The risk functions may be distinct for each source agent/country/intermediary/currency/mode and intermediary/demand market/currency/country/mode combination and are given, respectively, by:

$$\hat{r}^{il}_{jhm} = \hat{r}^{il}_{jhm}(x^{il}_{jhm},\eta^{il}_{jhm}), \quad \forall i,l,j,h,m, \tag{32}$$

$$r^j_{kh\hat{l}m} = r^j_{kh\hat{l}m}(y^j_{kh\hat{l}m},\eta^j_{kh\hat{l}m}), \quad \forall j,k,h,\hat{l},m. \tag{33}$$

Since a financial intermediary j is assumed to minimize his total risk, he is also faced with the optimization problem given by:

$$\text{Minimize} \quad \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2} \hat{r}^{il}_{jhm}(x^{il}_{jhm},\eta^{il}_{jhm}) + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2} r^j_{kh\hat{l}m}(y^j_{kh\hat{l}m},\eta^j_{kh\hat{l}m}) \tag{34}$$

subject to:

$$x_{jhm}^{il} \geq 0, \quad y_{kh\hat{l}m}^{j} \geq 0, \quad \forall i, l, h, m, k, \hat{l}, \tag{35}$$

$$0 \leq \eta_{jhm}^{il} \leq 1, \quad 0 \leq \eta_{kh\hat{l}m}^{j} \leq 1, \quad \forall i, l, h, m, k, \hat{l}. \tag{36}$$

As in the case of the source agents, intermediary j also tries to maximize his relationship values associated with the source agents and with the demand markets. We assume, as given, a relationship value function $\hat{v}_{jhm}^{il}$ for intermediary j in dealing with source agent il in currency h through transaction mode m and a relationship value function $v_{kh\hat{l}m}^{j}$ for intermediary j associated with his transacting with consumers at demand market $kh\hat{l}$ through mode m. The relationship value functions are assumed to be continuously differentiable and concave. They are assumed to be functions of the corresponding relationship levels and given, respectively, by

$$\hat{v}_{jhm}^{il} = \hat{v}_{jhm}^{il}(\eta_{jhm}^{il}), \quad \forall i, l, j, h, m, \tag{37}$$

$$v_{kh\hat{l}m}^{j} = v_{kh\hat{l}m}^{j}(\eta_{kh\hat{l}m}^{j}), \quad \forall j, k, h, \hat{l}, m. \tag{38}$$

Finally, financial intermediary j tries to maximize his total relationship value, given mathematically by the optimization problem:

$$\text{Maximize} \quad \sum_{i=1}^{I} \sum_{l=1}^{L} \sum_{h=1}^{H} \sum_{m=1}^{2} \hat{v}_{jhm}^{il}(\eta_{jhm}^{il}) + \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{\hat{l}=1}^{L} \sum_{m=1}^{2} v_{kh\hat{l}m}^{j}(\eta_{kh\hat{l}m}^{j}) \tag{39}$$

subject to:

$$0 \leq \eta_{jhm}^{il} \leq 1, \quad 0 \leq \eta_{kh\hat{l}m}^{j} \leq 1, \quad \forall i, l, j, h, m, k, \hat{l}. \tag{40}$$

The Multicriteria Decision-Making Problem Faced by a Financial Intermediary

We are now ready to construct the multicriteria decision-making problem faced by an intermediary which combines with appropriate individual weights the criteria of net revenue maximization given by (28); risk minimization, given by (34), and total relationship value maximization, given by (39). In particular, we let intermediary j assign a nonnegative weight δ^j to the total risk and a nonnegative weight γ^j to the total relationship value. The weight associated with net revenue maximization is set equal to 1 and serves as the numeraire (as in the case of the source agents). Let U^j denote the multicriteria objective function associated with intermediary j with his multicriteria decision-making problem expressed as:

$$\text{Maximize} \quad U^j = \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}(\rho^{j*}_{2kh\hat{l}m}+e^*_h)y^j_{kh\hat{l}m} - c_j(x^1) - \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}\hat{c}^{il}_{jhm}(x^{il}_{jhm},\eta^{il}_{jhm})$$

$$-\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}c^j_{kh\hat{l}m}(y^j_{kh\hat{l}m},\eta^j_{kh\hat{l}m}) - \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}\hat{b}^{il}_{jhm}(\eta^{il}_{jhm})$$

$$-\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}b^j_{kh\hat{l}m}(\eta^j_{kh\hat{l}m}) - \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}(\rho^{il*}_{1jhm}+e^*_h)x^{il}_{jhm}$$

$$-\delta^j(\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}\hat{r}^{il}_{jhm}(x^{il}_{jhm},\eta^{il}_{jhm}) + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}r^j_{kh\hat{l}m}(y^j_{kh\hat{l}m},\eta^j_{kh\hat{l}m}))$$

$$+\gamma^j(\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}\hat{v}^{il}_{jhm}(\eta^{il}_{jhm}) + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}v^j_{kh\hat{l}m}(\eta^j_{kh\hat{l}m})) \tag{41}$$

subject to:

$$\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}y^j_{kh\hat{l}m} \le \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}x^{il}_{jhm}, \tag{42}$$

$$x^{il}_{jhm} \ge 0, \quad y^j_{kh\hat{l}m} \ge 0, \quad \forall i,l,k,h,\hat{l},m, \tag{43}$$

$$0 \le \eta^{il}_{jhm} \le 1, \quad 0 \le \eta^j_{kh\hat{l}m} \le 1, \quad \forall i,l,h,m,k,\hat{l}. \tag{44}$$

Here we assume that the financial intermediaries can compete, with the governing optimality/equilibrium concept underlying noncooperative behavior being that of Nash (1950, 1951), which states that each decision-maker (intermediary) will determine his optimal strategies, given the optimal ones of his competitors. The optimality conditions for all financial intermediaries simultaneously, under the above stated assumptions, can be compactly expressed as (cf. Gabay and Moulin (1980), Dafermos and Nagurney (1987), and Nagurney and Ke (2001, 2003)): determine $(x^{1*}, y^*, \eta^{1*}, \eta^{3*}, \lambda^*) \in \mathcal{K}^2$, such that

$$\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}\left[\delta^j\frac{\partial\hat{r}^{il}_{jhm}(x^{il*}_{jhm},\eta^{il*}_{jhm})}{\partial x^{il}_{jhm}} + \frac{\partial c_j(x^{1*})}{\partial x^{il}_{jhm}} + \rho^{il*}_{1jhm} + e^*_h + \frac{\partial\hat{c}^{il}_{jhm}(x^{il*}_{jhm},\eta^{il*}_{jhm})}{\partial x^{il}_{jhm}} - \lambda^*_j\right]$$

$$\times\left[x^{il}_{jhm} - x^{il*}_{jhm}\right]$$

$$+\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}\left[\delta^j\frac{\partial r^j_{kh\hat{l}m}(y^{j*}_{kh\hat{l}m},\eta^{j*}_{kh\hat{l}m})}{\partial y^j_{kh\hat{l}m}}+\frac{\partial c^j_{kh\hat{l}m}(y^{j*}_{kh\hat{l}m},\eta^{j*}_{kh\hat{l}m})}{\partial y^j_{kh\hat{l}m}}-\rho^{j*}_{2kh\hat{l}m}-e^*_h+\lambda^*_j\right]$$
$$\times\left[y^j_{kh\hat{l}m}-y^{j*}_{kh\hat{l}m}\right]$$
$$+\sum_{j=1}^{J}\sum_{i=1}^{I}\sum_{l=}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}\left[\delta^j\frac{\partial \hat{r}^{il}_{jhm}(x^{il*}_{jhm},\eta^{il*}_{jhm})}{\partial \eta^{il}_{jhm}}+\frac{\partial \hat{c}^{il}_{jhm}(x^{il*}_{jhm},\eta^{il*}_{jhm})}{\partial \eta^{il}_{jhm}}-\gamma^j\frac{\partial \hat{v}^{il}_{jhm}(\eta^{il*}_{jhm})}{\partial \eta^{il}_{jhm}}\right.$$
$$\left.+\frac{\partial \hat{b}^{il}_{jhm}(\eta^{il*}_{jhm})}{\partial \eta^{il}_{jhm}}\right]\times\left[\eta^{il}_{jhm}-\eta^{il*}_{jhm}\right]$$
$$+\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}\left[\delta^j\frac{\partial r^j_{kh\hat{l}m}(y^{j*}_{kh\hat{l}m},\eta^{j*}_{kh\hat{l}m})}{\partial \eta^j_{kh\hat{l}m}}+\frac{\partial c^j_{kh\hat{l}m}(y^{j*}_{kh\hat{l}m},\eta^{j*}_{kh\hat{l}m})}{\partial \eta^j_{kh\hat{l}m}}-\gamma^j\frac{\partial v^j_{kh\hat{l}m}(\eta^{j*}_{kh\hat{l}m})}{\partial \eta^j_{kh\hat{l}m}}\right.$$
$$\left.+\frac{\partial b^j_{khl}(\eta^{j*}_{kh\hat{l}m})}{\partial \eta^j_{kh\hat{l}m}}\right]\times\left[\eta^j_{kh\hat{l}m}-\eta^{j*}_{kh\hat{l}m}\right]$$
$$+\sum_{j=1}^{J}\left[\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}x^{il*}_{jhm}-\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}y^{j*}_{kh\hat{l}m}\right]\times\left[\lambda_j-\lambda^*_j\right]\geq 0,\quad \forall(x^1,y,\eta^1,\eta^3,\lambda)\in\mathcal{K}^2, \tag{45}$$

where

$$\mathcal{K}^2\equiv\left[(x^1,y,\eta^1,\eta^3,\lambda)|x^{il}_{jhm}\geq 0,\ y^j_{kh\hat{l}m}\geq 0,\ 0\leq\eta^{il}_{jhm}\leq 1,\ 0\leq\eta^j_{kh\hat{l}m}\leq 1,\ \lambda_j\geq 0,\right.$$
$$\left.\forall i,l,j,h,m,k,\hat{l}\right]. \tag{46}$$

Here λ_j denotes the Lagrange multiplier associated with constraint (42) and λ is the column vector of all the intermediaries' Lagrange multipliers. These Lagrange multipliers can also be interpreted as shadow prices. Indeed, according to the fifth term in (45), λ^*_j serves as the price to "clear the market" at intermediary j.

Inequality (45) provides us with conditions under which optimal virtual and/or physical financial transactions between intermediaries and source agents occur and optimal conditions under which virtual transactions between source agents and demand markets occur. Furthermore, it formulates the optimality conditions under which the relationship levels associated with intermediaries interacting with either the source agents or the demand markets will take on positive values; in other words, a relationship exists.

The Consumers at the Demand Markets and the Equilibrium Conditions

We now describe the consumers located at the demand markets. The consumers take into account in making their consumption decisions not only the price charged for the financial product by the agents with source of funds and intermediaries but also their transaction costs associated with obtaining the product.

Let $\hat{c}^j_{kh\hat{l}m}$ denote the transaction cost associated with obtaining product k in currency h in country $\hat{l}$ via mode m from intermediary j and recall that $y^j_{kh\hat{l}m}$ is the amount of the financial product k in currency h flowing between intermediary j and consumers in country $\hat{l}$ via mode m. We assume that the transaction cost is measured in the base currency, is continuous, and of the general form:

$$\hat{c}^j_{kh\hat{l}m} = \hat{c}^j_{kh\hat{l}m}(x^2, y, \eta^2, \eta^3), \quad \forall j, k, h, \hat{l}, m. \tag{47}$$

Hence, the cost of transacting between an intermediary and a demand market via a specific mode, from the perspective of the consumers, can depend upon the volume of financial flows transacted either physically and/or electronically from intermediaries as well as from source agents and the associated relationship levels. As in the case of the source agents and the financial intermediaries, higher relationship levels potentially reduce transaction costs, which means that they can lead to quantifiable cost reductions. The generality of this cost function structure enables the modeling of competition on the demand side. Moreover, it allows for information exchange between the consumers at the demand markets who may inform one another as to their relationship levels which, in turn, can affect the transaction costs.

In addition, let $\hat{c}^{il}_{kh\hat{l}}$ denote the transaction cost associated with obtaining the financial product k in currency h in country $\hat{l}$ electronically from source agent il, where we assume that the transaction cost is continuous and of the general form:

$$\hat{c}^{il}_{kh\hat{l}} = \hat{c}^{il}_{kh\hat{l}}(x^2, y, \eta^2, \eta^3), \quad \forall i, l, k, h, \hat{l}. \tag{48}$$

Hence, the transaction cost associated with transacting directly with source agents is of a form of the same level of generality as the transaction costs associated with transacting with the financial intermediaries.

Let $\rho_{3kh\hat{l}}$ denote the price of the financial product k in currency h and in country $\hat{l}$, and defined in the base currency, and group all such prices into the column vector $\rho_3 \in R_+^{KHL}$. Denote the demand for product k in currency h in country $\hat{l}$ by $d_{kh\hat{l}}$ and assume, as given, the continuous demand functions:

$$d_{kh\hat{l}} = d_{kh\hat{l}}(\rho_3), \quad \forall k, h, \hat{l}. \tag{49}$$

Thus, according to (49), the demand of consumers for the financial product in a currency and country depends, in general, not only on the price of the product at that demand market (and currency and country) but also on the prices of the other products at the other demand markets (and in other countries and currencies). Consequently, consumers at a demand market, in a sense, also compete with consumers at other demand markets.

The consumers take the price charged by the intermediary, which was denoted by $\rho^{j*}_{2kh\hat{l}m}$ for intermediary j, product k, currency h, and country $\hat{l}$ via mode m, the price charged by source agent il, which was denoted by $\rho^{il*}_{1kh\hat{l}}$, and the rate of appreciation in the currency, plus the transaction costs, in making their consumption decisions. The equilibrium conditions for the consumers at demand market $kh\hat{l}$, thus, take the form: for all intermediaries: $j = 1, \ldots, J$ and all mode m; $m = 1, 2$:

$$\rho^{j*}_{2kh\hat{l}m} + e^*_h + \hat{c}^j_{kh\hat{l}m}(x^{2*}, y^*, \eta^{2*}, \eta^{3*}) \begin{cases} = \rho^*_{3kh\hat{l}}, & \text{if} \quad y^{j*}_{kh\hat{l}m} > 0 \\ \geq \rho^*_{3kh\hat{l}}, & \text{if} \quad y^{j*}_{kh\hat{l}m} = 0, \end{cases} \tag{50}$$

and for all source agents il; $i = 1, \ldots, I$ and $l = 1, \ldots, L$:

$$\rho^{il*}_{1kh\hat{l}} + e^*_h + \hat{c}^{il}_{kh\hat{l}}(x^{2*}, y^*, \eta^{2*}, \eta^{3*}) \begin{cases} = \rho^*_{3kh\hat{l}}, & \text{if} \quad x^{il*}_{kh\hat{l}} > 0 \\ \geq \rho^*_{3kh\hat{l}}, & \text{if} \quad x^{il*}_{kh\hat{l}} = 0. \end{cases} \tag{51}$$

In addition, we must have that

$$d_{kh\hat{l}}(\rho^*_3) \begin{cases} = \sum_{j=1}^{J} \sum_{m=1}^{2} y^{j*}_{kh\hat{l}m} + \sum_{i=1}^{I} \sum_{l=1}^{L} x^{il*}_{kh\hat{l}}, & \text{if} \quad \rho^*_{3kh\hat{l}} > 0 \\ \leq \sum_{j=1}^{J} \sum_{m=1}^{2} y^{j*}_{kh\hat{l}m} + \sum_{i=1}^{I} \sum_{l=1}^{L} x^{il*}_{kh\hat{l}}, & \text{if} \quad \rho^*_{3kh\hat{l}} = 0. \end{cases} \tag{52}$$

Conditions (50) state that consumers at demand market $kh\hat{l}$ will purchase the product from intermediary j, if the price charged by the intermediary for the product and the appreciation rate for the currency plus the transaction cost (from the perspective of the consumer) does not exceed the price that the consumers are willing to pay for the product in that currency and country, i.e., $\rho^*_{3kh\hat{l}}$. Note that, according to (50), if the transaction costs are identically equal to zero, then the price faced by the consumers for a given product is the price charged by the intermediary for the particular product and currency in the country plus the rate of appreciation in the currency. Condition (51) state the analogue, but for the case of electronic transactions with the source agents.

Condition (52), on the other hand, states that, if the price the consumers are willing to pay for the financial product at a demand market is positive, then the quantity of at the demand market is precisely equal to the demand.

In equilibrium, conditions (50), (51), and (52) will have to hold for all demand markets and these, in turn, can be expressed also as an inequality analogous to those in (21) and (45) and given by: determine $(x^{2*}, y^*, \rho_3^*) \in R_+^{(IL+2J+1)KHL}$, such that

$$\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}\left[\rho^{j*}_{2kh\hat{l}m} + e^*_h + \hat{c}^j_{kh\hat{l}m}(x^{2*}, y^*, \eta^{2*}, \eta^{3*}) - \rho^*_{3kh\hat{l}}\right] \times \left[y^j_{kh\hat{l}m} - y^{j*}_{kh\hat{l}m}\right]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\rho^{il*}_{1kh\hat{l}} + e^*_h + \hat{c}^{il}_{kh\hat{l}}(x^{2*}, y^*, \eta^{2*}, \eta^{3*}) - \rho^*_{3kh\hat{l}}\right] \times \left[x^{il}_{kh\hat{l}} - x^{il*}_{kh\hat{l}}\right]$$

$$+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\sum_{j=1}^{J}\sum_{m=1}^{2} y^{j*}_{kh\hat{l}m} + \sum_{i=1}^{I}\sum_{l=1}^{L} x^{il*}_{kh\hat{l}} - d_{kh\hat{l}}(\rho_3^*)\right] \times \left[\rho_{3kh\hat{l}} - \rho^*_{3kh\hat{l}}\right] \geq 0,$$

$$\forall (x^2, y, \rho_3) \in R_+^{(IL+2J+1)KHL}. \tag{53}$$

For further background, see Nagurney and Dong (2002).

The Equilibrium Conditions of the Supernetwork Integrating the International Financial Network and the Social Network

In equilibrium, the financial flows that the source agents in different countries transact with the intermediaries must coincide with those that the intermediaries actually accept from them. In addition, the amounts of the financial products that are obtained by the consumers in the different countries and currencies must be equal to the amounts that both the source agents and the intermediaries actually provide. Hence, although there may be competition between decision-makers at the same level of tier of nodes of the financial network there must be, in a sense, cooperation between decision-makers associated with pairs of nodes (through positive flows on the links joining them). Thus, in equilibrium, the prices and financial flows must satisfy the sum of the optimality conditions (21) and (45) and the equilibrium conditions (53). We make these relationships rigorous through the subsequent definition and variational inequality derivation below.

Definition 1: Supernetwork Integrating the International Financial Network and the Social Network

The equilibrium state of the supernetwork integrating the international financial network with the social network is one where the financial flows and relationship levels between the tiers of the network coincide and the financial flows, relationship levels, and prices satisfy the sum of conditions (21), (45), and (53).

The equilibrium state is equivalent to the following:

Theorem 1: Variational Inequality Formulation

The equilibrium conditions governing the supernetwork integrating the international financial network with the social network according to Definition 1 are equivalent to the solution of the variational inequality given by: determine $(x^{1*}, x^{2*}, y^*, \eta^{1*}, \eta^{2*}, \eta^{3*}, \lambda^*, \rho_3^*) \in \mathcal{K}$*, satisfying:*

$$\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}\Bigg[\alpha^{il}\frac{\partial r^{il}_{jhm}(x^{il*}_{jhm},\eta^{il*}_{jhm})}{\partial x^{il}_{jhm}}+\frac{\partial c^{il}_{jhm}(x^{il*}_{jhm},\eta^{il*}_{jhm})}{\partial x^{il}_{jhm}}+\delta^{j}\frac{\partial \hat{r}^{il}_{jhm}(x^{il*}_{jhm},\eta^{il*}_{jhm})}{\partial x^{il}_{jhm}}$$

$$+\frac{\partial c_j(x^{1*})}{\partial x^{il}_{jhm}}+\frac{\partial \hat{c}^{il}_{jhm}(x^{il*}_{jhm},\eta^{il*}_{jhm})}{\partial x^{il}_{jhm}}-\lambda^*_j\Bigg]\times\left[x^{il}_{jhm}-x^{il*}_{jhm}\right]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\Bigg[\alpha^{il}\frac{\partial r^{il}_{kh\hat{l}}(x^{il*}_{kh\hat{l}},\eta^{il*}_{kh\hat{l}})}{\partial x^{il}_{kh\hat{l}}}+\frac{\partial c^{il}_{kh\hat{l}}(x^{il*}_{kh\hat{l}},\eta^{il*}_{kh\hat{l}})}{\partial x^{il}_{kh\hat{l}}}+\hat{c}^{il}_{kh\hat{l}}(x^{2*},y^*,\eta^{2*},\eta^{3*})-\rho^*_{3kh\hat{l}}\Bigg]$$

$$\times\left[x^{il}_{kh\hat{l}}-x^{il*}_{kh\hat{l}}\right]$$

$$+\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}\Bigg[\delta^{j}\frac{\partial r^{j}_{kh\hat{l}m}(y^{j*}_{kh\hat{l}m},\eta^{j*}_{kh\hat{l}m})}{\partial y^{j}_{kh\hat{l}m}}+\frac{\partial c^{j}_{kh\hat{l}m}(y^{j*}_{kh\hat{l}m},\eta^{j*}_{kh\hat{l}m})}{\partial y^{j}_{kh\hat{l}m}}+\hat{c}^{j}_{kh\hat{l}m}(x^{2*},y^*,\eta^{2*},\eta^{3*})$$

$$+\lambda_j^* - \rho_{3kh\hat{l}}^*\Big] \times \Big[y_{kh\hat{l}m}^j - y_{kh\hat{l}m}^{j*}\Big]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}\Bigg[\frac{\partial c_{jhm}^{il}(x_{jhm}^{il*},\eta_{jhm}^{il*})}{\partial \eta_{jhm}^{il}} + \frac{\partial \hat{c}_{jhm}^{il}(x_{jhm}^{il*},\eta_{jhm}^{il*})}{\partial \eta_{jhm}^{il}} - \beta^{il}\frac{\partial v_{jhm}^{il}(\eta_{jhm}^{il*})}{\partial \eta_{jhm}^{il}}$$

$$-\gamma^j\frac{\partial \hat{v}_{jhm}^{il}(\eta_{jhm}^{il*})}{\partial \eta_{jhm}^{il}} + \alpha^{il}\frac{\partial r_{jhm}^{il}(x_{jhm}^{il*},\eta_{jhm}^{il*})}{\partial \eta_{jhm}^{il}} + \delta^j\frac{\partial \hat{r}_{jhm}^{il}(x_{jhm}^{il*},\eta_{jhm}^{il*})}{\partial \eta_{jhm}^{il}}$$

$$+\frac{\partial b_{jhm}^{il}(\eta_{jhm}^{il*})}{\partial \eta_{jhm}^{il}} + \frac{\partial \hat{b}_{jhm}^{il}(\eta_{jhm}^{il*})}{\partial \eta_{jhm}^{il}}\Bigg] \times \Big[\eta_{jhm}^{il} - \eta_{jhm}^{il*}\Big]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\Bigg[\frac{\partial c_{kh\hat{l}}^{il}(x_{kh\hat{l}}^{il*},\eta_{kh\hat{l}}^{il*})}{\partial \eta_{kh\hat{l}}^{il}} + \frac{\partial b_{kh\hat{l}}^{il}(\eta_{kh\hat{l}}^{il*})}{\partial \eta_{kh\hat{l}}^{il}} - \beta^{il}\frac{\partial v_{kh\hat{l}}^{il}(\eta_{kh\hat{l}}^{il*})}{\partial \eta_{kh\hat{l}}^{il}} + \alpha^{il}\frac{\partial r_{kh\hat{l}}^{il}(x_{kh\hat{l}}^{il*},\eta_{kh\hat{l}}^{il*})}{\partial \eta_{kh\hat{l}}^{il}}\Bigg]$$

$$\times \Big[\eta_{kh\hat{l}}^{il} - \eta_{kh\hat{l}}^{il*}\Big]$$

$$+\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}\Bigg[\delta^j\frac{\partial r_{kh\hat{l}m}^{j}(y_{kh\hat{l}m}^{j*},\eta_{kh\hat{l}m}^{j*})}{\partial \eta_{kh\hat{l}m}^{j}} + \frac{\partial c_{kh\hat{l}m}^{j}(y_{kh\hat{l}m}^{j*},\eta_{kh\hat{l}m}^{j*})}{\partial \eta_{kh\hat{l}m}^{j}} - \gamma^j\frac{\partial v_{kh\hat{l}m}^{j}(\eta_{kh\hat{l}m}^{j*})}{\partial \eta_{kh\hat{l}m}^{j}}$$

$$+\frac{\partial b_{khl}^{j}(\eta_{kh\hat{l}m}^{j*})}{\partial \eta_{kh\hat{l}m}^{j}}\Bigg] \times \Big[\eta_{kh\hat{l}m}^{j} - \eta_{kh\hat{l}m}^{j*}\Big]$$

$$+\sum_{j=1}^{J}\Bigg[\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2}x_{jhm}^{il*} - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}y_{kh\hat{l}m}^{j*}\Bigg] \times \Big[\lambda_j - \lambda_j^*\Big]$$

$$+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\Bigg[\sum_{j=1}^{J}\sum_{m=1}^{2}y_{kh\hat{l}m}^{j*} + \sum_{i=1}^{I}\sum_{l=1}^{L}x_{kh\hat{l}}^{il*} - d_{kh\hat{l}}(\rho_3^*)\Bigg] \times \Big[\rho_{3kh\hat{l}} - \rho_{3kh\hat{l}}^*\Big] \geq 0,$$

$$\forall (x^1, x^2, y, \eta^1, \eta^2, \eta^3, \lambda, \rho_3) \in \mathcal{K}, \tag{54}$$

where

$$\mathcal{K} \equiv \Big[(x^1, x^2, y, \eta^1, \eta^2, \eta^3, \lambda, \rho_3) | x_{jhm}^{il} \geq 0,\ \ x_{kh\hat{l}}^{il} \geq 0,\ \ y_{kh\hat{l}m}^{j} \geq 0,\ 0 \leq \eta_{jhm}^{il} \leq 1,\ 0 \leq \eta_{kh\hat{l}}^{il} \leq 1,$$

$$0 \leq \eta_{kh\hat{l}m}^{j} \leq 1,\ \rho_{3kh\hat{l}} \geq 0,\ \lambda_j \geq 0, \forall i, l, j, h, m, k, \hat{l},\ and\,(1)\,holds\Big]. \tag{55}$$

Proof: Summation of inequalities (21), (45), and (53), yields, after algebraic simplification, the variational inequality (54). □

We now put variational inequality (54) into standard form which will be utilized in the subsequent sections. For additional background on variational inequalities and their applications, see the book by Nagurney (1999). In particular, we have that variational inequality (54) can be expressed as:

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{56}$$

where $X \equiv (x^1, x^2, y, \eta^1, \eta^2, \eta^3, \lambda, \rho_3)$ and $F(X) \equiv (F_{iljhm}, F_{ilkh\hat{l}}, F_{jkh\hat{l}m}, \hat{F}_{iljhm}, \hat{F}_{ilkh\hat{l}}, \hat{F}_{jkh\hat{l}m}, F_j, F_{kh\hat{l}})$ with indices: $i = 1, \ldots, I$; $l = 1, \ldots, L$; $j = 1, \ldots, J$; $h = 1, \ldots, H$; $\hat{l} = 1, \ldots, L$; $m = 1, 2$, and the specific components of F given by the functional terms preceding the multiplication signs in (56), respectively. The term $\langle \cdot, \cdot \rangle$ denotes the inner product in N-dimensional Euclidean space.

We now describe how to recover the prices associated with the first two tiers of nodes in the international financial network. Clearly, the components of the vector ρ_3^* are obtained directly from the solution of variational inequality (56) as will be demonstrated explicitly through several numerical examples in Section 5. In order to recover the second tier prices associated with the intermediaries and the exchange rates one can (after solving variational inequality (56) for the particular numerical problem) *either* (cf. (50)) set $\rho^{j*}_{2kh\hat{l}m} + e^*_h = \left[\rho^*_{3kh\hat{l}} - \hat{c}^j_{kh\hat{l}m}(x^{2*}, y^*, \eta^{2*}, \eta^{3*})\right]$, for any $j, k, h, \hat{l}, m$ such that $y^{j*}_{kh\hat{l}m} > 0$, *or* (cf. (45)) for any $y^{j*}_{kh\hat{l}m} > 0$, set $\rho^{j*}_{2kh\hat{l}m} + e^*_h = \left[\delta^j \frac{\partial r^j_{kh\hat{l}m}(y^{j*}_{kh\hat{l}m}, \eta^{j*}_{kh\hat{l}m})}{\partial y^j_{kh\hat{l}m}} + \frac{\partial c^j_{khl}(y^{j*}_{kh\hat{l}m}, \eta^{j*}_{kh\hat{l}m})}{\partial y^j_{kh\hat{l}m}} + \lambda^*_j\right]$.

Similarly, from (21) we can infer that the top tier prices comprising the vector ρ_1^* can be recovered (once the variational inequality (56) is solved with particular data) thus: for any i, l, j, h, m, such that $x^{il*}_{jhm} > 0$, set $\rho^{il*}_{1jhm} + e^*_h = \left[\alpha^{il} \frac{\partial r^{il}_{jhm}(x^{il*}_{jhm}, \eta^{il*}_{jhm})}{\partial x^{il}_{jhm}} + \frac{\partial c^{il}_{jhm}(x^{il*}_{jhm}, \eta^{il*}_{jhm})}{\partial x^{il}_{jhm}}\right]$, or, equivalently, (cf. (45)), to $\left[\lambda^*_j - \delta^j \frac{\partial \hat{r}^{il}_{jhm}(x^{il*}_{jhm}, \eta^{il*}_{jhm})}{\partial x^{il}_{jhm}} - \frac{\partial c_j(x^{1*})}{\partial x^{il}_{jhm}} - \frac{\partial \hat{c}^{il}_{jhm}(x^{il*}_{jhm}, \eta^{il*}_{jhm})}{\partial x^{il}_{jhm}}\right]$.

In addition, in order to recover the first tier prices associated with the demand market and the exchange rates one can (after solving variational inequality (56) for the particular numerical problem) *either* (cf. (21)) set $\rho^{il*}_{1kh\hat{l}} + e^*_h = \left[\alpha^{il} \frac{\partial r^{il}_{kh\hat{l}}(x^{il*}_{kh\hat{l}}, \eta^{il*}_{kh\hat{l}})}{\partial x^{il}_{kh\hat{l}}} + \frac{\partial c^{il}_{kh\hat{l}}(x^{il*}_{kh\hat{l}}, \eta^{il*}_{kh\hat{l}})}{\partial x^{il}_{kh\hat{l}}}\right]$, for any $i, l, k, h, \hat{l}$ such that $x^{il*}_{kh\hat{l}} > 0$, *or* (cf. (51)) for any $x^{il*}_{kh\hat{l}} > 0$, set $\rho^{il*}_{1kh\hat{l}} + e^*_h = \left[\rho^*_{3kh\hat{l}} - \hat{c}^{il}_{kh\hat{l}}(x^{2*}, y^*, \eta^{2*}, \eta^{3*})\right]$.

Under the above pricing mechanism, the optimality conditions (21) and (45) as well as

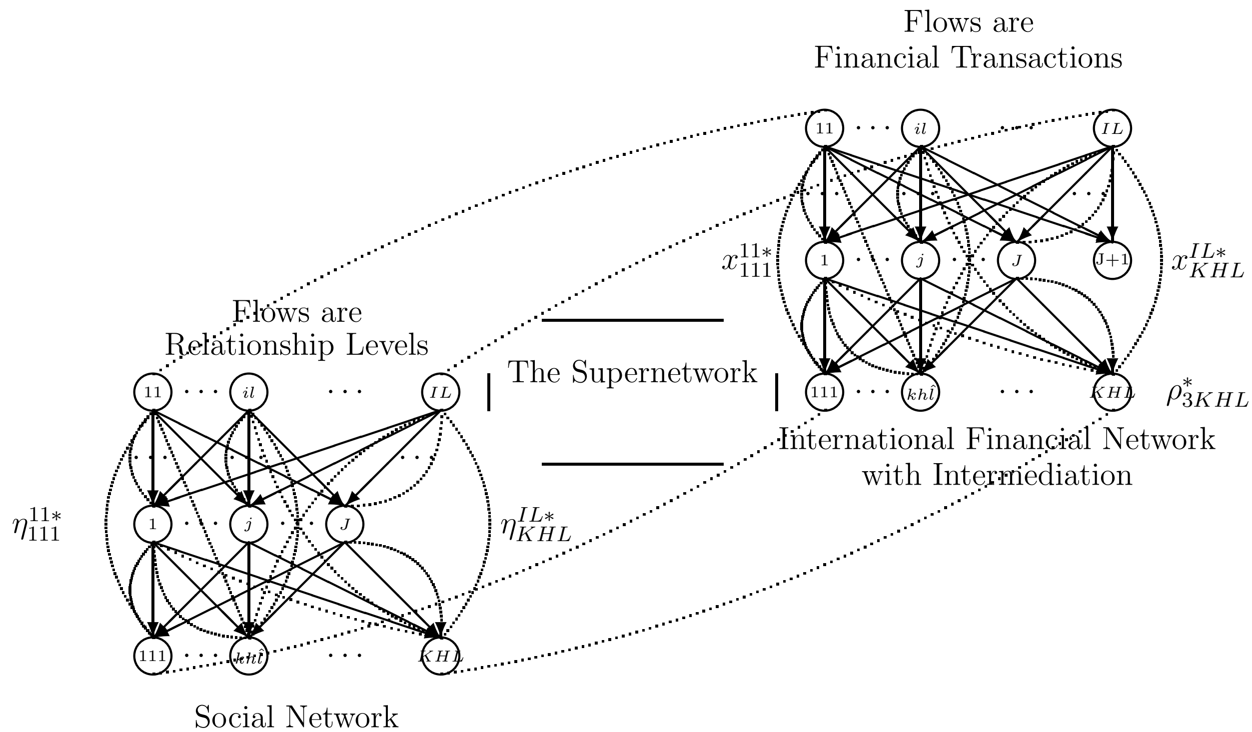

Figure 2: The Supernetwork at Equilibrium

the equilibrium conditions (53) also hold separately (as well as for each individual decision-maker).

In Figure 2, we display the supernetwork in equilibrium in which the equilibrium financial flows, relationship levels, and prices now appear. Note that, if the equilibrium values of the flows (be they financial or relationship levels) on links are identically equal to zero, then those links can effectively be removed from the supernetwork (in equilibrium). Moreover, the size of the equilibrium flows represent the "strength" of respective links (as discussed also in the social network/supply chain network equilibrium model of Wakolbinger and Nagurney (2004)). Thus, the supernetwork model developed here also provides us with the emergent integrated social and financial network structures. In the next section, we discuss the dynamic evolution of the financial flows, relationship levels, and prices until this equilibrium is achieved.

3. The Dynamic Adjustment Process

In this section, we describe the dynamics associated with the supernetwork model developed in Section 2 and formulate the corresponding dynamic model as a projected dynamical system (cf. Nagurney and Zhang (1996a), Nagurney and Ke (2003), Nagurney and Cruz (2004), and Nagurney, Wakolbinger, and Zhao (2004)). Importantly, the set of stationary points of the projected dynamical system which formulates the dynamic adjustment process will coincide with the set of solutions to the variational inequality problem (54). In particular, we describe the disequilibrium dynamics of the international financial flows, the relationship levels, as well as the prices.

The Dynamics of the Financial Flows from the Source Agents

Note that, unlike the financial flows (as well as the prices associated with the distinct nodal tiers of the network) between the intermediaries and the demand markets, the financial flows from the source agents are subject not only to nonnegativity constraints but also to budget constraints (cf. (1)). Hence, in order to guarantee that these constraints are not violated we need to introduce some additional machinery based on projected dynamical systems theory in order to describe the dynamics of these financial flows (see also, e.g., Nagurney and Siokos (1997), Nagurney and Zhang (1996a), Nagurney and Cruz (2004), and Nagurney, Cruz, and Matsypura (2003)).

In particular, we denote the rate of change of the vector of financial flows from source agent il by $\dot{x}^{il}$ and noting that the *best realizable direction* for the financial flows from source agent il must include the constraints, we have that:

$$\dot{x}^{il} = \Pi_{\mathcal{K}^{il}}(x^{il}, -F^{il}), \tag{57}$$

where Π_K is defined as (see also Nagurney and Zhang (1996a)):

$$\Pi_K(x, v) = \lim_{\delta \to 0} \frac{P_K(x + \delta v) - x}{\delta}, \tag{58}$$

and P_K is the norm projection defined by

$$P_K(x) = \text{argmin}_{x' \in K} \|x' - x\|. \tag{59}$$

The feasible set $\mathcal{K}^{il}$ is defined as: $\mathcal{K}^{il} \equiv \{x^{il} | x^{il} \in R_+^{2JH+KHL}$ and satisfies (1)$\}$, and F^{il} is the vector (see following (56)) with components: $F_{iljhm}, F_{ilkh\hat{l}}$ and with indices: $j = 1, \ldots, J$; $h = 1, \ldots, H$; $m = 1, 2$, and $k = 1, \ldots, K$. Hence, expression (57) reflects that the financial flow on a link emanating from a source agent will increase if the price (be it the market-clearing price associated with an intermediary or a demand market price) exceeds the various costs and weighted marginal risk; it will decrease if the latter exceeds the former.

The Dynamics of the Financial Products between the Intermediaries and the Demand Markets

The rate of change of the financial flow $y^j_{kh\hat{l}m}$, denoted by $\dot{y}^j_{kh\hat{l}m}$, is assumed to be equal to the difference between the price the consumers are willing to pay for the financial product at the demand market minus the price charged and the various transaction costs and the weighted marginal risk associated with the transaction. Here we also guarantee that the financial flows do not become negative. Hence, we may write: for every $j, k, h, \hat{l}, m$:

$$\dot{y}^j_{kh\hat{l}m} = \begin{cases} \rho_{3kh\hat{l}} - \delta^j \frac{\partial r^j_{kh\hat{l}m}(y^j_{kh\hat{l}m}, \eta^j_{kh\hat{l}m})}{\partial y^j_{kh\hat{l}m}} - \frac{\partial c^j_{kh\hat{l}m}(y^j_{kh\hat{l}m}, \eta^j_{kh\hat{l}m})}{\partial y^j_{kh\hat{l}m}} & \\ -\hat{c}^j_{kh\hat{l}m}(x^2, y, \eta^2, \eta^3) - \lambda_j, & \text{if} \quad y^j_{kh\hat{l}m} > 0 \\ & \\ \max\{0, \rho_{3kh\hat{l}} - \delta^j \frac{\partial r^j_{kh\hat{l}m}(y^j_{kh\hat{l}m}, \eta^j_{kh\hat{l}m})}{\partial y^j_{kh\hat{l}m}} - \frac{\partial c^j_{kh\hat{l}m}(y^j_{kh\hat{l}m}, \eta^j_{kh\hat{l}m})}{\partial y^j_{kh\hat{l}m}} & \\ -\hat{c}^j_{kh\hat{l}m}(x^2, y, \eta^2, \eta^3) - \lambda_j\}, & \text{if} \quad y^j_{kh\hat{l}m} = 0. \end{cases} \tag{60}$$

Hence, according, to (60), if the price that the consumers are willing to pay for the product (in the currency and country) exceeds the price that the intermediary charges and the various transaction costs and weighted marginal risk, then the volume of flow of the product to that demand market will increase; otherwise, it will decrease (or remain unchanged).

The Dynamics of the Relationship Levels between the Source Agents and the Financial Intermediaries

Now the dynamics of the relationship levels between the source agents in the various countries and the intermediaries are described. The rate of change of the relationship level η^{il}_{jhm}, denoted by $\dot{\eta}^{il}_{jhm}$, is assumed to be equal to the difference between the weighted relationship value for source agent il, intermediary j, currency h and mode m, and the sum of the marginal costs and the weighted marginal risks. Again, one must also guarantee that the relationship levels do not become negative. Moreover, they may not exceed the level equal to one. Hence, we can immediately write:

$$\dot{\eta}^{il}_{jhm} = \begin{cases} \beta^{il}\frac{\partial v^{il}_{jhm}(\eta^{il}_{jhm})}{\partial \eta^{il}_{jhm}} + \gamma^j \frac{\partial \hat{v}^{il}_{jhm}(\eta^{il}_{jhm})}{\partial \eta^{il}_{jhm}} - \frac{\partial c^{il}_{jhm}(x^{il}_{jhm},\eta^{il}_{jhm})}{\partial \eta^{il}_{jhm}} \\ -\frac{\partial \hat{c}^{il}_{jhm}(x^{il}_{jhm},\eta^{il}_{jhm})}{\partial \eta^{il}_{jhm}} - \alpha^{il}\frac{\partial r^{il}_{jhm}(x^{il}_{jhm},\eta^{il}_{jhm})}{\partial \eta^{il}_{jhm}} - \delta^j \frac{\partial \hat{r}^{il}_{jhm}(x^{il}_{jhm},\eta^{il}_{jhm})}{\partial \eta^{il}_{jhm}} \\ -\frac{\partial b^{il}_{jhm}(\eta^{il}_{jhm})}{\partial \eta^{il}_{jhm}} - \frac{\partial \hat{b}^{il}_{jhm}(\eta^{il*}_{jhm})}{\partial \eta^{il}_{jhm}}, & \text{if} \quad 0 < \eta^{il}_{jhm} < 1 \\ \\ \min\{1, \max\{0, \beta^{il}\frac{\partial v^{il}_{jhm}(\eta^{il}_{jhm})}{\partial \eta^{il}_{jhm}} + \gamma^j \frac{\partial \hat{v}^{il}_{jhm}(\eta^{il}_{jhm})}{\partial \eta^{il}_{jhm}} - \frac{\partial c^{il}_{jhm}(x^{il}_{jhm},\eta^{il}_{jhm})}{\partial \eta^{il}_{jhm}} \\ -\frac{\partial \hat{c}^{il}_{jhm}(x^{il}_{jhm},\eta^{il}_{jhm})}{\partial \eta^{il}_{jhm}} - \alpha^{il}\frac{\partial r^{il}_{jhm}(x^{il}_{jhm},\eta^{il}_{jhm})}{\partial \eta^{il}_{jhm}} - \delta^j \frac{\partial \hat{r}^{il}_{jhm}(x^{il}_{jhm},\eta^{il}_{jhm})}{\partial \eta^{il}_{jhm}} \\ -\frac{\partial b^{il}_{jhm}(\eta^{il}_{jhm})}{\partial \eta^{il}_{jhm}} - \frac{\partial \hat{b}^{il}_{jhm}(\eta^{il}_{jhm})}{\partial \eta^{il}_{jhm}}\}\}, & \text{otherwise,} \end{cases} \tag{61}$$

where $\dot{\eta}^{il}_{jhm}$ denotes the rate of change of the relationship level η^{il}_{jhm}.

This shows that if the sum of the weighted relationship values for the source agent and the intermediary are higher than the total marginal costs plus the total weighted marginal risk, then the level of relationship between that financial source agent and intermediary pair will increase. If it is lower, the relationship value will decrease.

The Dynamics of the Relationship Levels between the Source Agents and the Demand Markets

Here we describe the dynamics of the relationship levels between the source agents and the demand markets. The rate of change of the relationship level $\eta^{il}_{kh\hat{l}}$ in turn, responds to the difference between the weighted relationship value for source agent il and the sum of the marginal costs and weighted marginal risks . One also must guarantee that these relationship levels do not become negative (nor higher than one). Hence, one may write:

$$\dot{\eta}^{il}_{kh\hat{l}} = \begin{cases} \beta^{il}\frac{\partial v^{il}_{kh\hat{l}}(\eta^{il}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}} - \frac{\partial c^{il}_{kh\hat{l}}(x^{il}_{kh\hat{l}},\eta^{il}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}} - \frac{\partial b^{il}_{kh\hat{l}}(\eta^{il}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}} - \alpha^{il}\frac{\partial r^{il}_{kh\hat{l}}(x^{il}_{kh\hat{l}},\eta^{il}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}}, & \text{if} \quad 0 < \eta^{il}_{kh\hat{l}} < 1, \\ \min\{1, \max\{0, \beta^{il}\frac{\partial v^{il}_{kh\hat{l}}(\eta^{il}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}} - \frac{\partial c^{il}_{kh\hat{l}}(x^{il}_{kh\hat{l}},\eta^{il}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}} - \frac{\partial b^{il}_{kh\hat{l}}(\eta^{il}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}} \\ -\alpha^{il}\frac{\partial r^{il}_{kh\hat{l}}(x^{il}_{kh\hat{l}},\eta^{il}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}}\}\}, & \text{otherwise} \end{cases} \tag{62}$$

where $\dot{\eta}^{il}_{kh\hat{l}}$ denotes the rate of change of the relationship level $\eta^{il}_{kh\hat{l}}$. This shows that if the weighted relationship value for the source agent is higher than the total marginal costs plus the total weighted marginal risk, then the level of relationship between that financial source agent and demand market pair will increase. If it is lower, the relationship value will decrease. Of course, the bounds on the relationship levels must also hold.

The Dynamics of the Relationship Levels between the Financial Intermediaries and the Demand Markets

The dynamics of the relationship levels between the financial intermediaries and demand markets are now described. The rate of change of the relationship level product $\dot{\eta}^{j}_{kh\hat{l}m}$ transacted via mode m is assumed to be equal to the difference between the weighted relationship value for intermediary j and the sum of the marginal costs and weighted marginal risks, where, of course, one also must guarantee that the relationship levels do not become negative nor exceed one. Hence, one may write:

$$\dot{\eta}^{j}_{kh\hat{l}m} = \begin{cases} \gamma^{j}\frac{\partial v^{j}_{kh\hat{l}m}(\eta^{j}_{kh\hat{l}m})}{\partial \eta^{j}_{kh\hat{l}m}} - \delta^{j}\frac{\partial r^{j}_{kh\hat{l}m}(y^{j}_{kh\hat{l}m},\eta^{j}_{kh\hat{l}m})}{\partial \eta^{j}_{kh\hat{l}m}} - \frac{\partial c^{j}_{kh\hat{l}m}(y^{j}_{kh\hat{l}m},\eta^{j}_{kh\hat{l}m})}{\partial \eta^{j}_{kh\hat{l}m}} \\ -\frac{\partial b^{j}_{khl}(\eta^{j}_{kh\hat{l}m})}{\partial \eta^{j}_{kh\hat{l}m}}, & \text{if} \quad 0 < \eta^{j}_{kh\hat{l}m} < 1, \\ \min\{1, \max\{0, \gamma^{j}\frac{\partial v^{j}_{kh\hat{l}m}(\eta^{j*}_{kh\hat{l}m})}{\partial \eta^{j}_{kh\hat{l}m}} - \delta^{j}\frac{\partial r^{j}_{kh\hat{l}m}(y^{j}_{kh\hat{l}m},\eta^{j}_{kh\hat{l}m})}{\partial \eta^{j}_{kh\hat{l}m}} \\ -\frac{\partial c^{j}_{kh\hat{l}m}(y^{j}_{kh\hat{l}m},\eta^{j}_{kh\hat{l}m})}{\partial \eta^{j}_{kh\hat{l}m}} - \frac{\partial b^{j}_{khl}(\eta^{j}_{kh\hat{l}m})}{\partial \eta^{j}_{kh\hat{l}m}}\}\}, & \text{otherwise,} \end{cases} \tag{63}$$

where $\dot{\eta}^{j}_{kh\hat{l}m}$ denotes the rate of change of the relationship level $\eta^{j}_{kh\hat{l}m}$. Expression (63) reveals that if the weighted relationship value for the intermediary with the demand market is higher than the total marginal costs plus the total weighted marginal risk, then the level of relationship between that intermediary and demand market pair will increase. If it is lower, the relationship value will decrease.

Demand Market Price Dynamics

We assume that the rate of change of the price $\rho_{3kh\hat{l}}$, denoted by $\dot{\rho}_{3kh\hat{l}}$, is equal to the difference between the demand for the financial product at the demand market in the currency and country and the amount of the product actually available at that particular market. Hence, if the demand for the product at the demand market at an instant in time exceeds the amount available from the various intermediaries and source agents, then the price will increase; if the amount available exceeds the demand at the price, then the price will decrease. Moreover, it is guaranteed that the prices do not become negative. Thus, the dynamics of the price $\rho_{3kh\hat{l}}$ for each $k, h, \hat{l}$ can be expressed as:

$$\dot{\rho}_{3kh\hat{l}} = \begin{cases} d_{kh\hat{l}}(\rho_3) - \sum_{j=1}^{J}\sum_{m=1}^{2} y^j_{kh\hat{l}m} - \sum_{i=1}^{I}\sum_{l=1}^{L} x^{il}_{kh\hat{l}}, & \text{if} \quad \rho_{3kh\hat{l}} > 0 \\ \max\{0, d_{kh\hat{l}}(\rho_3) - \sum_{j=1}^{J}\sum_{m=1}^{2} y^j_{kh\hat{l}m} - \sum_{i=1}^{I}\sum_{l=1}^{L} x^{il}_{kh\hat{l}}\}, & \text{if} \quad \rho_{3kh\hat{l}} = 0. \end{cases} \tag{64}$$

The Dynamics of the Prices at the Intermediaries

The prices at the intermediaries, whether they are physical or virtual, must reflect supply and demand conditions as well. In particular, we let $\dot{\lambda}_j$ denote the rate of change in the market clearing price associated with intermediary j and we propose the following dynamic adjustment for every intermediary j:

$$\dot{\lambda}_j = \begin{cases} \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2} y^j_{kh\hat{l}m} - \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2} x^{il}_{jhm}, & \text{if} \quad \lambda_j > 0 \\ \max\{0, \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2} y^j_{kh\hat{l}m} - \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\sum_{m=1}^{2} x^{il}_{jhm}\}, & \text{if} \quad \lambda_j = 0. \end{cases} \tag{65}$$

Hence, if the financial flows from the source agents in the countries into an intermediary exceed the amount demanded at the demand markets from the intermediary, then the market-clearing price at that intermediary will decrease; if, on the other hand, the volume of financial flows into an intermediary is less than that demanded by the consumers at the demand markets (and handled by the intermediary), then the market-clearing price at that intermediary will increase.

The Projected Dynamical System

We now turn to stating the complete dynamic model. In the dynamic model the flows evolve according to the mechanisms described above; specifically, the financial flows from the source agents evolve according to (57) for all source agents il. The financial flows from the financial intermediaries to the demand markets evolve according to (60) for all financial intermediaries j, demand markets $kh\hat{l}$, and modes m. The relationship levels between source agents and financial intermediaries for all modes m evolve according to (61), the relationship levels between source agents il and demand markets $kh\hat{l}$ evolve according to (62), and the relationship levels between financial intermediaries j and demand markets $kh\hat{l}$ for all modes m evolve according to (63). Furthermore, the prices associated with the intermediaries evolve according to (64) for all intermediaries j, and the demand market prices evolve according to (65) for all k.

Let X and $F(X)$ be as defined following (56) and recall the feasible set $\mathcal{K}$. Then the dynamic model described by (57), (60)–(65) can be rewritten as a *projected dynamical system* (Nagurney and Zhang (1996a)) defined by the following initial value problem:

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X_0, \tag{66}$$

where $\Pi_{\mathcal{K}}$ is the projection operator of $-F(X)$ onto $\mathcal{K}$ at X (cf. (58)) and $X_0 = (x^{10}, x^{20}, y^0, \eta^{10}, \eta^{20}, \eta^{30}, \lambda^0, \rho_3^0)$ is the initial point corresponding to the initial financial flow and price pattern.

The trajectory of (66) describes the dynamic evolution of the relationship levels on the social network, the financial product transactions on the financial network, the demand market prices and the Lagrange multipliers or shadow prices associated with the intermediaries. The projection operation guarantees the constraints underlying the supernetwork system are not violated. Recall that the constraint set $\mathcal{K}$ consists not only of the conservation of flow constraints (cf. (1)) associated with the source agents but also the nonnegativity constraints associated with all the financial flows, the prices, as well as the relationships levels. Moreover, the relationship levels are assumed to not exceed the value of one.

Following Dupuis and Nagurney (1993) and Nagurney and Zhang (1996a), the following result is immediate.

Theorem 2: Set of Stationary Points Coincides with Set of Equilibrium Points

The set of stationary points of the projected dynamical system (66) coincides with the set of solutions of the variational inequality problem (54) and, thus, with the set of equilibrium points as defined in Definition 1.

With Theorem 2, we see that the dynamical system proposed in this Section provides the disequilibrium dynamics prior to the steady or equilibrium state of the international financial network. Hence, once, a stationary point of the projected dynamical system is reached, that is, when $\dot{X} = 0$ in (66), that point (consisting of financial flows, relationship levels, and prices) also satisfies variational inequality (54); equivalently, (56), and is, therefore, an international financial network equilibrium according to Definition 1.

The above described dynamics are very reasonable from an economic perspective and also illuminate that there must be cooperation between tiers of decision-makers although there may be competition within a tier.

We now state the following:

Theorem 3: Existence and Uniqueness of a Solution to the Initial Value Problem

Assume that $F(X)$ is Lipschitz continuous, that is, that

$$\|F(X') - F(X'')\| \leq \mathcal{L}\|X' - X''\|, \quad \forall X', X'' \in \mathcal{K}, \text{ where } \mathcal{L} > 0, \qquad (67).$$

Then, for any $X_0 \in \mathcal{K}$, there exists a unique solution $X_0(t)$ to the initial value problem (66).

Proof: Lipschitz continuity of the function F is sufficient for the result following Theorem 2.5 in Nagurney and Zhang (1996a).□

Under suitable conditions on the underlying functions (see also Nagurney and Dong (2002), Zhang and Nagurney (1995), and Nagurney, Wakolbinger, and Zhao (2004)), one can obtain stability results for the supernetwork. A similar result was obtained for a supply chain network model with electronic commerce and relationship levels in Wakolbinger and Nagurney (2004).

4. The Discrete-Time Algorithm

In this Section, we propose the Euler method for the computation of solutions to variational inequality (54); equivalently, the stationary points of the projected dynamical system (66). The Euler method is a special case of the general iterative scheme introduced by Dupuis and Nagurney (1993) for the solution of projected dynamical systems. Besides providing a solution to variational inequality problem (54), this algorithm also yields a time discretization of the continuous-time adjustment process of the projected dynamical system (66). This discretization may also be interpreted as a discrete-time adjustment process. Conditions for convergence of this algorithm are given in Dupuis and Nagurney (1993) and in Nagurney and Zhang (1996a). In Section 5, we apply this algorithm to several numerical examples.

The Euler Method

The statement of the Euler method is the following: At iteration $\mathcal{T}$ compute

$$X_{\mathcal{T}} = P_{\mathcal{K}}(X_{\mathcal{T}-1} - a_{\mathcal{T}-1} F(X_{\mathcal{T}-1})), \tag{68}$$

where $P_{\mathcal{K}}$ denotes the projection operator in the Euclidean sense (cf. (57) and Nagurney (1999)) onto the closed convex set $\mathcal{K}$ and $F(X)$ is defined following (56)). We discuss the sequence of positive terms $a_{\mathcal{T}}$ below. The complete statement of the method in the context of the dynamic supernetwork model is as follows:

Step 0: Initialization

Set $(x^{10}, x^{20}, y^0, \eta^{10}, \eta^{20}, \eta^{30}, \lambda^0, \rho_3^0) \in \mathcal{K}$. Let $\mathcal{T} = 1$ and set the sequence $\{a_{\mathcal{T}}\}$ so that $\sum_{\mathcal{T}=1}^{\infty} a_{\mathcal{T}} = \infty$, $a_{\mathcal{T}} > 0$, $a_{\mathcal{T}} \to 0$, as $\mathcal{T} \to \infty$ (such a sequence is required for convergence of the algorithm).

Step 1: Computation

Compute $(x^{1\mathcal{T}}, x^{2\mathcal{T}}, y^{\mathcal{T}}, \eta^{1\mathcal{T}}, \eta^{2\mathcal{T}}, \eta^{3\mathcal{T}}, \lambda^{\mathcal{T}}, \rho_3^{\mathcal{T}}) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}\Bigg[x_{jhm}^{il\mathcal{T}} + a_{\mathcal{T}}(\alpha^{il}\frac{\partial r_{jhm}^{il}(x_{jhm}^{il\mathcal{T}-1}, \eta_{jhm}^{il\mathcal{T}-1})}{\partial x_{jhm}^{il}} + \frac{\partial c_{jhm}^{il}(x_{jhm}^{il\mathcal{T}-1}, \eta_{jhm}^{il\mathcal{T}-1})}{\partial x_{jhm}^{il}}$$

$$+\delta^j \frac{\partial \hat{r}^{il}_{jhm}(x^{il\mathcal{T}-1}_{jhm}, \eta^{il\mathcal{T}-1}_{jhm})}{\partial x^{il}_{jhm}} + \frac{\partial c_j(x^{1\mathcal{T}-1})}{\partial x^{il}_{jhm}} + \frac{\partial \hat{c}^{il}_{jhm}(x^{il\mathcal{T}-1}_{jhm}, \eta^{il\mathcal{T}-1}_{jhm})}{\partial x^{il}_{jhm}} - \lambda^{\mathcal{T}-1}_j) - x^{il\mathcal{T}-1}_{jhm}\Bigg]$$

$$\times \left[x^{il}_{jhm} - x^{il\mathcal{T}}_{jhm}\right]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\Bigg[x^{il\mathcal{T}}_{kh\hat{l}} + a_{\mathcal{T}}(\alpha^{il}\frac{\partial r^{il}_{kh\hat{l}}(x^{il\mathcal{T}-1}_{kh\hat{l}}, \eta^{il\mathcal{T}-1}_{kh\hat{l}})}{\partial x^{il}_{kh\hat{l}}} + \frac{\partial c^{il}_{kh\hat{l}}(x^{il\mathcal{T}-1}_{kh\hat{l}}, \eta^{il\mathcal{T}-1}_{kh\hat{l}})}{\partial x^{il}_{kh\hat{l}}}$$

$$+\hat{c}^{il}_{kh\hat{l}}(x^{2\mathcal{T}-1}, y^{\mathcal{T}-1}, \eta^{2\mathcal{T}-1}, \eta^{3\mathcal{T}-1}) - \rho^{\mathcal{T}-1}_{3kh\hat{l}}) - x^{il\mathcal{T}-1}_{kh\hat{l}}\Bigg]$$

$$\times \left[x^{il}_{kh\hat{l}} - x^{il\mathcal{T}}_{kh\hat{l}}\right]$$

$$+\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}\Bigg[y^{j\mathcal{T}}_{kh\hat{l}m} + a_{\mathcal{T}}(\delta^j\frac{\partial r^j_{kh\hat{l}m}(y^{j\mathcal{T}-1}_{kh\hat{l}m}, \eta^{j\mathcal{T}-1}_{kh\hat{l}m})}{\partial y^j_{kh\hat{l}m}} + \frac{\partial c^j_{kh\hat{l}m}(y^{j\mathcal{T}-1}_{kh\hat{l}m}, \eta^{j\mathcal{T}-1}_{kh\hat{l}m})}{\partial y^j_{kh\hat{l}m}}$$

$$+\hat{c}^j_{kh\hat{l}m}(x^{2\mathcal{T}-1}, y^{\mathcal{T}-1}, \eta^{2\mathcal{T}-1}, \eta^{3\mathcal{T}-1}) + \lambda^{\mathcal{T}-1}_j - \rho^{\mathcal{T}-1}_{3kh\hat{l}}) - y^{j\mathcal{T}-1}_{kh\hat{l}m}\Bigg]$$

$$\times \left[y^j_{kh\hat{l}m} - y^{j\mathcal{T}}_{kh\hat{l}m}\right]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{m=1}^{2}\Bigg[\eta^{il\mathcal{T}}_{jhm} + a_{\mathcal{T}}(\frac{\partial c^{il}_{jhm}(x^{il\mathcal{T}-1}_{jhm}, \eta^{il\mathcal{T}-1}_{jhm})}{\partial \eta^{il}_{jhm}} + \frac{\partial \hat{c}^{il}_{jhm}(x^{il\mathcal{T}-1}_{jhm}, \eta^{il\mathcal{T}-1}_{jhm})}{\partial \eta^{il}_{jhm}}$$

$$-\beta^{il}\frac{\partial v^{il}_{jhm}(\eta^{il\mathcal{T}-1}_{jhm})}{\partial \eta^{il}_{jhm}} - \gamma^j\frac{\partial \hat{v}^{il}_{jhm}(\eta^{il\mathcal{T}-1}_{jhm})}{\partial \eta^{il}_{jhm}} + \alpha^{il}\frac{\partial r^{il}_{jhm}(x^{il\mathcal{T}-1}_{jhm}, \eta^{il\mathcal{T}-1}_{jhm})}{\partial \eta^{il}_{jhm}}$$

$$+\delta^j\frac{\partial \hat{r}^{il}_{jhm}(x^{il\mathcal{T}-1}_{jhm}, \eta^{il\mathcal{T}-1}_{jhm})}{\partial \eta^{il}_{jhm}} + \frac{\partial b^{il}_{jhm}(\eta^{il\mathcal{T}-1}_{jhm})}{\partial \eta^{il}_{jhm}} + \frac{\partial \hat{b}^{il}_{jhm}(\eta^{il\mathcal{T}-1}_{jhm})}{\partial \eta^{il}_{jhm}}) - \eta^{il\mathcal{T}-1}_{jhm}\Bigg]$$

$$\times \left[\eta^{il}_{jhm} - \eta^{il\mathcal{T}}_{jhm}\right]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\Bigg[\eta^{il\mathcal{T}}_{kh\hat{l}} + a_{\mathcal{T}}(\frac{\partial c^{il}_{kh\hat{l}}(x^{il\mathcal{T}-1}_{kh\hat{l}}, \eta^{il\mathcal{T}-1}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}} + \frac{\partial b^{il}_{kh\hat{l}}(\eta^{il\mathcal{T}-1}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}} - \beta^{il}\frac{\partial v^{il}_{kh\hat{l}}(\eta^{il\mathcal{T}-1}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}}$$

$$+\alpha^{il}\frac{\partial r^{il}_{kh\hat{l}}(x^{il\mathcal{T}-1}_{kh\hat{l}}, \eta^{il\mathcal{T}-1}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}}) - \eta^{il\mathcal{T}-1}_{kh\hat{l}}\Bigg] \times \left[\eta^{il}_{kh\hat{l}} - \eta^{il\mathcal{T}}_{kh\hat{l}}\right]$$

$$+\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\sum_{m=1}^{2}\Bigg[\eta^{j\mathcal{T}}_{kh\hat{l}m} + a_{\mathcal{T}}(\delta^j\frac{\partial r^j_{kh\hat{l}m}(y^{j\mathcal{T}-1}_{kh\hat{l}m}, \eta^{j\mathcal{T}-1}_{kh\hat{l}m})}{\partial \eta^j_{kh\hat{l}m}} + \frac{\partial c^j_{kh\hat{l}m}(y^{j\mathcal{T}-1}_{kh\hat{l}m}, \eta^{j\mathcal{T}-1}_{kh\hat{l}m})}{\partial \eta^j_{kh\hat{l}m}}$$

$$-\gamma^j\frac{\partial v^j_{kh\hat{l}m}(\eta^{j\mathcal{T}-1}_{kh\hat{l}m})}{\partial \eta^j_{kh\hat{l}m}} + \frac{\partial b^j_{khl}(\eta^{j\mathcal{T}-1}_{kh\hat{l}m})}{\partial \eta^j_{kh\hat{l}m}}) - \eta^{j\mathcal{T}-1}_{kh\hat{l}m}\Bigg] \times \left[\eta^j_{kh\hat{l}m} - \eta^{j\mathcal{T}}_{kh\hat{l}m}\right]$$

$$+\sum_{j=1}^{J}\left[\lambda_j^{\mathcal{T}}+a_{\mathcal{T}}(\sum_{m=1}^{2}\left[\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}x_{jhm}^{il\mathcal{T}-1}-\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}y_{kh\hat{l}m}^{j\mathcal{T}-1}\right])-\lambda_j^{\mathcal{T}-1}\right]\times\left[\lambda_j-\lambda_j^{\mathcal{T}}\right]$$

$$+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\rho_{3kh\hat{l}}^{\mathcal{T}}+a_{\mathcal{T}}(\sum_{j=1}^{J}\sum_{m=1}^{2}y_{kh\hat{l}m}^{j\mathcal{T}-1}+\sum_{i=1}^{I}\sum_{\hat{l}=1}^{L}x_{kh\hat{l}}^{il\mathcal{T}-1}-d_{kh\hat{l}}(\rho_3^{\mathcal{T}-1}))-\rho_{3kh\hat{l}}^{\mathcal{T}-1}\right]$$

$$\times\left[\rho_{3kh\hat{l}}-\rho_{3kh\hat{l}}^{\mathcal{T}}\right]\geq 0,\quad\forall(x^1,x^2,y,\eta^1,\eta^2,\eta^3,\lambda,\rho_3)\in\mathcal{K},\tag{69}$$

Step 2: Convergence Verification

If $|x_{jhm}^{il\mathcal{T}}-x_{jhm}^{il\mathcal{T}-1}|\leq\epsilon$, $|x_{kh\hat{l}}^{il\mathcal{T}}-x_{kh\hat{l}}^{il\mathcal{T}-1}|\leq\epsilon$, $|y_{kh\hat{l}m}^{j\mathcal{T}}-y_{kh\hat{l}m}^{j\mathcal{T}-1}|\leq\epsilon$, $|\eta_{jhm}^{il\mathcal{T}}-\eta_{jhm}^{il\mathcal{T}-1}|\leq\epsilon$, $|\eta_{kh\hat{l}}^{il\mathcal{T}}-\eta_{kh\hat{l}}^{il\mathcal{T}-1}|\leq\epsilon$, $|\eta_{kh\hat{l}m}^{j\mathcal{T}}-\eta_{kh\hat{l}m}^{j\mathcal{T}-1}|\leq\epsilon$, $|\lambda_j^{\mathcal{T}}-\lambda_j^{\mathcal{T}-1}|\leq\epsilon$, $|\rho_{3kh\hat{l}}^{\mathcal{T}}-\rho_{3kh\hat{l}}^{\mathcal{T}-1}|\leq\epsilon$, for all $i=1,\cdots,I$; $l=1,\ldots,L$; $\hat{l};\hat{l}=1,\ldots,L$; $m=1,2$; $j=1,\cdots,J$; $h=1,\ldots,H$; $k=1,\cdots,K$, with $\epsilon>0$, a pre-specified tolerance, then stop; otherwise, set $\mathcal{T}:=\mathcal{T}+1$, and go to Step 1.

Due to the simplicity of the feasible set $\mathcal{K}$ the solution of (69) is accomplished exactly and in closed form. In (69) the variational subproblem of the variables (x^1,x^2) can be solved using exact equilibration (cf. Dafermos and Sparrow (1969), Nagurney (1999)). The other variables can be obtained using the following explicit formulae:

Computation of the Financial Products from the Intermediaries

In particular, compute, at iteration $\mathcal{T}$, the $y_{kh\hat{l}m}^{j\mathcal{T}}$s, according to:

$$y_{kh\hat{l}m}^{j\mathcal{T}}=\max\{0,y_{kh\hat{l}m}^{j\mathcal{T}-1}-a_{\mathcal{T}}(\delta^j\frac{\partial r_{kh\hat{l}m}^j(y_{kh\hat{l}m}^{j\mathcal{T}-1},\eta_{kh\hat{l}m}^{j\mathcal{T}-1})}{\partial y_{kh\hat{l}m}^j}+\frac{\partial c_{kh\hat{l}m}^j(y_{kh\hat{l}m}^{j\mathcal{T}-1},\eta_{kh\hat{l}m}^{j\mathcal{T}-1})}{\partial y_{kh\hat{l}m}^j}$$

$$+\hat{c}_{kh\hat{l}m}^j(x^{2\mathcal{T}-1},y^{\mathcal{T}-1},\eta^{2\mathcal{T}-1},\eta^{3\mathcal{T}-1})+\lambda_j^{\mathcal{T}-1}-\rho_{3kh\hat{l}}^{\mathcal{T}-1})\},\quad\forall j,k,h,\hat{l},m.\tag{70}$$

Computation of the Relationship Levels

At iteration $\mathcal{T}$ compute the $\eta_{jhm}^{il\mathcal{T}}s$ according to:

$$\eta_{jhm}^{il\mathcal{T}}=\min\{1,\max\{0,\eta_{jhm}^{il\mathcal{T}-1}-a_{\mathcal{T}}(\frac{\partial c_{jhm}^{il}(x_{jhm}^{il\mathcal{T}-1},\eta_{jhm}^{il\mathcal{T}-1})}{\partial\eta_{jhm}^{il}}+\frac{\partial\hat{c}_{jhm}^{il}(x_{jhm}^{il\mathcal{T}-1},\eta_{jhm}^{il\mathcal{T}-1})}{\partial\eta_{jhm}^{il}}$$

$$
-\beta^{il}\frac{\partial v^{il}_{jhm}(\eta^{il\mathcal{T}-1}_{jhm})}{\partial \eta^{il}_{jhm}} - \gamma^j\frac{\partial \hat{v}^{il}_{jhm}(\eta^{il\mathcal{T}-1}_{jhm})}{\partial \eta^{il}_{jhm}} + \alpha^{il}\frac{\partial r^{il}_{jhm}(x^{il\mathcal{T}-1}_{jhm},\eta^{il\mathcal{T}-1}_{jhm})}{\partial \eta^{il}_{jhm}}
$$

$$
+\delta^j\frac{\partial \hat{r}^{il}_{jhm}(x^{il\mathcal{T}-1}_{jhm},\eta^{il\mathcal{T}-1}_{jhm})}{\partial \eta^{il}_{jhm}} + \frac{\partial b^{il}_{jhm}(\eta^{il\mathcal{T}-1}_{jhm})}{\partial \eta^{il}_{jhm}} + \frac{\partial \hat{b}^{il}_{jhm}(\eta^{il\mathcal{T}-1}_{jhm})}{\partial \eta^{il}_{jhm}})\}\}, \quad \forall i,l,j,h,m. \tag{71}
$$

Furthermore, at iteration $\mathcal{T}$ compute the $\eta^{il\mathcal{T}}_{kh\hat{l}}$ according to:

$$
\eta^{il\mathcal{T}}_{kh\hat{l}} = \min\{1, \max\{0, \eta^{il\mathcal{T}-1}_{kh\hat{l}} - a_{\mathcal{T}}(\frac{\partial c^{il}_{kh\hat{l}}(x^{il\mathcal{T}-1}_{kh\hat{l}},\eta^{il\mathcal{T}-1}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}} + \frac{\partial b^{il}_{kh\hat{l}}(\eta^{il\mathcal{T}-1}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}} - \beta^{il}\frac{\partial v^{il}_{kh\hat{l}}(\eta^{il\mathcal{T}-1}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}}
$$

$$
+\alpha^{il}\frac{\partial r^{il}_{kh\hat{l}}(x^{il\mathcal{T}-1}_{kh\hat{l}},\eta^{il\mathcal{T}-1}_{kh\hat{l}})}{\partial \eta^{il}_{kh\hat{l}}})\}\}, \quad \forall i,l,k,h,\hat{l}. \tag{72}
$$

At iteration $\mathcal{T}$ compute the $\eta^{j\mathcal{T}}_{kh\hat{l}m}$ according to:

$$
\eta^{j\mathcal{T}}_{kh\hat{l}m} = \min\{1, \max\{0, \eta^{j\mathcal{T}}_{kh\hat{l}m} - a_{\mathcal{T}}(\delta^j\frac{\partial r^j_{kh\hat{l}m}(y^{j\mathcal{T}-1}_{kh\hat{l}m},\eta^{j\mathcal{T}-1}_{kh\hat{l}m})}{\partial \eta^j_{kh\hat{l}m}} + \frac{\partial c^j_{kh\hat{l}m}(y^{j\mathcal{T}-1}_{kh\hat{l}m},\eta^{j\mathcal{T}-1}_{kh\hat{l}m})}{\partial \eta^j_{kh\hat{l}m}}
$$

$$
-\gamma^j\frac{\partial v^j_{kh\hat{l}m}(\eta^{j\mathcal{T}-1}_{kh\hat{l}m})}{\partial \eta^j_{kh\hat{l}m}} + \frac{\partial b^j_{khl}(\eta^{j\mathcal{T}-1}_{kh\hat{l}m})}{\partial \eta^j_{kh\hat{l}m}})\}\}, \quad \forall j,k,h,\hat{l},m. \tag{73}
$$

Computation of the Shadow Prices

At iteration $\mathcal{T}$, compute the $\gamma^{\mathcal{T}}_j$s according to:

$$
\gamma^{\mathcal{T}}_j = \max\{0, \gamma^{\mathcal{T}-1}_j - a_{\mathcal{T}}(\sum_{m=1}^{2}\left[\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H} x^{il\mathcal{T}-1}_{jhm} - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} y^{j\mathcal{T}-1}_{kh\hat{l}m}\right])\}, \quad \forall j, \tag{74}
$$

Computation of the Demand Market Prices

Finally, at iteration $\mathcal{T}$ compute the demand market prices, the $\rho^{\mathcal{T}}_{3kh\hat{l}}$s, according to:

$$
\rho^{\mathcal{T}}_{3kh\hat{l}} = \max\{0, \rho^{\mathcal{T}-1}_{3kh\hat{l}} - a_{\mathcal{T}}(\sum_{j=1}^{J}\sum_{m=1}^{2} y^{j\mathcal{T}-1}_{kh\hat{l}m} + \sum_{i=1}^{I}\sum_{l=1}^{L} x^{il\mathcal{T}-1}_{kh\hat{l}} - d_{kh\hat{l}}(\rho^{\mathcal{T}-1}_3))\}, \quad \forall k,h,\hat{l}. \tag{75}
$$

As one can see in the discrete-time adjustment process(es) described above the algorithm is initialized with a vector of financial flows, relationship levels, and prices. For example, the relationship levels may be set to zero (and the same holds for the prices, initially). The financial flows, shadow prices, and the demand market prices are computed in the financial network of the supernetwork. The financial product transactions between intermediaries and demand markets are computed according to (70). The relationship levels are computed in the social network of the supernetwork according to (71), (72), and (73), respectively. Finally, the shadow prices are computed according to (74) and the demand market prices are computed according to (75).

The dynamic supernetwork system will then evolve according to the discrete-time adjustment process (70) through (75) until a stationary/equilibrium point of the projected dynamical system (66) (equivalently, and a solution to variational inequality (54)) is achieved. Once the convergence tolerance has been reached then the equilibrium conditions according to Definition 1 are satisfied as one can see from (70) through (75).

In the next Section, we apply the Euler method to solve several international financial network examples. Convergence results for this algorithm may be found in Dupuis and Nagurney (1993) and for a variety of applications in Nagurney and Zhang (1996a).

5. Numerical Examples

In this section, we applied the Euler method to several supernetwork examples as discussed in the preceding sections. The Euler method was implemented in FORTRAN and the computer system used was a Sun system located at the University of Massachusetts at Amherst. For the solution of the induced network subproblems in the (x^1, x^2) variables we utilized the exact equilibration algorithm (see Dafermos and Sparrow (1969), Nagurney (1999), and the references therein). The other variables were determined explicitly and in closed form as described in the preceding section.

The convergence criterion used was that the absolute value of the flows, prices, and relationship levels between two successive iterations differed by no more than 10^{-4}. The sequence $\{a_{\mathcal{T}}\}$ used for all the examples was: $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots$ in the algorithm.

We assumed in all the examples that the risk was represented through variance-covariance matrices for both the source agents in the countries and for the financial intermediaries (see also Nagurney and Cruz (2004)). We initialized the Euler method as follows: We set $x^{il}_{jh1} = \frac{S^{il}}{JH}$ for each source agent i and country l and for all j and h. All the other variables were initialized to zero.

Example 1

The first numerical example consisted of one country, two source agents, two currencies, two intermediaries, and two financial products. Hence, $L = 1$, $I = 2$, $H = 2$, $J = 2$, and $K = 2$. In this example, the electronic transactions were only between the source agents and the demand markets. In addition, for simplicity, we considerd the possibility of the existence of relationship levels only between the source agents and the intermediaries, and the intermediaries and demand markets.

The data for the first example were constructed for easy interpretation purposes. The financial holdings of the two source agents were: $S^{11} = 20$ and $S^{21} = 20$. The variance-covariance matrices were equal to the identity matrices (appropriately dimensioned) for all source agents and all intermediaries. Note that since only physical transactions are allowed (except for as stated above), we have that $m = 1$.

The transaction cost functions faced by the source agents associated with transacting with the intermediaries (cf. (4)) were given by:

$$c^{il}_{jhm}(x^{il}_{jhm}, \eta^{il}_{jhm}) = .5(x^{il}_{jhm})^2 + 3.5x^{il}_{jhm} - \eta^{il}_{jhm}; \quad i = 1,2; l = 1; j = 1,2; h = 1,2; m = 1.$$

The analogous transaction costs but associated with the electronic transactions between source agents and demand markets (cf. (4)) were given by:

$$c^{il}_{kh\hat{l}}(x^{il}_{kh\hat{l}}) = .5(x^{il}_{kh\hat{l}})^2 + x_{kh\hat{l}}; \quad \forall i, l, \hat{l}, k, h.$$

Note that here we did not include relationship levels in the functional forms.

The handling costs of the intermediaries, in turn (see (27)), were given by:

$$c_j(x^1) = .5(\sum_{i=1}^{2}\sum_{h=1}^{2} x^{i1}_{jh1})^2; \quad j = 1,2.$$

The transaction costs of the intermediaries associated with transacting with the source agents were (cf. (25)) given by:

$$\hat{c}^{il}_{jhm}(x^{il}_{jhm}, \eta^{il}_{jhm}) = 1.5{x^{il}_{jhm}}^2 + 3x^{il}_{jhm}; \; i = 1,2; l = 1; j = 1,2; h = 1,2; m = 1.$$

The transaction costs, in turn, associated with the electronic transactions at the demand markets (from the perspective of the consumers (cf. (48)) were given by:

$$\hat{c}^{il}_{kh\hat{l}}(x^2, y, \eta^2, \eta^3) = .1x^{il}_{kh\hat{l}} + 1, \quad \forall i, l, \hat{l}, k, h.$$

The demand functions at the demand markets (refer to (49)) were:

$$d_{111}(\rho_3) = -2\rho_{3111} - 1.5\rho_{3121} + 1000, \quad d_{121}(\rho_3) = -2\rho_{3121} - 1.5\rho_{3111} + 1000,$$

$$d_{211}(\rho_3) = -2\rho_{3211} - 1.5\rho_{3221} + 1000, \quad d_{221}(\rho_3) = -2\rho_{3221} - 1.5\rho_{3211} + 1000,$$

and the transaction costs between the intermediaries and the consumers at the demand markets (see (47)) were given by:

$$\hat{c}^j_{kh\hat{l}m}(x^2, y, \eta^2, \eta^3) = y^j_{kh\hat{l}m} + 5 - \eta^j_{kh\hat{l}m}; \; k = 1,2; h = 1,2; \hat{l} = 1; m = 1.$$

The relationship value functions (14) and (38) were given by:

$$v^{il}_{jhm}(\eta^{il}_{jhm}) = \eta^{il}_{jhm}, \quad \forall i, l, j, h, m; \quad v^{j}_{kh\hat{l}m}(\eta^{j}_{kh\hat{l}m}) = \eta^{j}_{kh\hat{l}m}, \quad \forall j, k, h, \hat{l}, m,$$

with all other relationship value functions being set equal to zero.

The relationship cost functions (cf. (2) and (24)) were as follows:

$$b^{il}_{jhm}(\eta^{il}_{jhm}) = 2\eta^{il}_{jhm}, \quad \forall i, l, j, h, m; \quad b^{j}_{kh\hat{l}m}(\eta^{j}_{kh\hat{l}m}) = \eta^{j}_{kh\hat{l}m}, \quad \forall j, k, h, \hat{l}, m.$$

Since all the weights associated with the criteria were set equal to one this means that the source agents as well as the intermediaries weight the criterion of risk minimization equally to that of net revenue maximization and relationship value maximization.

The Euler method converged in 2,998 iterations and yielded the following equilibrium financial flow pattern:

$$x^{1*} := x^{11*}_{111} = x^{11*}_{121} = x^{11*}_{211} = x^{11*}_{221} = x^{211}_{11*} = x^{22*}_{211} = x^{21*}_{211} = x^{21*}_{221} = .0662;$$

$$x^{2*} := x^{11*}_{111} = x^{11*}_{121} = x^{11*}_{211} = x^{11*}_{221} = x^{21*}_{111} = x^{21*}_{121} = x^{21*}_{211} = x^{21*}_{221} = 4.9938;$$

$$y^{*} := y^{1*}_{1111} = y^{1*}_{1211} = y^{1*}_{2111} = y^{1*}_{2211} = y^{2*}_{1111} = y^{2*}_{1211} = y^{2*}_{2111} = y^{2*}_{2211} = .0642.$$

Both source agents allocated the entirety of their funds to the instrument in the two currencies; thus, there was no non-investment.

The vector λ^* had components: $\lambda^*_1 = \lambda^*_2 = 272.7246$, and the computed demand prices at the demand markets were: $\rho^*_{3111} = \rho^*_{3121} = \rho^*_{3211} = \rho^*_{3221} = 282.8586$.

All the relationship levels were identically equal to zero.

Note that due to the lower transaction costs associated with electronic transactions directly between the source agents and the demand markets a sizeable portion of the financial funds were transacted in this manner.

Example 2

Example 2 was constructed from Example 1 as follows. We kept the data the same except that we increased the weights associated with the relationship levels of the source agents from 1 to 20. Hence, in this example, the source agents weight relationship levels much higher than in Example 1. The Euler method again converged requiring the same number of iterations as in Example 1, but now the relationship levels of all the source agents increased to the maximum possible value of 1. All other computed equilibrium values remained as in Example 1.

Example 3

The third example consisted of two countries with two source agents in each country; two currencies, two intermediaries, and two financial products. Hence, $L = 2$, $I = 2$, $H = 2$, $J = 2$, and $K = 2$.

The data for Example 3 was constructed for easy interpretation purposes and to create a baseline from which additional simulations could be conducted. In fact, we essentially "replicated" the data for the first country as it appeared in Example 1 in order to construct the data for the second country.

Specifically, the financial holdings of the source agents were: $S^{11} = 20$, $S^{21} = 20$, $S^{12} = 20$, and $S^{22} = 20$. The variance-covariance matrices were equal to the identity matrices (appropriately dimensioned) for all source agents in each country and for all intermediaries, respectively.

The transaction cost functions faced by the source agents associated with transacting with the intermediaries were given by:

$$c^{il}_{jhm}(x^{il}_{jhm}, \eta^{il}_{jhm}) = .5(x^{il}_{jhm})^2 + 3.5x^{il}_{jhm} - \eta^{il}_{jhm};\ i = 1,2; l = 1,2; j = 1,2; h = 1,2; m = 1.$$

The handling costs of the intermediaries (since the number of intermediaries is still equal to two) remained as in Example 1, that is, they were given by:

$$c_j(x^1) = .5(\sum_{i=1}^{2}\sum_{h=1}^{2} x^{i1}_{jh1})^2;\ j = 1,2.$$

The transaction costs of the intermediaries associated with transacting with the source agents in the two countries were given by:

$$\hat{c}^{il}_{jhm}(x^{il}_{jhm}) = 1.5(x^{il}_{jhm})^2 + 3x^{il}_{jhm};\ i = 1,2; l = 1,2; j = 1,2; h = 1,2; m = 1.$$

The demand functions at the demand markets were:

$$d_{111}(\rho_3) = -2\rho_{3111} - 1.5\rho_{3121} + 1000, \quad d_{121}(\rho_3) = -2\rho_{3121} - 1.5\rho_{3111} + 1000,$$

$$d_{211}(\rho_3) = -2\rho_{3211} - 1.5\rho_{3221} + 1000, \quad d_{221}(\rho_3) = -2\rho_{3221} - 1.5\rho_{3211} + 1000,$$

$$d_{112}(\rho_3) = -2\rho_{3112} - 1.5\rho_{3122} + 1000, \quad d_{122}(\rho_3) = -2\rho_{3122} - 1.5\rho_{3112} + 1000,$$

$$d_{212}(\rho_3) = -2\rho_{3212} - 1.5\rho_{3222} + 1000, \quad d_{222}(\rho_3) = -2\rho_{3222} - 1.5\rho_{3212} + 1000,$$

and the transaction costs between the intermediaries and the consumers at the demand markets were given by:

$$\hat{c}^j_{kh\hat{l}}(y^j_{kh\hat{l}m}, \eta^j_{kh\hat{l}m}) = y^j_{kh\hat{l}} - \eta^j_{kh\hat{l}} + 5; \;\; j = 1,2; k = 1,2; h = 1,2; \hat{l} = 1,2; m = 1.$$

The data for the electronic links were as in Example 1 and were replicated for the other source agents.

The variance-covariance matrices were redimensioned and were equal to the identity matrices. The weights associated with the risk functions were set equal to 1 for all the source agents and intermediaries.

The Euler method converged in 1,826 iterations and yielded the following equilibrium international financial flow pattern: only the electronic links had positive flows with all other flows being identically equal to 0.000. In particular, the financial holdings of the source agents in the different countries were equally allocated via electronic transactions directly to the demand markets with $x^{il*}_{kh\hat{l}} = 2.5000$ for all $i, l, k, h, \hat{l}$.

The vector λ^* had components: $\lambda^*_1 = \lambda^*_2 = 279.6194$, and the computed demand prices at the demand markets were: $\rho^*_{3kh\hat{l}} = 282.8578$, $\forall k, h, \hat{l}$. In this example, all the financial transactions were conducted electronically.

Example 4

Example 4 was constructed from Example 3 in a similar manner to that in which Example 2 was constructed from Example 1. In other words, we now increased the weight associated with all the source agents in boths countries from 1 to 20 regarding the criterion of relationship value maximization. The equilibrium pattern computed by the Euler method was identical to that obtained for Example 3 but now the relationship values associated with the source agents were all equal to their upper bound with a value of 1.

These examples, although stylized, have been presented to show both the model and the computational procedure. Obviously, different input data and dimensions of the problems solved will affect the equilibrium financial flow, price, and relationship level patterns. One now has a powerful tool with which to explore the effects of perturbations to the data as well as the effects of changes in the number of source agents, countries, currencies, and/or products, as well as the effects of the introduction of electronic transactions. Morever, this supernetwork framework allows for the tracking of the co-evolution of the international financial flows, the product prices, as well as the relationship levels over space and time.

6. Summary and Conclusions

In this paper, we developed a supernetwork model that integrated international financial networks with intermediation with social networks in which relationship levels were made explicit. Both networks had three tiers of decision-makers, consisting of: the sources of financial funds in the countries, the financial intermediaries, and the consumers associated with the demand markets for the financial products. We allowed for physical as well as electronic transactions between the decision-makers in the supernetwork. The relationship levels were allowed to affect not only the risk but also the transaction costs (by reducing them, in general) but did have associated costs. Moreover, we allowed for multicriteria decision-making behavior in which the source agents as well as the financial intermediaries were permitted to weight, in an individual fashion, their objective functions of net revenue maximization, total risk minimization, and total relationship value maximization.

We modeled the supernetwork in equilibrium, in which the international financial flows between the tiers as well as the relationship levels coincide and established the variational inequality formulation of the governing equilibrium conditions. We then proposed the underlying dynamics and the continuous-time adjusrtment process(es) and constructed its projected dynamical system representation. We established that the set of stationary points of the projected dynamical system coincides with the set of solutions of the variational inequality problem. We also provided conditions under which the dynamic trajectories of the financial flows, relationship levels, and prices are well-defined.

We proposed a discrete-time algorithm to approximate the continuous-time adjustment process and applied it to several simple numerical examples for completeness and illustrative purposes. The framework developed here further advances the work in financial equilibrium modeling and analysis, especially within a network context by explicitly considering the effect of relationship levels in financial networks and by establishing the optimal relationship levels as well as financial transaction quantities and prices. Finally, it also gives us insight into the optimal designs of the supernetworks.

This framework generalizes the recent work of Nagurney, Wakolbinger, and Zhao (2004) in the evolution and emergence of integrated social and financial networks with electronic transactions to the international dimension. In addition, it incorporates social networks into the international financial network framework of Nagurney and Cruz (2003).

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