**Multitiered Supply Chain Networks: Multicriteria Decision–Making under Uncertainty**

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**Abstract:** In this paper, we present a supply chain network model with multiple tiers of decision-makers, consisting, respectively, of manufacturers, distributors, and retailers, who can compete within a tier but may cooperate between tiers. We consider multicriteria decision-making for both the manufacturers and the distributors whereas the retailers are subject to decision-making under uncertainty since the demands associated with the product are random. We derive the optimality conditions for the decision-makers, establish the equilibrium conditions, and derive the variational inequality formulation. We then utilize the variational inequality formulation to provide both qualitative properties of the equilibrium product shipment, service level, and price pattern and to propose a computational procedure, along with convergence results. This is the first supply chain network model to capture both multicriteria decision-making and decision-making under uncertainty in an integrated equilibrium framework.

**Key Words:** supply chains, multicriteria decision-making, decision-making under uncertainty, network equilibrium, variational inequalities

## 1. Introduction

The topic of supply chain modeling, analysis, computation, as well as management has been the subject of a great deal of interest due to its theoretical challenges as well as practical importance. Indeed, there is now a vast body of literature on the topic (cf. Stadtler and Kilger (2002) and the references therein) with the associated research being both conceptual in nature (see, e.g., Poirier (1996, 1999), Mentzer (2000), Bovet (2000)), due to the complexity of such problems and the distinct decision-makers involved in the transactions, as well as analytical (cf. Federgruen and Zipkin (1986), Federgruen (1993), Slats et al. (1995), Bramel and Simchi-Levi (1997), Ganeshan et al. (1998), Daganzo (1999), Miller (2001), Hensher, Button, and Brewer (2001) and the references therein).

Recently, there has been a notable effort expended on the development of decentralized supply chain network models in order to formalize the study of the interactions among the various decision-makers. For example, Lee and Billington (1993) emphasized the need for the development of decentralized models that allow for a generalized network structure and simplicity in computation. Anupindi and Bassok (1996), in turn, focused on the challenges of systems consisting of decentralized retailers with information sharing. Lederer and Li (1997), on the other hand, modeled the competition among firms that produce goods or services for customers who are sensitive to delay time.

In addition, case-based analyses of supply chain management with consideration of the special geographical and business environments have been conducted. For example, Yan and Chang (1999) viewed the current supply chain management practices in the PC industry, notably, in the greater China area. Yan, Yu, and Cheng (2001), in turn, proposed a strategic model for supply chain design with logical constraints. Yu, Yan, and Cheng (2001a,b) modeled and analyzed information sharing-based supply chain partnerships and identified conditions for obtaining benefits from such partnerships. Yan and Hao (2001) proposed a decision-making model for production loading planning for a textile company under a quick response requirement.

In this paper, we present a model for multicriteria decision-making in a multitiered supply chain network within an equilibrium context, in which we also accommodate, within a single framework, random demands at the multiple consumer markets. The first approach to model, analyze, and solve multitiered supply chain network equilibrium problems was proposed by Nagurney, Dong, and Zhang (2002). However, in that model all of the decision-makers were faced with a single criterion each (e.g., profit maximization in the case of manufacturers and retailers) and the demands were assumed known with certainty. Dong, Zhang, and Nagurney (2002), in turn, did consider multicriteria decision-making within a supply chain context but only considered two tiers of decision-makers and also assumed that the demands were known with certainty. Dong, Zhang, and Nagurney (2003) recently introduced random demands into a model of supply chain network competition. However, that model only considered two tiers of single criterion decision-makers. Recent research, from a supernetwork perspective, for supply chain network modeling, analysis, and computation in which electronic commerce is also incorporated, can be found in Nagurney, et al. (2002). For further background on supply chain networks and associated networks, see the book by Nagurney and Dong (2002), and the references therein.

For definiteness, we now clearly spell out the novelty of our new model. Specifically, in this paper, we assume that the manufacturers are involved in the production of a homogeneous product which is then shipped to the distributors. The manufacturers seek to determine their optimal production and shipment quantities, given the production costs and the transaction costs associated with conducting business with the different distributors as well as the prices that the distributors are willing to pay for the product and shipment alternative combinations. The manufacturers are assumed to have two objectives and these are profit maximization and market share maximization with the weight associated with the latter criterion being distinct for each manufacturer.

The distributors, in turn, seek to determine the "optimal" quantity of the product to obtain from each of the manufacturers, the "most-economic" shipment pattern, as well as the "optimal" service level. In particular, the distributors have three criteria and these are: the maximization of profit, the minimization of transportation time, and the maximization of service level. Each of the distributors can weight these criteria in a different manner. The shipment alternatives are represented by links characterized by specific transportation cost and transportation time functions. Hence, a specific shipment option or link may have a low associated transportation time but a high transportation cost, whereas another may have a high associated transportation time and a low cost. A higher price may be charged to have the products shipped faster to the distributors or, in contrast, manufacturers may accept a lower price if they select a slower and, presumably, cheaper, shipping mode. The service level indicates the percentage of time that the product is not out of stock. Usually, a higher service level implies a greater stock size on average, and thus, larger order quantities and less frequent order times. In other words, with a higher service level, the stock (inventory) holding cost goes up, and the transportation time and cost go down. Therefore, the service level set by each distributor significantly impacts the holding cost as well as the transportation time and cost.

As already mentioned, we do not require that the retailers know their demand functions with certainty, but, rather, assume that they possess some knowledge such as the density function of the random demand functions, based on historical data and/or forecasted data. This relaxation of assumptions to-date, hence, makes for a more realistic model. Importantly, even with this extension, we are able to not only derive the optimality conditions for the manufacturers, the distributors, and the retailers, but also to establish that the governing equilibrium conditions in the random demand case satisfy a finite-dimensional variational inequality. Furthermore, we provide reasonable conditions on the underlying functions in order to establish qualitative properties of the equilibrium price, service level, and product shipment pattern. In addition, we give conditions that, if satisfied, guarantee convergence of the proposed algorithmic scheme.

We note that Mahajan and Ryzin (2001) considered retailers under uncertain demand and focused on inventory competition. However, they assumed that the price of the product is exogenous. In this paper, in contrast, we assume competition, uncertain demand, and provide a means to determine the equilibrium prices both at the retailers and at the manufacturers. Lippman and McCardle (1997), in turn, developed a model of inventory competition for firms but assumed an aggregated random demand. In this paper, we allow each retailer to handle his own uncertain demand and to engage in competition, which seems closer to actual practice. More recently, Iida (2002) presented a production-inventory model with uncertain production capacity and uncertain demand.

The paper is organized as follows. In Section 2, we present the supply chain network model with random demands, derive the optimality conditions for the distinct tiers of multi-criteria decision-makers, and provide the equilibrium conditions, which are then shown to be equivalent to a finite-dimensional variational inequality problem. In Section 3, we establish certain qualitative properties of the chain network model, in particular, the existence and uniqueness of an equilibrium, under reasonable assumptions. In Section 4, we describe the computational procedure, along with convergence results. In Section 5, we summarize our results and present the conclusions.

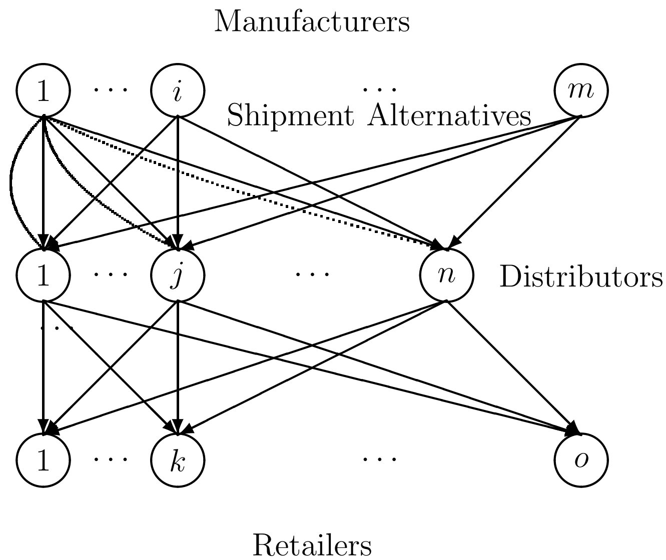


Figure 1: The Multitiered Network Structure of the Supply Chain with Multicriteria Decision-Makers

## 2. The Multitiered, Multicriteria Supply Chain Network Model with Uncertain Demands

In this Section, we develop the supply chain network model with manufacturers, distributors, and retailers. The behavior of the various decision-makers is first described, along with the optimality conditions. The integrated model is then constructed along with the variational inequality formulation of the governing equilibrium conditions. The network depiction of the supply chain in equilibrium is given in Figure 1.

### The Manufacturers

We assume that a homogeneous product is produced by $m$ manufacturers, is shipped to $n$ distributors, and is distributed to $o$ retailers at the demand markets. We denote a typical manufacturer by $i$, a typical distributor by $j$, and a typical retailer by $k$. Each manufacturer can ship the product to each distributor using one (or more) of $r$ possible shipment alternatives, which can represent mode/route alternatives. We denote a typical shipment alternative by $l$. Let $q_{ijl}$ denote the quantity of the product produced by manufacturer $i$ and shipped to distributor $j$ via shipment alternative $l$. For simplicity of presentation, we now define certain vectors. Let $q_i = \sum_{j=1}^n \sum_{l=1}^r q_{ijl}$ be the total production of manufacturer $i$ and let $q$ be the $m$-dimensional vector with components: $\{q_1, \ldots, q_m\}$. Let $q_j = \sum_{i=1}^m \sum_{l=1}^r q_{ijl}$ denote the product shipments to distributor $j$ from all the manufacturers. Finally, let $Q^1$ denote the $mnr$-dimensional vector of all the product shipments from all the manufacturers to all the distributors, with components: $\{q_{111}, \ldots, q_{mnr}\}$. We assume that all the vectors are column vectors.

Each manufacturer $i$ is assumed to be faced with a production cost function $f_i$, which may depend, in general, on the entire vector of production outputs. We associate with manufacturer $i$, who is transacting with distributor $j$ and using shipment alternative $l$, a transaction cost $c_{1ijl}$.

The total costs incurred by a manufacturer $i$, thus, are equal to the sum of the manufacturer's production cost plus the transaction costs. The revenue, in turn, is equal to the sum over all the distributors and shipment alternatives of the price of the product charged to each distributor and associated with a particular shipment alternative times the quantity. If we let $\rho^*_{1ijl}$ denote the (endogenous) price charged for the product by manufacturer $i$ to distributor $j$ and associated with shipment alternative $l$, we can express the criterion of profit maximization for manufacturer $i$ as:

$$\text{Maximize} \quad \sum_{j=1}^n \sum_{l=1}^r \rho^*_{1ijl} q_{ijl} - f_i(q) - \sum_{j=1}^n \sum_{l=1}^r c_{1ijl}(q_{ijl}) \tag{1}$$

subject to $q_{ijl} \geq 0$, for all $j, l$. We discuss how the supply price $\rho^*_{1ijl}$ is determined later in this Section. In addition to the criterion of profit maximization, we also assume that each manufacturer seeks to maximize his production output in an endeavor to gain market share. Therefore, the second criterion of each manufacturer $i$ can be expressed mathematically as:

$$\text{Maximize} \quad \sum_{j=1}^n \sum_{l=1}^r q_{ijl} \tag{2}$$

subject to $q_{ijl} \geq 0$, for all $j, l$.

Each manufacturer $i$ associates a nonnegative weight $\alpha_i$ with the output maximization criterion, the weight associated with the profit maximization criterion serving as the numeraire and being set equal to 1. Hence, we can construct a value function for each manufacturer (cf. Fishburn (1970), Chankong and Haimes (1983), Yu (1985), Keeney and Raiffa (1993)) using a constant additive weight value function. Consequently, the multicriteria decision-making problem for manufacturer $i$ is transformed into:

$$\text{Maximize} \quad \sum_{j=1}^{n}\sum_{l=1}^{r}\rho_{1ijl}^{*}q_{ijl} - f_i(q) - \sum_{j=1}^{n}\sum_{l=1}^{r}c_{1ijl}(q_{ijl}) + \alpha_i \sum_{j=1}^{n}\sum_{l=1}^{r}q_{ijl} \tag{3}$$

subject to:

$$q_{ijl} \geq 0, \forall j, l.$$

**The Optimality Conditions of the Manufacturers**

The manufacturers are assumed to compete in a noncooperative manner in the sense of Cournot (1838) and Nash (1950, 1951), seeking to determine their own optimal production and shipment quantities. If the production cost functions for each manufacturer is continuously differentiable and convex, as is each manufacturer's transaction cost function, then the optimality conditions (see Dafermos and Nagurney (1987), Nagurney (1999) and Gabay and Moulin (1980)) take the form of a variational inequality problem given by: determine $Q^{1^*} \in R_{+}^{mnr}$, such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{r}\left[\frac{\partial f_i(q^*)}{\partial q_{ijl}} + \frac{\partial c_{1ijl}(q_{ijl}^*)}{\partial q_{ijl}} - \alpha_i - \rho_{1ijl}^*\right] \times \left[q_{ijl} - q_{ijl}^*\right] \geq 0, \quad \forall Q^1 \in R_{+}^{mnr}. \tag{4}$$

The optimality conditions (4) have a nice economic interpretation as follows. A manufacturer will ship a positive amount of the product to a distributor via a particular shipment alternative (and the flow on the corresponding link will be positive) if the price that the distributor is willing to pay for the product, $\rho_{1ijl}^*$, is precisely equal to the manufacturer's marginal production and transaction costs associated with that distributor and with that shipment alternative, discounted by the market share weight. If the manufacturer's marginal production and transaction costs discounted by the market share weight exceed the price that the distributor is willing to pay for the product and shipment alternative, then the flow on the transaction link will be zero.

**The Distributors**

The distributors are positioned in the supply chain to have transactions with both their suppliers, who are the $m$ manufacturers, and with their customers, who are the $o$ retailers.

Let $q_{jk}$ denote the amount of the product purchased from distributor $j$ by the retailer $k$. Let the vector $Q^2$ consist of all the quantities of the product shipped to the consumers and be the $no$-dimensional vector with components: $\{q_{11}, \ldots, q_{no}\}$. Let $\rho^*_{2j}$ denote the price charged by distributor $j$. This price, as we will show, will be endogenously determined in the integrated model. Then, the total revenue of distributor $j$ is $\rho^*_{2j} \sum_{k=1}^{o} q_{jk}$.

The costs that a typical distributor $j$ is faced with include the price of obtaining the product from the manufacturers, the transaction cost, which includes the transportation cost, and which is denoted by $c_{2ijl}$, and the holding cost, $h_j$, for carrying, displaying, and maintaining the product stock. Since a distributor must decide upon an ordering system in terms of the order quantity and the order time (order point), the service level, denoted by $s_j$, is one of the main factors, financially, in addition to the ordering cost and carrying cost. Here we define the service level to be the percentage of time that the product is in stock when consumers come to buy it (cf. Arnold (2000)). Usually, a higher service level implies a greater stock size, on the average, and, thus, larger order quantities and less frequent order times (Beamon (1998)). We assume that the holding cost depends, in general, on $j$'s shipment pattern $q_j$, and his service level $s_j$, that is, $h_j = h_j(s_j, q_j)$. Distributor $j$'s transaction cost associated with obtaining the product from manufacturer $i$ via transportation alternative $l$, in turn, is $c_{2ijl} = c_{2ijl}(s_j, q_j)$. We denote the transportation time of shipping the product from manufacturer $i$ to distributor $j$ via alternative $l$ by $t_{ijl}$, which, in general, may depend upon $q_j$ as well as on the service level $s_j$ as discussed above. Therefore, we have that $t_{ijl} = t_{ijl}(s_j, q_j)$.

In a multicriteria decision-making setting, we assume that distributor $j$ has three criteria. They are: to maximize the profit, to minimize the total transportation time in getting the product from the manufacturers, and to maximize the service level. As mentioned earlier, by increasing his service level $s_j$, distributor $j$ is, indeed, in a sense, improving his service reputation, since the customers shopping at retail outlet $j$ have a lower chance of experiencing a stockout of the product when distributor $j$ adopts a higher service level. The model assumes that distributor $j$ associates with his value function a weight of 1 to his profit attribute in dollar value, a weight of $\beta_{1j}$ to his transportation time attribute, as the conversion rate of time to dollar value, and a weight of $\beta_{2j}$ to his service level attribute, as the conversion rate of service reputation to dollar value.

Therefore, the multicriteria decision-making problem for distributor $j$; $j = 1, ..., n$, can be transformed directly into the following optimization problem:

$$\text{Maximize} \quad \rho^*_{2j} \sum_{k=1}^{o} q_{jk} - \beta_{1j} \sum_{i=1}^{m} \sum_{l=1}^{r} t_{ijl}(s_j, q_j) + \beta_{2j} s_j$$

$$-\sum_{i=1}^{m}\sum_{l=1}^{r}\rho_{1ijl}^{*}q_{ijl}-\sum_{i=1}^{m}\sum_{l=1}^{r}c_{2ijl}(s_j,q_j)-h_j(s_j,q_j) \tag{5}$$

subject to

$$\sum_{k=1}^{o}q_{jk}\leq\sum_{i=1}^{m}\sum_{l=1}^{r}q_{ijl}, \tag{6}$$

$$s_j\leq 1, \tag{7}$$

and

$$q_{ijl}\geq 0,\quad q_{jk}\geq 0,\quad s_j\geq 0,\quad \forall i,l,k,$$

where $\gamma_j$ is the Lagrange multiplier associated with inequality (6) and $\eta_j$ is the Lagrange multiplier associated with inequality (7).

The objective function (5) represents a value function for distributor $j$, with $\beta_{1j}$ having the interpretation as the conversion rate of time into dollar value and $\beta_{2j}$ having the interpretation as the conversion rate of service level into dollar value. Constraint (6) expresses that retailers cannot purchase more from a distributor than is held in stock. Constraint (7) and the nonnegativity constraint on the level of service specify that the service level is set between 0 to 100 percent.

Subsequently, we derive the optimality conditions for this problem.

**The Optimality Conditions of the Distributors**

Suppose that the distributors compete in a noncooperative manner so that each maximizes his profit given the actions of the other distributors. We now consider the optimality conditions of the distributors assuming that each distributor is faced with the optimization problem (5), (6) and (7).

Assuming that all the holding costs $h_j$, the distributor transaction costs $c_{2ijl}$, and the transportation times for shipping the product from the manufacturers to the distributors $t_{ijl}$ are all continuous and convex, then the optimality conditions for the optimization problem (5), (6) and (7) for all the distributors satisfy the variational inequality: determine nonnegative $(Q^{1^*},Q^{2^*},\gamma^*,\eta^*,s^*)$ satisfying (7) such that:

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\gamma_j^*-\rho_{2j}^*\right]\times\left[q_{jk}-q_{jk}^*\right]$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{r}\left[\frac{\partial h_j(s_j^*,q_j^*)}{\partial q_{ijl}}+\rho_{1ijl}^*+\frac{\partial c_{2ijl}(s_j^*,q_j^*)}{\partial q_{ijl}}+\beta_{1j}\frac{\partial t_{ijl}(s_j^*,q_j^*)}{\partial q_{ijl}}-\gamma_j^*\right]\times\left[q_{ijl}-q_{ijl}^*\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{r}q_{ijl}^{*}-\sum_{k=1}^{o}q_{jk}^{*}\right]\times\left[\gamma_j-\gamma_j^{*}\right]+\sum_{j=1}^{n}\left[1-s_j^{*}\right]\times\left[\eta_j-\eta_j^{*}\right]$$

$$+\sum_{j=1}^{n}\left[\frac{\partial h_j(s_j^{*},q_j^{*})}{\partial s_j}+\sum_{i=1}^{m}\sum_{l=1}^{r}\frac{\partial c_{2ijl}(s_j^{*},q_j^{*})}{\partial s_j}+\sum_{i=1}^{m}\sum_{l=1}^{r}\beta_{1j}\frac{\partial t_{ijl}(s_j^{*},q_j^{*})}{\partial s_j}+\eta_j^{*}-\beta_{2j}\right]\times\left[s_j-s_j^{*}\right]\geq 0, \tag{8}$$

for all nonnegative $Q^1, Q^2, s, \gamma, \eta$ satisfying (7), where recall that $\gamma_j$ is the Lagrange multiplier for (6) and $\eta_j$ is the Lagrange multiplier for (7). For further background on such a derivation, see Bertsekas and Tsitsiklis (1992). In this derivation, as in the derivation of inequality (4), we have not had the prices charged be variables. They become endogenous variables in the integrated model.

We now highlight the economic interpretation of the distributors' optimality conditions. The third term in inequality (8) reveals that the Lagrange multiplier $\gamma_j^*$ can be interpreted as the market clearing price at distributor $j$'s outlet, that is, if $\gamma_j^*$ is positive, then, at equilibrium, the total inflow of the product should equal the total outflow at distributor $j$. We now turn to the first term in inequality (8). The first term implies that, if retailer $k$ purchase the product at distributor $j$, that is, $q_{jk}^* > 0$, then the price charged by distributor $j$, $\rho_{2j}^*$, should be equal to $\gamma_j^*$, which is the price to clear the market at distributor $j$. From the second term in (8), in turn, we see that if distributor $j$ adopts transportation alternative $l$ to ship the product from manufacturer $i$, that is, $q_{ijl}^* > 0$, then the total marginal cost for obtaining the product from manufacturer $i$ via alternative $l$ should be equal to $\gamma_j^*$, the price to clear market at distributor $j$. On the other hand, if this marginal cost for obtaining the product from manufacturer $i$ via alternative $l$ is greater than the market clearing price at distributor $j$, then distributor $j$ will not use alternative $l$ to ship the product from manufacturer $i$. The fourth term in (8) suggests that $\eta_j^*$ is the equilibrium shadow price for the deviation of the service level, $s_j^*$, from one hundred percent, since $\eta_j$ is defined to be the Lagrange multiplier for constraint (7). Finally, the fifth term in (8) argues that in order to have a positive service level $s_j^*$ at distributor $j$, the sum of the generalized marginal handling cost, which includes the marginal holding cost, the marginal transaction cost, the weighted transportation time (converted to its dollar value), and the shadow price for the deviation of service level from being one hundred percent, should be equal to the weight $\beta_{2j}$ assigned to the service level. This is because this term should economically measure the importance of increasing service level against other marginal costs.

**The Retailers**

The retailers, in turn, must decide how much to order from the distributors in order to cope with the random demand while still seeking to maximize their profits. A retailer $k$ is also faced with what we term a *generalized* cost, which may include, for example, the display and storage cost associated with the product as well as the trasaction cost to obtain the product from various distributors. We denote this cost by $c_k$ and, in the simplest case, we would have that $c_k$ is a function of $v_k = \sum_{j=1}^n q_{jk}$, that is, the generalozed cost of a retailer is a function of how much of the product he has obtained from the various distributors. However, for the sake of generality, and to enhance the modeling of competition, we allow the function to, in general, depend also on the amounts of the product held by other retailers and, therefore, we may write:

$$c_k = c_k(Q^2), \quad \forall k. \tag{9}$$

Let $\rho_{3k}$ denote the demand price of the product associated with retailer $k$. We assume that $\hat{d}_k(\rho_{3k})$ is the demand for the product at the demand price of $\rho_{3k}$ at retail outlet $k$, where $\hat{d}_k(\rho_{3k})$ is a random variable with a density function of $\mathcal{F}_k(x, \rho_{3k})$, with $\rho_{3k}$ serving as a parameter. Hence, we assume that the density function may vary with the demand price. Let $P_k$ be the probability distribution function of $\hat{d}_k(\rho_{3k})$, that is, $P_k(x, \rho_{3k}) = P_k(\hat{d}_k \leq x) = \int_0^x \mathcal{F}_k(x, \rho_{3k})dx$.

Retailer $k$ can sell to the consumers no more than the minimum of his supply or his demand, that is, the actual sale of $k$ cannot exceed $\min\{v_k, \hat{d}_k\}$. Let

$$\Delta_k^+ \equiv \max\{0, v_k - \hat{d}_k\} \tag{10}$$

and

$$\Delta_k^- \equiv \max\{0, \hat{d}_k - v_k\}, \tag{11}$$

where $\Delta_k^+$ is a random variable representing the excess supply (inventory), whereas $\Delta_k^-$ is a random variable representing the excess demand (shortage).

Note that the expected values of excess supply and excess demand of retailer $k$ are scalar functions of $v_k$ and $\rho_{3k}$. In particular, let $e_k^+$ and $e_k^-$ denote, respectively, the expected values: $E(\Delta_k^+)$ and $E(\Delta_k^-)$, that is,

$$e_k^+(v_k, \rho_{3k}) \equiv E(\Delta_k^+) = \int_0^{v_k} (v_k - x)\mathcal{F}_k(x, \rho_{3k})dx, \tag{12}$$

$$e_k^-(v_k, \rho_{3k}) \equiv E(\Delta_k^-) = \int_{v_k}^{\infty} (x - v_k)\mathcal{F}_k(x, \rho_{3k})dx. \tag{13}$$

Assume that the unit penalty of having excess supply at retail outlet $k$ is $\lambda_k^+$ and that the unit penalty of having excess demand is $\lambda_k^-$, where the $\lambda_k^+$ and the $\lambda_k^-$ are assumed to be nonnegative. Then, the expected total penalty of retailer $k$ is given by

$$E(\lambda_k^+ \Delta_k^+ + \lambda_k^- \Delta_k^-) = \lambda_k^+ e_k^+(v_k, \rho_{3k}) + \lambda_k^- e_k^-(v_k, \rho_{3k}). \tag{14}$$

Assuming, as already mentioned, that the retailers are also profit-maximizers, the expected revenue of retailer $k$ is $E(\rho_{3k} \min\{v_k, \hat{d}_k\})$. Hence, the optimization problem of a retailer $k$ can be expressed as:

$$\text{Maximize}\, E(\rho_{3k} \min\{v_k, \hat{d}_k\}) - E(\lambda_k^+ \Delta_k^+ + \lambda_k^- \Delta_k^-) - c_k(Q^2) - \sum_{j=1}^{n} \rho_{2j}^* q_{jk} \tag{15}$$

subject to:

$$q_{ik} \geq 0, \quad q_{jk} \geq 0, \text{ for all } i \text{ and } j. \tag{16}$$

Objective function (15) expresses that the expected profit of retailer $k$, which is the difference between the expected revenues and the sum of the expected penalty, the generalized cost, and the payouts to the manufacturers and to the distributors, should be maximized.

Applying now the definitions of $\Delta_k^+$, and $\Delta_k^-$, we know that $\min\{v_k, \hat{d}_k\} = \hat{d}_k - \Delta_k^-$. Therefore, the objective function (14) can be expressed as

$$\text{Maximize} \quad \rho_{3k} d_k(\rho_{3k}) - (\rho_{3k} + \lambda_k^-) e_k^-(v_k, \rho_{3k}) - \lambda_k^+ e_k^+(v_k, \rho_{3k}) - c_k(Q^2) - \sum_{j=1}^{n} \rho_{2j}^* q_{jk} \tag{17}$$

where $d_j(\rho_{3k}) \equiv E(\hat{d}_k)$ is a scalar function of $\rho_{3k}$.

**The Optimality Conditions of the Retailers**

We now consider the optimality conditions of the retailers assuming that each retailer is faced with the optimization problem (15), subject to (16), which represents the nonnegativity assumption on the variables. Here, we also assume that the retailers compete in a noncooperative manner so that each maximizes his profits, given the actions of the other retailers. Note that, at this point, we consider that retailers seek to determine the amount that they wish to obtain from the distributors. First, however, we make the following derivation and introduce the necessary notation:

$$\frac{\partial e_k^-(v_k,\rho_{3k})}{\partial q_{jk}} = \frac{\partial e_k^-(v_k,\rho_{3k})}{\partial q_{jk}} = P_k(v_k,\rho_{3k}) - 1 = P_k(\sum_{j=1}^{n} q_{jk},\rho_{3k}) - 1. \tag{18}$$

Assuming that the generalized cost for each retailer is continuous and convex, then the optimality conditions for all the retailers satisfy the variational inequality: determine $(Q^{2^*}) \in R_+^{no}$, satisfying:

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\lambda_k^+ P_k(v_k^*,\rho_{3k}) - (\lambda_k^- + \rho_{3k})(1-P_k(v_k^*,\rho_{3k})) + \frac{\partial c_k(Q^{2^*})}{\partial q_{jk}} + \rho_{2j}^*\right] \times \left[q_{jk} - q_{jk}^*\right] \geq 0,$$

$$\forall Q^2 \in R_+^{no}. \tag{19}$$

We now highlight the economic interpretation of the retailers' optimality conditions. In inequality (19), we can infer that, if a distributor $j$ transacts with a retailer $k$ resulting in a positive flow of the product between the two, then the selling price at retail outlet $k$, $\rho_{3k}$, with the probability of $(1 - P_k(\sum_{j=1}^{n} q_{jk}^*, \rho_{3k}))$, that is, when the demand is not less then the total order quantity, is precisely equal to the retailer $k$'s payment to the distributor, $\rho_{2j}^*$, plus his marginal cost of trasacting/handling the product and the penalty of having excess demand with probability of $P_k(\sum_{j=1}^{o} q_{jk}^*, \rho_{3k})$, (which is the probability when actual demand is less than the order quantity), subtracted by the penalty of having shortage with probability of $(1 - P_k(\sum_{j=1}^{o} q_{jk}^*, \rho_{3k}))$ (when the actual demand is greater than the order quantity).

**The Market Equilibrium Conditions**

We now turn to a discussion of the market equilibrium conditions. Subsequently, we construct the equilibrium conditions for the entire supply chain.

The equilibrium conditions associated with the transactions that take place between the retailers and the consumers are the stochastic economic equilibrium conditions, which, mathematically, take on the following form: For any retailer $k$; $k = 1, \ldots, o$:

$$\hat{d}_k(\rho_{3k}^*) \begin{cases} \leq \sum_{j=1}^{o} q_{jk}^* & \textbf{a.e.}, \text{ if } \rho_{3k}^* = 0 \\ = \sum_{j=1}^{o} q_{jk}^* & \textbf{a.e.}, \text{ if } \rho_{3k}^* > 0, \end{cases} \tag{20}$$

where **a.e.** means that the corresponding equality or inequality holds almost everywhere. Conditions (20) state that, if the demand price at outlet $k$ is positive, then the quantities purchased by the retailer from the manufacturers and from the distributors in the aggregate is equal to the demand, with exceptions of zero probability. These conditions correspond to the well-known economic equilibrium conditions (cf. Nagurney (1999) and the references therein). Related equilibrium conditions, were proposed in Nagurney, Dong, and Zhang (2002) and Dong, Zhang, and Nagurney (2003).

Equilibrium conditions (20) are equivalent to the following variational inequality problem, after taking the expected value and summing over all retailers $k$: determine $\rho_3^* \in R_+^o$ satisfying

$$\sum_{k=1}^{o}(\sum_{j=1}^{n} q_{jk}^* - d_j(\rho_{3k}^*)) \times [\rho_{3k} - \rho_{3k}^*] \geq 0, \qquad \forall \rho_3 \in R_+^o, \tag{21}$$

where $\rho_3$ is the $o$-dimensional column vector with components: $\{\rho_{31}, \ldots, \rho_{3o}\}$.

**The Equilibrium Conditions of the Supply Chain Network**

In equilibrium, we must have that the sum of the optimality conditions for all manufacturers, as expressed by inequality (4), the optimality conditions for the distributors, as expressed by condition (8), the optimality conditions for all retailers, as expressed by inequality (19), and the market equilibrium conditions, as expressed by inequality (21) must be satisfied. Hence, the shipments shipped from the manufacturers to the distributors, must be equal to those accepted by the distributors, and, finally, the shipments from the distributors to the retailers must coincide with those accepted by the retailers. We state this explicitly in the following definition:

**Definition 1: Supply Chain Equilibrium**

*A product shipment, price, and service level pattern* $(Q^{1^*}, Q^{2^*}, \gamma^*, \rho_3^*, s^*, \eta^*) \in \mathcal{K}$, *where* $\mathcal{K} \equiv R_+^{mnr} \times R_+^{no} \times R_+^n \times R_+^o \times [0,1]^n \times R_+^n$ *is said to be a supply chain equilibrium if it satisfies the optimality conditions for all manufacturers, for all retailers, and for all consumers, given, respectively, by (4), (8), (19),and (21), simultaneously.*

**Theorem 1: Variational Inequality Formulation**

*A supply chain network is in equilibrium, according to Definition 1 if and only if it satisfies the variational inequality problem: Determine* $(Q^{1^*}, Q^{2^*}, \gamma^*, \rho_3^*, s^*, \eta^*) \in \mathcal{K}$, *such that:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{r}\left[\frac{\partial f_i(q^*)}{\partial q_{ijl}} + \frac{\partial c_{1ijl}(q_{ijl}^*)}{\partial q_{ijl}} - \alpha_i + \frac{\partial h_j(s_j^*, q_j^*)}{\partial q_{ijl}} + \frac{\partial c_{2ijl}(s_j^*, q_j^*)}{\partial q_{ijl}}\right.$$

$$+\beta_{1j}\frac{\partial t_{ijl}(s_j^*, q_j^*)}{\partial q_{ijl}} - \gamma_j^*\Bigg] \times \left[q_{ijl} - q_{ijl}^*\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\Bigg[\gamma_j^* + \lambda_k^+ P_k(\sum_{j=1}^{n} q_{jk}^*, \rho_{3k}) - (\lambda_k^- + \rho_{3k})(1 - P_k(\sum_{j=1}^{n} q_{jk}^*, \rho_{3k}))$$

$$+\frac{\partial c_k(Q^{2*})}{\partial q_{jk}}\Bigg] \times \left[q_{jk} - q_{jk}^*\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{r} q_{ijl}^* - \sum_{k=1}^{o} q_{jk}^*\right] \times \left[\gamma_j - \gamma_j^*\right]$$

$$+\sum_{k=1}^{o}\left[\sum_{j=1}^{n} q_{jk}^* - d_k(\rho_{3k}^*)\right] \times [\rho_{3k} - \rho_{3k}^*]$$

$$+\sum_{j=1}^{n}\Bigg[\frac{\partial h_j(s_j^*, q_j^*)}{\partial s_j} + \sum_{i=1}^{m}\sum_{l=1}^{r}\frac{\partial c_{2ijl}(s_j^*, q_j^*)}{\partial s_j}$$

$$+\sum_{i=1}^{m}\sum_{l=1}^{r}\beta_{1j}\frac{\partial t_{ijl}(s_j^*, q_j^*)}{\partial s_j} + \eta_j^* - \beta_{2j}\Bigg] \times \left[s_j - s_j^*\right]$$

$$+\sum_{j=1}^{n}\left[1 - s_j^*\right] \times \left[\eta_j - \eta_j^*\right] \geq 0, \quad \forall (Q^1, Q^2, \gamma, \rho_3, s, \eta) \in \mathcal{K}, \tag{22}$$

*where $\gamma$ is the n-dimensional column vector with component $j$ given by $\gamma_j$.*

For easy reference in the subsequent sections, variational inequality problem (22) can be rewritten in standard variational inequality form (cf. Nagurney (1999)) as follows:

$$\langle F(X^*)^T, X - X^* \rangle \geq 0, \quad \forall X = (Q^1, Q^2, \gamma, \rho_3, s, \eta) \in \mathcal{K}, \tag{23}$$

and $F(X) \equiv (F_{ijl}^1, F_{jk}^2, F_j^3, F_k^4, F_j^5, F_j^6)_{i=1,\ldots,m, j=1,\ldots,n, k=1,\ldots,o, l=1,\ldots,r}$, where the terms of $F$ correspond to the terms preceding the multiplication signs in inequality (22), and $\langle \cdot, \cdot \rangle$ denotes the inner product in Euclidean space.

## 3. Qualitative Properties

In this Section, we provide some qualitative properties of the solution to variational inequality (22). In particular, we derive existence and uniqueness results. We also investigate properties of the function $F$ (cf. (23)) that enters the variational inequality of interest here.

Since the feasible set is not compact, we cannot derive existence simply from the assumption of the continuity of the functions. Nevertheless, we can impose a rather weak condition to guarantee the existence of a solution pattern. Let

$$\mathcal{K}_b \equiv \{(Q^1, Q^2, \gamma, \rho_3, s, \eta) | 0 \leq (Q^1, Q^2, \gamma, \rho_3, s, \eta) \leq b\}, \tag{24}$$

where $b = (b_1, b_2, b_3, b_4, 1, b_6) \geq 0$ and $Q^1 \leq b_1; Q^2 \leq b_2; \gamma \leq b_3; \rho_3 \leq b_4; s \leq 1; \eta \leq b_6$, mean that each of the right-hand sides is a uniform upper bound for all the components of the corresponding vectors. Then $\mathcal{K}_b$ is a bounded closed convex subset of $\mathcal{K}$. Thus, the following variational inequality

$$\langle F(X^b)^T, X - X^b \rangle \geq 0, \quad \forall X^b \in \mathcal{K}_b, \tag{25}$$

admits at least one solution $X^b \in \mathcal{K}_b$, from the standard theory of variational inequalities, since $\mathcal{K}_b$ is compact and $F$ is continuous. Following Kinderlehrer and Stampacchia (1980) (see also Theorem 1.5 in Nagurney (1999)), we then have:

**Theorem 2**

*Variational inequality (22) admits a solution if and only if there exists a $b > 0$, such that variational inequality (22) admits a solution in $\mathcal{K}_b$ with*

$$Q^1 < b_1; Q^2 < b_2; \gamma < b_3; \rho_3 < b_4; s \leq 1; \eta < b_6. \tag{26}$$

**Theorem 3**

*Suppose that there exist positive constants $M$, $N$, $R$ with $R > 0$, such that:*

$$\frac{\partial f_i(q)}{\partial q_{ijl}} + \frac{\partial c_{1ijl}(q_{ijl})}{\partial q_{ijl}} + \frac{\partial h_j(s_j, q_j)}{\partial q_{ijl}} + \frac{\partial c_{2ijl}(s_j, q_j)}{\partial q_{ijl}} + \beta_{1j} \frac{\partial t_{ijl}(s_j, q_j)}{\partial q_{ijl}} \geq M,$$
$$\forall q_{ijl} \quad \textit{with} \quad q_{ijl} \geq N, \quad \forall i, j, l, \tag{27}$$

$$\gamma_j + \lambda_k^+ P_k(\sum_{j=1}^n q_{jk}, \rho_{3k}) - (\lambda_k^- + \rho_{3k})(1 - P_k(\sum_{j=1}^n q_{jk}, \rho_{3k})) + \frac{\partial c_k(Q^2)}{\partial q_{jk}} \geq M,$$

$$\forall Q^2 \quad with \quad q_{jk} \geq N, \quad \forall j, k, \tag{28}$$

$$d_k(\rho_3) \leq N, \quad \forall \rho_3 \quad with \quad \rho_{3k} > R, \quad \forall k. \tag{29}$$

*Then, variational inequality (22) admits at least one solution.*

Assumptions (27), (28), and (29) are reasonable from an economics perspective. In particular, according to (27), when the product shipment between a manufacturer and a distributor via a certain shipment alternative is large, we can expect the corresponding sum of the marginal costs associated with the production, the shipment, and the holding of the product and the marginal time to exceed a positive lower bound. The rationale of assumption (28), in turn, can be seen through the following. If the amount of the product purchased by retailer $k$ at distributor $j$ is large, the transportation cost and the transportation time associated with obtaining the product at the distributor can also be expected to exceed a lower bound. Moreover, according to assumption (29), if the price of the product at the retailer is high, we can expect that the demand for the product will be bounded from above at that market.

We now recall the concept of additive production cost, which was introduced by Zhang and Nagurney (1996) in the stability analysis of dynamic spatial oligopolies, and has also been employed in qualitative analysis by Nagurney, Dong, and Zhang (2002) for the study of supply chain networks in which the decision-makers are faced with single criteria to optimize. Additive production costs will be assumed in Theorems 4, 5, and 6.

**Definition 2: Additive Production Cost**

*Suppose that for each manufacturer $i$, the production cost $f_i$ is additive, that is*

$$f_i(q) = f_i^1(q_i) + f_i^2(\bar{q}_i), \tag{30}$$

*where $f_i^1(q_i)$ is the internal production cost that depends solely on the manufacturer's own output level $q_i$, which may include the production operation and the facility maintenance, etc., and $f_i^2(\bar{q}_i)$ is the interdependent part of the production cost that is a function of all the other manufacturer' output levels $\bar{q}_i = (q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_m)$ and reflects the impact of the other manufacturers' production patterns on manufacturer $i$'s production cost. This interdependent part of the production cost may describe the competition for the resources, the cost of the raw materials, etc.*

We now establish additional qualitative properties of the function $F$ that enters the variational inequality problem, as well as uniqueness of the equilibrium pattern. Monotonicity and

Lipschitz continuity of $F$ will be utilized in the subsequent section for proving convergence of the algorithmic scheme.

**Lemma 1**

*Let $g_k(v_k, \rho_{3k})^T = (P_k(v_k, \rho_{3k}) - \rho_{3k}(1 - P_k(v_k, \rho_{3k})), v_k - \rho_{3k}))$, where $P_k$ is a probability distribution with the density function of $\mathcal{F}_k(x, \rho_{3k})$. Then $g_k(v_k, \rho_{3k})$ is monotone, that is,*

$$[-\rho'_{3k}(1 - P_k(v'_k, \rho'_{3k})) + \rho''_{3k}(1 - P_k(v''_k, \rho''_{3k}))] \times [q'_{jk} - q''_{jk}]$$

$$+[v'_k - d_k(\rho'_{3k}) - v''_k + d_k(\rho''_{3k})] \times [\rho'_{3k} - \rho''_{3k}] \geq 0, \quad \forall (v'_k, \rho'_{3k}), (v''_k, \rho''_{3k}) \in R^2_+ \tag{31}$$

*if and only if $d'_k(\rho_{3k}) \leq -(4\rho_{3k}\mathcal{F}_k)^{-1}(P_k + \rho_{3k}\frac{\partial P_k}{\partial \rho_{3k}})^2$.*

**Proof:** In order to prove that $g_k(s_k, \rho_{3k})$ is monotone with respect to $v_k$ and $\rho_{3k}$, we only need to show that its Jacobian matrix is positive semidefinite, which will be the case if all eigenvalues of the symmetric part of the Jacobian matrix are nonnegative real numbers.

The Jacobian matrix of $g_k$ is

$$\nabla g_k(v_k, \rho_{3k}) = \begin{bmatrix} \rho_{3k}\mathcal{F}_k(v_k, \rho_{3k}) & -1 + P_k(v_k, \rho_{3k}) + \rho_{3k}\frac{\partial P_k(v_k, \rho_{3k})}{\partial \rho_{3k}} \\ 1 & -d'_k(\rho_{3k}) \end{bmatrix}, \tag{32}$$

and its symmetric part is

$$\frac{1}{2}[\nabla g_k(v_k, \rho_{3k}) + \nabla^T g_k(v_k, \rho_{3k})] = \begin{bmatrix} \rho_{3k}\mathcal{F}_k(v_k, \rho_{3k}), & \frac{1}{2}\left(\rho_{3k}\frac{\partial P_k}{\partial \rho_{3k}} + P_k(v_k, \rho_{3k})\right) \\ \frac{1}{2}\left(\rho_{3k}\frac{\partial P_k}{\partial \rho_{3k}} + P_k(v_k, \rho_{3k})\right), & -d'_k(\rho_{3k}) \end{bmatrix}. \tag{33}$$

The two eigenvalues of (33) are

$$\gamma_{min}(v_k, \rho_{3k}) = \frac{1}{2}\left[(\rho_{3k}\mathcal{F}_k - d'_k) - \sqrt{(\alpha_k\mathcal{F}_k - d'_k)^2 + (\rho_{3k}\frac{\partial P_k}{\partial \rho_{3k}} + P_k)^2 + 4\rho_{3k}\mathcal{F}_k d'_k}\right], \tag{34}$$

$$\gamma_{max}(s_k, \rho_{3k}) = \frac{1}{2}\left[(\rho_{3k}\mathcal{F}_k - d'_k) + \sqrt{(\rho_{3k}\mathcal{F}_k - d'_k)^2 + (\rho_{3k}\frac{\partial P_k}{\partial \rho_{3k}} + P_k)^2 + 4\rho_{3k}\mathcal{F}_k d'_k}\right]. \tag{35}$$

Moreover, since what is inside the square root in both (34) and (35) can be rewritten as

$$(\rho_{3k}\mathcal{F}_k + d'_k)^2 + \left(\rho_{3k}\frac{\partial P_k}{\partial \rho_{3k}} + P_k\right)^2$$

and can be seen as being nonnegative, both eigenvalues are real. Furthermore, under the condition of the lemma, $d'_k$ is non-positive, so the first item in (34) and in (35) is nonnegative. The condition further implies that the second item in (34) and in (35), the square root part, is not greater than the first item, which guarantees that both eigenvalues are nonnegative real numbers. $\square$

The condition of Lemma 1 states that the expected demand function of a retailer is a nonincreasing function with respect to the demand price and its first order derivative has an upper bound.

**Theorem 4: Monotonicity**

*Suppose that the production cost functions $f_i; i=1,...,m$, are additive, as defined in Definition 2, and that the $f_i^1; i=1,...,m$, are convex functions. In addition, suppose that*
*(i). the $c_{1ijl}$, $h_j, c_{2ijl}$ and $t_{ijl}$ are all convex functions in the shipment $q_{ijl}, \forall i,j,l$;*
*(ii). the $c_k(Q^2)$ are monotone increasing functions with respect to $q_{jk}$, $\forall j,k$;*
*(iii). the $d_k$ are monotone decreasing functions of the prices $\rho_{3k}$, for all $k$; and $d'(\rho_{3k}) \leq -(4\rho_{3k})\mathcal{F}_k)^{-1}(P_k + \rho_{3k}\frac{\partial P}{\partial \rho_{3k}})^2$ and, finally,*
*(iv). the $h_j$, is a family of increasing convex function of the service levels $s_j$, $\forall j$, while the distributor transaction costs $c_{2ijl}$ and the transportation times $t_{ijl}$ are a family of decreasing and concave functions of the service levels $s_j, \forall i,j,l$. Then the vector function $F$ that enters the variational inequality (23) is monotone, that is,*

$$\langle (F(X') - F(X''))^T, X' - X'' \rangle \geq 0, \quad \forall X', X'' \in \mathcal{K}. \tag{36}$$

**Proof:** Analogous to the proof of Theorem 4 of Dong, Zhang, and Nagurney (2003).

Moreover, we have the following theorem.

**Theorem 5: Strict Monotonicity**

*Assume all the conditions of Theorem 4. In addition, suppose that*
*(i). one of the families of the vector functions $c_{1ijl}$, $h_j, c_{2ijl}$, or $t_{ijl}$ is strictly convex in shipment $q_{ijl}$;*
*(ii). the $d_k$ are strictly monotone decreasing functions of the prices $\rho_{3k}$, for all $k$; and $d'(\rho_{3k}) < -(4\rho_{3k})\mathcal{F}_k)^{-1}(P_k + \rho_{3k}\frac{\partial P}{\partial \rho_{3k}})^2$ ;*

*(iii). either the holding costs, $h_j, \forall j$, are increasing and strictly convex functions of the service levels $s_j, \forall j$, or one of the transaction cost functions $c_{2ijl}, \forall i,j,l$, and $t_{ijl}, \forall i,j,l$, is a family of decreasing and strictly concave functions of the service levels $s_j, \forall j$.*
*Then the vector function $F$ that enters the variational inequality (23) is strictly monotone, with respect to $(Q^1, Q^2, \rho_3, s)$, that is, for any two $X', X''$ with $(Q^{1'}, Q^{2'}, \rho_3', s') \neq (Q^{1''}, Q^{2''}, \rho_3'', s'')$:*

$$\langle (F(X') - F(X''))^T, X' - X'' \rangle > 0. \tag{37}$$

**Theorem 6: Uniqueness**

*Assuming the conditions of Theorem 5, there must be a unique production shipment pattern $Q^{1^*}$ and $Q^{2^*}$, a unique retailer price vector $\rho_3^*$, and a unique service vector $s^*$ satisfying the equilibrium conditions of the multitiered, multicriteria supply chain network. In other words, if the variational inequality (22) admits a solution, that should be the only solution in $Q^1, Q^2, \rho_3, s$.*

**Theorem 7: Lipschitz Continuity**

*The function that enters the variational inequality problem (22) is Lipschitz continuous, that is,*

$$\|F(X') - F(X'')\| \leq L\|X' - X''\|, \quad \forall X', X'' \in \mathcal{K}, \tag{38}$$

*under the following conditions:*

*(i). $c_{1ijl}$, $h_j, c_{2ijl}$ and $t_{ijl}$ have bounded second-order derivatives, $\forall i,j,l$, with respect to $q$;*
*(ii). $d_k$, $\forall k$, have bounded second-order derivatives, with respect to $q_{jk}$;*
*(iii). $d_k$ have bounded second-order derivatives, $\forall k$;*
*(iv). $h_j$, $\forall j; c_{2ijl}$, $\forall i,j,k; t_{ijl}$, $\forall i,j,k$, have bounded second-order partial derivatives with respect to $s$.*

**Proof:** Since the probability function $P_j$ is always less than or equal to 1, for each retailer $j$, the result is direct by applying a mid-value theorem from calculus to the vector function $F$ that enters the variational inequality problem (22). □

## 4. The Algorithm

Now we present the modified projection method of Korpelevich (1977) which can be applied to solve the variational inequality problem in standard form (22). The statement of the modified projection method is as follows, where $\mathcal{T}$ denotes an iteration counter:

**Modified Projection Method**

**Step 0: Initialization**

Set $X^0 \in \mathcal{K}$. Let $\mathcal{T} = 1$ and let $\alpha$ be a scalar such that $0 < \alpha \leq \frac{1}{L}$, where $L$ is the Lipschitz continuity constant (cf. (38)).

**Step 1: Computation**

Compute $\bar{X}^{\mathcal{T}}$ by solving the variational inequality subproblem:

$$\langle (\bar{X}^{\mathcal{T}} + \alpha F(X^{\mathcal{T}-1}) - X^{\mathcal{T}-1})^T, X - \bar{X}^{\mathcal{T}} \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{39}$$

**Step 2: Adaptation**

Compute $X^{\mathcal{T}}$ by solving the variational inequality subproblem:

$$\langle (X^{\mathcal{T}} + \alpha F(\bar{X}^{\mathcal{T}}) - X^{\mathcal{T}-1})^T, X - X^{\mathcal{T}} \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{40}$$

**Step 3: Convergence Verification**

If max $|X_l^{\mathcal{T}} - X_l^{\mathcal{T}-1}| \leq \epsilon$, for all $l$, with $\epsilon > 0$, a prespecified tolerance, then stop; else, set $\mathcal{T} =: \mathcal{T} + 1$, and go to Step 1.

**Theorem 8: Convergence**

*Assume that the function that enters the variational inequality (22) has at least one solution and satisfies the conditions in Theorem 4 and in Theorem 7. Then the modified projection method described above converges to the solution of the variational inequality.*

**Proof:** According to Korpelevich (1977), the modified projection method converges to the solution of the variational inequality problem of the form (23), provided that the function $F$ that enters the variational inequality is monotone and Lipschitz continuous and that a solution exists. Existence of a solution follows from Theorem 3. Monotonicity follows Theorem 4. Lipschitz continuity, in turn, follows from Theorem 7. □

Recall that Figure 1 depicts the multitiered network structure of the supply chain in equilibrium. The vectors of prices $\rho_1^*$, $\gamma^*$, and $\rho_3^*$ are associated with the respective tiers of nodes on the network, whereas the components of the vector of the equilibrium product shipments $Q^{1^*}$ correspond to the flows on the links joining the manufacturer nodes with the distributor nodes. The components of $\rho_1^*$ and $\rho_2^*$ can be determined as discussed following (4) and (8), respectively, whereas the components of $\rho_3^*$ are explicit in the solution of the variational inequality (22). The components of the vector of equilibrium product shipments $Q^{2^*}$, in turn, correspond to the flows on the links joining the distributor nodes with the demand market nodes.

In future work, we will apply this computational procedure to a variety of supply chain network problems.

## 5. Summary and Conclusions

This paper develops a mathematical model for the study of multitiered supply chain networks with multicriteria decision-makers and considers manufacturers, distributors, and retailers, respectively, in different tiers of the supply chain network. To cope with the practical concern of uncertain demand, the model assumes that each retailer is faced with demand that is a random function of his retail price. The optimality conditions for the decision-makers at each tier of the network are derived and are interpreted economically. These conditions are then integrated into a single variational inequality formulation that governs the equilibrium conditions of the entire supply chain network. The analytic properties of the variational inequality formulation are investigated. In particular, we show that, under reasonable conditions, there exists a unique production and shipment pattern for the manufacturers, a unique set of service levels for the distributors, a unique consumption pattern of the multiclass consumers, and a unique demand price pattern. A computational procedure is also proposed for the determination of the equilibrium shipment, price, and service level pattern, along with convergence results.

This research generalizes existing supply chain network models to include multicriteria decision-making and decision-making under uncertainty within a unified equilibrium framework.

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