Spatial Equilibration in Transport Networks

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1. Introduction

Transport networks are complex, large-scale spatial systems, and come in a variety of forms, ranging from road networks to air, rail, and waterway networks. They provide the foundation for the functioning of our economies and societies through the movement of people, goods, and services, and allow for the connectivity of residential locations with places of employment, schools, leisure activities, and retail outlets. From an economic perspective, the supply in such network systems is represented by the underlying network topology and the cost characteristics whereas the demand is represented by the users of the transportation system. An equilibrium occurs when the number of trips between an origin (e.g., residence/place of employment) and destination (place of employment/residence) equals the travel demand given by the market price, typically, represented by the travel time for the trips.

The study of transport networks and their efficient management dates to ancient times. For example, Romans imposed controls over chariot traffic during different times of day in order to deal with the congestion (see Banister and Button (1993)). From an economic perspective, some of the earliest contributions to the subject date to Pigou (1920), who considered a two-node, two-link transportation network, identified congestion as a problem, and recognized that distinct behavioral concepts regarding route selection may prevail (see also Knight (1924)).

The formal study of transport networks has challenged transportation scientists, economists, operations researchers, and engineers for several reasons: the above-mentioned size and scope of the systems involved; the behavior of the users of the network which may vary according to the application setting, thereby leading to different optimality/equilibrium concepts; distinct classes of users may perceive the cost of utilizing the network in an individual fashion; congestion is playing an increasing role in numerous transport networks; and there may be interactions between transport and other foundational networks, such as telecommunications networks.

For example, to help one fix the size and scope of modern-day transport networks, we point out that the topology of the Chicago Regional Transportation Network consists of 12,982 nodes, 39,018 links, and 2,297,945 origin/destination pairs of nodes between which travelers choose their routes (cf. Bar-Gera (1999)), whereas in the Southern California Association of Governments' model there are 25,428 nodes, 99,240 links, 3,217 origin/destination pairs, and 6 distinct classes of users (Wu, Florian, and He (2000)).

Road congestion results in $100 billion in lost productivity in the US alone with the figure being approximately $150 billion in Europe with the number of cars expected to increase by 50 percent by 2010 and to double by 2030 (see Nagurney (2000) and the references therein). Moreover, in many of today's transport networks, the "noncooperative" behavior of users aggravates the congestion problem. For example, in the case of urban transport networks, travelers select their routes from an origin to a destination so as to minimize their own travel cost or travel time, which although optimal from a user's perspective (user-optimization) may not be optimal from a societal one (system-optimization) where a decision-maker or central controller has control of the flows on the network and seeks to allocate the flows so as to minimize the total cost in the network. Hence, before making any policy decisions on transport networks one needs to identify the underlying behavioral mechanisms regarding route selection.

This point is richly illustrated through the famous Braess (1968) paradox example, in which it is assumed that the underlying behavioral principle is that of user-optimization and travelers select their routes accordingly. In the Braess network, the addition of a new road with no change in travel demand results in all travelers in the network incurring a higher travel cost. Hence, they are all worse off after the addition of the new road. Actual practical instances of such a phenomenon have been identified in New York City and in Stuttgart, Germany. In 1990, 42nd Street in New York was closed for Earth Day, and the traffic flow in the area improved (see Kolata (1990)). In Stuttgart, in turn, a new road was added to the downtown, but the traffic flow worsened and, following complaints, the new road was torn down (cf. Bass (1992)). Interestingly, this phenomenon is also relevant to telecommunications networks (see Korilis, Lazar, and Orda (1999)) and, specifically, to the Internet (cf. Cohen and Kelly (1990)).

The coupling of transportation networks with telecommunication networks through electronic commerce, notably, through business to business and business to consumer commerce, and through Intelligent Transportation Systems is further transforming the economic landscape and affecting the movement of people, goods, services, as well as information (see Nagurney and Dong (2002a)). Telecommunication networks are assuming many of the characteristics of transport networks, including large-size, noncooperative behavior of the users, as well as congestion. In fact, telecommunication networks, in a sense, may be interpreted as transport networks on which the flows correspond to information (rather than vehicles, etc.).

In this chapter, we recall the foundations of the equilibration of transport networks and trace the evolution of modeling frameworks for their study. The exposition is meant to be accessible to practitioners and to students, as well as to researchers in transport and to those interested in related network topics. Technical derivations and further supporting documentation are referred to in the citations. Further useful material and a supplementary chronological perspective of developments on this topic can be found in the review articles of Florian (1986), Boyce, LeBlanc, and Chon (1988), and Florian and Hearn (1995); in the books by Beckmann, McGuire, and Winsten (1956), Sheffi (1985), Patriksson (1994), Ran and Boyce (1996), Nagurney (1999, 2000), Nagurney and Dong (2002a), and in the volumes edited by Florian (1976, 1984), Volmuller and Hamerslag (1984), Lesort (1996), Marcotte and Nguyen (1998), Gendreau and Marcotte (2002), and Taylor (2002).

2. Basic Decision-Making Concepts and Models

Half a century ago, Wardrop (1952) explicitly recognized alternative possible behaviors of users of transport networks, notably, urban transport networks and stated two principles, which are commonly named after him:

First Principle: The journey times of all routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

Second Principle: The average journey time is minimal.

The first principle corresponds to the behavioral principle in which travelers seek to (unilaterally) determine their minimal costs of travel whereas the second principle corresponds to the behavioral principle in which the total cost in the network is minimal.

Beckmann, McGuire, and Winsten (1956) were the first to rigorously formulate these conditions mathematically, as had Samuelson (1952) in the framework of spatial price equilibrium problems in which there were, however, no congestion effects. Specifically, Beckmann, McGuire, and Winsten (1956) established the equivalence between the *traffic network equilibrium* conditions, which state that all used paths connecting an origin/destination (O/D) pair will have equal and minimal travel times (or costs) (corresponding to Wardrop's first principle), and the Kuhn-Tucker (1951) conditions of an appropriately constructed optimization problem, under a symmetry assumption on the underlying functions. Hence, in this case, the equilibrium link and path flows could be obtained as the solution of a mathematical programming problem. Their approach made the formulation, analysis, and subsequent computation of solutions to traffic network problems based on actual transportation networks realizable.

Dafermos and Sparrow (1969) coined the terms *user-optimized* (U-O) and *system-optimized* (S-O) transportation networks to distinguish between two distinct situations in which, respectively, users act unilaterally, in their own self-interest, in selecting their routes, and in which users select routes according to what is optimal from a societal point of view, in that the total cost in the system is minimized. In the latter problem, marginal total costs rather than average costs are equilibrated. The former problem coincides with Wardrop's first principle, and the latter with Wardrop's second principle.

Table 1: Distinct Behavior on Transportation Networks

User-Optimization	System-Optimization
⇓	⇓
Equilibrium Principle:	**Optimality Principle:**
User travel costs on used paths for each O/D pair are equalized and minimal.	Marginals of the total travel cost on used paths for each O/D pair are equalized and minimal.

See Table 1 for the two distinct behavioral principles underlying transportation networks. The concept of "system-optimization" is also relevant to other types of "routing models" in transportation, as well as in communications (cf. Bertsekas and Gallager (1992)), including those concerned with the routing of freight and computer messages, respectively. Dafermos and Sparrow (1969) also provided explicit computational procedures, that is, *algorithms*, to compute the solutions to such network problems in the case where the user travel cost on a link was an increasing (in order to handle congestion) function of the flow on the particular link, and linear.

2.1 System-Optimization versus User-Optimization

In this section, the basic transport network models are first reviewed, under distinct assumptions of their operation and distinct behavior of the users of the network. The models are classical and are due to Beckmann, McGuire, and Winsten (1956) and Dafermos and Sparrow (1969). In subsequent sections, we present more general models in which the user link cost functions are no longer separable and are also asymmetric. For such models we also provide the variational inequality formulations of the governing equilibrium conditions, since, in such cases, the conditions can no longer be reformulated as the Kuhn-Tucker conditions of a convex optimization problem.

For definiteness, and for easy reference, we present the classical system-optimized network model in Section 2.1.1 and then the classical user-optimized network model in Section 2.1.2.

2.1.1 The System-Optimized Problem

Consider a general network $\mathcal{G} = [\mathcal{N}, \mathcal{L}]$, where $\mathcal{N}$ denotes the set of nodes, and $\mathcal{L}$ the set of directed links. Let a denote a link of the network connecting a pair of nodes, and let p denote a path consisting of a sequence of links connecting an origin/destination (O/D) pair. In transport networks, nodes correspond to origins and destinations, as well as to intersections. Links, on the other hand, correspond to roads/streets in the case of urban transportation networks and to railroad segments in the case of train networks. A path in its most basic setting, thus, is a sequence of "roads" which comprise a route from an origin to a destination. In the telecommunication context, however, nodes can correspond to switches or to computers and links to telephone lines, cables, microwave links, etc. Note that here we consider *paths*, rather than *routes*, since the former subsumes the latter. Moreover, the network concepts presented here are sufficiently general to abstract not only transport decision-making but also combined location-transport decision-making, which we return to later. In addition, in the setting of *supernetworks* (see Nagurney and Dong (2002a)), a path is viewed more broadly and need not be limited to a route-type decision but may, in fact, correspond to not only transport but also to telecommunications decision-making, or a combination thereof, as in the case of teleshopping and/or telecommuting.

Let P_ω denote the set of paths connecting the origin/destination (O/D) pair of nodes ω. Let P denote the set of all paths in the network and assume that there are J origin/destination pairs of nodes in the set Ω. Let x_p represent the flow on path p and let f_a denote the flow on link a. The path flows on the network are grouped into the column vector $x \in R_+^{n_P}$, where n_P denotes the number of paths in the network. The link flows, in turn, are grouped into the column vector $f \in R_+^n$, where n denotes the number of links in the network.

The following conservation of flow equation must hold:

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in \mathcal{L}, \tag{1}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. Expression (1) states that the flow on a link a is equal to the sum of all the path flows on paths p that contain (traverse) link a.

Moreover, if one lets d_ω denote the demand associated with O/D pair ω, then one must have that

$$d_\omega = \sum_{p \in P_\omega} x_p, \quad \forall \omega \in \Omega, \tag{2}$$

where $x_p \geq 0$, $\forall p \in P$; that is, the sum of all the path flows between an origin/destination pair ω must be equal to the given demand d_ω.

Let c_a denote the user link cost associated with traversing link a, and let C_p denote the user cost associated with traversing the path p. Assume that the user link cost function is given by the *separable* function

$$c_a = c_a(f_a), \quad \forall a \in \mathcal{L}, \tag{3}$$

where c_a is assumed to be continuous and an increasing function of the link flow f_a in order to model the effect of the link flow on the cost.

The total cost on link a, denoted by $\hat{c}_a(f_a)$, hence, is given by:

$$\hat{c}_a(f_a) = c_a(f_a) \times f_a, \quad \forall a \in \mathcal{L}, \tag{4}$$

that is, the total cost on a link is equal to the user link cost on the link times the flow on the link. Here the cost is interpreted in a general sense. From a transportation engineering perspective, however, the cost on a link is assumed to typically coincide with the travel time on a link.

As noted earlier, in the system-optimized problem, there exists a central controller who seeks to minimize the total cost in the network system, where the total cost is expressed as

$$\sum_{a \in \mathcal{L}} \hat{c}_a(f_a), \tag{5}$$

and the total cost on a link is given by expression (4).

The system-optimization problem is, thus, given by:

$$\text{Minimize} \quad \sum_{a \in \mathcal{L}} \hat{c}_a(f_a) \tag{6}$$

subject to:

$$\sum_{p \in P_\omega} x_p = d_\omega, \quad \forall \omega \in \Omega, \tag{7}$$

$$f_a = \sum_{p \in P} x_p, \quad \forall a \in \mathcal{L}, \tag{8}$$

$$x_p \geq 0, \quad \forall p \in P. \tag{9}$$

The constraints (7) and (8), along with (9), are commonly referred to in network terminology as *conservation of flow equations*. In particular, they guarantee that the flow in the network, that is, the users (whether these are travelers or computer messages, for example) do not "disappear from the network," and, hence, are "conserved."

The total cost on a path, denoted by $\hat{C}_p$, is the user cost on a path times the flow on a path, that is,

$$\hat{C}_p = C_p x_p, \quad \forall p \in P, \tag{10}$$

where the user cost on a path, C_p, is given by the sum of the user costs on the links that comprise the path, that is,

$$C_p = \sum_{a \in \mathcal{L}} c_a(f_a)\delta_{ap}, \quad \forall a \in \mathcal{L}. \tag{11}$$

In view of (8), one may express the cost on a path p as a function of the path flow variables and, hence, an alternative version of the above system-optimization problem can be stated in path flow variables only, where one has now the problem:

$$\text{Minimize} \quad \sum_{p \in P} C_p(x) x_p \tag{12}$$

subject to constraints (7) and (9).

System-Optimality Conditions

Under the assumption of increasing user link cost functions, the objective function in the S-O problem is convex, and the feasible set consisting of the linear constraints is also convex. Therefore, the optimality conditions, that is, the Kuhn-Tucker conditions are: For each O/D pair $\omega \in \Omega$, and each path $p \in P_\omega$, the flow pattern x (and link flow pattern f), satisfying (7)–(9) must satisfy:

$$\hat{C}'_p \begin{cases} = \mu_\omega, & \text{if} \quad x_p > 0 \\ \geq \mu_\omega, & \text{if} \quad x_p = 0, \end{cases} \tag{13}$$

where $\hat{C}'_p$ denotes the marginal of the total cost on path p, given by:

$$\hat{C}'_p = \sum_{a \in \mathcal{L}} \frac{\partial \hat{c}_a(f_a)}{\partial f_a} \delta_{ap}, \tag{14}$$

evaluated in (13) at the solution and μ_ω is the Largrange multiplier associated with constraint (7) for that O/D pair ω.

Observe that conditions (13) may be rewritten so that there exists an ordering of the paths for each O/D pair whereby all used paths (that is, those with positive flow) have equal and minimal marginal total costs and the unused paths (that is, those with zero flow) have higher (or equal) marginal total costs than those of the used paths. Hence, in the S-O problem, according to the optimality conditions (13), it is the marginal of the total cost on each used path connecting an O/D pair which is equalized and minimal (see also, e.g., Dafermos and Sparrow (1969)).

2.1.2 The User-Optimized Problem

We now describe the user-optimized network problem, also commonly referred to in the transportation literature as the *traffic assignment* problem or the *traffic network equilibrium* problem. Again, as in the system-optimized problem of Section 2.1.1, the network $\mathcal{G} = [\mathcal{N}, \mathcal{L}]$, the demands associated with the origin/destination pairs, as well as the user link cost functions are assumed as given. Recall that user-optimization follows Wardrop's first principle.

Network Equilibrium Conditions

In the case of the user-optimization problem one seeks to determine the path flow pattern x^* (and the link flow pattern f^*) which satisfies the conservation of flow equations (7), (8), and the nonnegativity assumption on the path flows (9), and which also satisfies the network equilibrium conditions given by the following statement. For each O/D pair $\omega \in \Omega$ and each path $p \in P_\omega$:

$$C_p \begin{cases} = \lambda_\omega, & \text{if} \quad x_p^* > 0 \\ \geq \lambda_\omega, & \text{if} \quad x_p^* = 0. \end{cases} \tag{15}$$

Hence, in the user-optimization problem there is no explicit optimization concept, since now users of the transport network system act independently, in a noncooperative manner, until they cannot improve on their situations unilaterally and, thus, an equilibrium is achieved, governed by the above equilibrium conditions. Indeed, conditions (15) are simply a restatement of Wardrop's (1952) first principle mathematically and mean that only those paths connecting an O/D pair will be used which have equal and minimal user costs. In (15) the minimal cost for a given O/D pair is denoted by λ_ω and its value is obtained once the equilibrium flow pattern is determined.

Otherwise, a user of the network could improve upon his situation by switching to a path with lower cost. User-optimization represents decentralized decision-making, whereas system-optimization represents centralized decision-making. See also Table 1.

In order to obtain a solution to the above problem, Beckmann, McGuire, and Winsten (1956) established that the solution to the equilibrium problem, in the case of user link cost functions (cf. (3)) in which the cost on a link only depends on the flow on that link could be obtained by solving the following optimization problem:

$$\text{Minimize} \quad \sum_{a \in \mathcal{L}} \int_0^{f_a} c_a(y)dy \tag{16}$$

subject to:

$$\sum_{p \in P_\omega} x_p = d_\omega, \quad \forall \omega \in \Omega, \tag{17}$$

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in \mathcal{L}, \tag{18}$$

$$x_p \geq 0, \quad \forall p \in P. \tag{19}$$

Note that the conservation of flow equations are identical in both the user-optimized network problem (see (17)–(19)) and the system-optimized problem (see (7)–(9)). The behavior of the individual decision-makers termed "users," however, is different. Users of the network system, which generate the flow on the network now act independently, and are not controlled by a centralized controller.

The objective function given by (16) is simply a device constructed to obtain a solution using general purpose convex programming algorithms. It does not possess the economic meaning of the objective function encountered in the system-optimization problem given by (6), equivalently, by (12). Note that in the case of separable, as well as nonseparable but symmetric (which we come back to later), user link cost functions the λ_ω term in (15) corresponds to the Lagrange multiplier associated with the constraint (17) for that O/D pair ω. However, in the case of nonseparable and asymmetric functions there is no optimization reformulation of the traffic network equilibrium conditions (15) and the λ_ω term simply reflects the minimum user cost associated with the O/D pair ω at the equilibrium.

3. Models with Asymmetric Link Costs

There has been much research activity in the past several decades in terms of both the modeling and the development of methodologies to enable the formulation and computation of more general traffic (and related) network equilibrium models. Examples of general models include those that allow for multiple modes of transportation or multiple classes of users, who perceive cost on a link in an individual way. In this section, we consider network models in which the user cost on a link is no longer dependent solely on the flow on that link.

Assume that user link cost functions are now of a general form, that is, the cost on a link may depend not only on the flow on the link but on other link flows on the network, that is,

$$c_a = c_a(f), \quad \forall a \in \mathcal{L}. \tag{20}$$

In the case where the symmetry assumption exists, that is, $\frac{\partial c_a(f)}{\partial f_b} = \frac{\partial c_b(f)}{\partial f_a}$, for all links $a, b \in \mathcal{L}$, one can still reformulate the solution to the network equilibrium problem satisfying equilibrium conditions (15) as the solution to an optimization problem (cf. Dafermos (1972), and the references therein), albeit, again, with an objective function that is artificial and simply a mathematical device. However, when the symmetry assumption is no longer satisfied, such an optimization reformulation no longer exists and one must appeal to *variational inequality theory*. Models of traffic networks with asymmetric cost functions are important since they allow for the formulation, qualitative analysis, and, ultimately, given the state-of-the-art, solution to problems in which the cost on a link may depend on the flow on another link in a different way than the cost on the other link depends on that link's flow. Such a generalization allows for the more realistic treatment of intersections, two-way links, multiple modes of transport as well as distinct classes of users of the network.

Indeed, it was in the domain of such traffic network equilibrium problems that the theory of finite-dimensional variational inequalities realized its earliest success, beginning with the contributions of Smith (1979) and Dafermos (1980). For an introduction to the subject, as well as applications ranging from traffic network and spatial price equilibrium problems to financial equilibrium problems, see the book by Nagurney (1999). Below we present variational inequality formulations of both fixed demand and elastic demand traffic network equilibrium problems.

The system-optimization problem, in turn, in the case of nonseparable (cf. (20)) user link cost functions becomes (see also (6)–(9)):

$$\text{Minimize} \quad \sum_{a\in\mathcal{L}} \hat{c}_a(f), \tag{21}$$

subject to (7)–(9), where $\hat{c}_a(f) = c_a(f) \times f_a$, $\forall a \in \mathcal{L}$.

The system-optimality conditions remain as in (13), but now the marginal of the total cost on a path becomes, in this more general case:

$$\hat{C}'_p = \sum_{a,b\in\mathcal{L}} \frac{\partial \hat{c}_b(f)}{\partial f_a} \delta_{ap}, \quad \forall p \in P. \tag{22}$$

Variational Inequality Formulations of Fixed Demand Problems

As mentioned earlier, in the case where the user link cost functions are no longer symmetric, one cannot compute the solution to the U-O, that is, to the network equilibrium, problem using standard optimization algorithms. We emphasize, again, that such general cost functions are very important from an application standpoint since they allow for asymmetric interactions on the network. For example, allowing for asymmetric cost functions permits one to handle the situation when the flow on a particular link affects the cost on another link in a different way than the cost on the particular link is affected by the flow on the other link.

First, the definition of a variational inequality problem is recalled. For further background, theoretical formulations, derivations, and the proofs of the results below, see the books by Nagurney (1999) and by Nagurney and Dong (2002a) and the references therein. We provide the variational inequality of the network equilibrium conditions in path flows as well as in link flows.

Specifically, the variational inequality problem (finite-dimensional) is defined as follows:

Definition 1: Variational Inequality Problem

The finite-dimensional variational inequality problem, VI$(F,\mathcal{K})$*, is to determine a vector* $X^* \in \mathcal{K}$ *such that*

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{23}$$

where F *is a given continuous function from* $\mathcal{K}$ *to* R^N*,* $\mathcal{K}$ *is a given closed convex set, and* $\langle \cdot, \cdot \rangle$ *denotes the inner product in* R^N*.*

Variational inequality (23) is referred to as being in *standard form*. Hence, for a given problem, typically an *equilibrium* problem, one must determine the function F that enters the variational inequality problem, the vector of variables X, as well as the feasible set $\mathcal{K}$.

The variational inequality problem contains, as special cases, such well-known problems as systems of equations, optimization problems, and complementarity problems. Thus, it is a powerful unifying methodology for equilibrium analysis and computation.

A geometric interpretation of the variational inequality problem VI$(F,\mathcal{K})$ is given in Figure 1. In particular, $F(X^*)$ is "orthogonal" to the feasible set $\mathcal{K}$ at the point X^*.

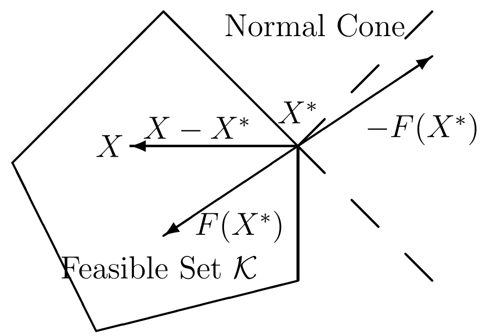

Figure 1: Geometric Interpretation of VI($F, \mathcal{K}$)

Theorem 1: Variational Inequality Formulation of Network Equilibrium with Fixed Demands – Path Flow Version

A vector $x^* \in K^1$ is a network equilibrium path flow pattern, that is, it satisfies equilibrium conditions (15) if and only if it satisfies the variational inequality problem:

$$\sum_{\omega \in \Omega} \sum_{p \in P_\omega} C_p(x^*) \times (x - x^*) \geq 0, \quad \forall x \in K^1, \tag{24}$$

or, in vector form:

$$\langle C(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K^1, \tag{25}$$

where C is the n_P-dimensional column vector of path user costs and K^1 is defined as: $K^1 \equiv \{x \geq 0,$ such that (17) holds$\}$.

Theorem 2: Variational Inequality Formulation of Network Equilibrium with Fixed Demands – Link Flow Version

A vector $f^* \in K^2$ is a network equilibrium link flow pattern if and only if it satisfies the *variational inequality problem:*

$$\sum_{a\in\mathcal{L}} c_a(f^*) \times (f_a - f_a^*) \geq 0, \quad \forall f \in K^2, \tag{26}$$

or, in vector form:

$$\langle c(f^*), f - f^* \rangle \geq 0, \quad \forall f \in K^2, \tag{27}$$

where c *is the* n*-dimensional column vector of link user costs and* K^2 *is defined as:* $K^2 \equiv \{f \,|\, \textit{there exists an}\, x \geq 0 \textit{ and satisfying}\, (17) \textit{ and}\, (18)\}$.

Note that one may put variational inequality (25) into standard form (23) by letting $F \equiv C$, $X \equiv x$, and $\mathcal{K} \equiv K^1$. Also, one may put variational inequality (27) into standard form where now $F \equiv c$, $X \equiv f$, and $\mathcal{K} \equiv K^2$.

Alternative variational inequality formulations of a problem are useful in devising other models, including dynamic versions, as well as for purposes of computation using different algorithms. In Section 5, we describe the relationship between variational inequality formulations and projected dynamical systems, in which the latter provides the disequilibrium dynamics prior to the attainment of the equilibrium, as formulated via the former.

The theory of variational inequalities (see Kinderlehrer and Stampacchia (1980) and Nagurney (1999)) allows one to qualitatively analyze the equilibrium patterns in terms of existence, uniqueness, as well as sensitivity and stability of solutions, and to apply rigorous algorithms for the numerical computation of the equilibrium patterns. Variational inequality algorithms usually resolve the variational inequality problem into series of simpler subproblems, which, in turn, are often optimization problems, which can then be effectively solved using a variety of algorithms, including the aforementioned equilibration algorithms of Dafermos and Sparrow (1969), which exploit network structure as well as the commonly used in practice Frank-Wolfe (1956) algorithm (see also LeBlanc, Morlok, and Pierskalla (1975)). In particular, projection methods as well as relaxation methods (see Dafermos (1980, 1982), Florian and Spiess (1982), Nagurney (1984, 1999), and Patriksson (1994)) have been successfully applied to compute solutions to variational inequality formulations of traffic network equilibrium problems.

We emphasize that the above network equilibrium framework is sufficiently general to also formalize the entire transportation planning process (consisting of origin selection, or destination selection, or both, in addition to route selection, in an optimal fashion) as path choices over an appropriately constructed *abstract* network or supernetwork. This was recognized by Dafermos in 1976 (in the context of separable link cost functions) in her development of such integrated traffic network equilibrium models in which location decisions are made simultaneous to transportation route decisions (see also Boyce (1980)). Further discussion can be found in that reference as well as in the books by Nagurney (1999, 2000) and Nagurney and Dong (2002a) who also developed more general models in which the costs (as described above) need not be separable nor asymmetric.

It is worth noting that the presentation of the variational inequality formulations of the fixed demand models given above was in the context of single mode (or single class) transport networks. We emphasize, however, that in view of the generality of the functions considered (cf. (20)), the modeling framework described above can also be adapted to multimodal/multiclass problems in which there are multiple modes of transport available and/or multiple classes of users, each of whom perceives the cost on the links of the network in an individual manner. Dafermos in (1972) demonstrated how, through a formal model, a multiclass traffic network could be cast into a single-class network through the construction of an expanded (and, again, *abstract*) network consisting of as many copies of the original network as there were classes. The application of such a transformation is also relevant to telecommunication networks.

Also, we note that here the focus is on deterministic network equilibrium problems. Some basic stochastic traffic network equilibrium models can be found in Sheffi (1985). Dial (1971) is credited with developing the first stochastic route choice model. Daganzo and Sheffi (1977), in turn, formulated a stochastic user-optimized traffic network model with route choice in which the equilibrium criterion could be succinctly stated as *no traveler can improve his or her perceived travel time by unilaterally changing routes*.

Finally, we emphasize that the dynamic models presented in Section 5 (although presented in a deterministic framework) have been analyzed qualitatively using tools from stochastic processes (cf. Dupuis and Nagurney (1993) and Nagurney and Zhang (1996).

Variational Inequality Formulations of Elastic Demand Problems

We now describe a general network equilibrium model with elastic demands due to Dafermos (1982). Specifically, it is assumed that one has associated with each O/D pair ω in the network a travel disutility function λ_ω, where here the general case is considered in which the disutility may depend upon the entire vector of demands, which are no longer fixed, but are now variables, that is,

$$\lambda_\omega = \lambda_\omega(d), \quad \forall \omega \in \Omega, \tag{28}$$

where d is the J-dimensional column vector of the demands.

The notation, otherwise, is as described earlier, except that here we also consider user link cost functions which are general, that is, of the form (20). The conservation of flow equations (see also (1) and (2)), in turn, are given by

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in \mathcal{L}, \tag{29}$$

$$d_\omega = \sum_{p \in P_\omega} x_p, \quad \forall \omega \in \Omega, \tag{30}$$

$$x_p \geq 0, \quad \forall p \in P. \tag{31}$$

Hence, in the elastic demand case, the demands in expression (30) are now variables and no longer given, as was the case for the fixed demand expression in (2).

Network Equilibrium Conditions in the Case of Elastic Demand

The network equilibrium conditions (see also (15)) now take on in the elastic demand case the following form: For every O/D pair $\omega \in \Omega$, and each path $p \in P_\omega$, a vector of path flows and demands (x^*, d^*) satisfying (30)–(31) (which induces a link flow pattern f^* through (29)) is a network equilibrium pattern if it satisfies:

$$C_p(x^*) \begin{cases} = \lambda_\omega(d^*), & \text{if} \quad x_p^* > 0 \\ \geq \lambda_\omega(d^*), & \text{if} \quad x_p^* = 0. \end{cases} \tag{32}$$

Equilibrium conditions (32) state that the costs on used paths for each O/D pair are equal and minimal and equal to the disutility associated with that O/D pair. Costs on unutilized paths can exceed the disutility. Condition (32) can be given an economic interpretation as described in the first paragraph of the Introduction. Observe that in the elastic demand model users of the network can forego travel altogether for a given O/D pair if the user costs on the connecting paths exceed the travel disutility associated with that O/D pair. We emphasize that this model, hence, allows one to ascertain the attractiveness of different O/D pairs based on the ultimate equilibrium demand associated with the O/D pairs. In addition, this model can also handle such situations as the equilibrium determination of employment location and route selection, or residential location and route selection, or residential and employment selection as well as route selection through the appropriate transformations via the addition of links and nodes, and given, respectively, functions associated with the residential locations, the employment locations, and the network overall (cf. Dafermos (1976), Nagurney (1999), and Nagurney and Dong (2002a)).

Also, we note that although the presentation of the elastic demand traffic network model has been in the case of a single mode of transport or class of user one can readily (with an accompanying increase in notation) explicitly introduce distinct modes to the above model as follows. One needs only to introduce subscripts to denote modes/classes, redefine all of the above vectors accordingly, and the conservation of flow equations, and state that (32) then must hold for each mode/class. In other words, in equilibrium, the used paths for a given mode and O/D pair must have minimal and equal user path costs, which in turn, must be equal to the travel disutility for that mode and O/D pair at the equilibrium demand. Of course, as described in the case of fixed demands, one can also have made as many copies as there are modes on the network in which case the above single-modal but extended elastic demand model would be equivalent to the multimodal one.

In the next two theorems, both the path flow version and the link flow version of the variational inequality formulations of the network equilibrium conditions (32) are presented. These are analogues of the formulations (24) and (25), and (26) and (27), respectively, for the fixed demand model.

Theorem 3: Variational Inequality Formulation of Network Equilibrium with Elastic Demands – Path Flow Version

A vector $(x^*, d^*) \in K^3$ *is a network equilibrium path flow pattern, that is, it satisfies equilibrium conditions (32) if and only if it satisfies the variational inequality problem:*

$$\sum_{\omega\in\Omega}\sum_{p\in P_\omega} C_p(x^*)\times(x-x^*) - \sum_{\omega\in\Omega}\lambda_\omega(d^*)\times(d_\omega - d^*_\omega) \geq 0, \quad \forall (x,d)\in K^3, \tag{33}$$

or, in vector form:

$$\langle C(x^*), x-x^*\rangle - \langle \lambda(d^*), d-d^*\rangle \geq 0, \quad \forall (x,d)\in K^3, \tag{34}$$

where λ *is the* J*-dimensional vector of disutilities and* K^3 *is defined as:* $K^3 \equiv \{x \geq 0,$ *such that* (30) *holds*$\}$.

Theorem 4: Variational Inequality Formulation of Network Equilibrium with Elastic Demands – Link Flow Version

A vector $(f^*, d^*) \in K^4$ *is a network equilibrium link flow pattern if and only if it satisfies the variational inequality problem:*

$$\sum_{a\in\mathcal{L}} c_a(f^*)\times(f_a - f^*_a) - \sum_{\omega\in\Omega}\lambda_\omega(d^*)\times(d_\omega - d^*_\omega) \geq 0, \quad \forall (f,d)\in K^4, \tag{35}$$

or, in vector form:

$$\langle c(f^*), f-f^*\rangle - \langle \lambda(d^*), d-d^*\rangle \geq 0, \quad \forall (f,d)\in K^4, \tag{36}$$

where $K^4 \equiv \{(f,d),$ *such that there exists an* $x\geq 0$ *satisfying* (29), $(31)\}$

Note that, under the symmetry assumption on the disutility functions, that is, if $\frac{\partial\lambda_w}{\partial d_\omega} = \frac{\partial\lambda_\omega}{\partial d_w}$, for all w, ω, in addition to such an assumption on the user link cost functions (see following (20)), one can obtain (see Beckmann, McGuire, and Winsten (1956)) an optimization reformulation of the network equilibrium conditions (32), which in the case of separable user link cost functions and disutility functions is given by:

$$\text{Minimize} \quad \sum_{a\in\mathcal{L}}\int_0^{f_a} c_a(y)dy - \sum_{\omega\in\Omega}\int_0^{d_\omega}\lambda_\omega(z)dz \tag{37}$$

subject to: (29)–(31).

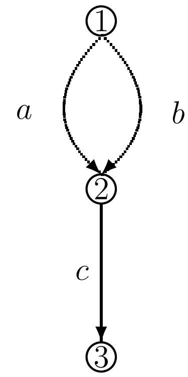

Figure 2: An Elastic Demand Example

We now present an example of a simple elastic demand network equilibrium problem.

Example 1

Consider the network depicted in Figure 2 in which there are three nodes: 1, 2, 3; three links: a, b, c; and a single O/D pair $\omega_1 = (1, 3)$. Let path $p_1 = (a, c)$ and path $p_2 = (b, c)$.

Assume that the user link cost functions are:

$$c_a(f) = 5f_a + 2f_b + 15, \quad c_b(f) = 7f_b + f_a + 15, \quad c_c(f) = 3f_c + f_a + f_b + 12,$$

and the disutility (or inverse demand) function is given by:

$$\lambda_{\omega_1}(d_{\omega_1}) = -2d_{\omega_1} + 114.$$

Observe that in this example, the user link cost functions are non-separable and asymmetric and, hence, the equilibrium conditions (32) cannot be reformulated as the solution to an optimization problem, but, rather, as the solution to the variational inequalities (33) (or (34)), or (35) (or (36)).

The U-O flow and demand pattern that satisfies equilibrium conditions (32) is: $x^*_{p_1} = 5$, $x^*_{p_2} = 4$, and $d^*_{\omega_1} = 9$, with associated link flow pattern: $f^*_a = 5$, $f^*_b = 4$, $f^*_c = 9$.

The incurred user costs on the paths are: $C_{p_1} = C_{p_2} = 96$, which is precisely the value of the disutility λ_{ω_1}. Hence, this flow and demand pattern satisfies equilibrium conditions (32). Indeed, both paths p_1 and p_2 are utilized and their user paths costs are equal to each other. In addition, these costs are equal to the disutility associated with the origin/destination pair that the two paths connect.

We note that the elastic demand model described above is related closely to the well-known spatial price equilibrium models of Samuelson (1952), Takayama and Judge (1971), and Florian and Los (1982). Indeed, as demonstrated by Dafermos and Nagurney (1985) in the context of a single commodity, and, subsequently, by Dafermos (1986) in the case of multiple commodities, spatial price equilibrium problems are *isomorphic* to traffic network equilibrium problems over appropriately constructed networks. Hence, the well-developed theory of traffic networks can be transferred to the study of commodity flows in the case of spatial price equilibrium in which the equilibrium production, consumption, and commodity trade flows are to be determined satisfying the equilibrium conditions that there will be a positive flow (in equilibrium) of the commodity between a pair of supply and demand markets if the supply price at the supply market plus the unit cost of transportation is equal to the demand price at the demand market. A variety of such models (both static and dynamic) and associated references can be found in the books by Nagurney (1999) and Nagurney and Zhang (1996).

Although the focus of this paper is on transport network equilibrium models in an urban setting, models of freight networks are closely related to those discussed above. Of course, one must distinguish the behavior of the operators of such networks and model the competition accordingly (see, e.g., Friesz and Harker (1985)).

4. Multiclass, Multicriteria Traffic Network Equilibrium Models

In this section, we describe multiclass, multicriteria network equilibrium models which can serve as an alternative to multimodal traffic network equilibrium models. These models are important since they allow for the individual weighting of distinct criteria associated with decision-making on networks and especially transport networks. Moreover, such models have been successful in formalizing decision-making surrounding transport/telecommunication network tradeoffs as in the case of telecommuting versus commuting decision-making and teleshopping versus shopping decision-making (cf. Nagurney and Dong (2002a, b, c), Nagurney, Dong, and Mokhtarian (2002a, b)).

Multicriteria traffic network models were introduced by Quandt (1967) and Schneider (1968) and explicitly consider that travelers may be faced with several criteria, notably, travel time and travel cost, in selecting their optimal routes of travel. The ideas were further developed by Dial (1979) who proposed an uncongested model and Dafermos (1981) who introduced congestion effects and derived an infinite-dimensional variational inequality formulation of her multiclass, multicriteria traffic network equilibrium problem, along with some qualitative properties. The paper by Nagurney and Dong (2002b) provides a chronology of citations, which highlights: the number of criteria treated by various authors, typically, travel time and travel cost; whether or not these functions are allowed to be flow-dependent or not, and the form (separable or general) handled. In addition, it notes the type of demand considered, that is, fixed or elastic, and whether the demand is class-dependent, and, if elastic, what form the demand functions take. Moreover, it provides the type of methodology used in the formulation and analysis such as, for example, an optimization approach, a finite-dimensional variational inequality approach, or infinite-dimensional approach, along with whether the citation contains algorithmic contributions and/or qualitative ones. We note that, in the case of infinite-dimensional variational inequality formulations, the number of classes is, usually, infinite, whereas in the case of finite-dimensional formulations, the number of classes is assumed to be finite. Additional citations, including literature exploring multicriteria traffic models used in practice, may be found in the book chapter by Leurent (1998).

In this section, for completeness, we recall the multiclass, multicriteria network equilibrium model with elastic demand developed by Nagurney and Dong (2002b). The model has the following novel and what we believe are significant features:

1. It includes weights associated with the two criteria of travel time and travel cost which are not only class-dependent but also, *explicitly*, link-dependent. Such weights may incorporate such subjective factors as the relative safety or risk associated with particular links, the relative comfort, or even the view.

2. It treats demand functions (rather than their inverses) which are very general and not separable functions. Specifically, the demand associated with a class and origin/destination (O/D) pair can depend not only on the travel disutility of different classes traveling between the particular O/D pair but can also be influenced by the disutilities of the classes traveling between other O/D pairs. Hence, the model has implications for locational choice (see, e. g., Beckmann, McGuire, and Winsten (1956), Boyce (1980), and Boyce, et al (1983)).

As in the transport network models described in Sections 2 and 3, we consider a general network $G = [\mathcal{N}, \mathcal{L}]$, where $\mathcal{N}$ denotes the set of nodes in the network and $\mathcal{L}$ the set of directed links. Let a denote a link of the network connecting a pair of nodes and let p denote a path, assumed to be acyclic, consisting of a sequence of links connecting an origin/destination pair of nodes. There are n links in the network and n_P paths. Let Ω denote the set of J O/D pairs. The set of paths connecting the O/D pair ω is denoted by P_ω and the entire set of paths in the network by P.

Assume now that there are k classes of travelers in the network with a typical class denoted by i. Let f_a^i denote the flow of class i on link a and let x_p^i denote the nonnegative flow of class i on path p. The relationship between the link flows by class and the path flows is:

$$f_a^i = \sum_{p \in P} x_p^i \delta_{ap}, \quad \forall i, \forall a, \tag{38}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. Hence, the flow of a class of traveler on a link is equal to the sum of the flows of the class on the paths that contain that link.

In addition, let f_a now denote the total flow on link a, where

$$f_a = \sum_{i=1}^{k} f_a^i, \quad \forall a \in \mathcal{L}. \tag{39}$$

Group the class link flows into the kn-dimensional column vector $\tilde{f}$ with components: $\{f_a^1, \ldots, f_n^1, \ldots, f_a^k, \ldots, f_n^k\}$ and the total link flows: $\{f_a, \ldots, f_n\}$ into the n-dimensional column vector f. Also, group the class path flows into the kn_P-dimensional column vector $\tilde{x}$ with components: $\{x_{p_1}^1, \ldots, x_{p_{n_P}}^k\}$.

We are now ready to describe the functions associated with the links. We assume, as given, a travel time function t_a associated with each link a in the network, where

$$t_a = t_a(f), \quad \forall a \in \mathcal{L}, \tag{40}$$

and a travel cost function c_a associated with each link a, that is,

$$c_a = c_a(f), \quad \forall a \in \mathcal{L}, \tag{41}$$

with both these functions assumed to be continuous. Note that here we allow for the general situation in which both the travel time and the travel cost can depend on the entire link flow pattern, whereas in Dafermos (1981) it was assumed that these functions were separable.

We assume that each class of traveler i has his own perception of the trade-off between travel time and travel cost which are represented by the nonnegative weights w^i_{1a} and w^i_{2a}. Here w^i_{1a} denotes the weight associated with class i's travel time on link a and w^i_{2a} denotes the weight associated with class i's travel cost on link a. The weights w^i_{1a} and w^i_{2a} are link-dependent and, hence, can incorporate such link-dependent factors as safety, comfort, and view. For example, in the case of a pleasant view on a link, travelers may weight the travel cost higher than the travel time on such a link. However, if a link has a rough surface or is noted for unsafe road conditions such as ice in the winter, travelers may then assign a higher weight to the travel time than the travel cost. Link-dependent weights provide a greater level of generality and flexibility in modeling travel decision-making than weights that are identical for the travel time and for the travel cost on all links for a given class.

We then construct the *generalized* cost/disutility of class i associated with link a, and denoted by u^i_a, as:

$$u^i_a = w^i_{1a} t_a + w^i_{2a} c_a, \quad \forall i, \forall a. \tag{42}$$

In view of (39), (40), and (41), we may write

$$u^i_a = u^i_a(\tilde{f}), \quad \forall i, \forall a, \tag{43}$$

and group the link generalized costs into the kn-dimensional column vector u with components: $\{u^1_a, \ldots, u^1_n, \ldots, u^k_a, \ldots, u^k_n\}$.

Observe that a possible weighting scheme would be: $w_{1a}^i = \psi_a^i$ and $w_{2a}^i = (1 - \psi_a^i)$ with ψ_a^i lying in the range from zero to one with $\psi_a^i = 1$ denoting a class of traveler who is only concerned with the travel time on a particular link a, and with $\psi_a^i = 0$ denoting a class of traveler only concerned about travel cost on link a; with weights within the range reflecting classes who perceive travel time and travel cost as per the disutility functions accordingly. Dafermos (1981) proposed such a weighting scheme in which $w_{1a}^i = \psi^i$ and $w_{2a}^i = (1 - \psi^i)$ for all links a and classes i. Such a weighting scheme has an interpretation of a weighted average, but is not link-dependent.

Let v_p^i denote the *generalized* cost of class i associated with traveling on path p, where

$$v_p^i = \sum_{a \in \mathcal{L}} u_a^i(\tilde{f})\delta_{ap}, \quad \forall i, \forall p. \tag{44}$$

Hence, the generalized cost, as perceived by a class, associated with traveling on a path is the sum of the generalized link costs on links comprising the path.

Let d_ω^i denote the travel demand of class i traveler between O/D pair ω, and let λ_ω^i denote the travel disutility associated with class i traveler traveling between the O/D pair ω. We group the travel demands into a kJ-dimensional column vector d and the O/D pair travel disutilities into a kJ-dimensional column vector λ.

The path flow vector $\tilde{x}$ induces the demand vector d with components

$$d_\omega^i = \sum_{p \in P_\omega} x_p^i, \quad \forall i, \forall \omega. \tag{45}$$

We assume that the travel demands are determined by the O/D travel disutilities, that is,

$$d_\omega^i = d_\omega^i(\lambda), \quad \forall i, \forall \omega, \tag{46}$$

and denote the kJ-dimensional row vector of demand functions by $d(\lambda)$.

Note that the travel demand function (46) is quite general and has choice location implications as well. For example, it allows the demand for a class associated with an O/D pair to depend not only on the travel disutilities of different classes associated with that O/D pair, but also on those associated with other O/D pairs.

Traffic Network Equilibrium Conditions

The traffic network equilibrium conditions in the case of elastic travel demands (see Beckmann, McGuire, and Winsten (1956), Dafermos and Nagurney (1984), and Nagurney (1999), Nagurney and Dong (2002b)), in the generalized context of the multiclass, multicriteria traffic network equilibrium problem take on the form: For each class i, for all O/D pairs $\omega \in \Omega$, and for all paths $p \in P_\omega$, the flow pattern $\tilde{x}^*$ is said to be in equilibrium if the following conditions hold:

$$v_p^i(\tilde{f}^*) \begin{cases} = \lambda_\omega^{i*}, & \text{if} \quad x_p^{i*} > 0 \\ \geq \lambda_\omega^{i*}, & \text{if} \quad x_p^{i*} = 0, \end{cases} \tag{47}$$

and

$$d_\omega^i(\lambda^*) \begin{cases} = \sum_{p \in P_\omega} x_p^{i*}, & \text{if} \quad \lambda_\omega^{i*} > 0 \\ \leq \sum_{p \in P_\omega} x_p^{i*}, & \text{if} \quad \lambda_\omega^{i*} = 0. \end{cases} \tag{48}$$

In other words, all utilized paths by a class connecting an O/D pair have equal and minimal generalized path costs. Meanwhile, if the travel disutility associated with traveling between O/D pair ω of class i is positive, then the market clears for that O/D pair and that class; that is, the sum of the path flows of that class of traveler on paths connecting that O/D pair is equal to the demand associated with that O/D pair; if the travel disutility is zero, then the sum of the path flows can exceed the demand of that class of traveler.

Hence, in the elastic demand framework, different classes of travelers can also choose their O/D pairs, in addition to their paths. Thus, this model allows one to capture the relative attractiveness of different O/D pairs as perceived by the distinct classes of travelers through the travel disutilities.

We define the feasible set $\mathcal{K}$ underlying the problem as $\mathcal{K} \equiv \{(\tilde{f}, d, \lambda) \,|\, \lambda \geq 0 \,\text{and}\, \exists \tilde{x} \geq 0, \text{such that} \,(38), (39), \text{and} \,(45)\, \text{hold}\}$.

Theorem 5: Variational Inequality Formulation of Multiclass, Multicriteria Network Equilibrium with Elastic Demands – Link Flow Version

A multiclass, multicriteria link flow, travel demand, and O/D travel disutility pattern $(\tilde{f}^*, d^*, \lambda^*) \in \mathcal{K}$ *is a traffic network equilibrium, that is, satisfies equilibrium conditions (47) and (48) if* *and only if it satisfies the variational inequality problem:*

$$\sum_{i=1}^{k}\sum_{a\in\mathcal{L}} u_a^i(\tilde{f}^*) \times (f_a^i - f_a^{i*}) - \sum_{i=1}^{k}\sum_{\omega\in W} \lambda_\omega^{i*} \times (d_\omega^i - d_\omega^{i*}) + \sum_{i=1}^{k}\sum_{\omega\in W} (d_\omega^{i*} - d_\omega^i(\lambda^*)) \times (\lambda_\omega^i - \lambda_\omega^{i*}) \geq 0,$$

$$\forall (\tilde{f}, d, \lambda) \in \mathcal{K}; \tag{49}$$

equivalently, in standard form:

$$\langle F(X^*, X - X^*\rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{50}$$

where $F \equiv (u, -\lambda, d - d(\lambda))$ *and* $X \equiv (\tilde{f}, d, \lambda)$*.*

In Nagurney and Dong (2002a) can be found network equilibrium models with multiple criteria and multiple classes in which there are a finite number of criteria associated with decision-making and the weights (as those above) are also link- and class-dependent. We emphasize that the qualitative analysis of such models is more challenging than of those presented in the preceding sections since the criterion functions are in terms of the total links flows whereas the generalized costs are constructed according to the classes and links. For qualitative properties as well as algorithmic procedures and numerical examples, we refer the reader to Nagurney and Dong (2002a, b, c) and to Nagurney, Dong, and Mokhtarian (2002a, b), and to the references therein.

6. Dynamics

In this Section, we summarize briefly how projected dynamical systems theory can be applied to the elastic demand traffic network equilibrium problem presented in Section 3 in order to provide the disequilibrium dynamics. Dupuis and Nagurney (1993) proved that, given a variational inequality problem, there is a naturally associated dynamical system, the set of stationary points of which coincides precisely with the set of solutions of the variational inequality problem. The dynamical system, termed a *projected dynamical system* by Zhang and Nagurney (1995), is non-classical in that its right-hand side, which is a projection operator, is discontinuous. Nevertheless, it can be qualitatively analyzed and approximated through discrete-time algorithms as described in Dupuis and Nagurney and also in the book by Nagurney and Zhang (1996). Importantly, projected dynamical systems theory provides insights into the travelers' dynamic behavior in making their trip decisions and in adjusting their route choices. Moreover, it provides for a powerful theory of stability analysis (cf. Zhang and Nagurney (1996, 1997)). Other approaches to dynamic traffic network problems can be found in Ran and Boyce (1996) and Mahmassani et al (1993). In particular, here we focus on the disequilibrium dynamics and on what can be viewed as the day to day adjustment until an equilibrium is reached.

Since users on a network select paths so as to reach their destinations from their origins, we consider variational inequality (34) as the basic one for the dynamical system equivalence. Specifically, we note that, in view of constraint (30), one may define $\hat{\lambda}(x) \equiv \lambda(d)$, in which case we may rewrite variational inequality (30) in the path flow variables x only, that is, we seek to determine $x^* \in R_+^{n_P}$, such that

$$\langle C(x^*) - \bar{\lambda}(x^*), x - x^* \rangle \geq 0, \quad \forall x \in R_+^{n_P}, \tag{51}$$

where $\bar{\lambda}(x)$ is the $n_{P_{\omega_1}}, \times n_{P_{\omega_2}} \times \ldots n_{P_{\omega_J}}$-dimensional column vector with components:

$$(\hat{\lambda}_{\omega_1}(x), \ldots, \hat{\lambda}_{\omega_1}(x), \ldots, \hat{\lambda}_{\omega_J}(x), \ldots, \hat{\lambda}_{\omega_J}(x)),$$

If we now let $X \equiv x$ and $F(X) \equiv C(x) - \bar{\lambda}(x)$ and $\mathcal{K} \equiv \{x | x \in R_+^{n_P}\}$, then, clearly, (51) can be put into standard form given by (23). The dynamical system, first presented by Dupuis and Nagurney (1993), whose stationary points correspond to solutions of (51), is given by:

$$\dot{x} = \Pi_{\mathcal{K}}(x, \bar{\lambda}(x) - C(x)), \quad x(0) = x_0 \in \mathcal{K}, \tag{52}$$

where the projection operator $\Pi_{\mathcal{K}}(x, v)$ is defined as:

$$\Pi_{\mathcal{K}}(x, v) = \lim_{\delta \to 0} \frac{(P_{\mathcal{K}}(x + \delta v) - x)}{\delta}, \tag{53}$$

and

$$P_{\mathcal{K}} = \text{argmin}_{z \in \mathcal{K}} \|z - x\|. \tag{54}$$

The dynamics described by (52) are as follows: the rate of change of flow on a path connecting an O/D pair is equal to the difference between the travel disutility for that O/D pair and the cost on that path at that instance in time. If the path cost exceeds the travel disutility, then the flow on the path will decrease; if it is less than the disutility, then the flow on that path will increase. The projection operator in (30) guarantees that the flow on the paths will not be negative, since this would violate feasibility. Hence, the path flows (and incurred travel demands) evolve from an initial path flow pattern at time zero given by $x(0)$ until a stationary point is reached, that is, when $\dot{x} = 0$; at which point we have that for that particular x^*:

$$\dot{x} = 0 = \Pi_{\mathcal{K}}(x^*, \bar{\lambda}(x^*) - C(x^*)), \tag{55}$$

and that x^* also solves variational inequality (51) and is, hence, a traffic network equilibrium satisfying the elastic demand equilibrium conditions (32).

Qualitative properties of the dynamic trajectories, as well as conditions for stability of the solutions as well as discrete-time algorithms can be found in Zhang and Nagurney (1995) and in Nagurney and Zhang (1996) and the references therein. In particular, we note that discrete-time algorithms such as those proposed in Nagurney and Zhang (1996) and the references therein provide for a time discretization of the continuous time trajectories and may also be interpreted as discrete-time adjustment processes.

In addition, dynamic but *within day* traffic network models (deterministic as well as stochastic) have received a lot of attention; see Ran and Boyce (1996) and the references therein.

6. Summary and New Directions

In this chapter, we have traced the evolution of the foundations of transport network equilibrium modeling and analysis with a focus on the principal methodological advances. In particular, we have attempted to set out in accessible fashion rigorous approaches to the formulation of a variety of traffic network equilibrium models and to establish relationships between the models as well as those that are closely linked such as spatial price equilibrium models.

We emphasize that this topic is a very active area of research as well as practice. We also would like to highlight and further emphasize that traffic network equilibrium modeling and analysis provides a powerful framework for decision-making on complex networks, in general. Indeed, given the interrelationships between telecommunication and transportation networks in today's Network Economy, we can expect further synergies and advances in the study of such foundational networks. Of particular promise are the areas of multicriteria decision-making on networks, multitiered networks, as well as multilevel networks (in the form of transportation/logistical/financial/informational networks), formally referred to as supernetworks. For further background, see Nagurney and Dong (2002a) and the references therein.

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