Global Supply Chain Networks and Risk Management: A Multi-Agent Framework

Anna Nagurney
Radcliffe Institute Fellow
Radcliffe Institute for Advanced Study
34 Concord Avenue
Harvard University
Cambridge, Massachusetts 02138
and
Department of Finance and Operations Management
Isenberg School of Management
University of Massachusetts
Amherst, Massachusetts 01003

Jose M. Cruz
Department of Operations and Information Management
School of Business
University of Connecticut
Storrs, Connecticut 06269
and

June Dong
Department of Marketing and Management
School of Business
State University of New York at Oswego
Oswego, New York 13126

Published in *Multiagent-Based Supply Chain Management*, (2006), pp 103-134,
B. Chaib-draa and J. P. Muller, Editors, Springer, Berlin, Germany.

Abstract: In this paper, we develop a global supply chain network model in which both physical and electronic transactions are allowed and in which supply-side risk as well as demand-side risk are included in the formulation. The model consists of three tiers of decision-makers/agents: the manufacturers, the distributors, and the retailers who may be located in the same or in different countries and may conduct their transactions in distinct currencies. We model the optimizing behavior of the various decision-makers, with the manufacturers and the distributors being multicriteria decision-makers, and concerned with both profit maximization and risk minimization. The retailers, in turn, are faced with random demands for the product. We derive the governing equilibrium conditions and establish the finite-dimensional variational inequality formulation. We provide qualitative properties of the equilibrium product flow and price pattern in terms of existence and uniqueness results and also establish conditions under which the proposed computational procedure is guaranteed to converge. Finally, we illustrate the global supply chain network model and the computational procedure through several numerical examples. This research illustrates the modeling, analysis, and computation of solutions to decentralized supply chain networks with multiple agents in the presence of electronic commerce and risk management in the global arena. Moreover, it highlights and applies some of the theoretical tools that are now available for such multi-agent problems.

Key words: e-commerce, global operations, financial/economic analysis, programming/optimization, supply management

1. Introduction

Growing competition and emphasis on efficiency and cost reduction, as well as the satisfaction of consumer demands, have brought new challenges for businesses in the global marketplace. At the same time that businesses and, in particular, supply chains have become increasingly globalized, the world environment has become filled with uncertainty. For example, recently, the threat of illness in the form of SARS (see Engardio et al. (2003)) has disrupted supply chains, as have terrorist threats (cf. Sheffi (2001)). On the other hand, innovations in technology and especially the availability of electronic commerce in which the physical ordering of goods (and supplies) (and, is some cases, even delivery) is replaced by electronic orders, offers the potential for reducing risks associated with physical transportation due to potential threats and disruptions in supply chains.

Indeed, the introduction of electronic commerce (e-commerce) has unveiled new opportunities for the management of supply chain networks (cf. Nagurney and Dong (2002) and the references therein) and has had an immense effect on the manner in which businesses order goods and have them transported. According to Mullaney et al. (2003) gains from electronic commerce could reach $450 billion a year by 2005, with consumer e-commerce in the United States alone expected to come close to the $108 billion predicted, despite a recession, terrorism, and war.

The importance of global issues in supply chain management and analysis has been emphasized in several papers (cf. Kogut and Kulatilaka (1994), Cohen and Malik (1997), Nagurney, Cruz, and Matsypura (2003)). Moreover, earlier surveys on supply chain analysis indicate that the research interest is growing rapidly (see Erenguc, Simpson, and Vakharia (1999) and Cohen and Huchzermeier (1997)). Nevertheless, the topic of supply chain risk management is fairly new and novel methodological approaches that capture both the operations as well as the financial aspects of such decision-making are sorely needed. In particular, the need to incorporate both supply-side and demand-side risk in supply chain decision-making and modeling is well-documented in the literature (see, e.g., Smeltzer and Siferd (1998), Agrawal and Seshadri (2000), Johnson (2001), Van Mieghem (2003), and Zsidisin (2003)).

Frameworks for risk management in a global supply chain context with a focus on centralized decision-making and optimization have been proposed by Huchzermeier and Cohen (1996) and Cohen and Mallik (1997) (see also the references therein). In this paper, in contrast, we build upon the recent work of Nagurney, Cruz, and Matsypura (2003) in the modeling of global supply chain networks with electronic commerce and that of Nagurney et al. (2002) and Dong, Zhang, and Nagurney (2002, 2004a,b) who introduced random demands in a decentralized supply chain network. In particular, we develop a global supply chain network model with both supply-side risk (handled as a multicriteria decision-making problem) and demand-side risk (formulated through the use of random demands).

The paper is organized as follows. In Section 2, we develop the global supply chain network model and derive the optimality conditions of the various decision-makers/agents. The model can handle as many manufacturers, countries, currencies, as well as distributors, and retailers, as required by the specific product application. Moreover, we establish that the governing equilibrium conditions can be formulated as a finite-dimensional variational inequality problem. We emphasize here that the concept of equilibrium, first explored in a general setting for supply chains by Nagurney, Dong, and Zhang (2002)), provides a valuable benchmark against which prices of the product at the various tiers of the network as well as product flows between the tiers can be compared.

In Section 3, we provide qualitative properties of the equilibrium pattern, and, under reasonable conditions, establish existence and uniqueness results. We also establish properties of the function that enters the variational inequality that allows us to establish convergence of the proposed algorithmic scheme in Section 4. In Section 5, we apply the algorithm to several global supply chain network examples for the computation of the equilibrium prices and shipments. The paper concludes with Section 6, in which we summarize our results and present suggestions for future research.

2. The Global Supply Chain Network Model with Risk Management

In this Section, we develop the global supply chain network model consisting of three tiers of decision-makers/agents. The multicriteria decision-makers on the supply side in the form of manufacturers and distributors are concerned not only with profit maximization but also with risk minimization. The demand-side risk, in turn, is represented by the uncertainty surrounding the random demands at the retailers. The model allows for not only physical transactions but also for electronic ones. For the structure of the global supply chain network, see Figure 1.

In particular, we consider L different countries with a typical country denoted by l, $\hat{l}$, $\bar{l}$ (since we need to distinguish a given country in a tier). There are I manufacturers in each country with a typical manufacturer i in country l denoted by il and associated with node il in the top tier of nodes in the global supply chain network depicted in Figure 1. Also, we consider J distributors in each country with a typical distributor j in country $\hat{l}$ being denoted by $j\hat{l}$ and associated with second tier node $j\hat{l}$ in the network. There are a total of JL distributors in the global supply chain network. A typical retailer k in country $\bar{l}$ dealing in currency h is denoted by $kh\bar{l}$ and is associated with the corresponding node in the bottom tier of the network. There are a total of KHL retailers in the global supply chain.

We assume a homogeneous product economy meaning that all manufacturers produce the same product which is then shipped to the distributors, who, in turn, distribute the product to the retailers. In order to include the influence of the Internet, we allow the manufacturers to transact either physically with the distributors, or directly, in an electronic manner, with the retailers. Hence, the links connecting the top and the bottom tiers of nodes in Figure 1 represent electronic links. Moreover, the retailers at the bottom tier of nodes of the global supply chain network can be either physical or virtual retailers.

The behavior of the various supply chain decision-makers/agents represented by the three tiers of nodes in Figure 1 is now described. We first focus on the manufacturers. We then turn to the distributors, and, subsequently, to the retailers.

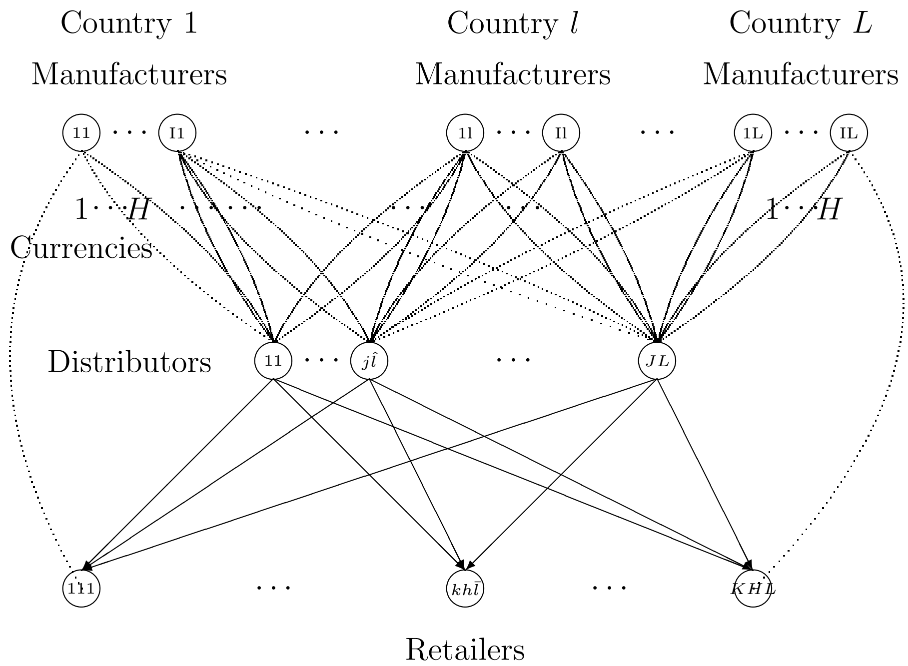

Figure 1: The Structure of the Global Supply Chain Network

The Behavior of the Manufacturers and their Optimality Conditions

Let q^{il} denote the nonnegative production output of manufacturer il. Group the production outputs of all manufacturers into the column vector $q \in R^{IL}_+$. Here it is assumed that each manufacturer il has a production cost function f^{il}, which can depend, in general, on the entire vector of production outputs, that is,

$$f^{il} = f^{il}(q), \quad \forall i, l. \tag{1}$$

Hence, the production cost of a particular manufacturer can depend not only on his production output but also on the production outputs of the other manufacturers. This allows one to model competition.

Note that in Figure 1, there are H distinct links between a manufacturer and distributor pair. Each of these links represents the possibility of a transaction between manufacturer il and distributor $j\hat{l}$ in a certain currency h. Let $c^{il}_{jh\hat{l}}$ denote the transaction cost (which we assume includes the cost of transportation and other expenses) that manufacturer il is faced with transacting with distributor $j\hat{l}$ in currency h. Let $c^{il}_{kh\bar{l}}$, in turn, denote the transaction cost (which also includes the cost of transportation and other expenses) that manufacturer il is faced with transacting directly with retailer k in currency h and country $\bar{l}$. These transaction costs may depend upon the volume of transactions between each such pair in a particular currency, and their form depends on the type of transaction. They are given, respectively, by:

$$c^{il}_{jh\hat{l}} = c^{il}_{jh\hat{l}}(q^{il}_{jh\hat{l}}), \quad \forall i, l, j, h, \hat{l} \tag{2a}$$

and

$$c^{il}_{kh\bar{l}} = c^{il}_{kh\bar{l}}(q^{il}_{kh\bar{l}}), \quad \forall i, l, k, h, \bar{l}. \tag{2b}$$

Obviously, not every product can be purchased and shipped over electronic distribution channels. For example, in some cases, the purchase may occur through the Internet but the delivery requires a physical (and not virtual) means of transport. The generality of the above transaction cost functions (2a) and (2b) can represent such combined (or aggregated) activities, as well.

The following conservation of flow equation must hold:

$$q^{il} = \sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} q^{il}_{jh\hat{l}} + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L} q^{il}_{kh\bar{l}}, \tag{3}$$

which states that the quantity of the product transacted by manufacturer il is equal to the amount produced by the manufacturer. For easy reference in the subsequent sections the product transactions between all pairs of manufacturers and distributors are grouped into the column vector denoted by $Q^1 \in R^{ILJHL}_+$. In addition, the product transactions between all pairs of manufacturers and retailers are grouped into the column vector denoted by $Q^2 \in R^{ILKHL}_+$. With this notation, one can express the production cost function of manufacturer il (cf. (1)) as a function of the vectors Q^1 and Q^2: $f^{il}(Q^1, Q^2)$.

It is assumed that each manufacturer seeks to maximize his profit which is the difference between his revenue and the total costs incurred. The revenue is equal to the product of the price of the product and the total quantity sold to all the distributors and all the retailers. Since we allow the transactions to take place in different currencies, the prices of the product in different currencies may be distinct. Let $\rho^{il*}_{1jh\hat{l}}$ denote the price associated with the product transacted between manufacturer il and distributor $j\hat{l}$ in currency h, and let $\rho^{il*}_{1kh\bar{l}}$ denote the price of the product associated with a transaction between manufacturer il and retailer k in currency h and country $\bar{l}$.

We now introduce the currency appreciation rate e_h, which is the appreciation (exchange) rate of currency h relative to the base currency and which is assumed to be known (see

Nagurney, Cruz, and Matsypura (2003) and Nagurney and Siokos (1997) for further details). This is necessary since the revenue of a given manufacturer needs to be expressed in a base currency. Hence, the total revenue of manufacturer il is given by:

$$\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}(\rho_{1jh\hat{l}}^{il*}\times e_h)q_{jh\hat{l}}^{il}+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}(\rho_{1kh\bar{l}}^{il*}\times e_h)q_{kh\bar{l}}^{il}.$$

The total costs incurred by the manufacturer il, in turn, are equal to the sum of the manufacturer's production costs and the total transaction costs. We assume that all the cost functions are in the base currency.

Hence, using the conservation of flow equation (3), the production cost functions, and the transaction cost functions (2a) and (2b), one can express the profit maximization criterion for manufacturer il as:

$$\text{Maximize} \quad \sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}(\rho_{1jh\hat{l}}^{il*}\times e_h)q_{jh\hat{l}}^{il}+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}(\rho_{1kh\bar{l}}^{il*}\times e_h)q_{kh\bar{l}}^{il}$$

$$-f^{il}(Q^1,Q^2)-\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}c_{jh\hat{l}}^{il}(q_{jh\hat{l}}^{il})-\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}c_{kh\bar{l}}^{il}(q_{kh\bar{l}}^{il}), \tag{4}$$

subject to: $q_{jh\hat{l}}^{il}\geq 0$, for all $j,h,\hat{l}$ and $q_{kh\bar{l}}^{il}\geq 0$, for all $k,h,\bar{l}$.

In addition to the criterion of profit maximization, we also assume that each manufacturer is concerned with risk minimization. Here, for the sake of generality, we assume, as given, a risk function r^{il}, for manufacturer i in country l, which is assumed to be continuous and convex and a function of not only the product transactions associated with the particular manufacturer but also of those of other manufacturers. Hence, we assume that

$$r^{il}=r^{il}(Q^1,Q^2), \quad \forall i,l. \tag{5}$$

Note that according to (5) the risk as perceived by a manufacturer is dependent not only upon his product transactions but also on those of other manufacturers. Hence, the second criterion of manufacturer il can be expressed as:

$$\text{Minimize} \quad r^{il}(Q^1,Q^2), \tag{6}$$

subject to: $q^{il}_{jh\hat{l}} \geq 0$, for all $j, h, \hat{l}$ and $q^{il}_{kh\bar{l}} \geq 0$, for all $k, h, \bar{l}$. The risk function may be distinct for each manufacturer/country combination and can assume whatever form is necessary.

The Multicriteria Decision-Making Problem for a Manufacturer in a Particular Country

Each manufacturer il associates a nonnegative weight α^{il} with the risk minimization criterion (6), with the weight associated with the profit maximization criterion (4) serving as the numeraire and being set equal to 1. Hence, we can construct a *value function* for each manufacturer (cf. Fishburn (1970), Chankong and Haimes (1983), Yu (1985), Keeney and Raiffa (1993)) using a constant additive weight value function. Consequently, the multicriteria decision-making problem for manufacturer il is transformed into:

$$\text{Maximize} \quad \sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}(\rho^{il*}_{1jh\hat{l}} \times e_h)q^{il}_{jh\hat{l}} + \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}(\rho^{il*}_{1kh\bar{l}} \times e_h)q^{il}_{kh\bar{l}}$$

$$-f^{il}(Q^1, Q^2) - \sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L} c^{il}_{jh\hat{l}}(q^{il}_{jh\hat{l}}) - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L} c^{il}_{kh\bar{l}}(q^{il}_{kh\bar{l}}) - \alpha^{il} r^{il}(Q^1, Q^2), \tag{7}$$

subject to: $q^{il}_{jh\hat{l}} \geq 0$, for all $j, h, \hat{l}$ and $q^{il}_{kh\bar{l}} \geq 0$, for all $k, h, \bar{l}$.

The manufacturers are assumed to compete in a noncooperative fashion. Also, it is assumed that the production cost functions and the transaction cost functions for each manufacturer are continuous and convex. The governing optimization/equilibrium concept underlying noncooperative behavior is that of Nash (1950, 1951), which states, in this context, that each manufacturer will determine his optimal production quantity and transactions, given the optimal ones of the competitors. Hence, the optimality conditions for all manufacturers *simultaneously* can be expressed as the following inequality (see also Gabay and Moulin (1980), Bazaraa, Sherali, and Shetty (1993), Nagurney (1999), Nagurney, Dong, and Zhang (2002)): determine the solution $(Q^{1*}, Q^{2*}) \in R^{IL(JHL+KHL)}_{+}$, which satisfies:

$$\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\frac{\partial f^{il}(Q^{1*}, Q^{2*})}{\partial q^{il}_{jh\hat{l}}} + \frac{\partial c^{il}_{jh\hat{l}}(q^{il*}_{jh\hat{l}})}{\partial q^{il}_{jh\hat{l}}} + \alpha^{il}\frac{\partial r^{il}(Q^{1*}, Q^{2*})}{\partial q^{il}_{jh\hat{l}}} - \rho^{il*}_{1jh\hat{l}} \times e_h\right] \times \left[q^{il}_{jh\hat{l}} - q^{il*}_{jh\hat{l}}\right]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\left[\frac{\partial f^{il}(Q^{1*}, Q^{2*})}{\partial q^{il}_{kh\bar{l}}} + \frac{\partial c^{il}_{kh\bar{l}}(q^{il*}_{kh\bar{l}})}{\partial q^{il}_{kh\bar{l}}} + \alpha^{il}\frac{\partial r^{il}(Q^{1*}, Q^{2*})}{\partial q^{il}_{kh\bar{l}}} - \rho^{il*}_{1kh\bar{l}} \times e_h\right]$$

$$\times \left[q^{il}_{kh\bar{l}} - q^{il*}_{kh\bar{l}}\right] \geq 0, \quad \forall (Q^1, Q^2) \in R_+^{IL(JHL+KHL)}. \tag{8}$$

The inequality (8), which is a *variational inequality* (cf. Nagurney (1999)) (for fixed prices and appreciation rates) has a nice economic interpretation. In particular, from the first term one can infer that, if there is a positive amount of the product transacted between a manufacturer and a distributor, then the marginal cost of production plus the marginal cost of transacting plus the weighted marginal risk associated with that transaction must be equal to the price (converted to the base currency) that the distributor is willing to pay for the product. If the marginal cost of production plus the marginal cost and the weighted marginal risk of transacting exceeds that price, then there will be zero volume of flow of the product between the two.

The second term in (8) has a similar interpretation; in particular, there will be a positive volume of transaction of the product from a manufacturer to a retailer if the marginal cost of production of the manufacturer plus the marginal cost of transacting with the retailer via the Internet and the weighted marginal risk is equal to the price (in the base currency) that the retailer is willing to pay for the product.

Note that, in the above framework, we explicitly allow each of the manufacturers to not only have distinct risk functions but also distinct weights associated with their respective risk functions. We emphasize that the study of risk has had a long and prominent history in the field of finance, dating to the work of Markowitz (1952, 1959), whereas the incorporation of risk into supply chain management has only been addressed fairly recently. Furthermore, we emphasize that by allowing the risk function to be general one can then construct an appropriate function pertaining to the specific situation and decision-maker. For example, in the field of finance, measurement of risk has included the use of variance-covariance matrices, yielding quadratic expressions for the risk (see also, e.g., Nagurney and Siokos (1997)). In addition, in finance, the bicriterion optimization problem of net revenue maximization and risk minimization is fairly standard (see also, e.g., Dong and Nagurney (2001)).

The Behavior of the Distributors and their Optimality Conditions

As was mentioned earlier, the distributors transact with both the manufacturers since they need to obtain the product for distribution, and with the retailers, who sell the product to the consumers. Similar to the manufacturers, and for the sake of modeling generality and flexibility, we assume that the distributors can transact in any of the H currencies.

Let $q^{j\hat{l}}_{kh\bar{l}}$ denote the amount of the product transacted between retailer $kh\bar{l}$ and distributor $j\hat{l}$ in currency h. We group these transaction quantities into the column vector $Q^3 \in R^{JLKHL}_+$.

A distributor $j\hat{l}$ is faced with certain expenses, which may include, for example, loading/unloading costs, storage costs, etc., associated with the product. We refer collectively to such costs as a *handling* cost and denote it by $c_{j\hat{l}}$. In a simple situation, one might have that

$$c_{j\hat{l}} = c_{j\hat{l}}(\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H} q^{il}_{jh\hat{l}}, \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L} q^{j\hat{l}}_{kh\bar{l}}), \tag{9a}$$

that is, the *handling* cost of a distributor is a function of how much of the product he has obtained and how much of the product he has transacted with the various retailers. However, for the sake of generality, and to enhance the modeling of competition, we follow Dong, Zhang, and Nagurney (2003) and allow the function to depend also on the amount of the product acquired and transacted by other distributors. Hence, we assume that, for all $j\hat{l}$:

$$c_{j\hat{l}} = c_{j\hat{l}}(Q^1, Q^3). \tag{9b}$$

Let $\rho^{j\hat{l}*}_{2kh\bar{l}}$ denote the price in currency h associated with the transaction between distributor $j\hat{l}$ and retailer $kh\bar{l}$. This price, as will be shown, will be endogenously determined in the model and will be, in the case of a positive volume of flow between a distributor-retailer pair, equal to a clearing-type price. The total amount of revenue the distributor obtains from his transactions is equal to the sum of the price (transformed into the base currency) and the amount of the product transacted with the various retailers in the distinct countries and currencies. Indeed, since transactions can be made in distinct currencies and with different retailers, who, in fact, may even be virtual, the total revenue of distributor $j\hat{l}$ can be expressed in the base currency as follows:

$$\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}(\rho^{j\hat{l}*}_{2kh\bar{l}} \times e_h) q^{j\hat{l}}_{kh\bar{l}}. \tag{10}$$

Assuming that the distributors are profit-maximizers, the profit maximization problem for distributor $j\hat{l}$ can be expressed as:

$$\text{Maximize} \quad \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}(\rho_{2kh\bar{l}}^{j\hat{l}*} \times e_h)q_{kh\bar{l}}^{j\hat{l}} - c_{j\hat{l}}(Q^1, Q^3) - \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}(\rho_{1jh\hat{l}}^{il*} \times e_h)q_{jh\hat{l}}^{il} \tag{11}$$

subject to:

$$\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L} q_{kh\bar{l}}^{j\hat{l}} \leq \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H} q_{jh\hat{l}}^{il}, \tag{12}$$

and the nonnegativity assumptions: $q_{jh\hat{l}}^{il} \geq 0$, for all $i, \bar{l}, h$ and $q_{kh\bar{l}}^{j\hat{l}} \geq 0$, for all $k, h, \bar{l}$.

Constraint (12) states that a distributor in a country cannot sell more of the product than he has obtained from the various manufacturers.

In addition, each distributor seeks to also minimize his risk associated with obtaining and shipping the product to the various retailers. Each distributor $j\hat{l}$ is faced with his own individual risk denoted by $r^{j\hat{l}}$ with the function being assumed to be continuous and convex and dependent on the transactions to and from all the distributors, that is,

$$r^{j\hat{l}} = r^{j\hat{l}}(Q^1, Q^3), \quad \forall j, \hat{l}. \tag{13}$$

The Multicriteria Decision-Making Problem for a Distributor in a Particular Country

We assume that each distributor associates a weight of 1 with the profit criterion (11) and a weight of $\beta^{j\hat{l}}$ with his risk level. Therefore, the multicriteria decision-making problem for distributor $j\hat{l}$; $j = 1, \ldots, J$; $\hat{l} = 1, \ldots, L$, can be transformed directly into the optimization problem:

$$\text{Maximize} \quad \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}(\rho_{2kh\bar{l}}^{j\hat{l}*} \times e_h)q_{kh\bar{l}}^{j\hat{l}} - c_{j\hat{l}}(Q^1, Q^3) - \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}(\rho_{1jh\hat{l}}^{il*} \times e_h)q_{jh\hat{l}}^{il} - \beta^{j\hat{l}}r^{j\hat{l}}(Q^1, Q^3) \tag{14}$$

subject to:

$$\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L} q_{kh\bar{l}}^{j\hat{l}} \leq \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H} q_{jh\hat{l}}^{il}, \tag{15}$$

and the nonnegativity constraints: $q_{jh\hat{l}}^{il} \geq 0$, and $q_{kh\bar{l}}^{j\hat{l}} \geq 0$, for all i, h, k, l, and $\bar{l}$.

Objective function (14) represents a value function for distributor $j\hat{l}$ with $\beta^{j\hat{l}}$ having the interpretation as a conversion rate in dollar value.

Here it is assumed that the distributors compete in a noncooperative manner, given the actions of the other distributors. Note that, at this point, we consider that the distributors seek to determine not only the optimal amounts purchased by the retailers, but, also, the amount that they wish to obtain from the manufacturers. In equilibrium, all the transactions between the tiers of global supply chain network will have to coincide.

Assuming that the handling cost $c_{j\hat{l}}$ for each distributor is a continuous and convex function, the optimality conditions of the distributors *simultaneously* can be stated as the following variational inequality: determine the solution $(Q^{1*}, Q^{3*}, \gamma^*) \in R_+^{JL(ILH+KHL+1)}$, which satisfies:

$$\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\left[\frac{\partial c_{j\hat{l}}(Q^{1*},Q^{3*})}{\partial q_{jh\hat{l}}^{il}} + \beta^{j\hat{l}}\frac{\partial r^{j\hat{l}}(Q^{1*},Q^{3*})}{\partial q_{jh\hat{l}}^{il}} + \rho_{1jh\hat{l}}^{il*} \times e_h - \gamma_{j\hat{l}}^*\right] \times \left[q_{jh\hat{l}}^{il} - q_{jh\hat{l}}^{il*}\right]$$

$$+\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\left[-\rho_{2kh\bar{l}}^{j\hat{l}*} \times e_h + \frac{\partial c_{j\hat{l}}(Q^{1*},Q^{3*})}{\partial q_{kh\bar{l}}^{j\hat{l}}} + \beta^{j\hat{l}}\frac{\partial r^{j\hat{l}}(Q^{1*},Q^{3*})}{\partial q_{kh\bar{l}}^{j\hat{l}}} + \gamma_{j\hat{l}}^*\right] \times \left[q_{kh\bar{l}}^{j\hat{l}} - q_{kh\bar{l}}^{j\hat{l}*}\right]$$

$$+\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}\left[\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H} q_{jh\hat{l}}^{il*} - \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L} q_{kh\bar{l}}^{j\hat{l}*}\right] \times \left[\gamma_{j\hat{l}} - \gamma_{j\hat{l}}^*\right] \geq 0,$$

$$\forall (Q^1, Q^3, \gamma) \in R_+^{JL(ILH+KHL+1)}, \tag{16}$$

where $\gamma_{j\hat{l}}$ is the Lagrange multiplier associated with constraint (12) for distributor $j\hat{l}$ and γ is the column vector of all the distributors' multipliers. In inequality (16), as in inequality (8), the prices charged are not variables.

The economic interpretation of the distributors' optimality conditions is now highlighted. From the second term in inequality (16), one has that, if retailer $kh\bar{l}$ purchases the product from a distributor $j\hat{l}$, that is, if the $q_{kh\bar{l}}^{j\hat{l}*}$ is positive, then the price charged by retailer $j\hat{l}$, converted into the base currency, $\rho_{2kh\bar{l}}^{j\hat{l}*} \times e_h$, is equal to the marginal handling cost plus the weighted marginal risk, plus the shadow price term $\gamma_{j\hat{l}}^*$, which, from the third term in the inequality, serves as the price to clear the market from distributor $j\hat{l}$. Furthermore, from the first term in inequality (16), one can infer that, if a manufacturer transacts with a distributor resulting in a positive flow of the product between the two, then the price $\gamma^*_{j\hat{l}}$ is precisely equal to distributor $j\hat{l}$'s payment to the manufacturer in a certain currency h transformed into the base currency, $\rho^{il*}_{1jh\hat{l}} \times e_h$, plus his marginal cost of handling the product associated with transacting with the particular manufacturer plus the weighted marginal risk associated with the transaction.

In the above derivations, we have considered supply-side risk from the perspective of the manufacturers as well as from the distributors. We now turn to the retailers and consider demand-side risk, which is modeled in a stochastic manner to represent uncertainty. Clearly, if the demands were known with certainty there would be no risk associated with them and the retailers and manufacturers could make their production and distribution decisions accordingly.

The Retailers and their Optimality Conditions

The retailers, in turn, must decide how much to order from the distributors and from the manufacturers in order to cope with the random demand while still seeking to maximize their profits. A retailer $kh\bar{l}$ is also faced with what we term a *handling* cost, which may include, for example, the display and storage cost associated with the product. We denote this cost by $c_{kh\bar{l}}$ and, in the simplest case, we would have that $c_{kh\bar{l}}$ is a function of $s_{kh\bar{l}} = \sum_{i=1}^{I}\sum_{l=1}^{L} q^{il}_{kh\bar{l}} + \sum_{j=1}^{J}\sum_{\hat{l}=1}^{L} q^{j\hat{l}}_{kh\bar{l}}$, that is, the holding cost of a retailer is a function of how much of the product he has obtained from transactions with the various manufacturers directly through orders via the Internet and from the various distributors. However, for the sake of generality, and to enhance the modeling of competition, we allow this function to depend, in general, also on the amounts of the product held by other retailers and, therefore, we may write:

$$c_{kh\bar{l}} = c_{kh\bar{l}}(Q^2, Q^3), \quad \forall k, h, \bar{l}. \tag{17}$$

Let $\rho_{3kh\bar{l}}$ denote the demand price of the product associated with retailer $kh\bar{l}$. We assume that $\hat{d}_{kh\bar{l}}(\rho_{3kh\bar{l}})$ is the demand for the product at the demand price of $\rho_{3kh\bar{l}}$ at retail outlet $kh\bar{l}$, where $\hat{d}_{kh\bar{l}}(\rho_{3kh\bar{l}})$ is a random variable with a density function of $F_{kh\bar{l}}(x, \rho_{3kh\bar{l}})$, with $\rho_{3kh\bar{l}}$ serving as a parameter. Hence, we assume that the density function may vary with the demand price. Let $P_{kh\bar{l}}$ be the probability distribution function of $\hat{d}_{kh\bar{l}}(\rho_{3kh\bar{l}})$, that is,

$P_{kh\bar{l}}(x, \rho_{3kh\bar{l}}) = P_{kh\bar{l}}(\hat{d}_{kh\bar{l}} \leq x) = \int_0^x F_{kh\bar{l}}(x, \rho_{3kh\bar{l}})dx$.

Retailer $kh\bar{l}$ can sell to the consumers no more than the minimum of his supply or his demand, that is, the actual sale of the product at retailer $kh\bar{l}$ cannot exceed $\min\{s_{kh\bar{l}}, \hat{d}_{kh\bar{l}}\}$. Let

$$\Delta^+_{kh\bar{l}} \equiv \max\{0, s_{kh\bar{l}} - \hat{d}_{kh\bar{l}}\} \tag{18}$$

and

$$\Delta^-_{kh\bar{l}} \equiv \max\{0, \hat{d}_{kh\bar{l}} - s_{kh\bar{l}}\}, \tag{19}$$

where $\Delta^+_{kh\bar{l}}$ is a random variable representing the excess supply (inventory), whereas $\Delta^-_{kh\bar{l}}$ is a random variable representing the excess excess demand (shortage).

Note that the expected values of excess supply and excess demand of retailer $kh\bar{l}$ are scalar functions of $s_{kh\bar{l}}$ and $\rho_{3kh\bar{l}}$. In particular, let $\pi^+_{kh\bar{l}}$ and $\pi^-_{kh\bar{l}}$ denote, respectively, the expected values $E(\Delta^+_{kh\bar{l}})$ and $E(\Delta^-_{kh\bar{l}})$, that is,

$$\pi^+_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}}) \equiv E(\Delta^+_{kh\bar{l}}) = \int_0^{s_{kh\bar{l}}} (s_{kh\bar{l}} - x)F_{kh\bar{l}}(x, \rho_{3kh\bar{l}})dx, \tag{20}$$

and

$$\pi^-_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}}) \equiv E(\Delta^-_{kh\bar{l}}) = \int_{s_{kh\bar{l}}}^{\infty} (x - s_{kh\bar{l}})F_{kh\bar{l}}(x, \rho_{3kh\bar{l}})dx. \tag{21}$$

Assume that retailer $kh\bar{l}$ is faced with certain penalties for having an excess or shortage in regards to the supply. Let $\lambda^+_{kh\bar{l}} \geq 0$ denote the unit penalty of having excess supply at retail outlet $kh\bar{l}$, and let $\lambda^-_{kh\bar{l}} \geq 0$ denote the unit penalty of having excess demand at outlet $kh\bar{l}$. The penalties are assumed to be in the base currency. Then the expected total penalty of retailer $kh\bar{l}$ can be expressed as:

$$E(\lambda^+_{kh\bar{l}}\Delta^+_{kh\bar{l}} + \lambda^-_{kh\bar{l}}\Delta^-_{kh\bar{l}}) = \lambda^+_{kh\bar{l}}\pi^+_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}}) + \lambda^-_{kh\bar{l}}\pi^-_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}}).$$

Assuming profit-maximizing behavior of the retailers, one can state the following optimization problem for retailer $kh\bar{l}$:

$$\text{Maximize} \quad E((\rho^*_{3kh\bar{l}} \times e_h) \cdot \min\{s_{kh\bar{l}}, \hat{d}_{kh\bar{l}}\}) - E(\lambda^+_{kh\bar{l}}\Delta^+_{kh\bar{l}} + \lambda^-_{kh\bar{l}}\Delta^-_{kh\bar{l}})$$

$$+c_{kh\bar{l}}(Q^2, Q^3) - \sum_{i=1}^{I}\sum_{l=1}^{L}(\rho^{il*}_{1kh\bar{l}} \times e_h)q^{il}_{kh\bar{l}} - \sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}(\rho^{j\hat{l}*}_{2kh\bar{l}} \times e_h)q^{j\hat{l}}_{kh\bar{l}}, \tag{22}$$

subject to: $q^{il}_{kh\bar{l}} \geq 0$, $q^{j\hat{l}}_{kh\bar{l}} \geq 0$, for all $i, l, j, \hat{l}$.

Objective function (22) expresses that the expected profit of retailer $kh\bar{l}$, which is the difference between the expected revenues and the sum of the expected penalty, the handling cost, and the payouts to the manufacturers and to the distributors, should be maximized.

Applying now the definitions of $\Delta^{+}_{kh\bar{l}}$, and $\Delta^{-}_{kh\bar{l}}$, we know that $\min\{s_{kh\bar{l}}, \hat{d}_{kh\bar{l}}\} = \hat{d}_{kh\bar{l}} - \Delta^{-}_{kh\bar{l}}$. Therefore, the objective function (22) can be expressed as

$$\text{Maximize} \quad (\rho^*_{3kh\bar{l}} \times e_h) d_{kh\bar{l}}(\rho^*_{3kh\bar{l}}) - (\rho^*_{3kh\bar{l}} \times e_h + \lambda^{-}_{kh\bar{l}})\pi^{-}_{kh\bar{l}}(s_{kh\bar{l}}, \rho^*_{3kh\bar{l}}) - \lambda^{+}_{kh\bar{l}}\pi^{+}_{kh\bar{l}}(s_{kh\bar{l}}, \rho^*_{3kh\bar{l}})$$

$$-c_{kh\bar{l}}(Q^2, Q^3) - \sum_{i=1}^{I}\sum_{l=1}^{L}(\rho^{il*}_{1kh\bar{l}} \times e_h) q^{il}_{kh\bar{l}} - \sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}(\rho^{j\hat{l}*}_{2kh\bar{l}} \times e_h) q^{j\hat{l}}_{kh\bar{l}}, \tag{23}$$

where $d_{kh\bar{l}}(\rho_{3kh\bar{l}}) \equiv E(\hat{d}_{kh\bar{l}})$ is a scalar function of $\rho_{3kh\bar{l}}$.

We now consider the optimality conditions of the retailers assuming that each retailer is faced with the optimization problem (23), subject to the nonnegativity assumption on the variables. Here, we also assume that the retailers compete in a noncooperative manner so that each maximizes his profits, given the actions of the other retailers. Note that, at this point, we consider that retailers seek to determine the amount that they wish to obtain from the manufacturers and from the distributors. First, however, we make the following derivation and introduce the necessary notation:

$$\frac{\partial \pi^{+}_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}})}{\partial q^{il}_{kh\bar{l}}} = \frac{\partial \pi^{+}_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}})}{\partial q^{j\hat{l}}_{kh\bar{l}}} = P_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}}) = P_{kh\bar{l}}(\sum_{i=1}^{I}\sum_{l=1}^{L} q^{il}_{kh\bar{l}} + \sum_{j=1}^{J}\sum_{\hat{l}=1}^{L} q^{j\hat{l}}_{kh\bar{l}}, \rho_{3kh\bar{l}}) \tag{24}$$

$$\frac{\partial \pi^{-}_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}})}{\partial q^{il}_{kh\bar{l}}} = \frac{\partial \pi^{-}_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}})}{\partial q^{j\hat{l}}_{kh\bar{l}}} = P_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}}) - 1$$

$$= P_{kh\bar{l}}(\sum_{i=1}^{I}\sum_{l=1}^{L} q^{il}_{kh\bar{l}} + \sum_{j=1}^{J}\sum_{\hat{l}=1}^{L} q^{j\hat{l}}_{kh\bar{l}}, \rho_{3kh\bar{l}}) - 1. \tag{25}$$

Assuming that the handling cost for each retailer is continuous and convex, then the optimality conditions for all the retailers satisfy the variational inequality, with $\rho^*_{3kh\bar{l}}$ denoting the equilibrium price at demand market $kh\bar{l}$: determine $(Q^{2*}, Q^{3*}) \in R_+^{(IL+JL)KHL}$, satisfying:

$$\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\Bigg\{\sum_{i=1}^{I}\sum_{l=1}^{L}\Big[\lambda^{+}_{kh\bar{l}} P_{kh\bar{l}}(s^*_{kh\bar{l}}, \rho^*_{3kh\bar{l}}) - (\lambda^{-}_{kh\bar{l}} + \rho^*_{3kh\bar{l}} \times e_h)(1 - P_{kh\bar{l}}(s^*_{kh\bar{l}}, \rho^*_{3kh\bar{l}}))$$

$$+\frac{\partial c_{kh\bar{l}}(Q^{2*},Q^{3*})}{\partial q^{il}_{kh\bar{l}}} + \rho^{il*}_{1kh\bar{l}} \times e_h\Bigg] \times \left[q^{il}_{kh\bar{l}} - q^{il*}_{kh\bar{l}}\right]$$

$$+\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}\Bigg[\lambda^{+}_{kh\bar{l}} P_{kh\bar{l}}(s^*_{kh\bar{l}},\rho^*_{3kh\bar{l}}) - (\lambda^{-}_{kh\bar{l}} + \rho^*_{3kh\bar{l}} \times e_h)(1 - P_{kh\bar{l}}(s^*_{kh\bar{l}},\rho^*_{3kh\bar{l}})) + \frac{\partial c_{kh\bar{l}}(Q^{2*},Q^{3*})}{\partial q^{j\hat{l}}_{kh\bar{l}}}$$

$$+\rho^{j\hat{l}*}_{2kh\bar{l}} \times e_h\Bigg] \times \left[q^{j\hat{l}}_{kh\bar{l}} - q^{j\hat{l}*}_{kh\bar{l}}\right]\Bigg\} \geq 0, \quad \forall (Q^2,Q^3) \in R_+^{(IL+JL)KHL}. \tag{26}$$

In this derivation, as in the derivation of inequalities (8) and (16), we have not had the prices charged be variables. They become endogenous variables in the integrated global supply chain network equilibrium model. A similar derivation but in the absence of electronic commerce (and in the case of only a two-tiered rather than a three-tiered supply chain network) was obtained in Dong, Zhang, and Nagurney (2004). See Dong, Zhang, and Nagurney (2003) for a three-tiered (single-country and currency) supply chain network model with random demands.

We now highlight the economic interpretation of the retailers' optimality conditions. In inequality (26), we can infer that, if a manufacturer il transacts with a retailer $kh\bar{l}$ resulting in a positive flow of the product between the two, then the price at retail outlet $kh\bar{l}$, $\rho^*_{3kh\bar{l}}$, with the probability of $(1-P_{kh\bar{l}}(\sum_{i=1}^{I}\sum_{l=1}^{L} q^{il*}_{kh\bar{l}} + \sum_{j=1}^{J}\sum_{\hat{l}=1}^{L} q^{j\hat{l}*}_{kh\bar{l}}, \rho^*_{3kh\bar{l}}))$, that is, when the demand is not less then the total order quantity, is precisely equal to the retailer $kh\bar{l}$'s payment to the manufacturer, $\rho^{il*}_{1kh\bar{l}} \times e_h$, plus his marginal cost of handling the product and the penalty of having excess demand with probability of $P_{kh\bar{l}}(\sum_{i=1}^{I}\sum_{l=1}^{L} q^{il*}_{kh\bar{l}} + \sum_{j=1}^{J}\sum_{\hat{l}=1}^{L} q^{j\hat{l}*}_{kh\bar{l}}, \rho^*_{3kh\bar{l}})$, (which is the probability when actual demand is less than the order quantity), subtracted by the penalty of having shortage with probability of $(1-P_{kh\bar{l}}(\sum_{i=1}^{I}\sum_{l=1}^{L} q^{il*}_{kh\bar{l}} + \sum_{j=1}^{J}\sum_{\hat{l}=1}^{L} q^{j\hat{l}*}_{kh\bar{l}}, \rho^*_{3kh\bar{l}}))$ (when the actual demand is greater than the order quantity).

Similarly, if a distributor $j\hat{l}$ transacts with a retailer $kh\bar{l}$ resulting in a positive flow of the product between the two, then the selling price at retail outlet $kh\bar{l}$, $\rho^*_{3kh\bar{l}}$, with the probability of $(1-P_{kh\bar{l}}(\sum_{i=1}^{I}\sum_{l=1}^{L} q^{il*}_{kh\bar{l}} + \sum_{j=1}^{J}\sum_{\hat{l}=1}^{L} q^{j\hat{l}*}_{kh\bar{l}}, \rho^*_{3kh\bar{l}}))$, that is, when the demand is not less then the total order quantity, is precisely equal to the retailer k's payment to the manufacturer, $\rho^{j\hat{l}*}_{2kh\bar{l}} \times e_h$, plus his marginal cost of handling the product and the penalty of having excess demand with probability of $P_{kh\bar{l}}(\sum_{i=1}^{I}\sum_{l=1}^{L} q^{il*}_{kh\bar{l}} + \sum_{j=1}^{J}\sum_{\hat{l}=1}^{L} q^{j\hat{l}*}_{kh\bar{l}}, \rho^*_{3kh\bar{l}})$, (which is the probability when actual demand is less than the order quantity), subtracted by the penalty of having shortage with probability of $(1-P_{kh\bar{l}}(\sum_{i=1}^{I}\sum_{l=1}^{L}q_{kh\bar{l}}^{il*}+\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}q_{kh\bar{l}}^{j\hat{l}*},\rho_{3kh\bar{l}}^{*}))$ (when the actual demand is greater than the order quantity).

The Equilibrium Conditions

We now turn to a discussion of the market equilibrium conditions. Subsequently, we construct the equilibrium conditions for the entire global supply chain network.

The equilibrium conditions associated with the transactions that take place between the retailers and the consumers are the stochastic economic equilibrium conditions, which, mathematically, take on the following form: for $k,h,\bar{l}$; $k=1,...,K$; $h=1,...,H$; $\bar{l}=1,...,L$:

$$\hat{d}_{kh\bar{l}}(\rho_{3kh\bar{l}}^{*})\begin{cases} =\sum_{i=1}^{I}\sum_{l=1}^{L}q_{kh\bar{l}}^{il*}+\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}q_{kh\bar{l}}^{j\hat{l}*} & \textbf{a.e.} \quad \text{if } \rho_{3kh\bar{l}}^{*}>0 \\ \leq\sum_{i=1}^{I}\sum_{l=1}^{L}q_{kh\bar{l}}^{il*}+\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}q_{kh\bar{l}}^{j\hat{l}*} & \textbf{a.e.} \quad \text{if } \rho_{3kh\bar{l}}^{*}=0, \end{cases} \tag{27}$$

where **a.e.** means that the corresponding equality or inequality holds almost everywhere.

Conditions (27) state that, if the demand price at outlet $kh\bar{l}$ is positive, then the quantities purchased by the retailer from the manufacturers and from the distributors in the aggregate are equal to the demand, with exceptions of zero probability. These conditions correspond to the well-known economic equilibrium conditions (cf. Nagurney (1999) and the references therein). Related equilibrium conditions, but without electronic transactions allowed, and in a single country context, were first proposed in Dong, Zhang, and Nagurney (2004a).

Equilibrium conditions (27) are equivalent to the following variational inequality problem, after taking the expected value of the demand and summing over all retailers $kh\bar{l}$: determine $\rho_3^{*}\in R_{+}^{KHL}$ satisfying:

$$\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\left[\sum_{i=1}^{I}\sum_{l=1}^{L}q_{kh\bar{l}}^{il*}+\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}q_{kh\bar{l}}^{j\hat{l}}-d_{kh\bar{l}}(\rho_{3kh\bar{l}}^{*})\right]\times[\rho_{3kh\bar{l}}-\rho_{3kh\bar{l}}^{*}]\geq 0, \qquad \forall\rho_3\in R_{+}^{KHL}, \tag{28}$$

where ρ_3 is the KHL-dimensional column vector with components: $\rho_{3111},...,\rho_{3kh\bar{l}}$.

The Equilibrium Conditions of the Global Supply Chain

In equilibrium, we must have that the sum of the optimality conditions for all manufacturers, as expressed by inequality (8), the optimality conditions of the distributors, as expressed by condition (16), the optimality conditions for all retailers, as expressed by inequality (26), and the market equilibrium conditions, as expressed by inequality (28) must be satisfied. Hence, the product transactions from the manufacturers to the retailers must be equal to the producttransactions that the retailers accept from the manufacturers. In addition, the product transactions from the manufacturers to the distributors, must be equal to those accepted by the distributors, and, finally, the product transactions from the distributors to the retailers must coincide with those accepted by the retailers. We state this explicitly in the following definition:

Definition 1: Global Supply Chain Network Equilibrium with Supply-Side and Demand-Side Risk

The equilibrium state of the global supply chain network with supply- and demand-side risk is one where the product transactions between the tiers of the decision-makers coincide and the product transactions and prices satisfy the sum of the optimality conditions (8), (16), and (26), and the conditions (28).

The summation of inequalities (8), (16), (26), and (28) (with the prices at the manufacturers, the distributors, and at the retailers denoted, respectively, by their values at the equilibrium, after algebraic simplification, yields the following result:

Theorem 1: Variational Inequality Formulation

A product transaction and price pattern $(Q^{1*}, Q^{2*}, Q^{3*}, \gamma^*, \rho_3^*) \in \mathcal{K}$ *is an equilibrium pattern of the global supply chain network model according to Definition 1 if and only if it satisfies the variational inequality problem:*

$$\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\frac{\partial f^{il}(Q^{1*},Q^{2*})}{\partial q^{il}_{jh\hat{l}}}+\frac{\partial c^{il}_{jh\hat{l}}(q^{il*}_{jh\hat{l}})}{\partial q^{il}_{jh\hat{l}}}+\frac{\partial c_{j\hat{l}}(Q^{1*},Q^{3*})}{\partial q^{il}_{jh\hat{l}}}+\alpha^{il}\frac{\partial r^{il}(Q^{1*},Q^{2*})}{\partial q^{il}_{jh\hat{l}}}\right.$$

$$+\beta^{j\hat{l}}\frac{\partial r^{j\hat{l}}(Q^{1*},Q^{3*})}{\partial q^{il}_{jh\hat{l}}}-\gamma^*_{j\hat{l}}\Bigg]\times\left[q^{il}_{jh\hat{l}}-q^{il*}_{jh\hat{l}}\right]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}[\frac{\partial f^{il}(Q^{1*},Q^{2*})}{\partial q^{il}_{kh\bar{l}}}+\frac{\partial c^{il}_{kh\bar{l}}(q^{il*}_{kh\bar{l}})}{\partial q^{il}_{kh\bar{l}}}+\frac{\partial c_{kh\bar{l}}(Q^{2*},Q^{3*})}{\partial q^{il}_{kh\bar{l}}}+\alpha^{il}\frac{\partial r^{il}(Q^{1*},Q^{2*})}{\partial q^{il}_{kh\bar{l}}}$$

$$+\lambda^+_{kh\bar{l}}P_{kh\bar{l}}(s^*_{kh\bar{l}},\rho^*_{3kh\bar{l}})-(\lambda^-_{kh\bar{l}}+\rho^*_{3kh\bar{l}}\times e_h)(1-P_{kh\bar{l}}(s^*_{kh\bar{l}},\rho^*_{3kh\bar{l}}))]\times\left[q^{il}_{kh\bar{l}}-q^{il*}_{kh\bar{l}}\right]$$

$$+\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\Big[\lambda^+_{kh\bar{l}}P_{kh\bar{l}}(s^*_{kh\bar{l}},\rho^*_{3kh\bar{l}})-(\lambda^-_{kh\bar{l}}+\rho^*_{3kh\bar{l}}\times e_h)(1-P_{kh\bar{l}}(s^*_{kh\bar{l}},\rho^*_{3kh\bar{l}}))$$

$$+\frac{\partial c_{j\hat{l}}(Q^{1*},Q^{3*})}{\partial q^{j\hat{l}}_{kh\bar{l}}}+\frac{\partial c_{kh\bar{l}}(Q^{2*},Q^{3*})}{\partial q^{j\hat{l}}_{kh\bar{l}}}+\beta^{j\hat{l}}\frac{\partial r^{j\hat{l}}(Q^{1*},Q^{3*})}{\partial q^{j\hat{l}}_{kh\bar{l}}}+\gamma^*_{j\hat{l}}\Bigg]\times\left[q^{j\hat{l}}_{kh\bar{l}}-q^{j\hat{l}*}_{kh\bar{l}}\right]$$

$$+\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\sum_{i=1}^{I}\sum_{l=1}^{L}q^{il*}_{jh\hat{l}}-\sum_{k=1}^{K}\sum_{\bar{l}=1}^{L}q^{j\hat{l}*}_{kh\bar{l}}\right]\times\left[\gamma_{j\hat{l}}-\gamma^*_{j\hat{l}}\right]$$

$$+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\left[\sum_{i=1}^{I}\sum_{l=1}^{L}q^{il*}_{kh\bar{l}}+\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}q^{j\hat{l}}_{kh\bar{l}}-d_{kh\bar{l}}(\rho^*_{3kh\bar{l}})\right]\times[\rho^*_{3kh\bar{l}}-\rho^*_{3kh\bar{l}}]\geq 0, \tag{29}$$

$$\forall(Q^1,Q^2,Q^3,\gamma,\rho_3)\in\mathcal{K},$$

where $\mathcal{K}\equiv\{(Q^1,Q^2,Q^3,\gamma,\rho_3)|(Q^1,Q^2,Q^3,\gamma,\rho_3)\in R_+^{IL(JHL+KHL)+JL(KHL+1)+KHL}\}$.

For easy reference in the subsequent sections, variational inequality problem (29) can be rewritten in standard variational inequality form (cf. Nagurney (1999)) as follows: determine $X^*\in\mathcal{K}$ satisfying:

$$\langle F(X^*),X-X^*\rangle\geq 0,\quad \forall X\in\mathcal{K}\equiv R_+^{IL(JHL+KHL)+JL(KHL+1)+KHL}, \tag{30}$$

where $X\equiv(Q^1,Q^2,Q^3,\gamma,\rho_3)$, and

$$F(X)\equiv(F^{il}_{jh\hat{l}},F^{il}_{kh\bar{l}},F^{j\hat{l}}_{kh\bar{l}},F_{j\hat{l}},F_{kh\bar{l}})_{i=1,\ldots,I;l=\hat{l}=\bar{l}=1,\ldots,L;j=1,\ldots,J;h=1,\ldots,H;k=1,\ldots,K},$$

with the specific components of F being given by the functional terms preceding the multiplication signs in (29). The term $\langle\cdot,\cdot\rangle$ denotes the inner product in N-dimensional Euclidean space.

Note that the variables in the model (and which can be determined from the solution of either variational inequality (29) or (30)) are: the equilibrium product transactions between manufacturers and the distributors given by Q^{1*}, the equilibrium product transactions transacted electronically between the manufacturers and the retailers denoted by Q^{2*}, and the equilibrium product transactions between the distributors and the retailers given by Q^{3*}, as well as the equilibrium demand prices ρ_3^* and the equilibrium shadow prices γ^*. We now discuss how to recover the prices ρ_1^* associated with the top tier of nodes of the global supply chain network and the prices ρ_2^* associated with the middle tier.

First, note that from (8), we have that (as already discussed briefly) that if $q_{jh\hat{l}}^{il*} > 0$, then the price $\rho_{1jh\hat{l}}^{il*} \times e_h = \frac{\partial f^{il}(Q^{1*},Q^{2*})}{\partial q_{jh\hat{l}}^{il}} + \frac{\partial c_{jh\hat{l}}^{il}(q_{jh\hat{l}}^{il*})}{\partial q_{jh\hat{l}}^{il}} + \alpha^{il} \frac{\partial r^{il}(Q^{1*},Q^{2*})}{\partial q_{jh\hat{l}}^{il}}$. Also, from (8) it follows that if $q_{kh\bar{l}}^{il*} > 0$, then the price $\rho_{1kh\bar{l}}^{il*} \times e_h = \frac{\partial f^{il}(Q^{1*},Q^{2*})}{\partial q_{kh\bar{l}}^{il}} + \frac{\partial c_{kh\bar{l}}^{il}(q_{kh\bar{l}}^{il*})}{\partial q_{kh\bar{l}}^{il}} + \alpha^{il} \frac{\partial r^{il}(Q^{1*},Q^{2*})}{\partial q_{kh\bar{l}}^{il}}$. Hence, the product is priced at the manufacturer's level according to whether it has been transacted physically or electronically; and also according to the distributor or retailer with whom the transaction has taken place. On the other hand, from (16) it follows that if $q_{kh\bar{l}}^{j\hat{l}*} > 0$, then $\rho_{2kh\bar{l}}^{j\hat{l}*} \times e_h = \frac{\partial c_{j\hat{l}}(Q^{1*},Q^{3*})}{\partial q_{kh\bar{l}}^{j\hat{l}}} + \gamma_{j\hat{l}}^* + \beta^{j\hat{l}} \frac{\partial r^{j\hat{l}}(Q^{1*},Q^{3*})}{\partial q_{kh\bar{l}}^{j\hat{l}}}$.

3. Qualitative Properties

In this Section, we provide some qualitative properties of the solution to variational inequality (29) (equivalently, variational inequality (30)). In particular, we derive existence and uniqueness results. We also investigate properties of the function F (cf. (30)) that enters the variational inequality of interest here.

Since the feasible set is not compact we cannot derive existence simply from the assumption of continuity of the functions. Nevertheless, we can impose a rather weak condition to guarantee existence of a solution pattern.

Let

$$\mathcal{K}_b = \{(Q^1, Q^2, Q^3, \gamma, \rho_3) | 0 \leq Q^m \leq b_m, m = 1, 2, 3; 0 \leq \gamma \leq b_4; 0 \leq \rho_3 \leq b_5\}, \tag{31}$$

where $b = (b_1, ..., b_5) \geq 0$ and $Q^m \leq b_m$; $\gamma \leq b_4$; $\rho_3 \leq b_5$ means that $q^{il}_{jh\hat{l}} \leq b_1$, $q^{il}_{kh\bar{l}} \leq b_2$, $q^{j\hat{l}}_{kh\bar{l}} \leq b_3$, and $\gamma_{j\hat{l}} \leq b_4$, $\rho_{3kh\bar{l}} \leq b_5$ for all i, l, $\hat{l}$, $\bar{l}$, j, k, h. Then $\mathcal{K}_b$ is a bounded closed convex subset of $R_+^{IL(JHL+KHL)+JL(KHL+1)+KHL}$. Thus the following variational inequality

$$\langle F(X^b), X - X^b \rangle \geq 0, \quad \forall X^b \in \mathcal{K}_b, \tag{32}$$

admits at least one solution $X^b \in \mathcal{K}_b$, since $\mathcal{K}_b$ is compact and F is continuous. Following Kinderlehrer and Stampacchia (1980)(see also Theorem 1.5 in Nagurney (1999)),we then have:

Theorem 2

Variational inequality (29) admits a solution if and only if there exists a $b > 0$, such that variational inequality (32) admits a solution in $\mathcal{K}_b$ with

$$Q^{1b} < b_1; \quad Q^{2b} < b_2; \quad Q^{3b} < b_3; \quad \gamma^b < b_4; \quad \rho_3^b < b_5 \tag{33}$$

Theorem 3: Existence

Suppose that there exist positive constants M, N, R with $R > 0$, such that:

$$\frac{\partial f^{il}(Q^1, Q^2)}{\partial q^{il}_{jh\hat{l}}} + \frac{\partial c^{il}_{jh\hat{l}}(q^{il}_{jh\hat{l}})}{\partial q^{il}_{jh\hat{l}}} + \frac{\partial c_{j\hat{l}}(Q^1, Q^3)}{\partial q^{il}_{jh\hat{l}}} + \alpha^{il}\frac{\partial r^{il}(Q^1, Q^2)}{\partial q^{il}_{jh\hat{l}}} + \beta^{j\hat{l}}\frac{\partial r^{j\hat{l}}(Q^1, Q^3)}{\partial q^{il}_{jh\hat{l}}} \geq M,$$

$$\forall Q^1 \quad with \quad q^{il}_{jh\hat{l}} \geq N \quad \forall i, l, j, h, \hat{l}; \tag{34a}$$

$$\frac{\partial f^{il}(Q^1, Q^2)}{\partial q^{il}_{kh\bar{l}}} + \frac{\partial c^{il}_{kh\bar{l}}(q^{il}_{kh\bar{l}})}{\partial q^{il}_{kh\bar{l}}} + \alpha^{il} \frac{\partial r^{il}(Q^1, Q^2)}{\partial q^{il}_{kh\bar{l}}} + \lambda^+_{kh\bar{l}} P_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}})$$

$$-(\lambda^-_{kh\bar{l}} + \rho_{3kh\bar{l}} \times e_h)(1 - P_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}})) + \frac{\partial c_{kh\bar{l}}(Q^2, Q^3)}{\partial q_{ikh}} \beta^{j\hat{l}} \frac{\partial r^{j\hat{l}}(Q^1, Q^3)}{\partial q^{j\hat{l}}_{kh\bar{l}}} \geq M,$$

$$\forall Q^2 \quad with \quad q^{il}_{kh\bar{l}} \geq N \quad \forall i, l, k, h, \bar{l}; \tag{34b}$$

$$\lambda^+_{kh\bar{l}} P_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}}) - (\lambda^-_{kh\bar{l}} + \rho_{3kh\bar{l}} \times e_h)(1 - P_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}})) + \frac{\partial c_{j\hat{l}}(Q^1, Q^3)}{\partial q^{j\hat{l}}_{kh\bar{l}}} + \frac{\partial c_{kh\bar{l}}(Q^2, Q^3)}{\partial q^{j\hat{l}}_{kh\bar{l}}} \geq M,$$

$$\forall Q^3 \quad with \quad q^{j\hat{l}}_{kh\bar{l}} \geq N \quad \forall j, \hat{l}, k, h, \bar{l}; \tag{34c}$$

and

$$d_{kh\bar{l}}(\rho_{3kh\bar{l}}) \leq N, \quad \forall \rho_3 \quad with \quad \rho_{3kh\bar{l}} \geq R, \quad \forall k, h, \bar{l}. \tag{35}$$

Then, variational inequality (30) admits at least one solution.

Proof: Follows using analogous arguments as the proof of existence for Proposition 1 in Nagurney and Zhao (1993) (see also existence proof in Nagurney, Dong, and Zhang (2002)). □

Assumptions (34a), (34b), (34c) and (35) can be economically justified as follows. In particular, when the volume of the product transacted, $q^{il}_{jh\hat{l}}$, between manufacturer il and distributor j in currency h, and that transacted between manufacturer il and retailer k in country $\bar{l}$, $q^{il}_{kh\bar{l}}$, are large, one can expect the corresponding sum of the marginal costs associated with the production, transaction, and holding plus the weighted marginal risk to exceed a positive lower bound, say, M. At the same time, the large $q^{il}_{jh\hat{l}}$ and $q^{il}_{kh\bar{l}}$ causes a greater $s_{kh\bar{l}}$, which, in turn, causes the probability distribution $P_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}})$ to be close to 1. Consequently, the sum of the middle two terms on the left-hand side of (36b), $\lambda^+_{kh\bar{l}} P_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}}) - (\lambda^-_{kh\bar{l}} + \rho_{3kh\bar{l}} \times e_h)(1 - P_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}}))$ is seen to be positive. Therefore, the left-hand sides of (34b) and (364c), respectively, are greater than or equal to the lower bound M. On the other hand, a high price $\rho_{3kh\bar{l}}$ at retailer k and country $\bar{l}$, will drive the demand at that retailer down, in line with the decreasing nature of any demand function, which ensures (35).

We now recall the concept of an *additive production cost*, which was introduced by Zhang and Nagurney (1996) in the stability analysis of dynamic spatial oligopolies, and has also been employed in the qualitative analysis of supply chains by Nagurney, Dong, and Zhang (2002).

Definition 2: Additive Production Cost

Suppose that for each manufacturer il, the production cost f^{il} *is additive, that is*

$$f^{il}(q) = f^{il^1}(q^{il}) + f^{il^2}(\bar{q}^{il}), \tag{36}$$

where $f^{il^1}(q^{il})$ *is the internal production cost that depends solely on the manufacturer's own output level* q^{il}*, which may include the production operation and the facility maintenance, etc., and* $f^{il^2}(\bar{q}^{il})$ *is the interdependent part of the production cost that is a function of all the other manufacturers' output levels* $\bar{q}^{il} = (q^{11}, \cdots, q^{il-1}, q^{il+1}, \cdots, q^{il})$ *and reflects the impact of the other manufacturers' production patterns on manufacturer il's cost. This interdependent part of the production cost may describe the competition for the resources, consumption of the homogeneous raw materials, etc.*

We now explore additional qualitative properties of the vector function F that enters the variational inequality problem. Specifically, we show that F is monotone as well as Lipschitz continuous. These properties are fundamental in establishing the convergence of the algorithmic scheme in the subsequent section.

Lemma 1

Let $g_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}})^T = (P_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}}) - \rho_{3kh\bar{l}}(1 - P_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}})),\ s_{kh\bar{l}} - \rho_{3kh\bar{l}}))$*, where* $P_{kh\bar{l}}$ *is a probability distribution with the density function of* $F_{kh\bar{l}}(x, \rho_{3kh\bar{l}})$*. Then* $g_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}})$ *is monotone, that is,*

$$[-\rho'_{3kh\bar{l}}(1 - P_{kh\bar{l}}(s'_{kh\bar{l}}, \rho'_{3kh\bar{l}})) + \rho''_{3kh\bar{l}}(1 - P_{kh\bar{l}}(s''_{kh\bar{l}}, \rho''_{3kh\bar{l}}))] \times [q^{j\hat{l}'}_{kh\bar{l}} - q^{j\hat{l}''}_{kh\bar{l}}]$$

$$+[s'_{kh\bar{l}} - d_{kh\bar{l}}(\rho'_{3kh\bar{l}}) - s''_{kh\bar{l}} + d_{kh\bar{l}}(\rho''_{3kh\bar{l}})] \times [\rho'_{3kh\bar{l}} - \rho''_{3k}] \geq 0, \quad \forall (s'_{kh\bar{l}}, \rho'_{3kh\bar{l}}), (s''_{kh\bar{l}}, \rho''_{3kh\bar{l}}) \in R^2_+ \tag{37}$$

if and only if $d'_{kh\bar{l}}(\rho_{3kh\bar{l}}) \leq -(4\rho_{3kh\bar{l}}F_{kh\bar{l}})^{-1}(P_{kh\bar{l}} + \rho_{3kh\bar{l}}\frac{\partial P_{kh\bar{l}}}{\partial \rho_{3kh\bar{l}}})^2$.

Proof: In order to prove that $g_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}})$ is monotone with respect to $s_{kh\bar{l}}$ and $\rho_{3kh\bar{l}}$, we only need to show that its Jacobian matrix is positive semidefinite, which will be the case if all all eigenvalues of the symmetric part of the Jacobian matrix are nonnegative real numbers.

The Jacobian matrix of $g_{kh\bar{l}}$ is

$$\nabla g_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}}) = \begin{bmatrix} \rho_{3kh\bar{l}} F_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}}) & -1 + P_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}}) + \rho_{3kh\bar{l}} \frac{\partial P_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}})}{\partial \rho_{3kh\bar{l}}} \\ 1 & -d'_{kh\bar{l}}(\rho_{3kh\bar{l}}) \end{bmatrix}, \tag{38}$$

and its symmetric part is

$$\frac{1}{2}[\nabla g_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}}) + \nabla^T g_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}})]$$

$$= \begin{bmatrix} \rho_{3kh\bar{l}} F_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}}), & \frac{1}{2}\left(\rho_{3kh\bar{l}} \frac{\partial P_{kh\bar{l}}}{\partial \rho_{3kh\bar{l}}} + P_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}})\right) \\ \frac{1}{2}\left(\rho_{3kh\bar{l}} \frac{\partial P_{kh\bar{l}}}{\partial \rho_{3kh\bar{l}}} + P_{kh\bar{l}}(s_{kh\bar{l}}, \rho_{3kh\bar{l}})\right), & -d'_{kh\bar{l}}(\rho_{3kh\bar{l}}) \end{bmatrix}. \tag{39}$$

The two eigenvalues of (39) are

$$\gamma_{min}(s_{kh\bar{l}}, \rho_{3kh\bar{l}}) = \frac{1}{2}\Bigg[(\rho_{3kh\bar{l}} F_{kh\bar{l}} - d'_{kh\bar{l}})$$

$$-\sqrt{(\rho_{3kh\bar{l}} F_{kh\bar{l}} - d'_{kh\bar{l}})^2 + (\rho_{3kh\bar{l}} \frac{\partial P_{kh\bar{l}}}{\partial \rho_{3kh\bar{l}}} + P_{kh\bar{l}})^2 + 4\rho_{3kh\bar{l}} F_{kh\bar{l}} d'_{kh\bar{l}}}\Bigg], \tag{40}$$

$$\gamma_{max}(s_{kh\bar{l}}, \rho_{3kh\bar{l}}) = \frac{1}{2}\Bigg[(\rho_{3kh\bar{l}} F_{kh\bar{l}} - d'_{kh\bar{l}})$$

$$+\sqrt{(\rho_{3kh\bar{l}} F_{kh\bar{l}} - d'_{kh\bar{l}})^2 + (\rho_{3kh\bar{l}} \frac{\partial P_{kh\bar{l}}}{\partial \rho_{3kh\bar{l}}} + P_{kh\bar{l}})^2 + 4\rho_{3kh\bar{l}} F_{kh\bar{l}} d'_{kh\bar{l}}}\Bigg]. \tag{41}$$

Moreover, since what is inside the square root in both (40) and (41) can be rewritten as

$$(\rho_{3kh\bar{l}} F_{kh\bar{l}} + d'_{kh\bar{l}})^2 + \left(\rho_{3kh\bar{l}} \frac{\partial P_{kh\bar{l}}}{\partial \rho_{3kh\bar{l}}} + P_{kh\bar{l}}\right)^2$$

and can be seen as being nonnegative, both eigenvalues are real. Furthermore, under the condition of the lemma, $d'_{kh\bar{l}}$ is non-positive, so the first item in (40) and in (41) is nonnegative. The condition further implies that the second item in (40) and in (41), the square root part, is not greater than the first item, which guarantees that both eigenvalues are nonnegative real numbers. □

The condition of Lemma 1 states that the expected demand function of a retailer is a nonincreasing function with respect to the demand price and its first order derivative has an upper bound.

Theorem 4: Monotonicity

The function F that enters the variational inequality problem (29) is monotone, if the condition assumed in Lemma 1 is satisfied for each k; $k = 1, \cdots, K$, and if the following conditions are also satisfied. Suppose that the production cost functions $f^{il}; i = 1, ..., I; l = 1, ..., L$, are additive, as defined in Definition 2, and that the f^{il^1}; $i = 1, ..., I$; $l = 1, ..., L$, are convex functions. If the $c^{il}_{jh\hat{l}}$, $c^{il}_{kh\bar{l}}$, $c_{kh\bar{l}}$, $c_{j\hat{l}}$, r^{il}, and $r^{j\hat{l}}$ functions are convex, for all i, l, $\hat{l}$, $\bar{l}$, j, k, h, then the vector function F that enters the variational inequality (30) is monotone, that is,

$$\langle F(X') - F(X''), X' - X'' \rangle \geq 0, \quad \forall X', X'' \in \mathcal{K}. \tag{42}$$

Proof: Let $X' = (Q^{1'}, Q^{2'}, Q^{3'}, \gamma', \rho_{3'})$, $X'' = (Q^{1''}, Q^{2''}, Q^{3''}, \gamma'', \rho''_3)$. Then, inequality (42) can been seen from the following:

$$\langle F(X') - F(X''), X' - X'' \rangle$$

$$= \sum_{i=1}^{I} \sum_{l=1}^{L} \sum_{j=1}^{J} \sum_{h=1}^{H} \sum_{\hat{l}=1}^{L} \left[\frac{\partial f^{il}(Q^{1'}, Q^{2'})}{\partial q^{il}_{jh\hat{l}}} - \frac{\partial f^{il}(Q^{1''}, Q^{2''})}{\partial q^{il}_{jh\hat{l}}} \right] \times \left[q^{il'}_{jh\hat{l}} - q^{il''}_{jh\hat{l}} \right]$$

$$+ \sum_{i=1}^{I} \sum_{l=1}^{L} \sum_{j=1}^{J} \sum_{h=1}^{H} \sum_{\hat{l}=1}^{L} \left[\frac{\partial c_{j\hat{l}}(Q^{1'}, Q^{3'})}{\partial q^{il}_{jh\hat{l}}} - \frac{\partial c_{j\hat{l}}(Q^{1''}, Q^{3''})}{\partial q^{il}_{jh\hat{l}}} \right] \times \left[q^{il'}_{jh\hat{l}} - q^{il''}_{jh\hat{l}} \right]$$

$$+ \sum_{i=1}^{I} \sum_{l=1}^{L} \sum_{j=1}^{J} \sum_{h=1}^{H} \sum_{\hat{l}=1}^{L} \left[\frac{\partial c^{il}_{jh\hat{l}}(q^{il'}_{jh\hat{l}})}{\partial q^{il}_{jh\hat{l}}} - \frac{\partial c^{il}_{jh\hat{l}}(q^{il''}_{jh\hat{l}})}{\partial q^{il}_{jh\hat{l}}} \right] \times \left[q^{il'}_{jh\hat{l}} - q^{il''}_{jh\hat{l}} \right]$$

$$+ \sum_{i=1}^{I} \sum_{l=1}^{L} \sum_{j=1}^{J} \sum_{h=1}^{H} \sum_{\hat{l}=1}^{L} \alpha^{il} \left[\frac{\partial r^{il}(Q^{1'}, Q^{2'})}{\partial q^{il}_{jh\hat{l}}} - \frac{\partial r^{il}(Q^{1''}, Q^{2''})}{\partial q^{il}_{jh\hat{l}}} \right] \times \left[q^{il'}_{jh\hat{l}} - q^{il''}_{jh\hat{l}} \right]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\beta^{j\hat{l}}\left[\frac{\partial r^{j\hat{l}}(Q^{1'},Q^{3'})}{\partial q^{il}_{jh\hat{l}}}-\frac{\partial r^{j\hat{l}}(Q^{1''},Q^{3''})}{\partial q^{il}_{jh\hat{l}}}\right]\times\left[q^{il'}_{jh\hat{l}}-q^{il''}_{jh\hat{l}}\right]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\left[\frac{\partial f^{il}(Q^{1'},Q^{2'})}{\partial q^{il}_{kh\bar{l}}}-\frac{\partial f_i(Q^{1''},Q^{2''})}{\partial q^{il}_{kh\bar{l}}}\right]\times\left[q^{il'}_{kh\bar{l}}-q^{il''}_{kh\bar{l}}\right]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\left[\frac{\partial c_{kh\bar{l}}(Q^{2'},Q^{3'})}{\partial q^{il}_{kh\bar{l}}}-\frac{\partial c_{kh\bar{l}}(Q^{2''},Q^{3''})}{\partial q^{il}_{kh\bar{l}}}\right]\times\left[q^{il'}_{kh\bar{l}}-q^{il''}_{kh\bar{l}}\right]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\left[\frac{\partial c^{il}_{kh\bar{l}}(q^{il'}_{kh\bar{l}})}{\partial q_{ik}}-\frac{\partial c^{il}_{kh\bar{l}}(q^{il''}_{kh\bar{l}})}{\partial q^{il}_{kh\bar{l}}}\right]\times\left[q^{il'}_{kh\bar{l}}-q^{il''}_{kh\bar{l}}\right]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\alpha^{il}\left[\frac{\partial r^{il}(Q^{1'},Q^{2'})}{\partial q^{il}_{kh\bar{l}}}-\frac{\partial r^{il}(Q^{1''},Q^{2''})}{\partial q^{il}_{kh\bar{l}}}\right]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}[\lambda^{+}_{kh\bar{l}}P_{kh\bar{l}}(s'_{kh\bar{l}},\rho'_{3kh\bar{l}})-\lambda^{+}_{kh\bar{l}}P_{kh\bar{l}}(s''_{kh\bar{l}},\rho''_{3kh\bar{l}})]\times[q^{il'}_{kh\bar{l}}-q^{il''}_{kh\bar{l}}]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}[-\lambda^{-}_{kh\bar{l}}(1-P_{kh\bar{l}}(s'_{kh\bar{l}},\rho'_{3kh\bar{l}}))+\lambda^{-}_{kh\bar{l}}(1-P_{kh\bar{l}}(s''_{kh\bar{l}},\rho''_{3kh\bar{l}}))]\times[q^{il'}_{kh\bar{l}}-q^{il''}_{kh\bar{l}}]$$

$$+\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}[-\rho'_{3kh\bar{l}}\times e_h(1-P_{kh\bar{l}}(s'_{kh\bar{l}},\rho'_{3kh\bar{l}}))+\rho''_{3kh\bar{l}}\times e_h(1-P_{kh\bar{l}}(s''_{kh\bar{l}},\rho''_{3kh\bar{l}}))]\times[q^{il'}_{kh\bar{l}}-q^{il''}_{kh\bar{l}}]$$

$$+\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\left[\frac{\partial c_{j\hat{l}}(Q^{1'},Q^{3'})}{\partial q^{j\hat{l}}_{kh\bar{l}}}-\frac{\partial c_{j\hat{l}}(Q^{1''},Q^{3''})}{\partial q^{j\hat{l}}_{kh\bar{l}}}\right]\times\left[q^{j\hat{l}'}_{kh\bar{l}}-q^{j\hat{l}''}_{kh\bar{l}}\right]$$

$$+\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\left[\frac{\partial c_{kh\bar{l}}(Q^{2'},Q^{3'})}{\partial q^{j\hat{l}}_{kh\bar{l}}}-\frac{\partial c_{kh\bar{l}}(Q^{2''},Q^{3''})}{\partial q^{j\hat{l}}_{kh\bar{l}}}\right]\times\left[q^{j\hat{l}'}_{kh\bar{l}}-q^{j\hat{l}''}_{kh\bar{l}}\right]$$

$$+\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\beta^{j\hat{l}}\left[\frac{\partial r^{j\hat{l}}(Q^{1'},Q^{3'})}{\partial q_{jk}}-\frac{\partial r^{j\hat{l}}(Q^{1''},Q^{3''})}{\partial q^{j\hat{l}}_{kh\bar{l}}}\right]\times\left[q^{j\hat{l}'}_{kh\bar{l}}-q^{j\hat{l}''}_{kh\bar{l}}\right]$$

$$+\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}[\lambda^{+}_{kh\bar{l}}P_{kh\bar{l}}(s'_{kh\bar{l}},\rho'_{3kh\bar{l}})-\lambda^{+}_{kh\bar{l}}P_{kh\bar{l}}(s''_{kh\bar{l}},\rho''_{3kh\bar{l}})]\times[q^{j\hat{l}'}_{kh\bar{l}}-q^{j\hat{l}''}_{kh\bar{l}}]$$

$$+\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}[-\lambda^{-}_{kh\bar{l}}(1-P_{kh\bar{l}}(s'_{kh\bar{l}},\rho'_{3kh\bar{l}}))+\lambda^{-}_{kh\bar{l}}(1-P_{kh\bar{l}}(s''_{kh\bar{l}},\rho''_{3kh\bar{l}}))]\times[q^{j\hat{l}'}_{kh\bar{l}}-q^{j\hat{l}''}_{kh\bar{l}}]$$

$$+\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}[-\rho'_{3kh\bar{l}}\times e_h(1-P_{kh\bar{l}}(s'_{kh\bar{l}},\rho'_{3kh\bar{l}}))+\rho''_{3kh\bar{l}}\times e_h(1-P_{kh\bar{l}}(s''_{kh\bar{l}},\rho''_{3kh\bar{l}}))]\times[q^{j\hat{l}'}_{kh\bar{l}}-q^{j\hat{l}''}_{kh\bar{l}}]$$

$$+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}[s'_{kh\bar{l}} - d_{kh\bar{l}}(\rho'_{3kh\bar{l}}) - s''_{kh\bar{l}} + d_{kh\bar{l}}(\rho''_{3kh\bar{l}})] \times [\rho'_{3kh\bar{l}} - \rho''_{3kh\bar{l}}]$$

$$= (I) + (II) + (III) + \cdots + (XV) + \cdots + (XIX). \tag{43}$$

Since the f^{il}; $i = 1, ..., I$; $l = 1, ..., L$, are additive, and the f^{il^1}; $i = 1, ..., I$; $l = 1, ..., L$ are convex functions, one has

$$(I) + (VI) = \sum_{i=1}^{I}\sum_{l=1}^{L}\left\{\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\frac{\partial f^{il^1}(Q^{1'}, Q^{2'})}{\partial q^{il}_{jh\hat{l}}} - \frac{\partial f^{il^1}(Q^{1''}, Q^{2''})}{\partial q^{il}_{jh\hat{l}}}\right] \times \left[q^{il'}_{jh\hat{l}} - q^{il''}_{jh\hat{l}}\right]\right.$$

$$\left.+\sum_{k=1}^{k}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\left[\frac{\partial f^{il}(Q^{1'}, Q^{2'})}{\partial q^{il}_{kh\bar{l}}} - \frac{\partial f^{il}(Q^{1''}, Q^{2''})}{\partial q^{il}_{kh\bar{l}}}\right] \times \left[q^{il'}_{kh\bar{l}} - q^{il''}_{kh\bar{l}}\right]\right\} \geq 0. \tag{44}$$

The convexity of $c_{j\hat{l}}$, for all $j, \hat{l}$; $c^{il}_{jh\hat{l}}$, for all $i, l, j, h, \hat{l}$; $c_{kh\bar{l}}$, for all $k, h, \bar{l}$, $c^{il}_{kh\bar{l}}$, for all i, l, k, h, r^{il}, for all i, l; and $r^{j\hat{l}}$, for all $j, \hat{l}$, gives, respectively,

$$(II) + (XIII) = \sum_{i=1}^{I}\sum_{l=1}^{L}\left\{\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\frac{\partial c_{j\hat{l}}(Q^{1'}, Q^{3'})}{\partial q^{il}_{jh\hat{l}}} - \frac{\partial c^{j\hat{l}}(Q^{1''}, Q^{3''})}{\partial q^{il}_{jh\hat{l}}}\right] \times \left[q^{il'}_{jh\hat{l}} - q^{il''}_{jh\hat{l}}\right]\right.$$

$$\left.+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\left[\frac{\partial c_{j\hat{l}}(Q^{1'}, Q^{3'})}{\partial q^{j\hat{l}}_{kh\bar{l}}} - \frac{\partial c_{j\hat{l}}(Q^{1''}, Q^{3''})}{\partial q^{j\hat{l}}_{kh\bar{l}}}\right] \times \left[q^{j\hat{l}}_{kh\bar{l}'} - q^{j\hat{l}''}_{kh\bar{l}}\right]\right\} \geq 0, \tag{45}$$

$$(III) = \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\frac{\partial c^{il}_{jh\hat{l}}(q^{il'}_{jh\hat{l}})}{\partial q^{il}_{jh\hat{l}}} - \frac{\partial c^{il}_{jh\hat{l}}(q^{il''}_{jh\hat{l}})}{\partial q^{il}_{jh\hat{l}}}\right] \times \left[q^{il'}_{jh\hat{l}} - q^{il''}_{jh\hat{l}}\right] \geq 0, \tag{46}$$

$$(VII) + (XIV) = \sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\left\{\sum_{i=1}^{I}\sum_{l=1}^{L}\left[\frac{\partial c_{kh\bar{l}}(Q^{2'}, Q^{3'})}{\partial q^{il}_{kh\bar{l}}} - \frac{\partial c_{kh\bar{l}}(Q^{2''}, Q^{3''})}{\partial q^{il}_{kh\bar{l}}}\right] \times \left[q^{il'}_{kh\bar{l}} - q^{il''}_{kh\bar{l}}\right]\right.$$

$$\left.+\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}\left[\frac{\partial c_{kh\bar{l}}(Q^{2'}, Q^{3'})}{\partial q^{j\hat{l}}_{kh\bar{l}}} - \frac{\partial c_{kh\bar{l}}(Q^{2''}, Q^{3''})}{\partial q^{j\hat{l}}_{kh\bar{l}}}\right] \times \left[q^{j\hat{l}'}_{kh\bar{l}} - q^{j\hat{l}''}_{kh\bar{l}}\right]\right\} \geq 0, \tag{47}$$

$$(VIII) = \sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\left[\frac{\partial c^{il}_{kh\bar{l}}(q^{il'}_{kh\bar{l}})}{\partial q^{il}_{kh\bar{l}}} - \frac{\partial c^{il}_{kh\bar{l}}(q^{il''}_{kh\bar{l}})}{\partial q^{il}_{kh\bar{l}}}\right] \times \left[q^{il'}_{kh\bar{l}} - q^{il''}_{kh\bar{l}}\right] \geq 0, \tag{48}$$

$$(IV) + (IX) = \sum_{i=1}^{I}\sum_{l=1}^{L}\alpha^{il}\left\{\sum_{j=1}^{J}\sum_{h=1}^{H}\sum_{\hat{l}=1}^{L}\left[\frac{\partial r^{il}(Q^{1'}, Q^{2'})}{\partial q^{il}_{jh\hat{l}}} - \frac{\partial r^{il}(Q^{1''}, Q^{2''})}{\partial q^{il}_{jh\hat{l}}}\right] \times \left[q^{il'}_{jh\hat{l}} - q^{il''}_{jh\hat{l}}\right]\right.$$

$$\left.+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\left[\frac{\partial r^{il}(Q^{1'}, Q^{2'})}{\partial q^{il}_{kh\bar{l}}} - \frac{\partial r^{il}(Q^{1''}, Q^{2''})}{\partial q^{il}_{kh\bar{l}}}\right]\right\} \geq 0, \tag{49}$$

$$(V)+(XV)=\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}\beta^{j\hat{l}}\left\{\sum_{i=1}^{I}\sum_{l=1}^{L}\sum_{h=1}^{H}\left[\frac{\partial r^{j\hat{l}}(Q^{1'},Q^{3'})}{\partial q^{il}_{jh\hat{l}}}-\frac{\partial r_j(Q^{1''},Q^{3''})}{\partial q^{il}_{jh\hat{l}}}\right]\times\left[q^{il'}_{jh\hat{l}}-q^{il''}_{jh\hat{l}}\right]\right.$$

$$\left.+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}\left[\frac{\partial r^{j\hat{l}}(Q^{1'},Q^{3'})}{\partial q^{j\hat{l}}_{kh\bar{l}}}-\frac{\partial r^{j\hat{l}}(Q^{1''},Q^{3''})}{\partial q^{j\hat{l}}_{kh\bar{l}}}\right]\times\left[q^{j\hat{l}'}_{kh\bar{l}}-q^{j\hat{l}''}_{kh\bar{l}}\right]\right\}\geq 0. \quad (50)$$

Since the probability function $P_{kh\bar{l}}$ is an increasing function w.r.t. $s_{kh\bar{l}}$, for all k, and $s_{kh\bar{l}}=\sum_{i=1}^{I}\sum_{l=1}^{L}q^{il}_{kh\bar{l}}+\sum_{j=1}^{J}\sum_{\hat{l}=1}^{L}q^{j\hat{l}}_{kh\bar{l}}$, hence, we have the following:

$$(X)+(XVI)=\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}[\lambda^{+}_{kh\bar{l}}P_{kh\bar{l}}(s'_{kh\bar{l}},\rho'_{3kh\bar{l}})-\lambda^{+}_{kh\bar{l}}P_{kh\bar{l}}(s''_{kh\bar{l}},\rho''_{3kh\bar{l}})]\times[s'_{kh\bar{l}}-s''_{kh\bar{l}}]\geq 0, \quad (51)$$

$$(XI)+(XVII)=\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}[-\lambda^{-}_{kh\bar{l}}(1-P_{kh\bar{l}}(s'_{kh\bar{l}},\rho'_{3kh\bar{l}}))+\lambda^{-}_{kh\bar{l}}(1-P_{kh\bar{l}}(s''_{kh\bar{l}},\rho''_{3kh\bar{l}}))]$$

$$\times[s'_{kh\bar{l}}-s''_{kh\bar{l}}]\geq 0. \quad (52)$$

Since for each k, applying Lemma 1, we can see that $g_{kh\bar{l}}(s_{kh\bar{l}},\rho_{3kh\bar{l}})$ is monotone, hence, we have:

$$(XII)+(XVIII)+(XIX)$$

$$=\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}[-\rho'_{3kh\bar{l}}\times e_h(1-P_{kh\bar{l}}(s'_{kh\bar{l}},\rho'_{3kh\bar{l}}))+\rho''_{3kh\bar{l}}\times e_h(1-P_{kh\bar{l}}(s''_{kh\bar{l}},\rho''_{3kh\bar{l}}))]$$

$$\times[s'_{kh\bar{l}}-s''_{kh\bar{l}}]$$

$$+\sum_{k=1}^{K}\sum_{h=1}^{H}\sum_{\bar{l}=1}^{L}[s'_{kh\bar{l}}-d_{kh\bar{l}}(\rho'_{3kh\bar{l}})-s''_{kh\bar{l}}+d_{kh\bar{l}}(\rho''_{3kh\bar{l}})]\times[\rho'_{3kh\bar{l}}-\rho''_{3kh\bar{l}}], \quad (53)$$

which is reasonable to assume is also nonnegative. Therefore, we conclude that (44) is nonnegative in $\mathcal{K}$. The proof is complete. □

Theorem 5: Strict Monotonicity

The function F that enters the variational inequality problem (30) is strictly monotone, if the conditions mentioned in Lemma 1 for $g_{kh\bar{l}}(s_{kh\bar{l}},\rho_{3kh\bar{l}})$ are satisfied strictly for all k and if the following conditions are also satisfied. Suppose that the production cost functions $f^{il}; i=1,...,I; l=1...L$, are additive, as defined in Definition 2, and that the ${f^{il}}^1; i=1,...,I$; $l=1,...,L$, are strictly convex functions. If the $c^{il}_{jh\hat{l}}$, $c^{il}_{kh\bar{l}}$, $c_{kh\bar{l}}$, $c_{j\hat{l}}$, r^{il}, and $r^{j\hat{l}}$ functions are *strictly convex, for all i,l, $\hat{l}$, $\bar{l}$, j, k, h, then the vector function F that enters the variational inequality (30) is strictly monotone, that is,*

$$\langle F(X') - F(X''), X' - X'' \rangle > 0, \quad \forall X', X'' \in \mathcal{K}. \tag{54}$$

Theorem 6: Uniqueness

Under the conditions indicated in Theorem 5, the function F that enters the variational inequality (30) has a unique solution in $\mathcal{K}$.

From Theorem 6 it follows that, under the above conditions, the equilibrium product shipment pattern between the manufacturers and the retailers, as well as the equilibrium price pattern at the retailers, is unique.

Theorem 7: Lipschitz Continuity

The function F that enters the variational inequality problem (30) is Lipschitz continuous, that is,

$$\|F(X') - F(X'')\| \leq L\|X' - X''\|, \quad \forall X', X'' \in \mathcal{K}, \text{ with } L > 0, \tag{55}$$

under the following conditions:

(i). Each function f^{il}; $i = 1, ..., I$; $l = 1, ..., L$, is additive and has a bounded second order derivative;

(ii). The $c^{il}_{jh\hat{l}}$, $c^{il}_{kh\bar{l}}$, $c_{kh\bar{l}}$, $c_{j\hat{l}}$, r^{il}, and $r^{j\hat{l}}$ functions have bounded second order derivatives, for all i, j, k, h, l, $\hat{l}$, $\bar{l}$.

Proof: Since the probability function $P_{kh\bar{l}}$ is always less than or equal to 1, for each retailer $kh\bar{l}$, the result is direct by applying a mid-value theorem from calculus to the vector function F that enters the variational inequality problem (30). □

4. The Algorithm

In this Section, an algorithm is presented which can be applied to solve any variational inequality problem in standard form (see (30)), that is:

Determine $X^* \in \mathcal{K}$, satisfying:

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{56}$$

The algorithm is guaranteed to converge provided that the function F that enters the variational inequality is monotone and Lipschitz continuous (and that a solution exists). The algorithm is the modified projection method of Korpelevich (1977).

The statement of the modified projection method is as follows, where $\mathcal{T}$ denotes an iteration counter:

Modified Projection Method

Step 0: Initialization

Set $X^0 \in \mathcal{K}$. Let $\mathcal{T} = 1$ and let a be a scalar such that $0 < a \leq \frac{1}{L}$, where L is the Lipschitz continuity constant (cf. Korpelevich (1977)) (see (55)).

Step 1: Computation

Compute $\bar{X}^{\mathcal{T}}$ by solving the variational inequality subproblem:

$$\langle \bar{X}^{\mathcal{T}} + aF(X^{\mathcal{T}-1}) - X^{\mathcal{T}-1}, X - \bar{X}^{\mathcal{T}} \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{57}$$

Step 2: Adaptation

Compute $X^{\mathcal{T}}$ by solving the variational inequality subproblem:

$$\langle X^{\mathcal{T}} + aF(\bar{X}^{\mathcal{T}}) - X^{\mathcal{T}-1}, X - X^{\mathcal{T}} \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{58}$$

Step 3: Convergence Verification

If max $|X_l^{\mathcal{T}} - X_l^{\mathcal{T}-1}| \leq \epsilon$, for all l, with $\epsilon > 0$, a prespecified tolerance, then stop; else, set $\mathcal{T} =: \mathcal{T} + 1$, and go to Step 1.

We now state the convergence result for the modified projection method for this model.

Theorem 8: Convergence

Assume that the function that enters the variational inequality (29) (or (30)) has at least one solution and satisfies the conditions in Theorem 4 and in Theorem 7. Then the modified projection method described above converges to the solution of the variational inequality (29) or (30).

Proof: According to Korpelevich (1977), the modified projection method converges to the solution of the variational inequality problem of the form (30), provided that the function F that enters the variational inequality is monotone and Lipschitz continuous and that a solution exists. Existence of a solution follows from Theorem 3. Monotonicity follows Theorem 5. Lipschitz continuity, in turn, follows from Theorem 7. $\square$

We emphasize that, in view of the fact that the feasible set $\mathcal{K}$ underlying the global supply chain network model with supply and demand side risk is the nonnegative orthant, the projection operation encountered in (57) and (58) takes on a very simple form for computational purposes. Indeed, the product transactions as well as the product prices at a given iteration in both (57) and in (58) can be exactly and computed in closed form,

5. Numerical Examples

In this Section,we apply the modified projection method to several numerical examples. The algorithm was coded in FORTRAN and the computer used was a SUN system at the University of Massachusetts at Amherst. The convergence criterion utilized was that the absolute value of the product transactions and prices between two successive iterations differed by no more than 10^{-4}. The parameter a in the modified projection method (cf. (57) and (58)) was set to .01 for all the examples.

The structure of the global supply chain network for the examples is given in Figure 2. Specifically, we assumed that there were two countries, with two manufacturers in each country, and two distributors in each country. In addition, we assumed a single currency (for example, the euro) and two retailers in each country. Note that electronic transactions were permitted between the manufacturers and the retailers. Hence, we had that $I = 2$, $L = 2$, $J = 2$, $K = 2$, and $H = 1$ and, hence, $e_h = e_1 = 1$.

In all the examples, we assumed that the demands associated with the retail outlets followed a uniform distribution. In particular, we assumed that the random demand $\hat{d}_{kh\bar{l}}(\rho_{3kh\bar{l}})$, of retailer $kh\bar{l}$, is uniformly distributed in $\left[0, \frac{b_{kh\bar{l}}}{\rho_{3kh\bar{l}}}\right]$, with $b_{kh\bar{l}} > 0$; $k = 1,2$; $h = 1$, and $\bar{l} = 1,2$. Therefore, we have that

$$P_{kh\bar{l}}(x, \rho_{3kh\bar{l}}) = \frac{x\rho_{3kh\bar{l}}}{b_{kh\bar{l}}}; \quad k = 1,2; h = 1; \bar{l} = 1,2, \tag{59}$$

$$F_{kh\bar{l}}(x, \rho_{3kh\bar{l}}) = \frac{\rho_{3kh\bar{l}}}{b_{kh\bar{l}}}; \quad k = 1,2; h = 1; \bar{l} = 1,2; \tag{60}$$

$$d_{kh\bar{l}}(\rho_{3kh\bar{l}}) = E(\hat{d}_{kh\bar{l}}) = \frac{1}{2}\frac{b_{kh\bar{l}}}{\rho_{3kh\bar{l}}}; \quad k = 1,2; h = 1; \bar{l} = 1,2. \tag{61}$$

It is straightforward to verify that the expected demand function $d_{kh\bar{l}}(\rho_{3kh\bar{l}})$ associated with retailer $kh\bar{l}$ is a decreasing function of the price at the demand market in the particular country.

The modified projection method was initialized as follows: all variables were set equal to zero, except for the initial retail prices $\rho_{3kh\bar{l}}$, which were set to 1 for all $k, h, \bar{l}$.

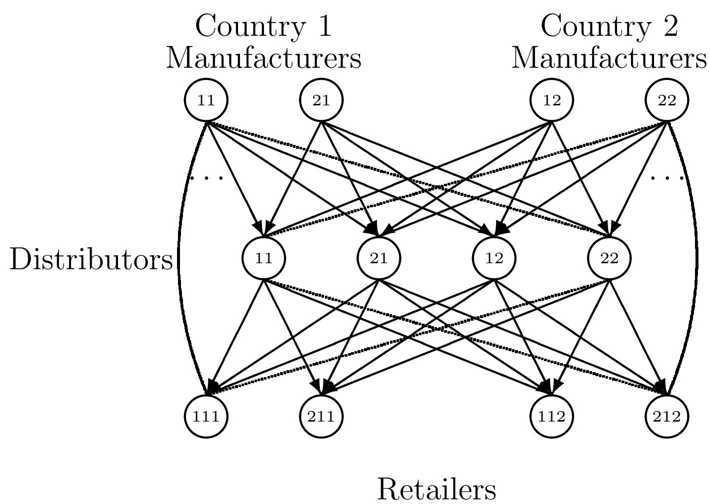

Figure 2: Global supply chain network for the numerical examples

Example 1

The data for this example were constructed for easy interpretation purposes. The production cost functions of the manufacturers in the two countries (cf. (1)) were given, respectively, by:

$$f^{11}(q) = 2.5(q^{11})^2 + q^{11}q^{21} + 2q^{11}, \quad f^{21}(q) = 2.5(q^{21})^2 + q^{11}q^{21} + 2q^{21},$$
$$f^{12}(q) = 2.5(q^{12})^2 + q^{12}q^{22} + 2q^{12}, \quad f^{22}(q) = 2.5(q^{12})^2 + q^{12}q^{22} + 2q^{22}.$$

The transaction costs faced by the manufacturers and associated with transacting with the distributors (cf. (2a)) were given by:

$$c^{il}_{jh\hat{l}} = .5(q^{il}_{jh\hat{l}})^2 + 3.5q^{il}_{jh\hat{l}}, \quad \text{for } i = 1,2; l = 1,2; j = 1,2; h = 1; \hat{l} = 1,2.$$

The transaction costs faced by the manufacturers but associated with transacting with the retailers electronically (cf. (2b)) were given by:

$$c^{il}_{kh\bar{l}} = .5(q^{il}_{kh\bar{l}})^2 + 5q^{il}_{kh\bar{l}}, \quad \text{for } i = 1,2; l = 1,2; k = 1,2; h = 1; \bar{l} = 1,2.$$

The handling costs of the distributors in the two countries, in turn, (cf.(9b), were given by:

$$c_{j\hat{l}} = .5(\sum_{i=1}^{2}\sum_{l=1}^{2} q^{il}_{jh\hat{l}})^2, \quad \text{for } j = 1,2; \hat{l} = 1,2.$$

The handling costs of the retailers (cf. (17)) were:

$$c_{kh\bar{l}} = .5(\sum_{j=1}^{2}\sum_{\hat{l}=1}^{2} q^{j\hat{l}}_{kh\bar{l}})^2, \quad \text{for } k = 1,2; h = 1; \bar{l} = 1,2.$$

The $b_{kh\bar{l}}$s (cf. (59) – (61)) were set to 100 for all $k, h, \bar{l}$. The weights associated with excess supply and with excess demand at the retailers were (see following (21)): $\lambda^+_{kh\bar{l}} = \lambda^-_{kh\bar{l}} = 1$ for $k = 1,2$; $h = 1$, and $\bar{l} = 1,2$. Thus, we assigned equal weights for each retailer in each country for excess supply and excess demand.

In Example 1, we set all the weights associated with risk minimization to zero, that is, we had that $\alpha^{il} = 0$ for $i = 1,2$ and $l = 1,2$ and $\beta^{j\hat{l}} = 0$ for $j = 1,2$ and $\hat{l} = 1,2$. This means that in the first example all the manufacturers and all the distributors were concerned with profit maximization exclusively.

The modified projection method converged and yielded the following equilibrium pattern. All physical transactions were equal to .186, that is, we had that $q^{il*}_{jh\hat{l}} = q^{j\hat{l}*}_{kh\bar{l}} = .186$ for all i, l, $j, h, \hat{l}$, and $k, \bar{l}$. All product transactions conducted electronically via the Internet, in turn, were equal to .177, that is, we had that $q^{il*}_{kh\bar{l}} = .177$ for all i, l, $k, h, \bar{l}$. Note that there was a larger volume of product transacted physically than electronically in this example.

The computed equilibrium prices, in turn, were as follows. The equilibrium prices at the distributors were: $\gamma^*_{j\hat{l}} = 15.091$ for $j = 1, 2$ and $\hat{l} = 1, 2$, whereas the demand market equilibrium prices were: $\rho^*_{3kh\bar{l}} = 32.320$ for $k = 1, 2$, $h = 1$, and $\bar{l} = 1, 2$. Note that, as expected, the demand market prices exceed the prices for the product at the distributor level. This is due to the fact that the prices increase as the product propagates down through the supply chain since costs accumulate.

Example 2

Example 2 was constructed from Example 1 as follows. All the data were as in Example 1 except that now we set the $b_{kh\bar{l}}$s =1000. This means (cf. (59) – (61)) that, in effect, the demand has increased for the product at all retailers in all countries.

The modified projection method converged and yielded the following new equilibrium pattern: the product transactions between the manufacturers and the distributors were: $q^{il*}_{jh\hat{l}} = .286$ for all i, l, $j, h, \hat{l}$; whereas the volumes of the product transacted electronically between the manufacturers and the retailers were: $q^{il*}_{kh\bar{l}} = 1.071$ for all i, l and $k, h, \bar{l}$. Hence, the volumes of electronic transactions exceeded the physical ones. Finally, the computed equilibrium product transactions between the distributors and the retailers were: $q^{j\hat{l}*}_{kh\bar{l}} = .286$ for all $j, \hat{l}$ and $k, h, \bar{l}$.

The computed equilibrium prices associated with the distributors, in turn, were: $\gamma^*_{j\hat{l}} = 39.487$ for $j = 1, 2$ and $\hat{l} = 1, 2$, whereas the equilibrium demand market prices were: $\rho^*_{3kh\bar{l}} = 90.395$ for $k = 1, 2$; $h = 1$, and $\bar{l} = 1, 2$.

Note that since the demand increased, the product transactions also increased. In this example, there were more transactions conducted electronically than physically. Also, observe that since demand increased, the demand prices also increased as did the prices at the distributors.

Example 3

Example 3 was constructed from Example 2 as follows. We kept the data as in Example 2 but we assumed now that the first manufacturer in the first country was a multicriteria decision-maker and concerned with risk minimization with his risk function being given by:

$$r^{11} = (\sum_{kh\bar{l}} q^{11}_{kh\bar{l}} - 2)^2,$$

that is, the manufacturer sought to achieve, in a sense, a certain goal target associated with his electronic transactions. The weight associated with his risk measure was $\alpha^{11} = 2$.

The modified projection method yielded the following new equilibrium pattern. The computed equilibrium product transactions between the first manufacturer in the first country and the distributors were now: $q^{11*}_{jh\hat{l}} = .569$, for $j = 1, 2$, $h = 1$, and $\hat{l} = 1, 2$. The analogous transactions, but from the second manufacturer in the first country were; $q^{21*}_{jh\hat{l}} = .216$. All other product transactions between the manufacturers and the distributors were equal to .215.

The equilibrium product transactions associated with electronic transactions were as follows. For the first manufacturer in the first country (who is now concerned with risk associated with electronic transactions) the product transactions were: $q^{11*}_{111} = .300$, $q^{11*}_{211} = .303$, $q^{11*}_{112} = .307$, and $q^{11*}_{212} = .307$. The analogous transactions but from the second manufacturer in the first country, were all 1.145, with the remainder of the electronic transactions at equilibrium equal to 1.143. Hence, the first manufacturer in the first country reduced the volume of his transactions conducted electronically since there was increased risk associated with such transactions.

The product transactions at equilibrium between the distributors and the retailers, in turn, were: $q^{j\hat{l}*}_{111} = .300$, for $j = 1, 2$ and $\hat{l} = 1, 2$; $q^{j\hat{l}*}_{211} = .303$ for $j = 1, 2$ and $\hat{l} = 1, 2$; $q^{j\hat{l}*}_{112} = .307$ and $q^{j\hat{l}*}_{212} = .307$ for $j = 1, 2$ and $\hat{l} = 1, 2$.

The equilibrium prices were now: $\gamma^*_{j\hat{l}} = 39.514$ for $j = 1, 2$; $\hat{l} = 1, 2$ whereas $\rho^*_{3kh\bar{l}} = 90.565$ for $k = 1, 2$; $h = 1$, and $\bar{l} = 1, 2$.

Obviously, the above examples are highly stylized but they, nevertheless, demonstrate the efficacy of the model and the computational procedure. One can now conduct numerous simulations by altering the data as well as adding decision-makers with their associated functions and weights.

6. Summary and Conclusions

This paper has developed a three-tiered global supply chain network equilibrium model consisting of manufacturers, distributors, and retailers with electronic commerce and with risk management. In particular, the manufacturers as well as the distributors are assumed to be multicriteria decision-makers and concerned not only with profit maximization but also with risk minimization. The demands for the product, in turn, are random.

The developed multi-agent framework permits for the handling of as many countries, as many manufacturers in each country, as many currencies in which the products can be obtained, as many retailers, and as many demand markets, as mandated by the specific application. Moreover, the generality of the multi-agent framework allows for the demand to have almost any distribution as long as it satisfies certain technical conditions. In addition, the retailers need not be country-specific and can transact either virtually or physically with both the manufacturers and the consumers.

Finite-dimensional variational inequality theory was used to formulate the derived equilibrium conditions, to study the model qualitatively, and also to obtain convergence results for the proposed algorithmic scheme. Finally, numerical examples were presented to illustrate the model and computational procedure.

This multi-agent framework generalizes the recent work of Nagurney, Cruz, and Matsypura (2003) to include supply-side risk modeled as a multicriteria decision-making problem from the perspective of both the manufacturers and the distributors, and demand-side risk, handled through random demands. The results herein provide a contribution to the burgeoning topic of the development of quantitative tools for multi-agent modeling, analysis, and computations that bridge finance and operations in the setting of supply chain modeling and analysis with a specific focus on the global arena.

Acknowledgments

The research of the authors was supported, in part, by NSF Grant No.: IIS-0002647, and that of the first two by a 2002 AT&T Industrial Ecology Faculty Fellowship. The first author also acknowledges support from the Radcliffe Institute for Advanced Study at Harvard University under its 2005-2006 Fellowship Program. The support is gratefully acknowledged.

The authors thanks Dr. Brahim Chaib-draa for editing this volume and for support in the preparation of this paper.

References

Agrawal, V. and S. Seshadri (2000), "Risk Intermediation in Supply Chains," *IIE Transactions* **32**, 819-831.

Bazaraa, M. S., H. D., Sherali and C. M. Shetty (1993), **Nonlinear Programming: Theory and Algorithms**, John Wiley & Sons, New York.

Chankong, V. and Y. Y. Haimes (1997), **Multiobjective Decision Making: Theory and Methodology**, North-Holland, New York.

Cohen, M. A. and S. Mallik (1997), "Global Supply Chains: Research and Applications," *Production and Operations Management* **6**, 193-210.

Cohen, M. A. and A. Huchzermeier (1998), "Global Supply Chain Management: A Survey of Research and Applications," in **Quantitative Models for Supply Chain Management**, S. Tayur, M. Magazine, and R. Ganeshan, editors, Kluwer Academic Publishers, New York, 669702.

Dong, J. and A. Nagurney (2001), "Bicriteria Decision Making and Financial Equilibrium: A Variational Inequality Perspective," *Computational Economics* **17**, 19-42.

Dong, J., D. Zhang and A. Nagurney (2002), "Supply Chain Networks with Multicriteria

Decision-Makers," in **Transportation and Traffic Theory in the 21st Century**, M. A. P. editor, Pergamon Press, Amsterdam, The Netherlands, 179-206.

Dong, J., D. Zhang and A. Nagurney (2004a), "Supply Chain Supernetworks with Random Demands," in **Urban and Regional Transportation Modeling: Essays in Honor of David E. Boyce**, D. -H. Lee, editor, Edward Elgar Publishers, Cheltenham, England, 289-313.

Dong, J., D. Zhang and A. Nagurney (2004b), "A Supply Chain Network Equilibrium Model with Random Demands," *European Journal of Operational Research* **156**, 194-212.

Engardio, P., M. Shari, A. Weintraub and C. Arnst (2003), Deadly Virus, *BusinessWeek Online*; see: `http://www.businessweek.com/magazine/content/03_15/b3828002_mx 046.htm`

Erenguc, S. S., N. C. Simpson and A. J. Vakharia (1999), "Integrated Production/Distribution Planning in Supply Chains: An Invited Review," *European Journal of Operational Research* **115**, 219-236.

Fishburn, P. C. (1970), **Utility Theory for Decision Making**, John Wiley & Sons, New York.

Gabay, D. and H. Moulin (1980), "On the Uniqueness and Stability of Nash Equilibria in Noncooperative Games," in: **Applied Stochastic Control of Econometrics and Management Science**, A. Bensoussan, P. Kleindorfer, and C. S. Tapiero, editors, North-Holland, Amsterdam, The Netherlands.

Huchzermeier, A. and M. A. Cohen (1996), "Valuing Operational Flexibility Under Exchange Rate Uncertainty," *Operations Research* **44**, 100-113.

Johnson, M. E. (2001), "Learning from Toys: Lessons in Managing Supply Chain Risk from the Toy Industry," *California Management Review* **43**, 106-124.

Keeney, R. L. and H. Raiffa (1993), **Decisions with Multiple Objectives: Preferences and Value Tradeoffs**, Cambridge University Press, Cambridge, England.

Kinderlehrer, D. and G. Stampacchia (1980), **An Introduction to Variational Inequalities and Their Application**, Academic Press, New York.

Kogut, B. and N. Kulatilaka (1994), "Options Thinking and Platform Investments: Investing in Opportunity," *California Management Review* **36**, 52-71.

Korpelevich, G. M. (1977), "The Extragradient Method for Finding Saddle Points and Other Problems," *Matekon* **13**, 35-49.

Markowitz, H. M. (1952), "Portfolio Selection," *The Journal of Finance* **7**, 77-91.

Markowitz, H. M. (1959), **Portfolio Selection: Efficient Diversification of Investments**, John Wiley & Sons, New York.

Mullany, T. J., H. Green, M. Arndt, R. D. Hof and L. Himelstein (2003), "The E-Biz Surprise," *Business Week Online*; see:
`http://www.businessweek.com/magazine/content/03_19/b3832601.htm`

Nagurney, A. (1999), **Network Economics: A Variational Inequality Approach**, second and revised edition, Kluwer Academic Publishers, Dordrecht, The Netherlands.

Nagurney, A., J. Cruz and D. Matsypura (2003), "Dynamics of Global Supply Chain Supernetworks," *Mathematical and Computer Modelling* **37**, 963-983.

Nagurney, A. and J. Dong (2002), **Supernetworks: Decision-Making for the Information Age**, Edward Elgar Publishers, Cheltenham, England.

Nagurney, A., J. Dong and D. Zhang (2002), "A Supply Chain Network Equilibrium Model," *Transportation Research E* **38**, 281-303.

Nagurney, A., J. Loo, J. Dong and D. Zhang (2002), "Supply Chain Networks and Electronic Commerce: A Theoretical Perspective," *Netnomics* **4**, 187-220.

Nagurney, A. and S. Siokos (1997), **Financial Networks: Statics and Dynamics**, Springer-Verlag, Heidelberg, Germany.

Nagurney, A. and L. Zhao (1993), "Networks and Variational Inequalities in the Formulation and Computation of Market Disequilibria: The Case of Direct Demand Functions," *Transportation Science* **27**, 4-15.

Nash, J. F. (1950), "Equilibrium Points in N-Person Games," in: *Proceedings of the National Academy of Sciences, USA* **36**, 48-49.

Nash, J. F. (1951), "Noncooperative Games," *Annals of Mathematics* **54**, 286-298.

Sheffi, Y. (2001), "Supply Chain Management under the Threat of International Terrorism," *The International Journal of Logistics Management* **12**, 1-11.

Smeltzer, L. R. and S. P. Siferd (1998), "Proactive Supply Management: The Management of Risk," *International Journal of Purchasing and Materials Management*, Winter, 38-45.

Van Mieghem, J. A. (2003), "Capacity Management, Investment and Hedging: Review and Recent Developments," *Manufacturing & Service Operations Management* **5**, 269-302.

Yu, P. L. (1985), **Multiple Criteria Decision Making – Concepts, Techniques, and Extensions**, Plenum Press, New York.

Zhang, D. and A. Nagurney (1996), "Stability Analysis of an Adjustment Process for Oligopolistic Market Equilibrium Modeled as a Projected Dynamical System," *Optimization* **36**, 263-285.

Zsidisin, G. A. (2003), "Managerial Perceptions of Supply Risk," *The Journal of Supply Chain Management: A Global Review of Purchasing and Supply*, Winter, 14-25.