Dynamics
of
Financial Networks with Intermediation

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Abstract:

In this paper, a dynamical systems framework is developed for the modeling, qualitative analysis, and computation of solutions to dynamic financial network problems with intermediation. An economy is considered consisting of three types of agents: those with sources of funds, such as firms and households; intermediary ones, such as, banks, savings institutions, insurance companies, investment companies, etc., and, finally, the consumers located at demand markets corresponding to the uses of funds, such as household loans, real estate loans, business loans, etc. We describe the behavior of the agents, identify the multi-tiered network, and propose the projected dynamical system, along with stability analysis results, that captures the adjustments of the financial flows and prices over space and time. A discrete-time adjustment process is also proposed and implemented in order to demonstrate the evolution of the flows and prices to the equilibrium solution.

1. Introduction

The conceptualization of financial systems as networks dates to Quesnay (1758) who depicted the circular flow of funds in an economy as a network. His basic idea was subsequently applied in the construction of flow of funds accounts, which are a statistical description of the flows of money and credit in an economy (cf. Board of Governors (1980), Cohen (1987), Nagurney and Hughes (1992)). However, since the flow of funds accounts are in matrix form, and, hence, two-dimensional, they fail to capture the dynamic behavior on a micro level of the various financial agents/sectors in an economy, such as banks, households, insurance companies, etc. Moreover, as noted by the Board of Governors (1980), "the generality of the matrix tends to obscure certain structural aspects of the financial system that are of continuing interest in analysis," with the structural concepts of concern including financial intermediation.

Thore, in (1980), recognized some of the shortcomings of financial flow of funds accounts and developed, instead, network models of linked portfolios with financial intermediation, using decentralization/decomposition theory. Note that, intermediation is typically associated with financial businesses, including banks, savings institutions, investment and insurance companies, etc., and the term implies borrowing for the purpose of lending, rather than for nonfinancial purposes. He also constructed some basic intertemporal models. However, the intertemporal models were not fully developed and the computational techniques at that time were not sufficiently advanced for computational purposes.

Thore (1969) had earlier introduced networks, along with the mathematics, for the study of systems of linked portfolios (see also Charnes and Cooper (1967)) in the context of credit networks and made use of linear programming. Storoy, Thore, and Boyer (1975), in turn, presented a network model of the interconnection of capital markets and demonstrated how decomposition theory of mathematical programming could be exploited for the computation of equilibrium. The utility functions facing a sector were no longer restricted to being linear functions.

Nagurney, Dong, and Hughes (1992) developed a multi-sector, multi-instrument financial equilibrium model and recognized the network structure underlying the subproblems encountered in their proposed decomposition scheme, which was based on finite-dimensional variational inequality theory. Nagurney and Ke Ke (2001), in turn, presented a general but, static, model of financial intermediation. In this paper, we build upon that work as well as the contribution of Dong, Zhang, and Nagurney (1996). These authors were the first to develop a dynamic model of general financial equilibrium with multiple sectors and multiple financial instruments, in which both the prices and the flows were endogeneous. They formulated the model as a *projected dynamical system* (cf. Dupuis and Nagurney (1993), Zhang and Nagurney (1995), and Nagurney and Zhang (1996)) and then established stability analysis results. This work was later extended to the international domain by Nagurney and Siokos (1997, 1998).

In this paper, we address the dynamics of the financial economy which explicitly includes financial intermediaries along with the "sources" and "uses" of financial funds. The perspective is an equilibrium one since the equilibrium state serves as a valuable benchmark. Tools are provided for studying the disequilibrium dynamics as well as the equilibrium state. Also, we consider transaction costs in the model, since they bring a greater degree of realism to the study of financial intermediation. Transaction costs have been studied to-date in multi-sector, multi-instrument financial equilibrium models by Nagurney and Dong (1995, 1996 a,b) but without considering the more general dynamic intermediation setting.

This paper is organized as follows. In Section 2, the dynamic financial model is developed with three distinct types of agents, the network structure of the problem is identified, and the disequilibrium dynamics are proposed. Transaction costs are introduced and associated with transactions conducted between agents located at distinct tiers of the networks. The problem is formulated as a projected dynamical system and a discussion of the stationary/equilibrium point is given. In Section 3, some qualitative properties of the dynamic trajectories are provided, along with stability analysis results. In Section 4, a discrete-time algorithm is proposed, which is a time discretization of the continuous adjustment process given in Section 2. The algorithm resolves the network problem into subproblems, each of which can be solved exactly and in closed form. In Section 5, the discrete-time adjustment process is implemented and applied to several numerical examples to determine the equilibrium flows and prices. The paper concludes with a summary and conclusions in Section 6.

2. The Dynamic Financial Network Model with Intermediation

In this Section, the dynamic financial network model is developed. The model consists of: agents with sources of funds, agents who are intermediaries, as well as agents who are consumers located at the demand markets. Specifically, we consider m agents with sources of financial funds, such as households and businesses, involved in the allocation of their financial resources among a portfolio of financial instruments which can be obtained by transacting with distinct n financial intermediaries, such as banks, insurance and investment companies, etc. The financial intermediaries, in turn, in addition to transacting with the source agents, also determine how to allocate the incoming financial resources among distinct uses, as represented by o demand markets with a demand market corresponding to, for example, the market for real estate loans, household loans, or business loans, etc.

The financial network is now described and depicted graphically in Figure 1. The top tier of nodes consists of the agents with sources of funds, with a typical source agent denoted by i and associated with node i. The middle tier of nodes consists of the intermediaries, with a typical intermediary denoted by j and associated with node j in the network. The bottom tier of nodes consists of the demand markets, with a typical demand market denoted by k and corresponding to node k.

For simplicity of notation, we assume that there are L instruments associated with each intermediary. Hence, from each source of funds node, there are L links connecting such a node with an intermediary node with the l-th such link corresponding to the l-th financial instrument available from the intermediary. In addition, we allow the option of non-investment in the available financial instruments and to denote this option, we then also construct an additional link from each source node to middle tier node $n+1$, which represents non-investment. From each intermediary node, we then construct o links, one to each "use" node or demand market in the bottom tier of nodes in the network to denote the transaction between the intermediary and the consumers at the demand market.

Let x_{ijl} denote the nonnegative amount of the funds that source i "invests" in financial instrument l obtained from intermediary j. We group the financial flows associated with source agent i, which are associated with the links emanating from the top tier node i to the intermediary nodes, into the column vector $x_i \in R^{nL}_+$. We assume that each source has, at

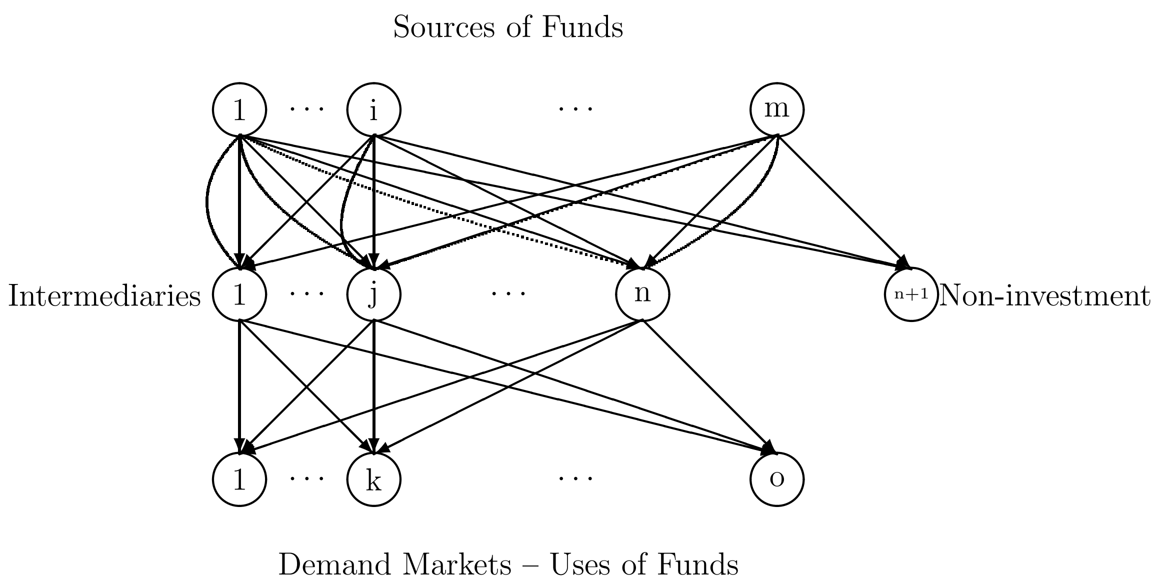

Figure 1: The Network Structure of the Financial Economy with Intermediation and with Non-investment Allowed

his disposal, an amount of funds S_i and we denote the unallocated portion of this amount (and flowing on the link joining node i with node $n+1$) by s_i. We then group the x_is of all the source agents into the column vector $x \in R_+^{mnL}$.

We associate a distinct financial product k with each demand market, bottom-tiered node k and let y_{jk} denote the amount of the financial product obtained by consumers at demand market k from intermediary j. We group these "consumption" quantities into the column vector $y \in R_+^{no}$. The intermediaries convert the incoming financial flows x into the outgoing financial flows y.

The notation for the prices is now given. Note that there will be prices associated with each of the tiers of nodes in the financial network. Let ρ_{1ijl} denote the price associated with instrument l as quoted by intermediary j to source agent i and group the first tier prices into the column vector $\rho_1 \in R_+^{mnL}$. Also, let ρ_{2j} denote the price charged by intermediary j and group all such prices into the column vector $\rho_2 \in R_+^n$. Finally, let ρ_{3k} denote the price of the financial product at the third or bottom-tiered node k, and group all such prices into the column vector $\rho_3 \in R_+^o$.

We now turn to describing the dynamics by which the source agents adjust the amounts they allocate to the various instruments over time, the dynamics by which the intermediaries adjust their transactions, and those by which the consumers obtain the financial products at the demand markets. In addition, we describe the dynamics by which the prices adjust over time. The dynamics are derived from the bottom tier of nodes of the financial network on up since, as mentioned previously, we assume that it is the demand for the financial products (and the corresponding prices) that actually drives the economic dynamics. We first present the price dynamics and then the dynamics underlying the financial flows.

The Demand Market Price Dynamics

We begin by describing the dynamics underlying the prices of the financial products associated with the demand markets (see the bottom-tiered nodes in the financial network). We assume, as given, a demand function d_k, which can depend, in general, upon the entire vector of prices ρ_3, that is,

$$d_k = d_k(\rho_3), \quad \forall k. \tag{1}$$

Moreover, we assume that the rate of change of the price ρ_{3k}, denoted by $\dot{\rho}_{3k}$, is equal to the difference between the demand at the demand market k, as a function of the demand market prices, and the amount available from the intermediaries at the demand market. Hence, if the demand for the product at the demand market (at an instant in time) exceeds the amount available, the price of the financial product at that demand market will increase; if the amount available exceeds the demand at the price, then the price at the demand market will decrease. Furthermore, we guarantee that the prices do not become negative. Thus, the dynamics of the price ρ_{3k} associated with the commodity at demand market k can be expressed as:

$$\dot{\rho}_{3k} = \begin{cases} d_k(\rho_3) - \sum_{j=1}^{n} y_{jk}, & \text{if} \quad \rho_{3k} > 0 \\ \max\{0, d_k(\rho_3) - \sum_{j=1}^{n} y_{jk}\}, & \text{if} \quad \rho_{3k} = 0. \end{cases} \tag{2}$$

The Dynamics of the Prices at the Intermediaries

The prices charged for the financial funds at the intermediaries, in turn, must reflect supply and demand conditions as well (and as we shall shortly see also reflect profit-maximizing behavior on the part of the intermediaries who seek to determine how much of the financial flows they obtain from the different sources of funds). In particular, we assume that the price associated with intermediary j, ρ_{2j}, and computed at node j lying in the second tier of nodes of the financial network, evolves over time according to:

$$\dot{\rho}_{2j} = \begin{cases} \sum_{k=1}^{o} y_{jk} - \sum_{i=1}^{m}\sum_{l=1}^{L} x_{ijl}, & \text{if} \quad \rho_{2j} > 0 \\ \max\{0, \sum_{k=1}^{o} y_{jk} - \sum_{i=1}^{m}\sum_{l=1}^{L} x_{ijl}\}, & \text{if} \quad \rho_{2j} = 0. \end{cases} \tag{3}$$

Hence, if the amount of the financial funds desired to be transacted by the consumers (at an instant in time) exceeds that available at the intermediary, then the price charged at the intermediary will increase; if the amount available is greater than that desired by the consumers, then the price charged at the intermediary will decrease. As in the case of the demand market prices, we guarantee that the prices charged by the intermediaries remain nonnegative.

Precursors to the Dynamics of the Financial Flows

We first introduce some preliminaries that will allow us to develop the dynamics of the financial flows over the links of the financial network. In particular, we discuss the utility-maximizing behavior of the source agents and the intermediaries.

We assume that each such source agent's and each intermediary agent's utility can be defined as a function of the expected future portfolio value, where the expected value of the future portfolio is described by two characteristics: the expected mean value and the uncertainty surrounding the expected mean. Here, the expected mean portfolio value is assumed to be equal to the market value of the current portfolio. Each agent's uncertainty, or assessment of risk, in turn, is based on a variance-covariance matrix denoting the agent's assessment of the standard deviation of the prices for each instrument/product. The variance-covariance matrix associated with source agent i's assets is denoted by Q^i and is of dimension $nL \times nL$, and is associated with vector x_i, whereas intermediary agent j's variance-covariance matrix is denoted by Q^j, is of dimension $o \times o$, and is associated with the vector y_j. For further discussion of such assumptions, see the books by Nagurney and Siokos (1997) and Markowitz (1959) and the references therein.

Optimizing Behavior of the Source Agents

We denote the total transaction cost associated with source agent i transacting with intermediary j to obtain financial instrument l by c_{ijl} and assume that:

$$c_{ijl} = c_{ijl}(x_{ijl}), \quad \forall i, j, l. \tag{4}$$

The total transaction costs incurred by source agent i, thus, are equal to the sum of all the agent's transaction costs. His revenue, in turn, is equal to the sum of the price (rate of return) that the the agent can obtain for the financial instrument times the total quantity obtained/purchased of that instrument. Recall that ρ_{1ijl} denotes the price associated with instrument l/agent i/intermediary j.

We assume that each such source agent seeks to maximize net return while, simultaneously, minimizing the risk, with source agent i's utility function denoted by U^i. Moreover, we assume that the variance-covariance matrix Q^i is positive-semidefinite and that the transaction cost functions are continuously differentiable and convex. Hence, we can express the optimization problem facing source agent i as:

$$\text{Maximize} \quad U_i(x_i) = \quad \sum_{j=1}^{n}\sum_{l=1}^{L}\rho_{1ijl}x_{ijl} - \sum_{j=1}^{n}\sum_{l=1}^{L}c_{ijl}(x_{ijl}) - {x_i}^T Q^i x_i, \tag{5}$$

subject to $x_{ijl} \geq 0$, for all j, l, and to the constraint:

$$\sum_{j=1}^{n}\sum_{l=1}^{L} x_{ijl} \leq S^i, \tag{6}$$

that is, the allocations of source agent i's funds among the financial instruments made available by the different intermediaries cannot exceed his holdings. Note that the utility function given in (5) is concave for each source agent i. Note that (6) allows a source agent to not invest in any of the instrument. Indeed, as we shall show through numerical examples in Section 5, this constraint has important financial implications.

Clearly, in the case of *unconstrained* utility maximization, $\nabla_{x_i} U_i = (\frac{\partial U_i}{\partial x_{i11}}, \ldots, \frac{\partial U_i}{\partial x_{inL}})$, represents agent $i's$ idealized direction, with the jl-component of $\nabla_{x_i} U_i$ given by:

$$(\rho_{1ijl} - 2Q^i_{z_{jl}} \cdot x_i - \frac{\partial c_{ijl}(x_{ijl})}{\partial x_{ijl}}), \tag{7}$$

where $Q^i_{z_{jl}}$ denotes the z_{jl}-th row of Q^i, where z_{jl} is the indicator defined as: $z_{jl} = (l-1)n+j$. We return later to describe how the constraints are explicitly incorporated into the dynamics.

Optimizing Behavior of the Intermediaries

The intermediaries (cf. Figure 1), in turn, are involved in transactions both with the source agents, as well as with the users of the funds, that is, with the ultimate consumers associated with the markets for the distinct types of loans/products at the bottom tier of the network. Thus, an intermediary conducts transactions both with the "source" agents as well as with the consumers at the demand markets.

An intermediary j is faced with what we term a *handling/conversion* cost, which may include, for example, the cost of converting the incoming financial flows into the financial loans/products associated with the demand markets. We denote this cost by c_j and, in the simplest case, we would have that c_j is a function of $\sum_{i=1}^m \sum_{l=1}^L x_{ijl}$, that is, the holding/conversion cost of an intermediary is a function of how much he has obtained from the various source agents. For the sake of generality, however, we allow the function to, in general, depend also on the amounts held by other intermediaries and, therefore, we may write:

$$c_j = c_j(x), \quad \forall j. \tag{8}$$

The intermediaries also have associated transaction costs in regards to transacting with the source agents, which we assume can be dependent on the type of instrument. We denote the transaction cost associated with intermediary j transacting with source agent i associated with instrument l by $\hat{c}_{ijl}$ and we assume that it is of the form

$$\hat{c}_{ijl} = \hat{c}_{ijl}(x_{ijl}), \quad \forall i, j, l. \tag{9}$$

Recall that the intermediaries convert the incoming financial flows x into the outgoing financial flows y. We assume that an intermediary j incurs a transaction cost c_{jk} associated with transacting with demand market k, where

$$c_{jk} = c_{jk}(y_{jk}), \quad \forall j, k. \tag{10}$$

The intermediaries associate a price with the financial funds, which recall is denoted by ρ_{2j}, for intermediary j. Assuming that the intermediaries are also utility maximizers with the utility functions for each being comprised of net revenue maximization as well as risk minimization, then the utility maximization problem for intermediary agent j with his utility function denoted by U^j, can be expressed as:

$$\text{Maximize} \quad U_j(x_j, y_j)$$

$$= \sum_{i=1}^{m}\sum_{l=1}^{L} \rho_{2j} x_{ijl} - c_j(x) - \sum_{i=1}^{m}\sum_{l=1}^{L} \hat{c}_{ijl}(x_{ijl}) - \sum_{k=1}^{o} c_{jk}(y_{jk}) - \sum_{i=1}^{m}\sum_{l=1}^{L} \rho_{1ijl} x_{ijl} - {y_j}^T Q^j y_j \tag{11}$$

subject to: the nonnegativity constraints: $x_{ijl} \geq 0$, and $y_{jk} \geq 0$, for all i, l and k. Here, for convenience, we have let $x_j = (x_{1j1}, \ldots, x_{mjL})$. Objective function (11) expresses that the difference between the revenues minus the handling cost and the transaction costs and the payout to the source agents should be maximized, whereas the risk should be minimized. We assume that the variance-covariance matrix Q^j is positive-semidefinite and that the transaction cost functions are continuously differentiable and convex. Hence, the utility function given in (11) is concave for each intermediary j.

Ignoring, for the time being, the constraints, $\nabla_{x_j} U_j = (\frac{\partial U_j}{\partial x_{1j1}}, \ldots, \frac{\partial U_j}{\partial x_{mjL}})$ represents agent j's idealized direction in terms of x_j, whereas $\nabla_{y_j} U_j = (\frac{\partial U_j}{\partial y_{j1}}, \ldots, \frac{\partial U_j}{\partial y_{jo}})$ represents his idealized direction in terms of y_j. Note that the il-th component of $\nabla_{x_j} U_j$ is given by:

$$(\rho_{2j} - \rho_{1ijl} - \frac{\partial c_j(x)}{\partial x_{ijl}} - \frac{\partial \hat{c}_{ijl}(x_{ijl})}{\partial x_{ijl}}), \tag{12}$$

whereas the jk-th component of $\nabla_{y_j} U_j$ is given by:

$$(-\frac{\partial c_{jk}(y_{jk})}{\partial y_{jk}} - 2Q^j_k \cdot y_j). \tag{13}$$

However, since both source agent i and intermediary j must agree in terms of the x_{ijl}s, the direction (7) must coincide with that in (12), yielding, after algebraic simplification:

$$(\rho_{2j} - 2Q^i_{z_{jl}} \cdot x_i - \frac{\partial c_{ijl}(x_{ijl})}{\partial x_{ijl}} - \frac{\partial c_j(x)}{\partial x_{ijl}} - \frac{\partial \hat{c}_{ijl}(x_{ijl})}{\partial x_{ijl}}). \tag{14}$$

The Dynamics of the Financial Flows Between the Source Agents and the Intermediaries

We are now ready to express the dynamics of the financial flows between the source agents and the intermediaries. In particular, we define the feasible set $K_i \equiv \{x_i | x_{ijl} \geq 0, \forall i, j, l, \text{ and } (6) \text{ holds}\}$. Let also K be the Cartesian product fiven by $K \equiv \Pi_{i=1}^{m} K_i$ and define F_{ijl}^1 as minus the term in (14) with $F_i^1 = (F_{i11}^1, \ldots, F_{inL}^1)$. Then the *best realizable* direction for the vector of financial instruments x_i can be mathematically expressed as:

$$\dot{x}_i = \pi_{K_i}(x_i, -F_i^1), \tag{15}$$

where $\pi_{\mathcal{K}}(X, v)$ is defined as (cf. Dupuis and Nagurney (1993) and Nagurney and Zhang (1996); see also Dong, Zhang, and Nagurney (1996)):

$$\pi_{\mathcal{K}}(X, v) = \lim_{\delta \to 0} \frac{P_{\mathcal{K}}(X + \delta v) - X}{\delta}, \tag{16}$$

and $P_{\mathcal{K}}$ is the norm projection defined by

$$P_{\mathcal{K}}(X) = \text{Argmin}_{X' \in \mathcal{K}} \|X' - X\|. \tag{17}$$

The Dynamics of the Financial Flows Between the Intermediaries and the Demand Markets

In terms of the financial flows between the intermediaries and the demand markets, both the intermediaries and the consumers must be in agreement as to the financial flows y. The consumers take into account in making their consumption decisions not only the price charged for the financial product by the intermediaries but also their transaction costs associated with obtaining the product.

Let $\hat{c}_{jk}$ denote the transaction cost associated with obtaining the product at demand market k from intermediary j. We assume that this unit transaction cost is continuous and of the general form:

$$\hat{c}_{jk} = \hat{c}_{jk}(y), \quad \forall j, k. \tag{18}$$

The consumers take the price charged by the intermediaries, which, recall was denoted by ρ_{2j} for intermediary j, plus the unit transaction cost, in making their consumption decisions.

From the perspective of the consumers at the demand markets, we can expect that an idealized direction in terms of the evolution of the financial flow of a product between an intermediary/demand market pair would be:

$$(\rho_{3k} - \hat{c}_{jk}(y) - \rho_{2j}). \tag{19}$$

On the other hand, as already derived above, we can expect that the intermediaries would adjust the volume of the product to a demand market according to (13). Combining now (13) and (19), and guaranteeing that the financial products do not assume negative quantities, yields the following dynamics:

$$\dot{y}_{jk} = \begin{cases} \rho_{3k} - \hat{c}_{jk}(y) - \rho_{2j} - \frac{\partial c_{jk}(y_{jk})}{\partial y_{jk}} - 2Q^j_k \cdot y_j, & \text{if} \quad y_{jk} > 0 \\ \max\{0, \rho_{3k} - \hat{c}_{jk}(y) - \rho_{2j} - \frac{\partial c_{jk}(y_{jk})}{\partial y_{jk}} - 2Q^j_k \cdot y_j, & \text{if} \quad y_{jk} = 0. \end{cases} \tag{20}$$

The Projected Dynamical System

Consider now the dynamic model in which the demand prices evolve according to (2) for all demand markets k, the prices at the intermediaries evolve according to (3) for all intermediaries j; the financial flows between the source agents and the intermediaries evolve according to (15) for all source agents i, and the financial products between the intermediaries and the demand markets evolve according to (20) for all intermediary/demand market pairs j, k.

Let now X denote the aggregate column vector (x, y, ρ_2, ρ_3) in the feasible set $\mathcal{K} \equiv K \times R^{no+n+o}_+$. Define the column vector $F(X) \equiv (F^1, F^2, F^3, F^4)$, where F^1 is as has been defined previously; $F^2 = (F^2_{11}, \ldots, F^2_{no})$, with component $F^2_{jk} = (2Q^j_k \cdot y_j + \frac{\partial c_{jk}(y_{jk})}{\partial y_{jk}} + \hat{c}_{jk}(y) + \rho_{2j} - \rho_{3k})$, $\forall j, k$; $F^3 = (F^3_1, \ldots, F^3_n)$, where $F^3_j \equiv (\sum_{i=1}^m \sum_{l=1}^L x_{ijl} - \sum_{k=1}^o y_{jk})$, and $F^4 = (F^4_1, \ldots, F^4_o)$, with $F^4_k \equiv (\sum_{j=1}^n y_{jk} - d_k(\rho_3))$.

Then the dynamic model described by (2), (3), (15), and (20) for all k, j, i, l can be rewritten as the *projected dynamical system* (PDS) (cf. Nagurney and Zhang (1996)) defined by the following initial value problem:

$$\dot{X} = \pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X_0, \tag{21}$$

where, as defined in (16), $\pi_{\mathcal{K}}$ is the projection operator of $-F(X)$ onto $\mathcal{K}$ at X and $X_0 = (x^0, y^0, \rho_2^0, \rho_3^0)$ is the initial point corresponding to the initial financial flows and the initial prices. The trajectory of (21) describes the dynamic evolution of and the dynamic interactions among the prices and the financial flows.

The dynamical system (21) is non-classical in that the right-hand side is discontinuous in order to guarantee that the constraints, which in the context of the above model are not only nonnegativity constraints on the variables, but also a form of budget constraints. Such dynamical systems were introduced by Dupuis and Nagurney (1993) and to-date have been used to model a variety of applications ranging from dynamic traffic network problems (cf. Nagurney and Zhang (1997)) and oligopoly problems (see Nagurney, Dupuis, and Zhang (1994)) and spatial price equilibrium problems (cf. Nagurney, Takayama, and Zhang (1995)). Here we apply this methodology, for the first time, to study financial systems in the presence of intermediation. A variety of dynamic financial models, but without intermediation, formulated as projected dynamical systems can be found in the book by Nagurney and Siokos (1997).

A Stationary/Equilibrium Point

We now discuss the stationary point of the projected dynamical system (21). Recall that a stationary point is that point when $\dot{X} = 0$ and, hence, in the context of our model, when there is no change in the financial flows in the financial network and no change in the prices. Moreover, as established in Dupuis and Nagurney (1993), since the feasible set $\mathcal{K}$ is a polyhedron and convex, the set of stationary points of the projected dynamical system of the form given in (21) coincides with the set of solutions to the variational inequality problem given by: Determine $X^* \in \mathcal{K}$, such that

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{22}$$

where in the model $F(X)$ and X are as defined above and $\langle \cdot, \cdot \rangle$ denotes the inner product in N-dimensional Euclidean space where here $N = mnL + no + n + o$. In particular, variational inequality (22) here takes the form: Determine $(x^*, y^*, \rho_2^*, \rho_3^*) \in \mathcal{K}$, satisfying:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{l=1}^{L} \left[2Q^i_{z_{jl}} \cdot x_i^* + \frac{\partial c_{ijl}(x^*_{ijl})}{\partial x_{ijl}} + \frac{\partial c_j(x^*)}{\partial x_{ijl}} + \frac{\partial \hat{c}_{ijl}(x^*_{ijl})}{\partial x_{ijl}} - \rho^*_{2j} \right] \times \left[x_{ijl} - x^*_{ijl} \right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[2Q_k^j \cdot y_j^* + \frac{\partial c_{jk}(y_{jk}^*)}{\partial y_{jk}} + \hat{c}_{jk}(y^*) + \rho_{2j}^* - \rho_{3k}^*\right] \times \left[y_{jk} - y_{jk}^*\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{L} x_{ijl}^* - \sum_{k=1}^{o} y_{jk}^*\right] \times \left[\rho_{2j} - \rho_{2j}^*\right]$$

$$+\sum_{k=1}^{o}\left[\sum_{j=1}^{n} y_{jk}^* - d_k(\rho_3^*)\right] \times [\rho_{3k} - \rho_{3k}^*] \geq 0, \quad \forall (x, y, \rho_2, \rho_3) \in \mathcal{K}, \tag{23}$$

where $\mathcal{K} \equiv \{K \times R_+^{no+n+o}\}$.

We now discuss the equilibrium conditions. First, note that if the rate of change of the demand price $\dot{\rho}_{3k} = 0$, then from (2) we can conclude that:

$$d_k(\rho_3^*) \begin{cases} = \sum_{j=1}^{n} y_{jk}^*, & \text{if} \quad \rho_{3k}^* > 0 \\ \leq \sum_{j=1}^{n} y_{jk}^*, & \text{if} \quad \rho_{3k}^* = 0. \end{cases} \tag{24}$$

Condition (24) states that, if the price the consumers are willing to pay for the financial product at a demand market is positive, then the quantity consumed by the consumers at the demand market is precisely equal to the demand. If the demand is less than the amount of the product available, then the price for that product is zero. This condition holds for all demand market prices in equilibrium.

Note that condition (24) also follows directly from variational inequality (23) if we set $x = x^*$; $y = y^*$, and $\rho_2 = \rho_2^*$, and make the substitution into (23) and note that the demand prices must be nonnegative.

Observe now that if the rate of change of a price charged by an intermediary is zero, that is, $\dot{\rho}_{2j} = 0$, then (3) implies that

$$\sum_{i=1}^{m}\sum_{l=1}^{L} x_{ijl}^* - \sum_{k=1}^{o} y_{jk}^* \begin{cases} = 0, & \text{if} \quad \rho_{2j}^* > 0 \\ \geq 0, & \text{if} \quad \rho_{2j}^* = 0. \end{cases} \tag{25}$$

In other words, if the price for the financial funds at an intermediary is positive, then the market for the funds "clears" at the intermediary, that is, the supply of funds, as given by

$\sum_{i=1}^{m}\sum_{l=1}^{L}x^*_{ijl}$ is equal to the demand of funds, $\sum_{k=1}^{o}y^*_{jk}$ at the intermediary. If the supply exceeds the demand, then the price at the intermediary will be zero. These are well-known economic equilibrium conditions as are those given in (24). Of course, condition (25) could also be recovered from variational inequality (23) by setting $x = x^*$, $y = y^*$, and $\rho_3 = \rho_3^*$, and making the substitution into (23) and noting that these prices must be nonnegative. In equilibrium, conditions (25) holds for all intermediary prices.

On the other hand, if we set $\dot{x}_i = 0$ (cf. (15)), for all i; $\dot{y}_{jk} = 0$ for all j, k (cf. (20)), we obtain that the following *equilibrium* conditions, which must be satisfied simultaneously.

Optimality Conditions for all Source Agents:

The optimality conditions for all source agents i, since each K_i is closed and convex, and the objective function (5) is concave, can be expressed as (assuming a given ρ^*_{1jl}, for all i, j, l, which we return to later (see also Bazaraa, Sherali, and Shetty (1993) and Bertsekas and Tsitsiklis (1992)) as:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{L}\left[2Q^i_{z_{jl}}\cdot x^*_i + \frac{\partial c_{ijl}(x^*_{ijl})}{\partial x_{ijl}} - \rho^*_{1ijl}\right]\times\left[x_{ijl} - x^*_{ijl}\right] \geq 0, \quad \forall x \in K. \tag{26}$$

Optimality Conditions for All Intermediary Agents:

The optimality conditions for all the intermediaries j, with objective functions of the form (11), which are concave, can be expressed as:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{L}\left[\frac{\partial c_j(x^*)}{\partial x_{ijl}} + \rho^*_{1ijl} + \frac{\partial \hat{c}_{ijl}(x^*_{ijl})}{\partial x_{ijl}} - \rho^*_{2j}\right]\times\left[x_{ijl} - x^*_{ijl}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[2Q^j_k\cdot y_j + \frac{\partial c_{jk}(y^*_{jk})}{\partial y_{jk}} + \rho^*_{2j}\right]\times\left[y_{jk} - y^*_{jk}\right] \geq 0, \quad \forall x \in R^{mnL}_+, \forall y \in R^{no}_+. \tag{27}$$

Note that (27) provides a means for recovering the top-tiered prices, ρ^*_1. Indeed, for each such y^*_{jk} we can set $y^*_{jk} = \rho^*_{2j} - \frac{\partial c_j(x^*)}{\partial x_{ijl}} - \frac{\partial \hat{c}_{ijl}(x^*_{ijl})}{\partial x_{ijl}}$. We do precisely this in Section 5, when we present numerical examples.

Equilibrium Conditions for Consumers at the Demand Markets

Also, the equilibrium conditions for consumers at demand market k, thus, take the form: for all intermediaries: $j; j = 1, \ldots, n$:

$$\rho^*_{2j} + \hat{c}_{jk}(y^*) \begin{cases} = \rho^*_{3k}, & \text{if} \quad y^*_{jk} > 0 \\ \geq \rho^*_{3k}, & \text{if} \quad y^*_{jk} = 0, \end{cases} \tag{28a}$$

with (28a) holding for all demand markets k, which is equivalent to $y^* \in R^{no}_+$ satisfying:

$$\sum_{j=1}^{n} \sum_{k=1}^{o} (\rho^*_{2j} + \hat{c}_{jk}(y^*) - \rho^*_{3k})) \times (y_{jk} - y^*_{jk}) \geq 0, \quad \forall y \in R^{no}_+. \tag{28b}$$

Conditions (28) simply state that consumers at demand market k will purchase the product from intermediary j, if the price charged by the intermediary for the product plus the transaction cost (from the perspective of the consumers) does not exceed the price that the consumers are willing to pay for the product, i.e., ρ^*_{3k}.

In equilibrium, optimality conditions (26), (27), as well as (28), (25), and (24) must hold simultaneously and these define the equilibrium state. This is equivalent to the vector $(x^*, y^*, \rho^*_2, \rho^*_3)$ satisfying variational inequality (23) or to being a stationary point of the projected dynamical system (21). Indeed, for (26), (27), and (28) to hold simultaneously, the sum of the first two terms in (23) must be greater than or equal to zero, whereas for (25) to hold the third term in (23) must be greater than or equal to zero, and, finally, for (24) to hold, the fourth term in (23) must be greater than or equal to zero.

In Nagurney and Ke Ke (2001) a variational inequality of the form (23) was derived in a manner entirely different from that given above for a static financial network model with intermediation, but with a slightly different feasible set since therein it was assumed that the constraints (6) had to be tight. In that paper, a different notation was also utilized for the prices at the intermediary nodes. Here the approach is more general since we derive the model by considering the disequilibrium dynamics.

3. Qualitative Properties

In this Section, we provide some qualitative properties of the dynamical financial network model with intermediation developed in Section 2. In particular, we provide conditions for establishing the existence of a unique trajectory to the initial value problem (21) and a global stability analysis result.

We now recall two results obtained by Nagurney and Ke Ke (2001), which we can utilize to obtain qualitative properties of the dynamical system (21) due to the similarity of variational inequality (23) and the one in the quoted paper. In particular, we have:

Theorem 1: Monotonicity (Nagurney and Ke Ke (2001))

If the c_{ijl}, c_j, and $\hat{c}_{ijl}$ and c_{jk} functions are convex, the $\hat{c}_{jk}$ functions are monotone increasing, the d_k functions are monotone decreasing functions of the demand market prices, for all i, j, k, l, and the variance-covariance matrices Q^i and Q^j are positive-semidefinite for all i and j, then the vector function F that enters the variational inequality (23) is monotone, that is,

$$\langle F(X') - F(X''), X' - X'' \rangle \geq 0, \quad \forall X', X'' \in \mathcal{K}. \tag{29}$$

Theorem 2: Lipschitz Continuity (Nagurney and Ke Ke (2001))

The function that enters the variational inequality problem (23) is Lipschitz continuous, that is,

$$\|F(X') - F(X'')\| \leq L\|X' - X''\|, \quad \forall X', X'' \in \mathcal{K}, \tag{30}$$

under the following conditions:

(i). c_{ijl}, c_j, $\hat{c}_{ijl}$ and c_{jk} have bounded second-order derivatives, for all i, j, l, k;

(ii). $\hat{c}_{jk}$ and d_k have bounded first-order derivatives.

We now state a fundamental property of the projected dynamical system (21).

Theorem 3: Existence and Uniqueness

Assume the conditions of Theorem 2. Then, for any $X_0 \in \mathcal{K}$*, there exists a unique solution* $X_0(t)$ *to the initial value problem (21).*

Proof: Follows from Theorem 2.5 in Nagurney and Zhang (1996). □

We now turn to addressing the stability (see also Zhang and Nagurney (1995) and Nagurney and Zhang (2001)) of the financial network system through the initial value problem (21). We first recall the following:

Definition 1: Stability of the System

The system defined by (21) is stable if, for every X_0 *and every equilibrium point* X^**, the Euclidean distance* $\|X^* - X_0(t)\|$ *is a monotone nonincreasing function of time* t*.*

We state a global stability result in the next theorem.

Theorem 4: Stability of the System

Assume the conditions of Theorem 1. Then the dynamical system (21) underlying the financial network system with intermediation is stable.

Proof: Under the assumptions of Theorem 1, $F(X)$ is monotone and, hence, the conclusion follows directly from Theorem 4.1 of Zhang and Nagurney (1995). □

From the above results, we see that the dynamic financial network model with intermediation as given by (21) is well-defined and, moreover, the system is stable.

4. The Discrete-Time Adjustment Process

Note that the projected dynamical system (21) is a continuous time adjustment process. However, in order to further fix ideas and to provide a means of "tracking" the trajectory, we propose a discrete-time adjustment process. The discrete-time adjustment process is a special case of the general iterative scheme of Dupuis and Nagurney (1993) and is, in fact, an Euler method, where at iteration τ the process takes the form:

$$X^{\tau} = P_{\mathcal{K}}(X^{\tau-1} - \alpha_{\tau-1}F(X^{\tau-1})), \tag{31}$$

where recall that $P_{\mathcal{K}}$ denotes the operator of projection (in the sense of the least Euclidean distance (cf. Nagurney (1999)) onto the closed convex set $\mathcal{K}$ and $F(X)$ is as defined preceding (21). Specifically, the complete statement of this method in the context of our model takes the form:

Step 0: Initialization Step

Set $(x^0, y^0, \rho_2^0, \rho_3^0) \in \mathcal{K}$. Let $\tau = 1$ and set the sequence $\{a_\tau\}$ so that $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$.

Step 1: Computation Step

Compute $(x^\tau, y^\tau, \rho_2^\tau, \rho_3^\tau) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{L}\left[x_{ijl}^{\tau} + a_\tau(2Q_{z_{jl}}^{i} \cdot x_i^{\tau-1} + \frac{\partial c_{ijl}(x_{ijl}^{\tau-1})}{\partial x_{ijl}} + \frac{\partial c_j(x^{\tau-1})}{\partial x_{ijl}} + \frac{\partial \hat{c}_{ijl}({x_{ijl}}^{\tau-1})}{\partial x_{ijl}} - \rho_{2j}^{\tau-1}) - x_{ijl}^{\tau-1}\right]$$

$$\times \left[x_{ijl} - x_{ijl}^{\tau}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[y_{jk}^{\tau} + a_\tau(2Q_k^i \cdot y_j^{\tau-1} + \hat{c}_{jk}(y^{\tau-1}) + \frac{\partial c_{jk}(y_{jk}^{\tau-1})}{\partial y_{jk}} + \rho_{2j}^{\tau-1} - \rho_{3k}^{\tau-1}) - y_{jk}^{\tau-1}\right] \times \left[y_{jk} - y_{jk}^{\tau}\right]$$

$$+\sum_{j=1}^{n}\left[\rho_{2j}^{\tau} + a_\tau(\sum_{i=1}^{m}\sum_{l=1}^{L} x_{ijl}^{\tau-1} - \sum_{k=1}^{o} y_{jk}^{\tau-1}) - \rho_{2j}^{\tau-1}\right] \times \left[\rho_{2j} - \rho_{2j}^{\tau}\right]$$

$$+\sum_{k=1}^{o}\left[\bar{\rho}_{3k}^{\tau} + a_\tau(\sum_{j=1}^{n} y_{jk}^{\tau-1} - d_k(\rho_3^{\tau-1})) - \rho_{3k}^{\tau-1}\right] \times [\rho_{3k} - \rho_{3k}^{\tau}] \geq 0, \quad \forall (x, y, \rho_2, \rho_3) \in \mathcal{K}. \tag{32}$$

Step 2: Convergence Verification

If $|x_{ijl}^{\tau} - x_{ijl}^{\tau-1}| \leq \epsilon$, $|y_{jk}^{\tau} - y_{jk}^{\tau-1}| \leq \epsilon$, $|\rho_{2j}^{\tau} - \rho_{2j}^{\tau-1}| \leq \epsilon$, $|\rho_{3k}^{\tau} - \rho_{3k}^{\tau-1}| \leq \epsilon$, for all $i = 1, \cdots, m$; $j = 1, \cdots, n$; $l = 1, \ldots, L$; $k = 1, \cdots, o$, with $\epsilon > 0$, a pre-specified tolerance, then stop; otherwise, set $\tau := \tau + 1$, and go to Step 1.

Note that the variational inequality subproblem (32) encountered at each iteration of the discrete-time algorithm can be solved explicitly and in closed form since it is actually a quadratic programming problem and the feasible set is a Cartesian product consisting of the the product of K, which has a simple network structure, and the nonnegative orthants: R_+^{no}, R_+^n, and R_+^o, and corresponding to the variables x, y, ρ_2, and ρ_3, respectively. In fact, the subproblem in (32) in the x variables can be solved using exact equilibration (cf. Dafermos and Sparrow (1969) and Nagurney (1999)), whereas the remainder of the variables in (32), can be obtained by explicit formulae, which we provide below for convenience.

In particular, we compute, at iteration τ, y^τ, according to:

$$y_{jk}^{\tau} = \max\{0, y_{jk}^{\tau-1} - a_\tau(2Q_k^i \cdot y_j^{\tau-1} + \hat{c}_{jk}(y^{\tau-1}) + \frac{\partial c_{jk}(y_{jk}^{\tau-1})}{\partial y_{jk}} + \rho_{2j}^{\tau-1} - \rho_{3k}^{\tau-1})\}, \quad \forall j, k. \quad (33)$$

At iteration τ, we compute the ρ_2^τ according to:

$$\rho_{2j}^{\tau} = \max\{0, \rho_{2j}^{\tau-1} - a_\tau(\sum_{i=1}^{m}\sum_{l=1}^{L} x_{ijl}^{\tau-1} - \sum_{k=1}^{o} y_{jk}^{\tau-1})\}, \quad \forall j, \quad (34)$$

whereas the ρ_3^τ are computed explicitly and in closed form according to:

$$\rho_{3k}^{\tau} = \max\{0, \rho_{3k}^{\tau-1} - a_\tau(\sum_{j=1}^{n} y_{jk}^{\tau-1} - d_k(\rho_3^{\tau-1}))\}, \quad \forall k. \quad (35)$$

Note, that in the discrete-time adjustment process, the financial flows and prices are updated simultaneously at each iteration.

Convergence conditions for this method can be found in Dupuis and Nagurney (1993) and interpreted in a variety of distinct applications in Nagurney and Zhang (1996).

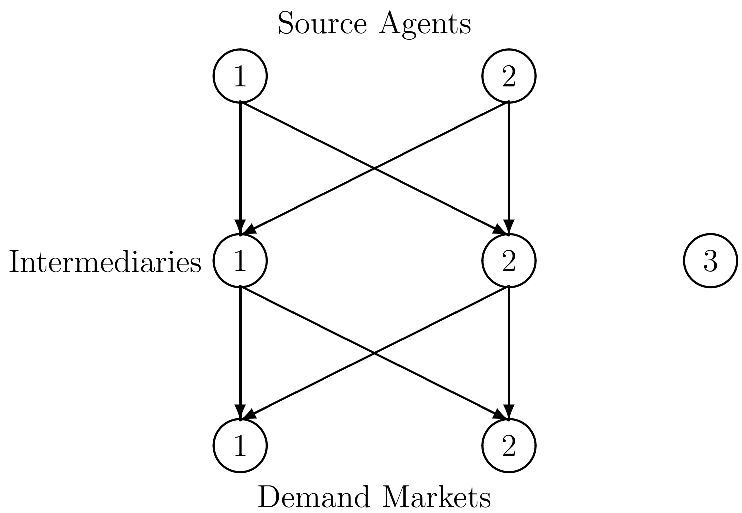

Figure 2: The Financial Network Structure of the Numerical Examples

5. Numerical Examples

In this Section, we apply the discrete-tme algorithm to several numerical examples. The algorithm was implemented in FORTRAN and the computer system used was a DEC Alpha system located at the University of Massachusetts at Amherst. For the solution of the induced network subproblems in x we utilized the exact equilibration algorithm (see Nagurney (1999) and the references therein).

The convergence criterion used was that the absolute value of the flows and prices between two successive iterations differed by no more than 10^{-4}. For the examples, the sequence $\{a_\tau\} = .1\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\}$, which is of the form given in the intialization step of the algorithm in the preceding section. The numerical examples had the network structure depicted in Figure 2 and consisted of two source agents, two intermediaries, and two demand markets, with a single financial instrument handled by each intermediary.

We initialized the algorithm as follows: Since there was a single financial instrument associated with each of the intermediaries, we set $x_{ij1} = \frac{S^i}{n}$ for each source agent i. All the other variables, that is, the initial vector y, ρ_2, and ρ_3 were all set to zero.

Example 1

The data for the first example were constructed for easy interpretation purposes. The supplies of the two source agents were: $S^1 = 10$ and $S^2 = 10$. The variance-covariance matrices Q^i and Q^j were equal to the identity matrices for all source agents i and all intermediaries j.

The transaction cost functions faced by the source agents associated with transacting with the intermediaries were given by:

$$c_{111}(x_{111}) = .5x_{111}^2 + 3.5x_{111}, \quad c_{121}(x_{121}) = .5x_{121}^2 + 3.5x_{121},$$

$$c_{211}(x_{211}) = .5x_{211}^2 + 3.5x_{211}, \quad c_{221}(x_{221}) = .5x_{221}^2 + 3.5x_{221}.$$

The handling costs of the intermediaries, in turn, were given by:

$$c_1(x) = .5(\sum_{i=1}^{2} .5x_{i11})^2, \quad c_2(x) = .5(\sum_{i=1}^{2} x_{i21})^2.$$

The transaction costs of the intermediaries associated with transacting with the source agents were, respectively, given by:

$$\hat{c}_{111}(x_{111}) = 1.5x_{111}^2 + 3x_{111}, \quad \hat{c}_{121}(x_{121}) = 1.5x_{121}^2 + 3x_{121},$$

$$\hat{c}_{211}(x_{211}) = 1.5x_{211}^2 + 3x_{211}, \quad \hat{c}_{221}(x_{221}) = 1.5x_{221}^2 + 3x_{221}.$$

The demand functions at the demand markets were:

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1000, \quad d_2(\rho_3) = -2\rho_{32} - 1.5\rho_{31} + 1000,$$

and the transaction costs between the intermediaries and the consumers at the demand markets were given by:

$$\hat{c}_{11}(y) = y_{11} + 5, \quad \hat{c}_{12}(y) = y_{12} + 5, \quad \hat{c}_{21}(y) = y_{21} + 5, \quad \hat{c}_{22}(y) = y_{22} + 5.$$

We assumed for this and the subsequent examples that the transaction costs as perceived by the intermediaries and associated with transacting with the demand markets were all zero, that is, $c_{jk}(y_{jk}) = 0$, for all j, k.

The discrete-time algorithm converged and yielded the following equilibrium pattern:

$$x^*_{111} = x^*_{121} = x^*_{211} = x^*_{221} = 5.000,$$

$$y^*_{11} = y^*_{12} = y^*_{21} = y^*_{22} = 5.000.$$

The vector ρ^*_2 had components: $\rho^*_{21} = \rho^*_{22} = 262.6664$, and the computed demand prices at the demand markets were: $\rho^*_{31} = \rho^*_{32} = 282.8106$.

It is easy to verify that the optimality/equilibrium conditions were satisfied with good accuracy. Note that in this example, constraint (6) was tight for both source agents, that is, $s^*_1 = s^*_2 = 0$, and, hence, there was zero flow on the links connecting node 3 with top tier nodes 1 and 2. Thus, it was optimal for both source agents to invest their entire financial holdings in each instrument made available by each of the two intermediaries.

The prices ρ_{1ijl} were as follows and were recovered according to the discussion following (27). The ρ^*_{1ijl}s were as follows: All ρ^*_{1ij1}s= 234.6664.

Example 2

We then constructed the following variant of Example 1. We kept the data identical to that in Example 1 except that we increased the supply for each source sector so that $S^1 = S^2 = 50$.

The discrete-time algorithm converged and yielded the following new equilibrium pattern:

$$x^*_{111} = x^*_{121} = x^*_{211} = x^*_{221} = 23.6832,$$

$$y^*_{11} = y^*_{12} = y^*_{21} = y^*_{22} = 23.7247.$$

The vector ρ^*_2 had components: $\rho^*_{21} = \rho^*_{22} = 196.0174$, and the demand prices at the demand markets were: $\rho^*_{31} = \rho^*_{32} = 272.1509$.

It is easy to verify that the optimality/equilibrium conditions, again, were satisfied with good accuracy. Note, however, that unlike the solution for Example 1, both source agent 1 and source agent 2 did not invest their entire financial holdings. Indeed, each opted to not invest the amount 23.7209 and this was the volume of flow on each of the two links ending in node 3 in Figure 2.

The prices ρ^*_{1ijl} were as follows and were recovered according to the discussion following (27). All the ρ^*_{1ij1}s= 74.6013. Note that since the supply of financial funds increased, the price for the instruments charged by the intermediaries decreased from 262.6664 to 196.1074. The demand prices at the demand markets also decreased, from 282.8106 to 272.1509.

Example 3

We then modified Example 2 as follows: The data were identical to that in Example 2 except that we modified the first diagonal term in the variance-covariance matrix Q^1 from 1 to 2.

The discrete-time algorithm converged, yielding the following new equilibrium pattern:

$$x^*_{111} = 18.8676, \quad x^*_{121} = 23.7285, \quad x^*_{211} = 25.1543, \quad x^*_{221} = 23.7267,$$

$$y^*_{11} = y^*_{12} = 22.0501, \quad y^*_{21} = y^*_{22} = 23.7592.$$

The vector ρ^*_2 had components: $\rho^*_{21} = 201.4985$, $\rho^*_{22} = 196.3633$, and the demand prices at the demand markets were: $\rho^*_{31} = \rho^*_{32} = 272.6178$.

The prices ρ_{1ijl} at equilibrium were as follows: respectively: $\rho^*_{1111} = 97.8737$, $\rho^*_{1121} = 74.7227$, $\rho^*_{1211} = 79.0138$, and $\rho^*_{1221} = 74.7281$.

Example 4

The fourth example was constructed from Example 3. In particular, we made a single change to the second demand function by modifying the fixed term 1000 to 1200. Thus, in effect, we increased the demand for the second financial product.

The discrete-time algorithm yielded the following equilibrium pattern:

$$x^*_{111} = 22.3404, \quad x^*_{121} = 27.6596, \quad x^*_{211} = 26.0638, \quad x^*_{221} = 23.9362,$$

$$y^*_{11} = 0.0000, \quad y^*_{12} = 48.6257, \quad y^*_{21} = 0.0000, \quad y^*_{22} = 51.8172.$$

The vector ρ^*_2 had components: $\rho^*_{21} = 248.4244$, $\rho^*_{22} = 238.8500$, and the demand prices at the demand markets were: $\rho^*_{31} = 200.6086$, $\rho^*_{32} = 399.2637$.

Hence, the constraints (6) were tight for each source agent and the flows on the links ending in node 3 in Figure 2 were equal to zero.

The prices ρ_{1ijl} at equilibrium were as follows: $\rho^*_{1111} = 129.9989$, $\rho^*_{1121} = 101.2755$, $\rho^*_{1211} = 118.887$, and $\rho^*_{1221} = 112.4457$.

Note that since now the demand for financial product 1 is identically equal to zero and, hence, there are zero flows on links joining intermediary nodes 1 and 2 to the bottom tier node 1.

6. Summary and Conclusions

In this paper, we developed a framework for the formulation, qualitative analysis, and computation of solutions to dynamic financial network problems with intermediation. The financial network consists of a multi-tierd network. In particular, we proposed a projected dynamical systems model by which agents in the form of source agents and intermediaries, who are assumed to be utility maximizers, with their utility functions comprised of net return and risk terms, determine their optimal allocations among financial instrument and products, in response to demand market prices and the prices associated with the intermediaries, which consumers also respond to. The prices adjust dynamically according to supply and demand conditions.

We discussed the stationary/equilibrium point, established qualitative properties of the dynamical trajectory, and also obtained a global stability analysis result.

We then proposed a discrete-time algorithm which is an approximation to the continuous-time adjustment process and applied it to several numerical examples. This research extends the work in general financial equilibrium modeling, analysis, and computation in a network framework (see, e.g., Nagurney and Siokos (1997) and Thore (1980)) to include financial intermediaries and the underlying dynamics associated with the financial flows and the prices. It, thus, brings the added dimension, lacking in financial flow of funds accounts, of the incorporation of the explicit dynamic behavior of the various financial agents as well as the price dynamics.

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