Identification of Critical Nodes and Links in Financial Networks with Intermediation and Electronic Transactions

Anna Nagurney and Qiang Qiang

Isenberg School of Management
University of Massachusetts
Amherst, Massachusetts 01003
nagurney@gbfin.umass.edu

Contribution for *Computational Methods in Financial Engineering*, E. J. Kontoghiorghes, B. Rustem, and P. Winker, Editors, Springer, Germany (2007).

Summary. In this paper, we propose a network performance/efficiency measure for the evaluation of financial networks with intermediation. The measure captures risk, transaction cost, price, transaction flow, revenue, and demand information in the context of the decision-makers' behavior in multitiered financial networks that also allow for electronic transactions. The measure is then utilized to define the importance of a financial network component, that is, a node or a link, or a combination of nodes and links. Numerical examples are provided in which the efficiency of the financial network is computed along with the importance ranking of the nodes and links. The results in this paper can be used to assess which nodes and links in financial networks are the most vulnerable in the sense that their removal will impact the efficiency of the network in the most significant way. Hence, the results in this paper have relevance to national security as well as implications for the insurance industry.

Key words: financial networks, financial intermediation, risk management, portfolio optimization, complex networks, supernetworks, critical infrastructure networks, electronic finance, network efficiency measure, network performance, network vulnerability, network disruptions, network security, network equilibrium, variational inequalities

1 Introduction

The study of financial networks dates to the 1750s when Quesnay (1758), in his *Tableau Economique*, conceptualized the circular flow of financial funds in an economy as a network. Copeland (1952) further explored the relationships among financial funds as a network and asked the question, *"Does money flow like water or electricity?"* The advances in information technology and globalization have further shaped today's financial world into a complex network, which is characterized by distinct sectors, the proliferation of new financial instruments, and with increasing international diversification of portfolios. Recently, financial networks have been studied using network models with multiple tiers of decision-makers, including intermediaries. For a detailed literature review of financial networks, please refer to the paper by Nagurney (2007) (see also Fei (1960), Charnes and Cooper (1967), Thore (1969), Thore and Kydland (1972), Thore (1980), Christofides, Hewins, and Salkin (1979), Crum and Nye (1981), Mulvey (1987), Nagurney and Hughes (1992), Nagurney, Dong and Hughes (1992), Nagurney and Siokos (1997), Nagurney and Ke (2001, 2003), Boginski, Butenko, and Pardalos (2003), Geunes and Pardalos (2003), Nagurney and Cruz (2003a, 2003b), Nagurney, Wakolbinger, and Zhao (2006), and the references therein). Furthermore, for a detailed discussion of optimization, risk modeling, and network equilibrium problems in finance and economics, please refer to the papers in the book edited by Kontoghiorghes, Rustem, and Siokos (2002).

Since today's financial networks may be highly interconnected and interdependent, any disruptions that occur in one part of the network may produce consequences in other parts of the network, which may not only be in the same region but many thousands of miles away in other countries. As pointed out by Sheffi (2005) in his book, one of the main characteristics of disruptions in networks is "the seemingly unrelated consequences and vulnerabilities stemming from global connectivity." For example, the unforgettable 1987 stock market crash was, in effect, a chain reaction throughout the world; it originated in Hong Kong, then propagated to Europe, and, finally, the United States. It is, therefore, crucial for the decision-makers in financial networks, including, managers, to be able to identify a network's vulnerable components in order to protect the functionality of the network. The management at Merrill Lynch well understood the criticality of their operations in the World Trade Center and made contingency plans. Right after the 9/11 terrorist attacks, they were able to switch their operations from the World Trade Center to the backup centers and the redundant trading floors near New York City. Therefore, the company managed to mitigate the losses for both its customers and itself (see Sheffi (2005)).

Notably, the analysis and the identification of the vulnerable components in networks have, recently, emerged as a major research theme, especially in the study of what are commonly referred to as *complex* networks, or, collectively, as *network science* (see the survey by Newman (2003)). However, in order to be able to evaluate the vulnerability and the reliability of a network, a measure that can quantifiably capture the efficiency of a network must be developed. In a series of papers, Latora and Marchiori (2001, 2003, 2004) discussed the network performance issue by measuring the "global efficiency" in a weighted network as compared to that of the simple non-weighted small-world network. The weight on each link is the geodesic distance between the nodes that the link connects. This measure has been applied by the above authors to evaluate the importance of network components in a variety of networks, including the (MBTA) Boston subway transportation network and the Internet (cf. Latora and Marchiori 2002, 2004).

However, the Latora-Marchiori network efficiency measure does not take into consideration the flow on networks, which we believe is a crucial indicator of network performance as well as network vulnerability. Indeed, flows represent the usage of a network and which paths and links have positive flows and the magnitude of these flows are relevant in the case of network disruptions. Qiang and Nagurney (2006) proposed a network efficiency measure that can be used to assess the network efficiency in the case of either fixed or elastic demands. The measure proposed by Qiang and Nagurney (2006) captures flow information and user/decision-maker behavior, and also allows one to determine the criticality of various nodes (as well as links) through the identification of their importance and ranking. In particular, Nagurney and Qiang (2007a, b) were able to demonstrate the applicability of the new measure, in the case of fixed demands, to, respectively, transportation networks, as well as to other critical infrastructure networks, including electric power generation and distribution networks, in the form of supply chains. Interestingly, the above network measure contains, as a special case, the Latora-Marchiori network efficiency measure but is more general because besides costs, it also captures flows and behavior on the network as established in Nagurney and Qiang (2007a, b).

As extremely important infrastructure networks, financial networks have a great impact on the global economy, and their study has recently also attracted attention from researchers in the area of complex networks. For example, Onnela, Kaski and Kertész (2004) studied a financial network in which the nodes are stocks and the edges are the correlations among the prices of stocks (see also, Kim and Jeong (2005)). Caldarelli et al. (2004) studied different financial networks, namely, board and director networks, and stock ownership networks and discovered that all these networks displayed scale-free properties (see also Boginski, Butenko, and Pardalos (2003)).

Several recent studies in finance, in turn, have analyzed the local consequences of catastrophes and the design of risk sharing/management mechanisms since the occurrence of major events such as 9/11 and Hurricane Katrina (see, for example, Gilli and Këllezi (2006), Loubergé, Këllezi, and Gilli (1999), Doherty (1997), Niehaus (2002), and the references therein).

Nevertheless, there is very little literature that addresses the vulnerability of financial networks. Robinson, Woodard, and Varnado (1998) discussed, from the policy-making point of view, how to protect the critical infrastructure in the US, including financial networks. Odell and Phillips (2001) conducted an empirical study to analyze the impact of the 1906 San Francisco earthquake on the bank loan rates in the financial network within San Francisco. To the best of our knowledge, however, there is no network efficiency measure to-date that has been applied to financial networks that captures both economic behavior as well as the underlying network/graph structure. In this paper, we propose a novel financial network efficiency measure, which is motivated by Qiang and Nagurney (2006). The measure is then applied to identify the importance and, hence, the vulnerability, of the financial network components.

The paper is organized as follows. In Section 2, we briefly recall the financial network model with intermediation of Liu and Nagurney (2006), which provides the basis for our financial network model with intermediation and electronic transactions. The financial network efficiency measure is developed in Section 3, along with the associated definition of the importance of network components. Section 4 then presents two financial network examples for which the efficiencies are computed and the node and link importance rankings determined. The paper concludes with Section 5.

2 The Financial Network Model with Intermediation and Electronic Transactions

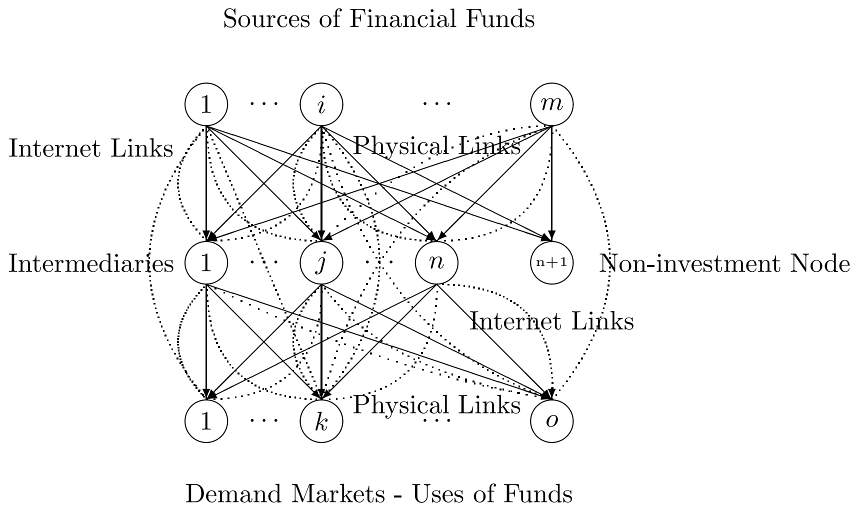

Fig. 1. The Structure of the Financial Network with Intermediation and with Electronic Transactions

In this Section, we recall the financial network model with intermediation and with electronic transactions in the case of known inverse demand functions associated with the financial products at the demand markets (cf. Liu and Nagurney (2006)).

The financial network consists of m sources of financial funds, n financial intermediaries, and o demand markets, as depicted in Figure 1. In the financial network model, the financial transactions are denoted by the links with the transactions representing electronic transactions delineated by hatched links. The majority of the notation for this model is given in Table 1.

Table 1. Notation for the Financial Network Model

Notation	Definition
S	m-dimensional vector of the amounts of funds held by the source agents with component i denoted by S^i
q_i	$(2n+o)$-dimensional vector associated with source agent i; $i=1,\ldots,m$ with components: $\{q_{ijl}; j=1,\ldots,n; l=1,2; q_{ik}; k=1,\ldots,o\}$
q_j	$(2m+2o)$-dimensional vector associated with intermediary j; $j=1,\ldots,n$ with components: $\{q_{ijl}; i=1,\ldots,m; l=1,2; q_{jkl}; k=1,\ldots,o; l=1,2\}$
Q^1	$2mn$-dimensional vector of all the financial transactions/flows for all source agents/intermediaries/modes with component ijl denoted by q_{ijl}
Q^2	mo-dimensional vector of the electronic financial transactions/flows between the sources of funds and the demand markets with component ik denoted by q_{ik}
Q^3	$2no$-dimensional vector of all the financial transactions/flows for all intermediaries/demand markets/modes with component jkl denoted by q_{jkl}
g	n-dimensional vector of the total financial flows received by the intermediaries with component j denoted by g_j, with $g_j \equiv \sum_{i=1}^{m}\sum_{l=1}^{2} q_{ijl}$
γ	n-dimensional vector of shadow prices associated with the intermediaries with component j denoted by γ_j
d	o-dimensional vector of market demands with component k denoted by d_k
$\rho_{3k}(d)$	the demand price (inverse demand) function at demand market k
V^i	the $(2n+o)\times(2n+o)$ dimensional variance-covariance matrix associated with source agent i
V^j	the $(2m+2o)\times(2m+2o)$ dimensional variance-covariance matrix associated with intermediary j
$c_{ijl}(q_{ijl})$	the transaction cost incurred by source agent i in transacting with intermediary j using mode l with the marginal transaction cost denoted by $\frac{\partial c_{ijl}(q_{ijl})}{\partial q_{ijl}}$
$c_{ik}(q_{ik})$	the transaction cost incurred by source agent i in transacting with demand market k with marginal transaction cost denoted by $\frac{\partial c_{ik}(q_{ik})}{\partial q_{ik}}$
$c_{jkl}(q_{jkl})$	the transaction cost incurred by intermediary j in transacting with demand market k via mode l with marginal transaction cost denoted by $\frac{\partial c_{jkl}(q_{jkl})}{\partial q_{jkl}}$
$c_j(Q^1) \equiv c_j(g)$	conversion/handling cost of intermediary j with marginal handling cost with respect to g_j denoted by $\frac{\partial c_j}{\partial g_j}$ and the marginal handling cost with respect to q_{ijl} denoted by $\frac{\partial c_j(Q^1)}{\partial q_{ijl}}$
$\hat{c}_{ijl}(q_{ijl})$	the transaction cost incurred by intermediary j in transacting with source agent i via mode l with the marginal transaction cost denoted by $\frac{\partial \hat{c}_{ijl}(q_{ijl})}{\partial q_{ijl}}$
$\hat{c}_{jkl}(Q^2,Q^3)$	the unit transaction cost associated with obtaining the product at demand market k from intermediary j via mode l
$\hat{c}_{ik}(Q^2,Q^3)$	the unit transaction cost associated with obtaining the product at demand market k from source agent i

All vectors are assumed to be column vectors. The equilibrium solutions throughout this paper are denoted by *.

The m agents or sources of funds at the top tier of the financial network in Figure 1 seek to determine the optimal allocation of their financial resources transacted either physically or electronically with the intermediaries or electronically with the demand markets. Examples of source agents include: households and businesses. The financial intermediaries, in turn, which can include banks, insurance companies, investment companies, etc., in addition to transacting with the source agents determine how to allocate the incoming financial resources among the distinct uses or financial products associated with the demand markets, which correspond to the nodes at the bottom tier of the financial network in Figure 1. Examples of demand markets are: the markets for real estate loans, household loans, business loans, etc. The transactions between the financial intermediaries and the demand markets can also take place physically or electronically via the Internet.

We denote a typical source agent by i; a typical financial intermediary by j, and a typical demand market by k. The mode of transaction is denoted by l with $l = 1$ denoting the physical mode and with $l = 2$ denoting the electronic mode.

We now describe the behavior of the decision-makers with sources of funds. We then discuss the behavior of the financial intermediaries and, finally, the consumers at the demand markets. Subsequently, we state the financial network equilibrium conditions and derive the variational inequality formulation governing the equilibrium conditions.

The Behavior of the Source Agents

The behavior of the decision-makers with sources of funds, also referred to as source agents is briefly recalled below (see Liu and Nagurney (2006)).

Since there is the possibility of non-investment allowed, the node $n + 1$ in the second tier in Figure 1 represents the "sink" to which the uninvested portion of the financial funds flows from the particular source agent or source node. We then have the following conservation of flow equations:

$$\sum_{j=1}^{n}\sum_{l=1}^{2} q_{ijl} + \sum_{k=1}^{o} q_{ik} \le S^i, \quad i = 1, \ldots, m, \tag{1}$$

that is, the amount of financial funds available at source agent i and given by S^i cannot exceed the amount transacted physically and electronically with the intermediaries plus the amount transacted electronically with the demand markets. Note that the "slack" associated with constraint (1) for a particular source agent i is given by $q_{i(n+1)}$ and corresponds to the uninvested amount of funds.

Let ρ_{1ijl} denote the price charged by source agent i to intermediary j for a transaction via mode l and, let ρ_{1ik} denote the price charged by source agent i for the electronic transaction with demand market k. The ρ_{1ijl} and ρ_{1ik} are endogenous variables and their equilibrium values ρ^*_{1ijl} and ρ^*_{1ik}; $i = 1, \ldots, m$; $j = 1, \ldots, n$; $l = 1, 2$, $k = 1, \ldots, o$ are determined once the complete financial network model is solved. As noted earlier, we assume that each source agent seeks to maximize his net revenue and to minimize his risk. For further background on risk management, see Rustem and Howe (2002). We assume as in Liu and Nagurney (2006) that the risk for source agent i is represented by the variance-covariance matrix V^i so that the optimization problem faced by source agent i can be expressed as:

$$\text{Maximize} \quad U^i(q_i) = \sum_{j=1}^{n}\sum_{l=1}^{2} \rho^*_{1ijl} q_{ijl} + \sum_{k=1}^{o} \rho^*_{1ik} q_{ik} - \sum_{j=1}^{n}\sum_{l=1}^{2} c_{ijl}(q_{ijl})$$

$$- \sum_{k=1}^{o} c_{ik}(q_{ik}) - q_i^T V^i q_i \tag{2}$$

subject to:

$$\sum_{j=1}^{n}\sum_{l=1}^{2} q_{ijl} + \sum_{k=1}^{o} q_{ik} \le S^i$$

$$q_{ijl} \ge 0, \quad \forall j, l,$$

$$q_{ik} \ge 0, \quad \forall k,$$

$$q_{i(n+1)} \ge 0.$$

The first four terms in the objective function (2) represent the net revenue of source agent i and the last term is the variance of the return of the portfolio, which represents the risk associated with the financial transactions.

We assume that the transaction cost functions for each source agent are continuously differentiable and convex, and that the source agents compete in a noncooperative manner in the sense of Nash (1950, 1951). The optimality conditions for all decision-makers with source of funds simultaneously coincide with the solution of the following variational inequality: determine $(Q^{1*}, Q^{2*}) \in \mathcal{K}^0$ such that:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^i_{z_{jl}} \cdot q_i^* + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \rho^*_{1ijl}\right] \times \left[q_{ijl} - q^*_{ijl}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^i_{z_{2n+k}} \cdot q_i^* + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}} - \rho^*_{1ik}\right] \times [q_{ik} - q^*_{ik}] \geq 0,$$

$$\forall (Q^1, Q^2) \in \mathcal{K}^0, \tag{3}$$

where $V^i_{z_{jl}}$ denotes the z_{jl}-th row of V^i and z_{jl} is defined as the indicator: $z_{jl} = (l-1)n + j$. Similarly, $V^i_{z_{2n+k}}$ denotes the z_{2n+k}-th row of V^i but with z_{2n+k} defined as the $2n+k$-th row, and the feasible set $\mathcal{K}^0 \equiv \{(Q^1, Q^2) | (Q^1, Q^2) \in R_+^{2mn+mo} \text{and (1) holds for all}\, i\}$.

The Behavior of the Financial Intermediaries

The behavior of the intermediaries in the financial network model of Liu and Nagurney (2006) is recalled below.

Let the endogenous variable ρ_{2jkl} denote the product price charged by intermediary j with ρ^*_{2jkl} denoting the equilibrium price, where $j = 1, \ldots, n$; $k = 1, \ldots, o$, and $l = 1, 2$. We assume that each financial intermediary also seeks to maximize his net revenue while minimizing his risk. Note that a financial intermediary, by definition, may transact either with decision-makers in the top tier of the financial network as well as with consumers associated with the demand markets in the bottom tier. Noting the conversion/handling cost as well as the various transaction costs faced by a financial intermediary and recalling that the variance-covariance matrix associated with financial intermediary j is given by V^j (cf. Table 1), we have that the financial intermediary is faced with the following optimization problem:

$$\text{Maximize} \quad U^j(q_j) = \sum_{k=1}^{o}\sum_{l=1}^{2}\rho^*_{2jkl}q_{jkl} - c_j(Q^1) - \sum_{i=1}^{m}\sum_{l=1}^{2}\hat{c}_{ijl}(q_{ijl})$$
$$-\sum_{k=1}^{o}\sum_{l=1}^{2}c_{jkl}(q_{jkl}) - \sum_{i=1}^{m}\sum_{l=1}^{2}\rho^*_{1ijl}q_{ijl} - q_j^T V^j q_j \tag{4}$$

subject to:

$$\sum_{k=1}^{o}\sum_{l=1}^{2}q_{jkl} \leq \sum_{i=1}^{m}\sum_{l=1}^{2}q_{ijl}, \tag{5}$$
$$q_{ijl} \geq 0, \quad \forall i, l,$$
$$q_{jkl} \geq 0, \quad \forall k, l.$$

The first five terms in the objective function (4) denote the net revenue, whereas the last term is the variance of the return of the financial allocations, which represents the risk to each financial intermediary. Constraint (5) guarantees that an intermediary cannot reallocate more of its financial funds among the demand markets than it has available.

Let γ_j be the Lagrange multiplier associated with constraint (5) for intermediary j. We assume that the cost functions are continuously differentiable and convex, and that the intermediaries compete in a noncooperative manner. Hence, the optimality conditions for all intermediaries simultaneously can be expressed as the following variational inequality: determine $(Q^{1*}, Q^{3*}, \gamma^*) \in R_+^{2mn+2no+n}$ satisfying:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^j_{z_{il}}\cdot q_j^* + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \rho^*_{1ijl} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \gamma_j^*\right] \times \left[q_{ijl} - q^*_{ijl}\right]$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V^j_{z_{kl}}\cdot q_j^* + \frac{\partial c_{jkl}(q^*_{jkl})}{\partial q_{jkl}} - \rho^*_{2jkl} + \gamma_j^*\right] \times \left[q_{jkl} - q^*_{jkl}\right]$$
$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2}q^*_{ijl} - \sum_{k=1}^{o}\sum_{l=1}^{2}q^*_{jkl}\right] \times \left[\gamma_j - \gamma_j^*\right] \geq 0,$$
$$\forall (Q^1, Q^3, \gamma) \in R_+^{2mn+2no+n}, \tag{6}$$

where $V^j_{z_{il}}$ denotes the z_{il}-th row of V^j and z_{il} is defined as the indicator: $z_{il} = (l-1)m + i$. Similarly, $V^j_{z_{kl}}$ denotes the z_{kl}-th row of V^j and

z_{kl} is defined as the indicator: $z_{kl} = 2m + (l-1)o + k$.

Additional background on risk management in finance can be found in Nagurney and Siokos (1997); see also the book by Rustem and Howe (2002).

The Consumers at the Demand Markets and the Equilibrium Conditions

By referring to the model of Liu and Nagurney (2006), we now assume, as given, the inverse demand functions $\rho_{3k}(d)$; $k = 1, \ldots, o$, associated with the demand markets at the bottom tier of the financial network. Recall that the demand markets correspond to distinct financial products. Of course, if the demand functions are invertible, then one may obtain the price functions simply by inversion.

The following conservation of flow equations must hold:

$$d_k = \sum_{j=1}^{n} \sum_{l=1}^{2} q_{jkl} + \sum_{i=1}^{m} q_{ik}, \quad k = 1, \ldots, o. \tag{7}$$

Equations (7) state that the demand for the financial product at each demand market is equal to the financial transactions from the intermediaries to that demand market plus those from the source agents.

The equilibrium condition for the consumers at demand market k are as follows: for each intermediary j; $j = 1, \ldots, n$ and mode of transaction l; $l = 1, 2$:

$$\rho^*_{2jkl} + \hat{c}_{jkl}(Q^{2*}, Q^{3*}) \begin{cases} = \rho_{3k}(d^*), \text{ if } & q^*_{jkl} > 0 \\ \geq \rho_{3k}(d^*), \text{ if } & q^*_{jkl} = 0. \end{cases} \tag{8}$$

In addition, we must have that, in equilibrium, for each source of funds i; $i = 1, \ldots, m$:

$$\rho^*_{1ik} + \hat{c}_{ik}(Q^{2*}, Q^{3*}) \begin{cases} = \rho_{3k}(d^*), \text{ if } & q^*_{ik} > 0 \\ \geq \rho_{3k}(d^*), \text{ if } & q^*_{ik} = 0. \end{cases} \tag{9}$$

Condition (8) states that, in equilibrium, if consumers at demand market k purchase the product from intermediary j via mode l, then the price the consumers pay is exactly equal to the price charged by the intermediary plus the unit transaction cost via that mode. However, if the sum of price charged by the intermediary and the unit transaction cost is greater than the price the consumers are willing to pay at the demand market, there will be no transaction between this intermediary/demand market pair via that mode. Condition (9) states the analogue but for the case of electronic transactions with the source agents.

In equilibrium, conditions (8) and (9) must hold for all demand markets. We can also express these equilibrium conditions using the following variational inequality: determine $(Q^{2*}, Q^{3*}, d^*) \in \mathcal{K}^1$, such that

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[\rho_{2jkl}^{*} + \hat{c}_{jkl}(Q^{2*}, Q^{3*})\right] \times \left[q_{jkl} - q_{jkl}^{*}\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\rho_{1ik}^{*} + \hat{c}_{ik}(Q^{2*}, Q^{3*})\right] \times [q_{ik} - q_{ik}^{*}]$$

$$-\sum_{k=1}^{o}\rho_{3k}(d^*) \times [d_k - d_k^*] \geq 0, \quad \forall (Q^2, Q^3, d) \in \mathcal{K}^1, \tag{10}$$

where $\mathcal{K}^1 \equiv \{(Q^2, Q^3, d) | (Q^2, Q^3, d) \in R_{+}^{2no+mo+o}$ and (7) holds.$\}$

The Equilibrium Conditions for Financial Network with Electronic Transactions

In equilibrium, the optimality conditions for all decision-makers with source of funds, the optimality conditions for all the intermediaries, and the equilibrium conditions for all the demand markets must be simultaneously satisfied so that no decision-maker has any incentive to alter his or her decision. We recall the equilibrium condition in Liu and Nagurney (2006) for the entire financial network with intermediation and electronic transactions as follows.

Definition 1: Financial Network Equilibrium with Intermediation and with Electronic Transactions

The equilibrium state of the financial network with intermediation is one where the financial flows between tiers coincide and the financial flows and prices satisfy the sum of conditions (3), (6), and (10).

We now define the feasible set:

$$\mathcal{K}^2 \equiv \{(Q^1, Q^2, Q^3, \gamma, d) | (Q^1, Q^2, Q^3, \gamma, d) \in R_+^{m+2mn+2no+n+o}$$
$$\text{and (1) and (7) hold}\}$$

and state the following theorem. For the proof of Theorem 1, please refer to the paper by Liu and Nagurney (2006).

Theorem 1: Variational Inequality Formulation

The equilibrium conditions governing the financial network model with intermediation are equivalent to the solution to the variational inequality problem given by: determine $(Q^{1^*}, Q^{2^*}, Q^{3^*}, \gamma^*, d^*) \in \mathcal{K}^2$ *satisfying:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{2}\left[2V^i_{z_{jl}} \cdot q_i^* + 2V^j_{z_{il}} \cdot q_j^* + \frac{\partial c_{ijl}(q^*_{ijl})}{\partial q_{ijl}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ijl}} + \frac{\partial \hat{c}_{ijl}(q^*_{ijl})}{\partial q_{ijl}} - \gamma_j^*\right]$$
$$\times \left[q_{ijl} - q^*_{ijl}\right]$$
$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[2V^i_{z_{2n+k}} \cdot q_i^* + \frac{\partial c_{ik}(q^*_{ik})}{\partial q_{ik}} + \hat{c}_{ik}(Q^{2*}, Q^{3*})\right] \times [q_{ik} - q^*_{ik}]$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\sum_{l=1}^{2}\left[2V^j_{z_{kl}} \cdot q_j^* + \frac{\partial c_{jkl}(q^*_{jkl})}{\partial q_{jkl}} + \hat{c}_{jkl}(Q^{2*}, Q^{3*}) + \gamma_j^*\right] \times \left[q_{jkl} - q^*_{jkl}\right]$$
$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{2} q^*_{ijl} - \sum_{k=1}^{n}\sum_{l=1}^{2} q^*_{jkl}\right] \times \left[\gamma_j - \gamma_j^*\right] - \sum_{k=1}^{o} \rho_{3k}(d^*) \times [d_k - d_k^*] \geq 0,$$
$$\forall (Q^1, Q^2, Q^3, \gamma, d) \in \mathcal{K}^2. \tag{11}$$

The variables in the variational inequality problem (11) are: the financial flows from the source agents to the intermediaries, Q^1; the direct financial flows via electronic transaction from the source agents to the demand markets, Q^2; the financial flows from the intermediaries to the demand markets, Q^3; the shadow prices associated with handling the product by the intermediaries, γ, and the prices at demand markets ρ_3. The solution to the variational inequality problem (11), $(Q^{0^*}, Q^{1^*}, Q^{2^*}, Q^{3^*}, \gamma^*, d^*)$, coincides with the equilibrium financial flow and price pattern according to Definition 1.

3 The Financial Network Efficiency Measure

In this Section, we propose the novel financial network efficiency measure and the associated network component importance definition. As stated in the Introduction, the financial network measure is motivated by the work of Qiang and Nagurney (2006). In the case of the financial network efficiency measure, we state the definitions directly within the context of financial networks, without making use of the transformation of the financial network model into a network equilibrium model with defined origin/destination pairs and paths as was done by Qiang and Nagurney (2006), who considered network equilibrium problems with a transportation focus (see also, Nagurney and Qiang (2007a, b) and Liu and Nagurney (2006)).

Definition 2: The Financial Network Efficiency Measure

The financial network efficiency measure, $\mathcal{E}$, for a given network topology G, and demand price functions $\rho_{3k}(d)$ $(k = 1, 2, \ldots, o)$, and available funds held by source agents S, is defined as follows:

$$\mathcal{E} = \frac{\sum_{k=1}^{o} \frac{d_k^*}{\rho_{3k}(d^*)}}{o}, \tag{12}$$

where o is the number of demand markets in the financial network, and d_k^* and $\rho_{3k}(d^*)$ denote the equilibrium demand and the equilibrium price for demand market k, respectively.

The financial network efficiency measure $\mathcal{E}$ defined in (12) is actually the average demand to price ratio. It measures the overall (economic) functionality of the financial network. When the network topology G, the demand price functions, and the available funds held by source agents are given, a financial network is considered to be more efficient if it can satisfy higher demand at lower prices.

By referring to the equilibrium conditions (8) and (9), we assume that if there is a positive transaction between a source agent or an intermediary with a demand market at the equilibrium, the price charged by the source agent or the intermediary plus the respective unit transaction costs is always positive. Furthermore, we assume that if the equilibrium demand at a demand market is zero, the demand market price (i.e., the inverse demand function value) is positive. Hence, the demand market prices will always be positive and the above network efficiency measure is well-defined.

The importance of the network components is analyzed, in turn, by studying their impact on the network efficiency through their removal. The network efficiency of a financial network can be expected to deteriorate when a critical network component is eliminated from the network. Such a component can include a link or a node or a subset of nodes and links depending on the financial network problem under study. Furthermore, the removal of a critical network component will cause more severe damage than that of a trivial one. Hence, the importance of a network component is defined as follows (cf. Qiang and Nagurney (2006)):

Definition 3: Importance of a Financial Network Component

The importance of a financial network component $g \in G$, $I(g)$, is measured by the relative financial network efficiency drop after g is removed from the network:

$$I(g) = \frac{\triangle \mathcal{E}}{\mathcal{E}} = \frac{\mathcal{E}(G) - \mathcal{E}(G-g)}{\mathcal{E}(G)} \tag{13}$$

where $G-g$ is the resulting financial network after component g is removed from network G.

It is worth pointing out that the above importance of the network components is well-defined even in a financial network with disconnected source agent/demand market pairs. In our financial network efficiency measure, the elimination of a transaction link is treated by removing that link from the network while the removal of a node is managed by removing the transaction links entering or exiting that node. In the case that the removal results in no transaction path connecting a source agent/demand market pair, we simply assign the demand for that source agent/demand market pair to an abstract transaction path with an associated cost of infinity. The above procedure(s) to handle disconnected agent/demand market pairs, will be illustrated in the numerical examples in Section 4, when we compute the importance of the financial network components and their rankings.

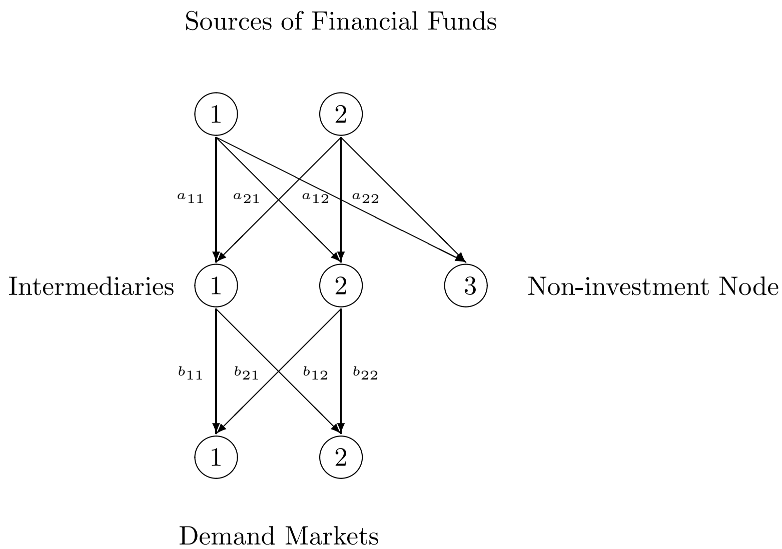

Fig. 2. The Financial Network Structure of the Numerical Examples

4 Numerical Examples

In order to further demonstrate the applicability of the financial network efficiency measure proposed in Section 3, we, in this Section, present two numerical financial network examples. For each example, our network efficiency measure is computed and the importance and the rankings of links and the nodes are also reported.

The examples consist of two source agents, two financial intermediaries, and two demand markets. These examples have the financial network structure depicted in Figure 2. For simplicity, we exclude the electronic transactions. The transaction links between the source agents and the intermediaries are denoted by a_{ij} where $i = 1, 2$; $j = 1, 2$. The transaction links between the intermediaries and the demand markets are denoted by b_{jk} where $j = 1, 2$; $k = 1, 2$. Since the non-investment portions of the funds do not participate in the actual transactions, we will not discuss the importance of the links and the node related to the non-investment funds. The examples below were solved using the Euler method (see, Nagurney and Zhang (1996, 1997), Nagurney and

Ke (2003), and Nagurney, Wakolbinger, and Zhao (2006)).

Example 1

The financial holdings for the two source agents in the first example are: $S^1 = 10$ and $S^2 = 10$. The variance-covariance matrices V^i and V^j are identity matrices for all the source agents $i = 1, 2$. We have suppressed the subscript l associated with the transaction cost functions since we have assumed a single (physical) mode of transaction only being available. Please refer to Table 1 for a compact exposition of the notation.

The transaction cost function of source agent 1 associated with his transaction with intermediary 1 is given by:

$$c_{11}(q_{11}) = 4q_{11}^2 + q_{11} + 1.$$

The other transaction cost functions of the source agents associated with the transactions with the intermediaries are given by:

$$c_{ij}(q_{ij}) = 2q_{ij}^2 + q_{ij} + 1, \quad \text{for } i = 1, 2; j = 1, 2$$

while i and j are not equal to 1 at the same time.

The transaction cost functions of the intermediaries associated with transacting with the sources agents are given by:

$$\hat{c}_{ij}(q_{ij}) = 3q_{ij}^2 + 2q_{ij} + 1, \quad \text{for } i = 1, 2; j = 1, 2.$$

The handling cost functions of the intermediaries are:

$$c_1(Q^1) = 0.5(q_{11} + q_{21})^2, \quad c_2(Q^1) = 0.5(q_{12} + q_{22})^2.$$

We assumed that in the transactions between the intermediaries and the demand markets, the transaction costs perceived by the intermediaries are all equal to zero, that is,

$$c_{jk} = 0, \quad \text{for } j = 1, 2; k = 1, 2.$$

The transaction costs between the intermediaries and the consumers at the demand markets, in turn, are given by:

$$\hat{c}_{jk} = q_{jk} + 2, \quad \text{for } j = 1, 2; k = 1, 2.$$

The demand price functions at the demand markets are:

$$\rho_{3k}(d) = -2d_k + 100, \quad \text{for } k = 1, 2.$$

The equilibrium financial flow pattern, the equilibrium demands, and the incurred equilibrium demand market prices are reported below.

For Q^{1*}, we have:

$$q_{11}^* = 3.27, \; q_{12}^* = 4.16, \; q_{21}^* = 4.36, \; q_{22}^* = 4.16.$$

For Q^{2*}, we have:

$$q_{11}^* = 3.81, \; q_{12}^* = 3.81, \; q_{21}^* = 4.16, \; q_{22}^* = 4.16.$$

Also, we have:

$$d_1^* = 7.97, \; d_2^* = 7.97,$$

$$\rho_{31}(d^*) = 84.06, \; \rho_{32}(d^*) = 84.06.$$

The financial network efficiency (cf. (12)) is:

$$\mathcal{E} = \frac{\frac{7.97}{84.06} + \frac{7.97}{84.06}}{2} = 0.0949.$$

The importance of the links and the nodes and their ranking are reported in Table 2 and 3, respectively.

Table 2. Importance and Ranking of the Links in Example 1

Link	Importance Value	Ranking
a_{11}	0.1574	3
a_{12}	0.2003	2
a_{21}	0.2226	1
a_{22}	0.2003	2
b_{11}	0.0304	5
b_{12}	0.0304	5
b_{21}	0.0359	4
b_{22}	0.0359	4

Table 3. Importance and Ranking of the Nodes in Example 1

Node	Importance Value	Ranking
Source Agent 1	0.4146	4
Source Agent 2	0.4238	3
Intermediary 1	0.4759	2
Intermediary 2	0.5159	1
Demand Market 1	0.0566	5
Demand Market 2	0.0566	5

Discussion

First note that, in Example 1, both source agents choose to not invest a portion of their financial funds. Given the cost structure and the demand price functions in the network of Example 1, the transaction link between source agent 2 and intermediary 1 is the most important link because it carries a large amount of financial flow, in equilibrium, and the removal of the link causes the highest efficiency loss assessed by the financial network efficiency measure. Similarly, because intermediary 2 handles the largest amount of financial input from the source agents, it is ranked as the most important node in the above network. On the other hand, since the transaction links between intermediary 1 to the demand markets 1 and 2 carry the least amount of equilibrium financial flow, they are the least important links.

Example 2

In the second example, the parameters are identical to those in Example 1, except for the following changes.

The transaction cost function of source agent 1 associated with his transaction with intermediary 1 is changed to:

$$c_{11}(q_{11}) = 2q_{11}^2 + q_{11} + 1$$

and the financial holdings of the source agents are changed, respectively, to $S_1 = 6$ and $S_2 = 10$.

The equilibrium financial flow pattern, the equilibrium demands, and the incurred equilibrium demand market prices are reported below.

For Q^{1*}, we have:

$$q_{11}^* = 3.00, \; q_{12}^* = 3.00, \; q_{21}^* = 4.48, \; q_{22}^* = 4.48.$$

For Q^{2*}, we have:

$$q_{11}^* = 3.74, \; q_{12}^* = 3.74, \; q_{21}^* = 3.74, \; q_{22}^* = 3.74.$$

Also, we have:

$$d_1^* = 7.48, \; d_2^* = 7.48,$$

$$\rho_{31}(d^*) = 85.04, \; \rho_{32}(d^*) = 85.04.$$

The financial network efficiency (cf. (12)) is:

$$\mathcal{E} = \frac{\frac{7.48}{85.04} + \frac{7.48}{85.04}}{2} = 0.0880 \tag{23}$$

The importance of the links and the nodes and their ranking are reported in Table 4 and 5, respectively.

Table 4. Importance and Ranking of the Links in Example 2

Link	Importance Value	Ranking
a_{11}	0.0917	2
a_{12}	0.0917	2
a_{21}	0.3071	1
a_{22}	0.3071	1
b_{11}	0.0211	3
b_{12}	0.0211	3
b_{21}	0.0211	3
b_{22}	0.0211	3

Table 5. Importance and Ranking of the Nodes in Example 2

Node	Importance Value	Ranking
Source Agent 1	0.3687	3
Source Agent 2	0.6373	1
Intermediary 1	0.4348	2
Intermediary 2	0.4348	2
Demand Market 1	-0.0085	4
Demand Market 2	-0.0085	4

Discussion

Note that, in Example 2, the first source agent has no funds non-invested. Given the cost structure and the demand price functions, since the transaction links between source agent 2 and intermediaries 1 and 2 carry the largest amount of equilibrium financial flow, they are ranked the most important. In addition, since source agent 2 allocates the largest amount of financial flow in equilibrium, it is ranked as the most important node. The negative importance value for demand markets 1 and 2 is due to the fact that the existence of each demand market brings extra flows on the transaction links and nodes and, therefore, increases the marginal transaction cost. The removal of one demand market has two effects: first, the contribution to the network efficiency of the removed demand market becomes zero; second, the marginal transaction cost on links/nodes decreases, which decreases the equilibrium prices and increases the demands at the other demand markets. If the efficiency loss caused by the removal of the demand markets is overcompensated by the improvement of the demand-price ratio of the other demand markets, the removed demand market will have a negative importance value. It simply implies that the "negative externality" caused by the demand market has a larger impact than the efficiency loss due to its removal.

5 Summary and Conclusions

In this paper, we proposed a novel financial network efficiency measure, which is motivated by the recent research of Qiang and Nagurney (2006) and Nagurney and Qiang (2007a, b) in assessing the importance of network components in the case of disruptions in network systems ranging from transportation networks to such critical infrastructure networks as electric power generation and distribution networks. The financial network measure determines the network efficiency by incorporating the economic behavior of the decision-makers, with the resultant equilibrium prices and transaction flows, coupled with the network topology. The financial network efficiency measure, along with the network component importance definition, provide valuable methodological tools for the evaluation of the financial network vulnerability and reliability. Furthermore, our measure is shown to be able to evaluate the importance of nodes and links in financial networks even when the source agent/demand market pairs become disconnected.

This research has implications for national security as well as for the insurance industry.

Acknowledgments

This research was supported, in part, by NSF Grant No.: IIS-0002647, under the Management of Knowledge Intensive Dynamic Systems (MKIDS) program. The first author also acknowledges support from the John F. Smith Memorial Fund at the University of Massachusetts at Amherst. The support provided is very much appreciated.

References

Boginski, V., Butenko, S. and Pardalos, P. M.: 2003, On structural properties of the market graph, in *Innovations in Financial and Economic Networks*, A. Nagurney, Editor, Edward Elgar Publishing, Cheltenham, England, 29-42.

Caldarelli, G., Battiston, S., Garlaschelli, D. and Catanzaro, M.: 2004, Emergence of complexity in financial networks, *Lecture Notes in Physics*, **650**, 399-423.

Charnes, A. and Cooper, W. W.: 1967, Some network characterizations for mathematical programming and accounting approaches to planning and control, *The Accounting Review*, **42**, 24-52.

Christofides, N., Hewins, R. D. and Salkin, G. R.: 1979, Graph theoretic approaches to foreign exchange operations, *Journal of Financial and Quantitative Analysis*, **14**, 481-500.

Copeland, M. A.: 1952, *A Study of Moneyflows in the United States*, National Bureau of Economic Research, New York, New York.

Crum, R. L. and Nye, D. J.: 1981, A network model of insurance company cash flow management, *Mathematical Programming Study*, **15**, 86-101.

Doherty, N. A.: 1997, Financial innovation in the management of catastrophe risk, *Journal of Applied Corporate Finance*, **10**, 84-95.

Fei, J. C. H.: 1960, The study of the credit system by the method of the linear graph, *The Review of Economics and Statistic*, **42**, 417-428.

Geunes, J. and Pardalos, P. M.: 2003, Network optimization in supply chain management and financial engineering: An annotated bibliography, *Networks*, **42**, 66-84.

Gilli, M. and Këllezi, E.: 2006, An application of extreme value theory for measuring financial risk, *Computational Economics*, **27**, 207-228.

Kim, D. H. and Jeong, H.: 2005, Systematic analysis of group identification in stock markets, *Physical Review E*, **72**, Article No. 046133.

Kontoghiorghes, E. J., Rustem, B. and Siokos, S., Editors: 2002, *Computational Methods in Decision-Making, Economics and Finance, Optimization Models*, Kluwer Academic Publishers, Boston, Massachusetts.

Latora, V. and Marchiori, M.: 2001, Efficient behavior of small-world networks, *Physical Review Letters*, **87**, Article No. 198701.

Latora, V. and Marchiori, M.: 2002, Is the Boston subway a small-world network? *Physica A: Statistical Mechanics and its Applications*, **314**, 109-113.

Latora, V. and Marchiori, M.: 2003, Economic small-world behavior in weighted networks, *The European Physical Journal B*, **32**, 249-263.

Latora, V. and Marchiori, M.: 2004, How the science of complex networks can help developing strategies against terrorism, *Chaos, Solitons and Fractals*, **20**, 69-75.

Liu, Z. and Nagurney, A.: 2006, Financial networks with intermediation and transportation network equilibria: A supernetwork equivalence and computational management reinterpretation of the equilibrium conditions with computations, *Computational Management Science*, in press.

Loubergé, H., Këllezi, E. and Gilli, M.: 1999, Using catastrophe-linked securities to diversify insurance risk: A financial analysis of cat-bonds, *The Journal of Insurance Issues*, **2**, 125-146.

Mulvey, J. M.: 1987, Nonlinear networks in finance, *Advances in Mathematical Programming and Financial Planning*, **20**, 187-217.

Nagurney, A.: 2007, Networks in finance, to appear in the *Handbook on Information Technology and Finance*, D. Seese, C. Weinhardt, and F. Schlottmann, Editors, Springer, Berlin, Germany.

Nagurney, A. and Cruz, J.: 2003a, International financial networks with electronic transactions, in *Innovations in Financial and Economic Networks*, A. Nagurney, Editor, Edward Elgar Publishing, Cheltenham, England, 136-168.

Nagurney, A. and Cruz, J.: 2003b, International financial networks with intermediation: modeling, analysis, and computations, *Computational Management Science*, **1**, 31-58.

Nagurney, A., Dong, J. and Hughes, M.: 1992, Formulation and computation of general financial equilibrium, *Optimization*, **26**, 339-354.

Nagurney, A. and Hughes, M.: 1992, Financial flow of funds networks, *Networks*, **22**, 145-161.

Nagurney, A. and Ke, K.: 2001, Financial networks with intermediation, *Quantitative Finance*, **1**, 309-317.

Nagurney, A. and Ke, K.: 2003, Financial networks with electronic transactions: modeling, analysis, and computations, *Quantitative Finance*, **3**, 71-87.

Nagurney, A. and Qiang, Q.: 2007a, A transportation network efficiency measure that captures flows, behavior, and costs with applications to network component importance identification and vulnerability, accepted for presentation at the POMS Conference, Dallas, Texas, May 4-7.

Nagurney, A. and Qiang, Q.: 2007b, A network efficiency measure with application to critical infrastructure networks, to appear in *Journal of Global Optimization*.

Nagurney, A. and Siokos, S.: 1997, *Financial Networks: Statics and Dynamics*, Springer-Verlag, Heidelberg, Germany.

Nagurney, A., Wakolbinger, T. and Zhao, L.: 2006, The evolution and emergence of integrated social and financial networks with electronic transactions: A dynamic supernetwork theory for the modeling, analysis, and computation of financial flows and relationship levels, *Computational Economics*, **27**, 353-393.

Nagurney, A. and Zhang, D.: 1996, *Projected Dynamical Systems and Variational Inequalities with Applications*, Kluwer Academic Publishers, Boston, Massachusetts.

Nagurney, A. and Zhang, D.: 1997, Projected dynamical systems in the formulation, stability analysis, and computation of fixed demand traffic network equilibria, *Transportation Science*, **31**, 147-158.

Nash, J. F.: 1950, Equilibrium points in N-person games, *Proceedings of the National Academy of Sciences, USA*, **36**, 48-49.

Nash, J. F.: 1951, Noncooperative games, *Annals of Mathematics*, **54**, 286-298.

Newman, M.: 2003, The structure and function of complex networks, *SIAM Review*, **45**, 167-256.

Niehaus, G.: 2002, The allocation of catastrophe risk, *Journal of Banking and Finance*, **26**, 585-596.

Odell, K. and Phillips, R. J.: 2001, Testing the ties that bind: financial networks and the 1906 San Francisco earthquake, working paper, Department of Economics, Colorado State University, Fort Collins, Colorado.

Onnela, J. P., Kaski, K. and Kertész, J.: 2004, Clustering and information in correlation based financial networks, *The European Physical Journal B*, **38**, 353-362.

Qiang, Q., and Nagurney, A.: 2007, A unified network performance measure with importance identification and the ranking of network components, to appear in *Optimization Letters*.

Quesnay, F.: 1758, *Tableau Economique*, reproduced in facsimile with an introduction by H. Higgs by the British Economic Society, 1895.

Robinson, C. P., Woodard, J. B. and Varnado, S. G.: 1998, Critical infrastructure: interlinked and vulnerable, *Issues in Science and Technology*, **15**, 61-67.

Rustem, B., and Howe, M.: 2002, *Algorithms for Worst-Case Design and Risk Management*, Princeton University Press, Princeton, New Jersey.

Sheffi, Y.: 2005, *The Resilient Enterprise: Overcoming Vulnerability for Competitive Advantage*, MIT Press, Cambridge, Massachusetts.

Thore, S.: 1969, Credit networks, *Economica*, **36**, 42-57.

Thore, S., and Kydland, F.: 1972, Dynamic flow-of-funds networks, in *Applications of Management Science in Banking and Finance*, S. Eilon and T. R. Fowkes, Editors, Epping, England, 259-276.

Thore, S.: 1980 *Programming the Network of Financial Intermediation*, Universitetsforlaget, Oslo, Norway.