Chapter 1

Financial Networks

Invited Chapter Prepared for Hanbook of Financial Engineering, C. Zopounidis, M. Doumpos, and P. M. Pardalos, Editors, Kluwer Academic Publishers (2005)

1.1 Introduction

Finance is a discipline that is concerned with the study of capital flows over space and time in the presence of risk. It has benefited from a plethora of mathematical and engineering tools that have been developed and utilized for the modeling, analysis, and computation of solutions in the present complex economic environment. Indeed, the financial landscape today is characterized by the existence of distinct sectors in economies, the proliferation of new financial instruments, with increasing diversification of portfolios internationally, various transaction costs, the increasing growth of electronic transactions, and a myriad of governmental policy interventions. Hence, rigorous methodological tools that can capture the complexity and richness of financial decision-making today and that can take advantage of powerful computer resources have never been more important and needed for financial quantitative analyses.

In this chapter, the focus is on financial networks as a powerful financial engineering tool and medium for the modeling, analysis, and solution of a spectrum of financial decision-making problems ranging from portfolio optimization to multi-sector, multi-instrument general financial equilibrium problems, dynamic multi-agent financial problems with intermediation, as well as the financial engineering of the integration of social networks with financial systems.

Note that throughout history, the emergence and evolution of various physical networks, ranging from transportation and logistical networks to telecommunication networks and the effects of human decision-making on such networks have given rise to the development of rich theories and scientific methodologies that are network-based (cf. Ford and Fulkerson (1962), Ahuja, Magnanti, and Orlin (1993), Nagurney (1999), and Guenes and Pardalos (2003)). The novelty of networks is that they are pervasive, providing the fabric of connectivity for our societies and economies, while, methodologically, network theory has developed into a powerful and dynamic medium for abstracting complex problems, which, at first glance, may not even appear to be networks, with associated nodes, links, and flows.

The topic of networks as a subject of scientific inquiry originated in the paper by Euler (1736), which is credited with being the earliest paper on *graph* theory. By a graph in this setting is meant, mathematically, a means of abstractly representing a system by its depiction in terms of vertices (or nodes) and edges (or arcs, equivalently, links) connecting various pairs of vertices. Euler was interested in determining whether it was possible to stroll around Königsberg (later called Kaliningrad) by crossing the seven bridges over the River Pregel exactly once. The problem was represented as a graph in which the vertices corresponded to land masses and the edges to bridges.

Quesnay (1758), in his *Tableau Economique*, conceptualized the circular flow of financial funds in an economy as a network and this work can be identified as the first paper on the topic of financial networks. Quesnay's basic idea has been utilized in the construction of financial flow of funds accounts, which are a statistical description of the flows of money and credit in an economy (see Cohen (1987)).

The concept of a network in economics, in turn, was implicit as early as the classical work of Cournot (1838), who not only seems to have first explicitly stated that a competitive price is determined by the intersection of supply and demand curves, but had done so in the context of two spatially separated markets in which the cost associated with transporting the goods was also included. Pigou (1920) studied a network system in the form of a transportation network consisting of two routes and noted that the decision-making behavior of the the users of such a system would lead to different flow patterns. Hence, the network of concern therein consists of the graph, which is directed, with the edges or links represented by arrows, as well as the resulting flows on the links.

Copeland (1952) recognized the conceptualization of the interrelationships among financial funds as a network and asked the question, "Does money flow like water or electricity?" Moreover, he provided a "wiring diagram for the main money circuit." Kirchhoff is credited with pioneering the field of electrical engineering by being the first to have systematically analyzed electrical circuits and with providing the foundations for the principal ideas of network flow theory. Interestingly, Enke in 1951 had proposed electronic circuits as a means of solving spatial price equilibrium problems, in which goods are produced, consumed, and traded, in the presence of transportation costs. Such analog computational devices, were soon to be superseded by digital computers along with advances in computational methodologies, that is, algorithms, based on mathematical programming.

In this chapter, we further elaborate upon historical breakthroughs in the use of networks for the formulation, analysis, and solution of financial problems. Such a perspective allows one to trace the methodological developments as well as the applications of financial networks and provides a platform upon which further innovations can be made. One of the principal goals of this chapter is to highlight some of the major developments in financial engineering in the context of financial networks. Methodological tools that will be utilized to formulate and solve the financial network problems in this chapter are drawn from optimization, variational inequalities, as well as projected dynamical systems theory. We begin with a discussion of financial optimization problems within a network context and then turn to a range of financial network equilibrium problems.

1.2 Financial Optimization Problems

Network models have been proposed for a wide variety of financial problems characterized by a single objective function to be optimized as in portfolio optimization and asset allocation problems, currency translation, and risk management problems, among others. This literature is now briefly overviewed with the emphasis on the innovative work of Markowitz (1952, 1959) that established a new era in financial economics and became the basis for many financial optimization models that exist and are used to this day.

Note that although many financial optimization problems (including the work by Markowitz) had an underlying network structure, and the advantages of network programming were becoming increasingly evident (cf. Charnes and Cooper (1958)), not many financial network optimization models were developed until some time later. Some exceptions are several early models due to Charnes and Miller (1957) and Charnes and Cooper (1961). It was not until the last years of the 1960s and the first years of the 1970s that the network setting started to be extensively used for financial applications.

Among the first financial network optimization models that appear in the literature were a series of currency translating models. Rutenberg (1970) suggested that the translation among different currencies could be performed through the use of arc multipliers. His network model was multiperiod with linear costs on the arcs (a characteristic common to the earlier financial networks models). The nodes of such generalized networks represented a particular currency in a specific period and the flow on the arcs the amount of cash moving from one period and/or currency to another. Christofides, Hewins, and Salkin (1979) and Shapiro and Rutenberg (1976), among others, introduced related financial network models. In most of these models, the currency prices were determined according to the amount of capital (network flow) that was moving from one currency (node) to the other.

Barr (1972) and Srinivasan (1974) used networks to formulate a series of cash management problems, with a major contribution being Crum's (1976) introduction of a generalized linear network model for the cash management of a multinational firm. The links in the network represented possible cash flow patterns and the multipliers incorporated costs, fees, liquidity changes, and exchange rates. A series of related cash management problems were modeled as network problems in subsequent years by Crum and Nye (1981) and Crum,

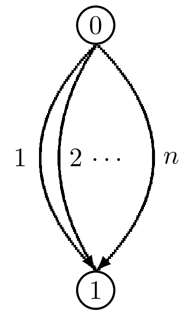

Figure 1.1: Network Structure of Classical Portfolio Optimization

Klingman, and Tavis (1983), and others. These papers further extended the applicability of network programming in financial applications. The focus was on linear network flow problems in which the cost on an arc was a linear function of the flow. Crum, Klingman, and Tavis (1979), in turn, demonstrated how contemporary financial capital allocation problems could be modeled as an integer generalized network problem, in which the flows on particular arcs were forced to be integers.

It is important to note that in many financial network optimization problems the objective function must be nonlinear due to the modeling of the risk function and, hence, typically, such financial problems lie in the domain of nonlinear, rather than linear, network flow problems. Mulvey (1987) presented a collection of nonlinear financial network models that were based on previous cash flow and portfolio models in which the original authors (see, e.g., Rudd and Rosenberg (1979) and Soenen (1979)) did not realize, and, thus, did not exploit the underlying network structure. Mulvey also recognized that the Markowitz (1952, 1959) mean-variance minimization problem was, in fact, a network optimization problem with a nonlinear objective function. The classical Markowitz models are now reviewed and cast into the framework of network optimization problems. See Figure 1.1 for the network structure of such problems. Additional financial network optimization models and associated references can be found in Nagurney and Siokos (1997) and in the volume edited by Nagurney (2003).

Markowitz's model was based on mean-variance portfolio selection, where the average and the variability of portfolio returns were determined in terms of the mean and covariance of the corresponding investments. The mean is a measure of an average return and the variance is a measure of the distribution of the returns around the mean return. Markowitz formulated the portfolio optimization problem as associated with risk minimization with the objective function:

$$\text{Minimize} \quad V = X^T Q X \tag{1.1}$$

subject to constraints, representing, respectively, the attainment of a specific return, a budget constraint, and that no short sales were allowed, given by:

$$R = \sum_{i=1}^{n} X_i r_i \tag{1.2}$$

$$\sum_{i=1}^{n} X_i = 1 \tag{1.3}$$

$$X_i \geq 0, \quad i = 1, \ldots, n. \tag{1.4}$$

Here n denotes the total number of securities available in the economy, X_i represents the relative amount of capital invested in security i, with the securities being grouped into the column vector X, Q denotes the $n \times n$ variance-covariance matrix on the return of the portfolio, r_i denotes the expected value of the return of security i, and R denotes the expected rate of return on the portfolio. Within a network context (cf. Figure 1.1), the links correspond to the securities, with their relative amounts $X_1, \ldots, X_n$ corresponding to the flows on the respective links: $1, \ldots, n$. The budget constraint and the nonnegativity assumption on the flows are the network conservation of flow equations. Since the objective function is that of risk minimization, it can be interpreted as the sum of the costs on the n links in the network. Observe that the network representation is abstract and does not correspond (as in the case of transportation and telecommunication) to physical locations and links.

Markowitz suggested that, for a fixed set of expected values r_i and covariances of the returns of all assets i and j, every investor can find an (R, V) combination that better fits his taste, solely limited by the constraints of the specific problem. Hence, according to the original work of Markowitz (1952), the efficient frontier had to be identified and then every investor had to select a portfolio through a mean-variance analysis that fitted his preferences.

A related mathematical optimization model (see Markowitz (1959)) to the one above, which can be interpreted as the investor seeking to maximize his returns while minimizing his risk can be expressed by the quadratic programming problem:

$$\text{Maximize} \quad \alpha R - (1 - \alpha) V \tag{1.5}$$

subject to:

$$\sum_{i=1}^{n} X_i = 1 \tag{1.6}$$

$$X_i \geq 0, \quad i = 1, \ldots, n, \tag{1.7}$$

where α denotes an indicator of how risk-averse a specific investor is. This model is also a network optimization problem with the network as depicted in Figure 1.1 with equations (1.6) and (1.7) again representing a conservation of flow equation.

A collection of versions and extensions of Markowitz's model can be found in Francis and Archer (1979), with $\alpha = 1/2$ being a frequently accepted value. A recent interpretation of the model as a multicriteria decision-making model along with theoretical extensions to multiple sectors can be found in Dong and Nagurney (2001), where additional references are available. References to multicriteria decision-making and financial applications can also be found in Doumpos, Zopounidis, and Pardalos (2000).

A segment of the optimization literature on financial networks has focused on variables that are stochastic and have to be treated as random variables in the optimization procedure. Clearly, since most financial optimization problems are of large size, the incorporation of stochastic variables made the problems more complicated and difficult to model and compute. Mulvey (1987) and Mulvey and Vladimirou (1989, 1991), among others, studied stochastic financial networks, utilizing a series of different theories and techniques (e.g., purchase power priority, arbitrage theory, scenario aggregation) that were then utilized for the estimation of the stochastic elements in the network in order to be able to represent them as a series of deterministic equivalents. The large size and the computational complexity of stochastic networks, at times, limited their usage to specially structured problems where general computational techniques and algorithms could be applied. See Rudd and Rosenberg (1979), Wallace (1986), Rockafellar and Wets (1991), and Mulvey, Simsek, and Pauling (2003) for a more detailed discussion on aspects of realistic portfolio optimization and implementation issues related to stochastic financial networks.

1.3 General Financial Equilibrium Problems

We now turn to networks and their utilization for the modeling and analysis of financial systems in which there is more than a single decision-maker, in contrast to the above financial optimization problems. It is worth noting that Quesnay (1758) actually considered a financial system as a network.

Thore (1969) introduced networks, along with the mathematics, for the study of systems of linked portfolios. His work benefited from that of Charnes and Cooper (1967) who demonstrated that systems of linked accounts could be represented as a network, where the nodes depict the balance sheets and the links depict the credit and debit entries. Thore considered credit networks, with the explicit goal of providing a tool for use in the study of the propagation of money and credit streams in an economy, based on a theory of the behavior of banks and other financial institutions. The credit network recognized that these sectors interact and its solution made use of linear programming. Thore (1970)) extended the basic network model to handle holdings of financial reserves in the case of uncertainty. The approach utilized two-stage linear programs under uncertainty introduced by Ferguson and Dantzig (1956) and Dantzig and Madansky (1961). See Fei (1960) for a graph theoretic approach to the credit system. More recently, Boginski, Butenko, and Pardalos (2003) presented a detailed study of the stock market graph, yielding a new tool for the analysis of market structure through the classification of stocks into different groups, along with an application to the US stock market.

Storoy, Thore, and Boyer (1975), in turn, developed a network representation of the interconnection of capital markets and demonstrated how decomposition theory of mathematical programming could be exploited for the computation of equilibrium. The utility functions facing a sector were no longer restricted to being linear functions. Thore (1980) further investigated network models of linked portfolios, financial intermediation, and decentralization/decomposition theory. However, the computational techniques at that time were not sufficiently well-developed to handle such problems in practice.

Thore (1984) later proposed an international financial network for the Euro dollar market and viewed it as a logistical system, exploiting the ideas of Samuelson (1952) and Takayama and Judge (1971) for spatial price equilibrium problems. In this paper, as in Thore's preceding papers on financial networks, the micro-behavioral unit consisted of the individual bank, savings and loan, or other financial intermediary and the portfolio choices were described in some optimizing framework, with the portfolios being linked together into a network with a separate portfolio visualized as a node and assets and liabilities as directed links.

Notably, the above-mentioned contributions focused on the use and application of networks for the study of financial systems consisting of multiple economic decision-makers. In such systems, equilibrium was a central concept, along with the role of prices in the equilibrating mechanism. Rigorous approaches that characterized the formulation of equilibrium and the corresponding price determination were greatly influenced by the Arrow-Debreu economic model (cf. Arrow (1951), Debreu (1951)). In addition, the importance of the inclusion of dynamics in the study of such systems was explicitly emphasized (see, also, Thore and Kydland (1972)).

The first use of finite-dimensional variational inequality theory for the computation of multi-sector, multi-instrument financial equilibria is due to Nagurney, Dong, and Hughes (1992), who recognized the network structure underlying the subproblems encountered in their proposed decomposition scheme. Hughes and Nagurney (1992) and Nagurney and Hughes (1992) had, in turn, proposed the formulation and solution of estimation of financial flow of funds accounts as network optimization problems. Their proposed optimization scheme fully exploited the special network structure of these problems. Nagurney and Siokos (1997) then developed an international financial equilibrium model utilizing finite-dimensional variational inequality theory for the first time in that framework.

Finite-dimensional variational inequality theory is a powerful unifying methodology in that it contains, as special cases, such mathematical programming problems as: nonlinear equations, optimization problems, and complementarity problems. To illustrate this methodology and its application in general financial equilibrium modeling and computation, we now present a multi-sector, multi-instrument model and an extension due to Nagurney, Dong, and Hughes (1992) and Nagurney (1994), respectively. For additional references to variational inequalities in finance, along with additional theoretical foundations, see Nagurney and Siokos (1997) and Nagurney (2001, 2003).

1.3.1 A Multi-Sector, Multi-Instrument Financial Equilibrium Model

Recall the classical mean-variance model presented in the preceding section, which is based on the pioneering work of Markowitz (1959). Now, however, assume that there are m sectors, each of which seeks to maximize his return and, at the same time, to minimize the risk of his portfolio, subject to the balance accounting and nonnegativity constraints. Examples of sectors include: households, businesses, state and local governments, banks, etc. Denote a typical sector by j and assume that there are liabilities in addition to assets held by each sector. Denote the volume of instrument i that sector j holds as an asset, by X_i^j, and group the (nonnegative) assets in the portfolio of sector j into the column vector $X^j \in R^n_+$. Further, group the assets of all sectors in the economy into the column vector $X \in R^{mn}_+$. Similarly, denote the volume of instrument i that sector j holds as a liability, by Y_i^j, and group the (nonnegative) liabilities in the portfolio of sector j into the column vector $Y^j \in R^n_+$. Finally, group the liabilities of all sectors in the economy into the column vector $Y \in R^{mn}_+$. Let r_i denote the nonnegative price of instrument i and group the prices of all the instruments into the column vector $r \in R^n_+$.

It is assumed that the total volume of each balance sheet side of each sector is exogenous. Recall that a *balance sheet* is a financial report that demonstrates the status of a company's assets, liabilities, and the owner's equity at a specific point of time. The left-hand side of a balance sheet contains the assets that a sector holds at a particular point of time, whereas the right-hand side accommodates the liabilities and owner's equity held by that sector at the same point of time. According to accounting principles, the sum of all assets is equal to the sum of all the liabilities and the owner's equity. Here, the term "liabilities" is used in its general form and, hence, also includes the owner's equity. Let S^j denote the financial volume held by sector j. Finally, assume that the sectors under consideration act in a perfectly competitive environment.

A Sector's Portfolio Optimization Problem

Recall that in the mean-variance approach for portfolio optimization, the minimization of a portfolio's risk is performed through the use of the variance-covariance matrix. Hence, the portfolio optimization problem for each sector j is the following:

$$\text{Minimize} \quad \begin{pmatrix} X^j \\ Y^j \end{pmatrix}^T Q^j \begin{pmatrix} X^j \\ Y^j \end{pmatrix} - \sum_{i=1}^{n} r_i \left(X_i^j - Y_i^j \right) \tag{1.8}$$

subject to:

$$\sum_{i=1}^{n} X_i^j = S^j \tag{1.9}$$

$$\sum_{i=1}^{n} Y_i^j = S^j \tag{1.10}$$

$$X_i^j \geq 0,\ Y_i^j \geq 0, \quad i = 1, 2, \ldots, n, \tag{1.11}$$

where Q^j is a symmetric $2n \times 2n$ variance-covariance matrix associated with the assets and liabilities of sector j. Moreover, since Q^j is a variance-covariance matrix, one can assume that it is positive definite and, as a result, the objective function of each sector's portfolio optimization problem, given by the above, is strictly convex.

Partition the symmetric matrix Q^j, as

$$Q^j = \begin{pmatrix} Q_{11}^j & Q_{12}^j \\ Q_{21}^j & Q_{22}^j \end{pmatrix},$$

where Q_{11}^j and Q_{22}^j are the variance-covariance matrices for only the assets and only the liabilities, respectively, of sector j. These submatrices are each of dimension $n \times n$. The submatrices Q_{12}^j and Q_{21}^j, in turn, are identical since Q^j is symmetric. They are also of dimension $n \times n$. These submatrices are, in fact, the symmetric variance-covariance matrices between the asset and the liabilities of sector j. Denote the i-th column of matrix $Q_{(\alpha\beta)}^j$, by $Q_{(\alpha\beta)i}^j$, where α and β can take on the values of 1 and/or 2.

Optimality Conditions

The necessary and sufficient conditions for an optimal portfolio for sector j, are that the vector of assets and liabilities, $(X^{j^*}, Y^{j^*}) \in K_j$, where K_j denotes the feasible set for sector j, given by (1.9) – (1.11), satisfies the following system of equalities and inequalities: For each instrument $i; i = 1, \ldots, n$, we must have that:

$$2(Q_{(11)i}^j)^T \cdot X^{j^*} + 2(Q_{(21)i}^j)^T \cdot Y^{j^*} - r_i^* - \mu_j^1 \geq 0,$$

$$2(Q_{(22)i}^j)^T \cdot Y^{j^*} + 2(Q_{(12)i}^j)^T \cdot X^{j^*} + r_i^* - \mu_j^2 \geq 0,$$

$$X_i^{j^*}\left[2(Q_{(11)i}^j)^T \cdot X^{j^*} + 2(Q_{(21)i}^j)^T \cdot Y^{j^*} - r_i^* - \mu_j^1\right] = 0,$$

$$Y_i^{j^*}\left[2(Q_{(22)i}^j)^T \cdot Y^{j^*} + 2(Q_{(12)i}^j)^T \cdot X^{j^*} + r_i^* - \mu_2^j\right] = 0,$$

where μ_j^1 and μ_j^2 are the Lagrange multipliers associated with the accounting constraints, (1.9) and (1.10), respectively.

Let $\mathcal{K}$ denote the feasible set for all the asset and liability holdings of all the sectors and all the prices of the instruments, where $\mathcal{K} \equiv \{K \times R_+^n\}$ and $K \equiv \prod_{i=1}^m K_j$. The network structure of the sectors' optimization problems is depicted in Figure 1.2.

Economic System Conditions

The economic system conditions, which relate the supply and demand of each financial instrument and the instrument prices, are given by: for each instrument $i; i = 1, \ldots, n$, an equilibrium asset, liability, and price pattern, $(X^*, Y^*, r^*) \in \mathcal{K}$, must satisfy:

$$\sum_{j=1}^{J}(X_i^{j^*} - Y_i^{j^*}) \begin{cases} = 0, & \text{if } r_i^* > 0 \\ \geq 0, & \text{if } r_i^* = 0. \end{cases} \tag{1.12}$$

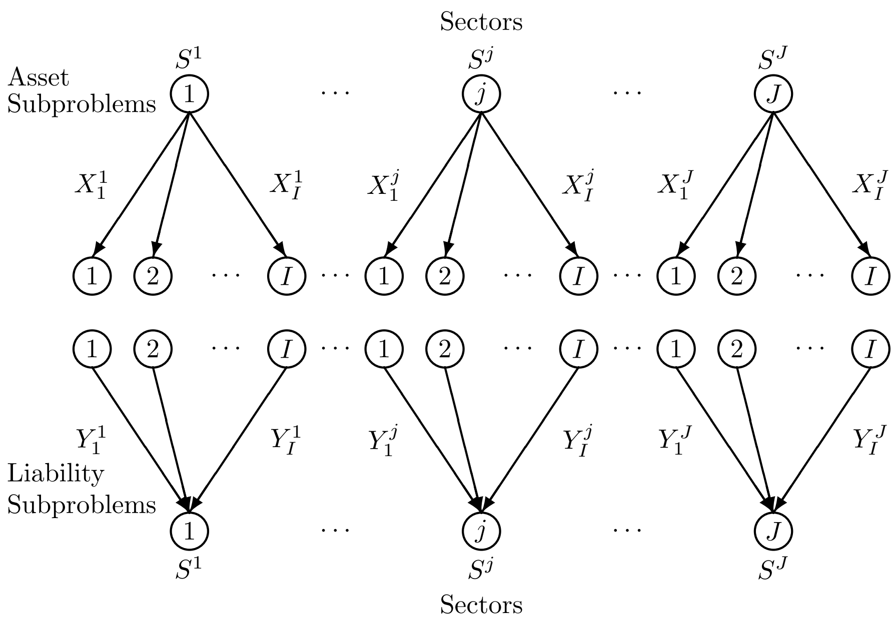

Figure 1.2: Network Structure of the Sectors' Optimization Problems

The definition of financial equilibrium is now presented along with the variational inequality formulation. For the derivation, see Nagurney, Dong, and Hughes (1992) and Nagurney and Siokos (1997). Combining the above optimality conditions for each sector with the economic system conditions for each instrument, we have the following definition of equilibrium.

Definition 1: Multi-Sector, Multi-Instrument Financial Equilibrium
A vector $(X^*, Y^*, r^*) \in \mathcal{K}$ is an equilibrium of the multi-sector, multi-instrument financial model if and only if it satisfies the optimality conditions and the economic system conditions (1.12), for all sectors j; $j = 1, \ldots, m$, and for all instruments i; $i = 1, \ldots, n$, simultaneously.

The variational inequality formulation of the equilibrium conditions, due to Nagurney, Dong, and Hughes (1992) is given by:

Theorem 1: Variational Inequality Formulation for the Quadratic Model
A vector of assets and liabilities of the sectors, and instrument prices, $(X^*, Y^*, r^*) \in \mathcal{K}$, is a financial equilibrium if and only if it satisfies the variational inequality problem:

$$\sum_{j=1}^{m}\sum_{i=1}^{n}\left[2(Q^j_{(11)i})^T \cdot X^{j^*} + 2(Q^j_{(21)i})^T \cdot Y^{j^*} - r_i^*\right] \times \left[X_i^j - X_i^{j^*}\right]$$

$$+\sum_{j=1}^{m}\sum_{i=1}^{n}\left[2(Q^j_{(22)i})^T \cdot Y^{j^*} + 2(Q^j_{(12)i})^T \cdot X^{j^*} + r_i^*\right] \times \left[Y_i^j - Y_i^{j^*}\right]$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{m}\left[X_i^{j^*} - Y_i^{j^*}\right] \times \left[r_i - r_i^*\right] \geq 0, \quad \forall (X, Y, r) \in \mathcal{K}. \tag{1.13}$$

For completeness, the *standard form* of the variational inequality is now presented. For additional background, see Nagurney (1999). Define the N-dimensional column vector $Z \equiv (X, Y, r) \in \mathcal{K}$, and the N-dimensional column vector $F(Z)$ such that:

$$F(Z) \equiv D \begin{pmatrix} X \\ Y \\ r \end{pmatrix} \quad \text{where} \quad D = \begin{pmatrix} 2Q & B \\ -B^T & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} Q^1_{11} & & & Q^1_{21} & & \\ & \ddots & & & \ddots & \\ & & Q^J_{11} & & & Q^J_{21} \\ Q^1_{12} & & & Q^1_{22} & & \\ & \ddots & & & \ddots & \\ & & Q^J_{12} & & & Q^J_{22} \end{pmatrix}_{2mn \times 2mn},$$

and

$$B^T = \begin{pmatrix} -\mathcal{I} & \dots & -\mathcal{I} & \mathcal{I} & \dots & \mathcal{I} \end{pmatrix}_{n\times mn},$$

and $\mathcal{I}$ is the $n \times n$-dimensional identity matrix.

It is clear that variational inequality problem (1.13) can be put into standard variational inequality form: determine $Z^* \in \mathcal{K}$, satisfying:

$$\langle F(Z^*)^T, Z - Z^* \rangle \geq 0, \quad \forall Z \in \mathcal{K}. \tag{1.14}$$

1.3.2 Model with Utility Functions

The above model is a special case of the financial equilibrium model due to Nagurney (1994) in which each sector j seeks to maximize his utility function,

$$U^j(X^j,^j, r) = u^j(X^j, Y^j) + r^T \cdot (X^j - Y^j),$$

which, in turn, is a special case of the model with a sector j's utility function given by the general form: $U^j(X^j, Y^j, r)$. Interestingly, it has been shown by Nagurney and Siokos (1997) that, in the case of utility functions of the form $U^j(X^j, Y^j, r) = u^j(X^j, Y^j) + r^T \cdot (X^j - Y^j)$, of which the above described quadratic model is an example, one can obtain the solution to the above variational inequality problem by solving the optimization problem:

$$\text{Maximize} \sum_{j=1}^{J} u^j(X^j, Y^j) \tag{1.15}$$

subject to:

$$\sum_{j=1}^{J}(X_i^j - Y_i^j) = 0, \quad i = 1, \dots, n \tag{1.16}$$

$$(X^j, Y^j) \in K_j, \quad j = 1, \dots, m, \tag{1.17}$$

with Lagrange multiplier r_i^* associated with the i-th "market clearing" constraint (1.16). Moreover, this optimization problem is actually a network optimization problem as revealed in Nagurney and Siokos (1997). The structure of the financial system in equilibrium is as depicted in Figure 1.3.

1.3.3 Computation of Financial Equilibria

In this section, an algorithm for the computation of solutions to the above financial equilibrium problems is recalled. The algorithm is the modified projection method of Korpelevich (1977). The advantage of this computational method in the context of the general financial equilibrium problems is that the original problem can be decomposed into a series of smaller and simpler subproblems of network structure, each of which can then be solved explicitly and in closed form. The realization of the modified projection method for the solution of the financial equilibrium problems with general utility functions is then presented.

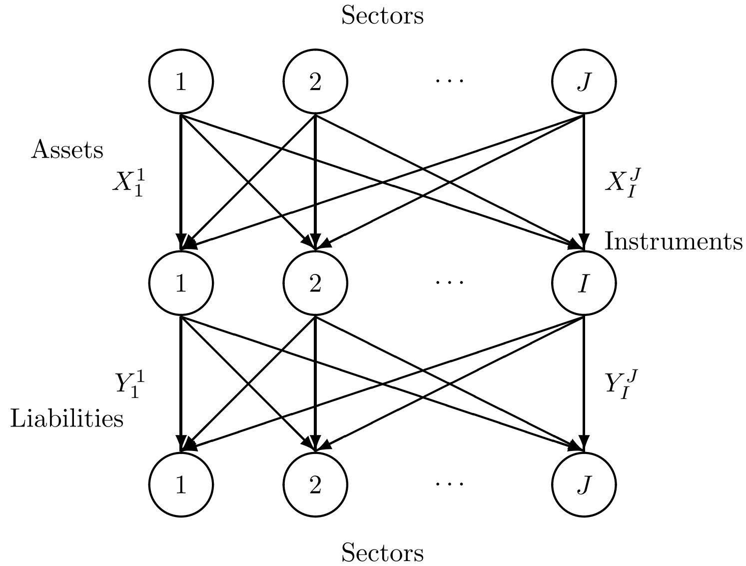

Figure 1.3: The Network Structure at Equilibrium

The modified projection method, can be expressed as:

Step 0: Initialization

Select $Z^0 \in \mathcal{K}$. Let $\tau := 0$ and let γ be a scalar such that $0 < \gamma \leq \frac{1}{L}$, where L is the Lipschitz constant (see Nagurney and Siokos (1997)).

Step 1: Computation

Compute $\bar{Z}^\tau$ by solving the variational inequality subproblem:

$$\langle (\bar{Z}^\tau + \gamma F(Z^\tau)^T - Z^\tau)^T, Z - \bar{Z}^\tau \rangle \geq 0, \quad \forall Z \in \mathcal{K}. \tag{1.18}$$

Step 2: Adaptation

Compute $Z^{\tau+1}$ by solving the variational inequality subproblem:

$$\langle (Z^{\tau+1} + \gamma F(\bar{Z}^\tau)^T - Z^\tau)^T, Z - Z^{\tau+1} \rangle \geq 0, \quad \forall Z \in \mathcal{K}. \tag{1.19}$$

Step 3: Convergence Verification

If max $|Z_b^{\tau+1} - Z_b^\tau| \leq \epsilon$, for all b, with $\epsilon > 0$, a prespecified tolerance, then stop; else, set $\tau := \tau + 1$, and go to Step 1.

For completeness, we now present the modified projection algorithm in which the function $F(Z)$ is in expanded form for the specific model.

The Modified Projection Method

Step 0: Initialization

Set $(X^0, Y^0, r^0) \in \mathcal{K}$. Let $\tau := 0$ and set γ so that $0 < \gamma \leq \frac{1}{L}$.

Step 1: Computation

Compute $(\bar{X}^\tau, \bar{Y}^\tau, \bar{r}^\tau) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\sum_{j=1}^{J} \sum_{i=1}^{I} \left[\bar{X}_i^{j\tau} + \gamma \left(-\frac{\partial U^j(X^{j\tau}, Y^{j\tau}, r^\tau)}{\partial X_i^j} \right) - X_i^{j\tau} \right] \times \left[X_i^j - \bar{X}_i^{j\tau} \right]$$

$$+ \sum_{j=1}^{J} \sum_{i=1}^{I} \left[\bar{Y}_i^{j\tau} + \gamma \left(-\frac{\partial U^j(X^{j\tau}, Y^{j\tau}, r^\tau)}{\partial Y_i^j} \right) - Y_i^{j\tau} \right] \times \left[Y_i^j - \bar{Y}_i^{j\tau} \right]$$

$$+ \sum_{i=1}^{I} \left[\bar{r}_i^\tau + \gamma \left[\sum_{j=1}^{J} \left(X_i^{j\tau} - Y_i^{j\tau} \right) \right] - r_i^\tau \right] \times \left[r_i - \bar{r}_i^\tau \right], \quad \forall (X, Y, r) \in \mathcal{K}.$$

Step 2: Adaptation

Compute $(X^{\tau+1}, Y^{\tau+1}, r^{\tau+1}) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\sum_{j=1}^{J} \sum_{i=1}^{I} \left[X_i^{j\tau+1} + \gamma \left(-\frac{\partial U^j(\bar{X}^{j\tau}, \bar{Y}^{j\tau}, \bar{r}^\tau)}{\partial X_i^j} \right) - X_i^{j\tau} \right] \times \left[X_i^j - X_i^{j\tau+1} \right]$$

$$+ \sum_{j=1}^{J} \sum_{i=1}^{I} \left[Y_i^{j\tau+1} + \gamma \left(-\frac{\partial U^j(\bar{X}^{j\tau}, \bar{Y}^{j\tau}, \bar{r}^\tau)}{\partial Y_i^j} \right) - Y_i^{j\tau} \right] \times \left[Y_i^j - Y_i^{j\tau+1} \right]$$

$$\sum_{i=1}^{I} \left[r_i^{\tau+1} + \gamma \left[\sum_{j=1}^{J} \left(\bar{X}_i^{j\tau} - \bar{Y}_i^{j\tau} \right) \right] - r_i^\tau \right] \times \left[r_i - r_i^{\tau+1} \right], \quad \forall\, (X, Y, r) \in \mathcal{K}.$$

Step 3: Convergence Verification:

If $\max_{i,j} |X_i^{j\tau+1} - X_i^{j\tau}| \leq \epsilon$; $\max_{i,j} |Y_i^{j\tau+1} - Y_i^{j\tau}| \leq \epsilon$; $\max_i |r_i^{\tau+1} - r_i^\tau| \leq \epsilon$, for all $i; i = 1, \ldots, I$, and $j; j = 1, \ldots, J$, with $\epsilon > 0$, a prespecified tolerance, then stop; else, set $\tau := \tau + 1$, and go to Step 1.

Convergence results are given in Nagurney, Dong, and Hughes (1992); see also Nagurney and Siokos (1997).

An interpretation of the modified projection method as an adjustment process is now provided. The interpretation of the algorithm as an adjustment process was given by Nagurney (1999). In particular, at an iteration, the sectors in the economy receive all the price information on every instrument from the previous iteration. They then allocate their capital according to their preferences.

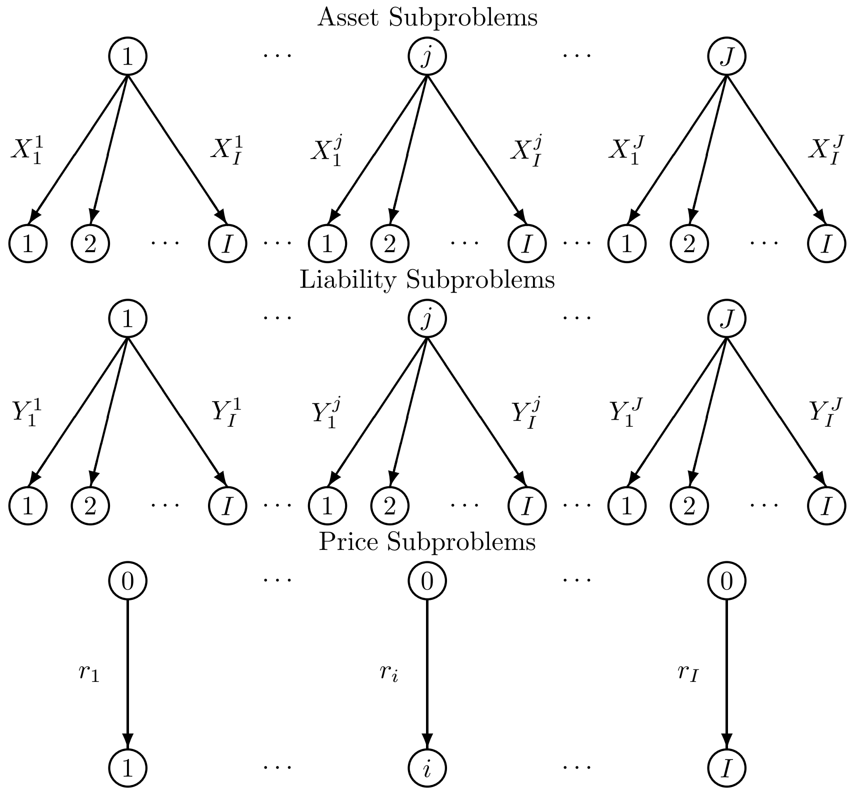

Figure 1.4: Financial Network Subproblems Induced by the Modified Projection Method

The market reacts on the decisions of the sectors and derives new instrument prices. The sectors then improve upon their positions through the adaptation step, whereas the market also adjusts during the adaptation step. This process continues until noone can improve upon his position, and the equilibrium is reached, that is, the above variational inequality is satisfied with the computed asset, liability, and price pattern.

The financial optimization problems in the computation step and in the adaptation step are equivalent to separable quadratic programming problems, of special network structure, as depicted in Figure 1.4. Each of these network subproblems structure can then be solved, at an iteration, simultaneously, and exactly in closed form. The exact equilibration algorithm (see, e.g., Nagurney and Siokos (1997)) can be applied for the solution of the asset and liability subproblems, whereas the prices can be obtained using explicit formulae.

A numerical example is now presented for illustrative purposes and solved using the modified projection method, embedded with the exact equilibration algorithm.

Example 1: A Numerical Example

Assume that there are two sectors in the economy and three financial instruments. Assume that the "size" of each sector is given by $S^1 = 1$ and $S^2 = 2$. The variance–covariance matrices of the two sectors are:

$$Q^1 = \begin{pmatrix} 1 & .25 & .3 & 0 & 0 & 0 \\ .25 & 1 & .1 & 0 & 0 & 0 \\ .3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & .2 & .3 \\ 0 & 0 & 0 & .2 & 1 & .5 \\ 0 & 0 & 0 & .3 & .5 & 1 \end{pmatrix}$$

and

$$Q^2 = \begin{pmatrix} 1 & 0 & .3 & 0 & 0 & 0 \\ 0 & 1 & .2 & 0 & 0 & 0 \\ .3 & .2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & .5 & 0 \\ 0 & 0 & 0 & .5 & 1 & .2 \\ 0 & 0 & 0 & 0 & .2 & 1 \end{pmatrix}.$$

The modified projection method was coded in FORTRAN. The variables were initialized as follows: $r_i^0 = 1$, for all i, with the financial volume S^j equally distributed among all the assets and among all the liabilities for each sector j. The γ parameter was set to 0.35. The convergence tolerance ε was set to 10^{-3}.

The modified projection method converged in 16 iterations and yielded the following equilibrium pattern:

Equilibrium Prices:

$$r_1^* = .34039, \quad r_2^* = .23805, \quad r_3^* = .42156,$$

Equilibrium Asset Holdings:

$$X_1^{1^*} = .27899, \quad X_2^{1^*} = .31803, \quad X_3^{1^*} = .40298,$$

$$X_1^{2^*} = .79662, \quad X_2^{2^*} = .60904, \quad X_3^{2^*} = .59434,$$

Equilibrium Liability Holdings:

$$Y_1^{1^*} = .37081, \quad Y_2^{1^*} = .43993, \quad Y_3^{1^*} = .18927,$$

$$Y_1^{2^*} = .70579, \quad Y_2^{2^*} = .48693, \quad Y_3^{2^*} = .80729.$$

The above results show that the algorithm yielded optimal portfolios that were feasible. Moreover, the market cleared for each instrument, since the price of each instrument was positive.

Other financial equilibrium models, including models with transaction costs, with hedging instruments such as futures and options, as well as, international financial equilibrium models, can be found in Nagurney and Siokos (1997), and the references therein.

Moreover, with projected dynamical systems theory (see the book by Nagurney and Zhang (1996)) one can trace the dynamic behavior prior to an equilibrium state (formulated as a variational inequality). In contrast to classical dynamical systems, projected dynamical systems are characterized by a discontinuous right-hand side, with the discontinuity arising due to the constraint set underlying the application in question. Hence, this methodology allows one to model systems dynamically which are subject to limited resources, with a principal constraint in finance being budgetary restrictions.

Dong, Zhang, and Nagurney (1996) were the first to apply the methodology of projected dynamical systems to develop a dynamic multi-sector, multi-instrument financial model, whose set of stationary points coincided with the set of solutions to the variational inequality model developed in Nagurney (1994); and then to study it qualitatively, providing stability analysis results. In the next section, the methodology of projected dynamical systems is illustrated in the context of a dynamic financial network model with intermediation (cf. Nagurney and Dong (2002)).

1.4 Dynamic Financial Networks with Intermediation

In this section, dynamic financial networks with intermediation are explored. As noted earlier, the conceptualization of financial systems as networks dates to Quesnay (1758) who depicted the circular flow of funds in an economy as a network. His basic idea was subsequently applied to the construction of flow of funds accounts, which are a statistical description of the flows of money and credit in an economy (cf. Board of Governors (1980), Cohen (1987), Nagurney and Hughes (1992)). However, since the flow of funds accounts are in matrix form, and, hence, two-dimensional, they fail to capture the dynamic behavior on a micro level of the various financial agents/sectors in an economy, such as banks, households, insurance companies, etc. Moreover, as noted by the Board of Governors (1980) on page 6 of that publication, "the generality of the matrix tends to obscure certain structural aspects of the financial system that are of continuing interest in analysis," with the structural concepts of concern including financial intermediation.

Thore (1980) recognized some of the shortcomings of financial flow of funds accounts and developed, instead, network models of linked portfolios with financial intermediation, using decentralization/decomposition theory. Note that intermediation is typically associated with financial businesses, including banks, savings institutions, investment and insurance companies, etc., and the term implies borrowing for the purpose of lending, rather than for nonfinancial purposes. Thore also constructed some basic intertemporal models. However, the intertemporal models were not fully developed and the computational techniques

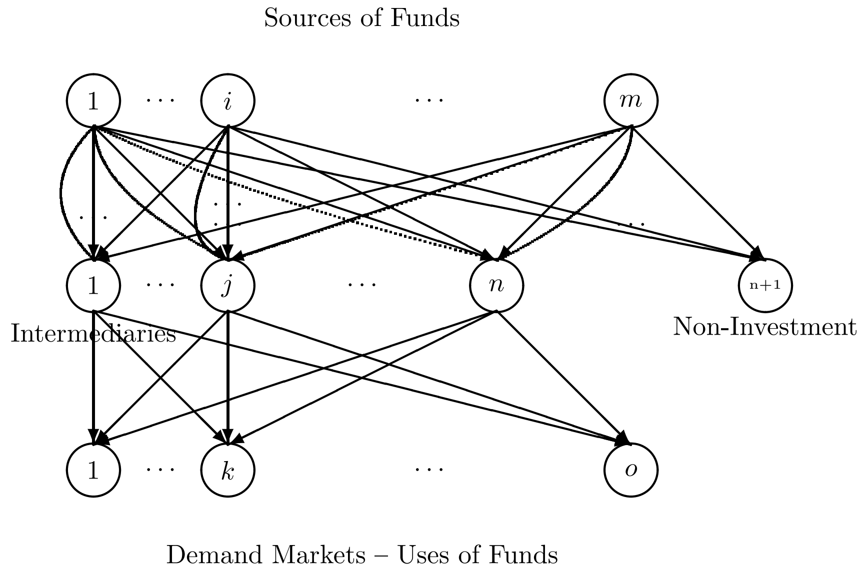

Figure 1.5: The Network Structure of the Financial Economy with Intermediation and with Non-Investment Allowed

at that time were not sufficiently advanced for computational purposes.

In this section, we address the dynamics of the financial economy which explicitly includes financial intermediaries along with the "sources" and "uses" of financial funds. Tools are provided for studying the disequilibrium dynamics as well as the equilibrium state. Also, transaction costs are considered, since they bring a greater degree of realism to the study of financial intermediation. Transaction costs had been studied earlier in multi-sector, multi-instrument financial equilibrium models by Nagurney and Dong (1996 a,b) but without considering the more general dynamic intermediation setting.

The dynamic financial network model is now described. The model consists of agents with sources of funds, agents who are intermediaries, as well as agents who are consumers located at the demand markets. Specifically, consider m agents with sources of financial funds, such as households and businesses, involved in the allocation of their financial resources among a portfolio of financial instruments which can be obtained by transacting with distinct n financial intermediaries, such as banks, insurance and investment companies, etc. The financial intermediaries, in turn, in addition to transacting with the source agents, also determine how to allocate the incoming financial resources among distinct uses, as represented by o demand markets with a demand market corresponding to, for example, the market for real estate loans, household loans, or business loans, etc. The financial network with intermediation is now described and depicted graphically in Figure 1.5.

The top tier of nodes in Figure 1.5 consists of the agents with sources of funds, with a typical source agent denoted by i and associated with node i. The middle tier of nodes in Figure 1.5 consists of the intermediaries, with a typical intermediary denoted by j and associated with node j in the network. The bottom tier of nodes consists of the demand markets, with a typical demand market denoted by k and corresponding to the node k.

For simplicity of notation, assume that there are L financial instruments associated with each intermediary. Hence, from each source of funds node, there are L links connecting such a node with an intermediary node with the l-th such link corresponding to the l-th financial instrument available from the intermediary. In addition, the option of non-investment in the available financial instruments is allowed and to denote this option, construct an additional link from each source node to the middle tier node $n+1$, which represents non-investment. Note that there are as many links connecting each top tier node with each intermediary node as needed to reflect the number of financial instruments available. Also, note that there is an additional abstract node $n+1$ with a link connecting each source node to it, which, as shall shortly be shown, will be used to "collect" the financial funds which are not invested. In the model, it is assumed that each source agent has a fixed amount of financial funds.

From each intermediary node, construct o links, one to each "use" node or demand market in the bottom tier of nodes in the network to denote the transaction between the intermediary and the consumers at the demand market.

Let x_{ijl} denote the nonnegative amount of the funds that source i "invests" in financial instrument l obtained from intermediary j. Group the financial flows associated with source agent i, which are associated with the links emanating from the top tier node i to the intermediary nodes in the logistical network, into the column vector $x_i \in R^{nL}_+$. Assume that each source has, at his disposal, an amount of funds S_i and denote the unallocated portion of this amount (and flowing on the link joining node i with node $n+1$) by s_i. Group then the x_is of all the source agents into the column vector $x \in R^{mnL}_+$.

Associate a distinct financial product k with each demand market, bottom-tiered node k and let y_{jk} denote the amount of the financial product obtained by consumers at demand market k from intermediary j. Group these "consumption" quantities into the column vector $y \in R^{no}_+$. The intermediaries convert the incoming financial flows x into the outgoing financial flows y.

The notation for the prices is now given. Note that there will be prices associated with each of the tiers of nodes in the network. Let ρ_{1ijl} denote the price associated with instrument l as quoted by intermediary j to source agent i and group the first tier prices into the column vector $\rho_1 \in R^{mnL}_+$. Also, let ρ_{2j} denote the price charged by intermediary j and group all such prices into the column vector $\rho_2 \in R^n_+$. Finally, let ρ_{3k} denote the price of the financial product at the third or bottom-tiered node k in the network, and group all such prices into the column vector $\rho_3 \in R^o_+$.

We now turn to describing the dynamics by which the source agents adjust the amounts they allocate to the various financial instruments over time, the dynamics by which the intermediaries adjust their transactions, and those by which the consumers obtain the financial products at the demand markets. In addition, the dynamics by which the prices adjust over time are described. The dynamics are derived from the bottom tier of nodes of the network on up since it is assumed that it is the demand for the financial products (and the corresponding prices) that actually drives the economic dynamics. The price dynamics are presented first and then the dynamics underlying the financial flows.

The Demand Market Price Dynamics

We begin by describing the dynamics underlying the prices of the financial products associated with the demand markets (see the bottom-tiered nodes). Assume, as given, a demand function d_k, which can depend, in general, upon the entire vector of prices ρ_3, that is,

$$d_k = d_k(\rho_3), \quad \forall k. \tag{1.20}$$

Moreover, assume that the rate of change of the price ρ_{3k}, denoted by $\dot{\rho}_{3k}$, is equal to the difference between the demand at the demand market k, as a function of the demand market prices, and the amount available from the intermediaries at the demand market. Hence, if the demand for the product at the demand market (at an instant in time) exceeds the amount available, the price of the financial product at that demand market will increase; if the amount available exceeds the demand at the price, then the price at the demand market will decrease. Furthermore, it is guaranteed that the prices do not become negative. Thus, the dynamics of the price ρ_{3k} associated with the product at demand market k can be expressed as:

$$\dot{\rho}_{3k} = \begin{cases} d_k(\rho_3) - \sum_{j=1}^{n} y_{jk}, & \text{if} \quad \rho_{3k} > 0 \\ \max\{0, d_k(\rho_3) - \sum_{j=1}^{n} y_{jk}\}, & \text{if} \quad \rho_{3k} = 0. \end{cases} \tag{1.21}$$

The Dynamics of the Prices at the Intermediaries

The prices charged for the financial funds at the intermediaries, in turn, must reflect supply and demand conditions as well (and as shall be shown shortly also reflect profit-maximizing behavior on the part of the intermediaries who seek to determine how much of the financial flows they obtain from the different sources of funds). In particular, assume that the price associated with intermediary j, ρ_{2j}, and computed at node j lying in the second tier of nodes, evolves over time according to:

$$\dot{\rho}_{2j} = \begin{cases} \sum_{k=1}^{o} y_{jk} - \sum_{i=1}^{m} \sum_{l=1}^{L} x_{ijl}, & \text{if} \quad \rho_{2j} > 0 \\ \max\{0, \sum_{k=1}^{o} y_{jk} - \sum_{i=1}^{m} \sum_{l=1}^{L} x_{ijl}\}, & \text{if} \quad \rho_{2j} = 0, \end{cases} \tag{1.22}$$

where $\dot{\rho}_{2j}$ denotes the rate of change of the j-th intermediary's price. Hence, if the amount of the financial funds desired to be transacted by the consumers (at an instant in time) exceeds that available at the intermediary, then the price charged at the intermediary will increase; if the amount available is greater than that desired by the consumers, then the price charged at the intermediary will decrease. As in the case of the demand market prices, it is guaranteed that the prices charged by the intermediaries remain nonnegative.

Precursors to the Dynamics of the Financial Flows

First some preliminaries are needed that will allow the development of the dynamics of the financial flows. In particular, the utility-maximizing behavior of the source agents and that of the intermediaries is now discussed.

Assume that each such source agent's and each intermediary agent's utility can be defined as a function of the expected future portfolio value, where the expected value of the future portfolio is described by two characteristics: the expected mean value and the uncertainty surrounding the expected mean. Here, the expected mean portfolio value is assumed to be equal to the market value of the current portfolio. Each agent's uncertainty, or assessment of risk, in turn, is based on a variance-covariance matrix denoting the agent's assessment of the standard deviation of the prices for each instrument/product. The variance-covariance matrix associated with source agent i's assets is denoted by Q^i and is of dimension $nL \times nL$, and is associated with vector x_i, whereas intermediary agent j's variance-covariance matrix is denoted by Q^j, is of dimension $o \times o$, and is associated with the vector y_j.

Optimizing Behavior of the Source Agents

Denote the total transaction cost associated with source agent i transacting with intermediary j to obtain financial instrument l by c_{ijl} and assume that:

$$c_{ijl} = c_{ijl}(x_{ijl}), \quad \forall i, j, l. \tag{1.23}$$

The total transaction costs incurred by source agent i, thus, are equal to the sum of all the agent's transaction costs. His revenue, in turn, is equal to the sum of the price (rate of return) that the the agent can obtain for the financial instrument times the total quantity obtained/purchased of that instrument. Recall that ρ_{1ijl} denotes the price associated with agent i/intermediary j/instrument l.

Assume that each such source agent seeks to maximize net return while, simultaneously, minimizing the risk, with source agent i's utility function denoted by U^i. Moreover, assume that the variance-covariance matrix Q^i is positive semidefinite and that the transaction cost functions are continuously differentiable and convex. Hence, one can express the optimization problem facing source agent i as:

$$\text{Maximize} \quad U_i(x_i) = \sum_{j=1}^{n} \sum_{l=1}^{L} \rho_{1ijl} x_{ijl} - \sum_{j=1}^{n} \sum_{l=1}^{L} c_{ijl}(x_{ijl}) - {x_i}^T Q^i x_i, \tag{1.24}$$

subject to $x_{ijl} \geq 0$, for all j, l, and to the constraint:

$$\sum_{j=1}^{n} \sum_{l=1}^{L} x_{ijl} \leq S^i, \tag{1.25}$$

that is, the allocations of source agent i's funds among the financial instruments made available by the different intermediaries cannot exceed his holdings. Note that the utility function above is concave for each source agent i. A source agent may choose to not invest in any of the instruments. Indeed, as shall be illustrated through subsequent numerical examples, this constraint has important financial implications.

Clearly, in the case of *unconstrained* utility maximization, the gradient of source agent i's utility function with respect to the vector of variables x_i and denoted by $\nabla_{x_i}U_i$, where $\nabla_{x_i}U_i=(\frac{\partial U_i}{\partial x_{i11}}, \ldots, \frac{\partial U_i}{\partial x_{inL}})$, represents agent i's idealized direction, with the jl-component of $\nabla_{x_i}U_i$ given by:

$$(\rho_{1ijl} - 2Q^i_{z_{jl}} \cdot x_i - \frac{\partial c_{ijl}(x_{ijl})}{\partial x_{ijl}}), \tag{1.26}$$

where $Q^i_{z_{jl}}$ denotes the z_{jl}-th row of Q^i, and z_{jl} is the indicator defined as: $z_{jl} = (l-1)n + j$. We return later to describe how the constraints are explicitly incorporated into the dynamics.

Optimizing Behavior of the Intermediaries

The intermediaries, in turn, are involved in transactions both with the source agents, as well as with the users of the funds, that is, with the ultimate consumers associated with the markets for the distinct types of loans/products at the bottom tier of the financial network. Thus, an intermediary conducts transactions both with the "source" agents as well as with the consumers at the demand markets.

An intermediary j is faced with what is termed a *handling/conversion* cost, which may include, for example, the cost of converting the incoming financial flows into the financial loans/products associated with the demand markets. Denote this cost by c_j and, in the simplest case, one would have that c_j is a function of $\sum_{i=1}^m \sum_{l=1}^L x_{ijl}$, that is, the holding/conversion cost of an intermediary is a function of how much he has obtained from the various source agents. For the sake of generality, however, allow the function to, in general, depend also on the amounts held by other intermediaries and, therefore, one may write:

$$c_j = c_j(x), \quad \forall j. \tag{1.27}$$

The intermediaries also have associated transaction costs in regard to transacting with the source agents, which are assumed to be dependent on the type of instrument. Denote the transaction cost associated with intermediary j transacting with source agent i associated with instrument l by $\hat{c}_{ijl}$ and assume that it is of the form

$$\hat{c}_{ijl} = \hat{c}_{ijl}(x_{ijl}), \quad \forall i, j, l. \tag{1.28}$$

Recall that the intermediaries convert the incoming financial flows x into the outgoing financial flows y. Assume that an intermediary j incurs a transaction cost c_{jk} associated with transacting with demand market k, where

$$c_{jk} = c_{jk}(y_{jk}), \quad \forall j, k. \tag{1.29}$$

The intermediaries associate a price with the financial funds, which is denoted by ρ_{2j}, for intermediary j. Assuming that the intermediaries are also utility maximizers with the utility functions for each being comprised of net revenue maximization as well as risk minimization, then the utility maximization problem for intermediary agent j with his utility function denoted by U^j, can be expressed as:

$$\text{Maximize} \quad U_j(x_j, y_j) =$$

$$\sum_{i=1}^{m}\sum_{l=1}^{L}\rho_{2j}x_{ijl} - c_j(x) - \sum_{i=1}^{m}\sum_{l=1}^{L}\hat{c}_{ijl}(x_{ijl}) - \sum_{k=1}^{o}c_{jk}(y_{jk}) - \sum_{i=1}^{m}\sum_{l=1}^{L}\rho_{1ijl}x_{ijl} - {y_j}^T Q^j y_j, \tag{1.30}$$

subject to the nonnegativity constraints: $x_{ijl} \geq 0$, and $y_{jk} \geq 0$, for all i, l, and k. Here, for convenience, we have let $x_j = (x_{1j1}, \ldots, x_{mjL})$. The above bijective function expresses that the difference between the revenues minus the handling cost and the transaction costs and the payout to the source agents should be maximized, whereas the risk should be minimized. Assume now that the variance-covariance matrix Q^j is positive semidefinite and that the transaction cost functions are continuously differentiable and convex. Hence, the utility function above is concave for each intermediary j.

The gradient $\nabla_{x_j}U_j = (\frac{\partial U_j}{\partial x_{1j1}}, \ldots, \frac{\partial U_j}{\partial x_{mjL}})$ represents agent j's idealized direction in terms of x_j, ignoring the constraints, for the time being, whereas the gradient $\nabla_{y_j}U_j = (\frac{\partial U_j}{\partial y_{j1}}, \ldots, \frac{\partial U_j}{\partial y_{jo}})$ represents his idealized direction in terms of y_j. Note that the il-th component of $\nabla_{x_j}U_j$ is given by:

$$(\rho_{2j} - \rho_{1ijl} - \frac{\partial c_j(x)}{\partial x_{ijl}} - \frac{\partial \hat{c}_{ijl}(x_{ijl})}{\partial x_{ijl}}), \tag{1.31}$$

whereas the jk-th component of $\nabla_{y_j}U_j$ is given by:

$$(-\frac{\partial c_{jk}(y_{jk})}{\partial y_{jk}} - 2Q^j_k \cdot y_j). \tag{1.32}$$

However, since both source agent i and intermediary j must agree in terms of the x_{ijl}s, the direction (1.26) must coincide with that in (1.31), so adding both gives us a "combined force," which, after algebraic simplification, yields:

$$(\rho_{2j} - 2Q^i_{z_{jl}} \cdot x_i - \frac{\partial c_{ijl}(x_{ijl})}{\partial x_{ijl}} - \frac{\partial c_j(x)}{\partial x_{ijl}} - \frac{\partial \hat{c}_{ijl}(x_{ijl})}{\partial x_{ijl}}). \tag{1.33}$$

The Dynamics of the Financial Flows between the Source Agents and the Intermediaries

We are now ready to express the dynamics of the financial flows between the source agents and the intermediaries. In particular, define the feasible set $K_i \equiv \{x_i | x_{ijl} \geq 0, \forall i, j, l, \text{ and } (1.25) \text{ holds}\}$. Let also K be the Cartesian product given by $K \equiv \Pi_{i=1}^m K_i$ and define F^1_{ijl} as minus the term in (1.33) with $F^1_i =$

$(F^1_{i11}, \ldots, F^1_{inL})$. Then the *best realizable* direction for the vector of financial instruments x_i can be mathematically expressed as:

$$\dot{x}_i = \Pi_{K_i}(x_i, -F^1_i), \tag{1.34}$$

where $\Pi_K(Z, v)$ is defined as:

$$\Pi_K(Z, v) = \lim_{\delta \to 0} \frac{P_K(Z + \delta v) - Z}{\delta}, \tag{1.35}$$

and P_K is the norm projection defined by

$$P_K(Z) = \text{argmin}_{Z' \in K} \|Z' - Z\|. \tag{1.36}$$

The Dynamics of the Financial Flows between the Intermediaries and the Demand Markets

In terms of the financial flows between the intermediaries and the demand markets, both the intermediaries and the consumers must be in agreement as to the financial flows y. The consumers take into account in making their consumption decisions not only the price charged for the financial product by the intermediaries but also their transaction costs associated with obtaining the product.

Let $\hat{c}_{jk}$ denote the transaction cost associated with obtaining the product at demand market k from intermediary j. Assume that this unit transaction cost is continuous and of the general form:

$$\hat{c}_{jk} = \hat{c}_{jk}(y), \quad \forall j, k. \tag{1.37}$$

The consumers take the price charged by the intermediaries, which was denoted by ρ_{2j} for intermediary j, plus the unit transaction cost, in making their consumption decisions. From the perspective of the consumers at the demand markets, one can expect that an idealized direction in terms of the evolution of the financial flow of a product between an intermediary/demand market pair would be:

$$(\rho_{3k} - \hat{c}_{jk}(y) - \rho_{2j}). \tag{1.38}$$

On the other hand, as already derived above, one can expect that the intermediaries would adjust the volume of the product to a demand market according to (1.32). Combining now (1.32) and (1.38), and guaranteeing that the financial products do not assume negative quantities, yields the following dynamics:

$$\dot{y}_{jk} = \begin{cases} \rho_{3k} - \hat{c}_{jk}(y) - \rho_{2j} - \frac{\partial c_{jk}(y_{jk})}{\partial y_{jk}} - 2Q^j_k \cdot y_j, & \text{if} \quad y_{jk} > 0 \\ \max\{0, \rho_{3k} - \hat{c}_{jk}(y) - \rho_{2j} - \frac{\partial c_{jk}(y_{jk})}{\partial y_{jk}} - 2Q^j_k \cdot y_j\}, & \text{if} \quad y_{jk} = 0. \end{cases} \tag{1.39}$$

The Projected Dynamical System

Consider now the dynamic model in which the demand prices evolve according to (1.21) for all demand markets k, the prices at the intermediaries evolve according to (1.22) for all intermediaries j; the financial flows between the source agents and the intermediaries evolve according to (1.34) for all source agents i, and the financial products between the intermediaries and the demand markets evolve according to (1.39) for all intermediary/demand market pairs j, k.

Let now Z denote the aggregate column vector (x, y, ρ_2, ρ_3) in the feasible set $\mathcal{K} \equiv K \times R_+^{no+n+o}$. Define the column vector $F(Z) \equiv (F^1, F^2, F^3, F^4)$, where F^1 is as has been defined previously; $F^2 = (F^2_{11}, \ldots, F^2_{no})$, with component $F^2_{jk} \equiv (2Q^j_k \cdot y_j + \frac{\partial c_{jk}(y_{jk})}{\partial y_{jk}} + \hat{c}_{jk}(y) + \rho_{2j} - \rho_{3k})$, $\forall j, k$; $F^3 = (F^3_1, \ldots, F^3_n)$, where $F^3_j \equiv (\sum_{i=1}^m \sum_{l=1}^L x_{ijl} - \sum_{k=1}^o y_{jk})$, and $F^4 = (F^4_1, \ldots, F^4_o)$, with $F^4_k \equiv (\sum_{j=1}^n y_{jk} - d_k(\rho_3))$.

Then the dynamic model described by (1.21), (1.22), (1.34), and (1.39) for all k, j, i, l can be rewritten as the *projected dynamical system* defined by the following initial value problem:

$$\dot{Z} = \Pi_{\mathcal{K}}(Z, -F(Z)), \quad Z(0) = Z_0, \tag{1.40}$$

where, $\Pi_{\mathcal{K}}$ is the projection operator of $-F(Z)$ onto $\mathcal{K}$ at Z and $Z_0 = (x^0, y^0, \rho_2^0, \rho_3^0)$ is the initial point corresponding to the initial financial flows and the initial prices. The trajectory of (1.40) describes the dynamic evolution of and the dynamic interactions among the prices and the financial flows.

The dynamical system (1.40) is non-classical in that the right-hand side is discontinuous in order to guarantee that the constraints in the context of the above model are not only nonnegativity constraints on the variables, but also a form of budget constraints. Here this methodology is applied to study financial systems in the presence of intermediation. A variety of dynamic financial models, but without intermediation, formulated as projected dynamical systems can be found in the book by Nagurney and Siokos (1997).

A Stationary/Equilibrium Point

The stationary point of the projected dynamical system (1.40) is now discussed. Recall that a stationary point Z^* is that point that satisfies

$$\dot{Z} = 0 = \Pi_{\mathcal{K}}(Z^*, -F(Z^*)),$$

and, hence, in the context of the dynamic financial model with intermediation, when there is no change in the financial flows and no change in the prices. Moreover, as established in Dupuis and Nagurney (1993), since the feasible set $\mathcal{K}$ is a polyhedron and convex, the set of stationary points of the projected dynamical system of the form given in (1.40) coincides with the set of solutions to the variational inequality problem given by: determine $Z^* \in \mathcal{K}$, such that

$$\langle F(Z^*)^T, Z - Z^* \rangle \geq 0, \quad \forall Z \in \mathcal{K}, \tag{1.41}$$

where in the model $F(Z)$ and Z are as defined above and recall that $\langle \cdot, \cdot \rangle$ denotes the inner product in N-dimensional Euclidean space where here $N = mnL + no + n + o$.

Variational Inequality Formulation of Financial Equilibrium with Intermediation

In particular, variational inequality (1.41) here takes the form:
determine $(x^*, y^*, \rho_2^*, \rho_3^*) \in \mathcal{K}$, satisfying:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{L}\left[2Q^i_{z_{jl}} \cdot x_i^* + \frac{\partial c_{ijl}(x^*_{ijl})}{\partial x_{ijl}} + \frac{\partial c_j(x^*)}{\partial x_{ijl}} + \frac{\partial \hat{c}_{ijl}(x^*_{ijl})}{\partial x_{ijl}} - \rho^*_{2j}\right]$$

$$\times \left[x_{ijl} - x^*_{ijl}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[2Q^j_k \cdot y_j^* + \frac{\partial c_{jk}(y^*_{jk})}{\partial y_{jk}} + \hat{c}_{jk}(y^*) + \rho^*_{2j} - \rho^*_{3k}\right] \times \left[y_{jk} - y^*_{jk}\right]$$

$$+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{l=1}^{L} x^*_{ijl} - \sum_{k=1}^{o} y^*_{jk}\right] \times \left[\rho_{2j} - \rho^*_{2j}\right]$$

$$+\sum_{k=1}^{o}\left[\sum_{j=1}^{n} y^*_{jk} - d_k(\rho_3^*)\right] \times \left[\rho_{3k} - \rho^*_{3k}\right] \geq 0, \quad \forall (x, y, \rho_2, \rho_3) \in \mathcal{K}, \tag{1.42}$$

where $\mathcal{K} \equiv \{K \times R_+^{no+n+o}\}$ and $Q^i_{z_{jl}}$ is as was defined following (1.26).

We now discuss the equilibrium conditions. First, note that if the rate of change of the demand price $\dot{\rho}_{3k} = 0$, then from (1.21) one can conclude that:

$$d_k(\rho_3^*) \begin{cases} = \sum_{j=1}^{n} y^*_{jk}, & \text{if} \quad \rho^*_{3k} > 0 \\ \leq \sum_{j=1}^{n} y^*_{jk}, & \text{if} \quad \rho^*_{3k} = 0. \end{cases} \tag{1.43}$$

Condition (1.43) states that, if the price the consumers are willing to pay for the financial product at a demand market is positive, then the quantity consumed by the consumers at the demand market is precisely equal to the demand. If the demand is less than the amount of the product available, then the price for that product is zero. This condition holds for all demand market prices in equilibrium.

Note that condition (1.43) also follows directly from variational inequality (1.42) if one sets $x = x^*$, $y = y^*$, and $\rho_2 = \rho_2^*$, and make the substitution into (1.42) and note that the demand prices must be nonnegative.

Observe now that if the rate of change of a price charged by an intermediary is zero, that is, $\dot{\rho}_{2j} = 0$, then (1.22) implies that

$$\sum_{i=1}^{m}\sum_{l=1}^{L} x^*_{ijl} - \sum_{k=1}^{o} y^*_{jk} \begin{cases} = 0, & \text{if} \quad \rho^*_{2j} > 0 \\ \geq 0, & \text{if} \quad \rho^*_{2j} = 0. \end{cases} \tag{1.44}$$

Hence, if the price for the financial funds at an intermediary is positive, then the market for the funds "clears" at the intermediary, that is, the supply of funds, as given by $\sum_{i=1}^{m}\sum_{l=1}^{L} x^*_{ijl}$ is equal to the demand of funds, $\sum_{k=1}^{o} y^*_{jk}$ at the intermediary. If the supply exceeds the demand, then the price at the intermediary will be zero. These are well-known economic equilibrium conditions as are those given in (1.43). Of course, condition (1.44) could also be recovered from variational inequality (1.42) by setting $x = x^*$, $y = y^*$, and $\rho_3 = \rho_3^*$, and making the substitution into (1.42) and noting that these prices must be nonnegative. In equilibrium, condition (1.44) holds for all intermediary prices.

On the other hand, if one sets $\dot{x}_i = 0$ (cf. (1.34) and (1.40)), for all i and $\dot{y}_{jk} = 0$ for all j, k (cf. (1.39) and (1.40), one obtains the equilibrium conditions, which correspond, equivalently, to the first two summands in inequality (1.42) being greater than equal to zero. Expressed in another manner, we must have that the sum of the inequalities (1.45), (1.46), and (1.48) below must be satisfied.

Optimality Conditions for all Source Agents

Indeed, note that the optimality conditions for all source agents i, since each K_i is closed and convex, and the objective function (1.24) is concave, can be expressed as (assuming a given ρ^*_{1jl}, for all i, j, l:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{L}\left[2Q^i_{z_{jl}}\cdot x^*_i + \frac{\partial c_{ijl}(x^*_{ijl})}{\partial x_{ijl}} - \rho^*_{1ijl}\right]\times\left[x_{ijl} - x^*_{ijl}\right] \geq 0, \quad \forall x \in K. \tag{1.45}$$

Optimality Conditions for all Intermediary Agents

The optimality conditions for all the intermediaries j, with objective functions of the form (1.30), which are concave, and, given ρ_1^* and ρ_2^*, can, in turn, be expressed as:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{L}\left[\frac{\partial c_j(x^*)}{\partial x_{ijl}} + \rho^*_{1ijl} + \frac{\partial \hat{c}_{ijl}(x^*_{ijl})}{\partial x_{ijl}} - \rho^*_{2j}\right]\times\left[x_{ijl} - x^*_{ijl}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[2Q^j_k\cdot y_j + \frac{\partial c_{jk}(y^*_{jk})}{\partial y_{jk}}\right]\times\left[y_{jk} - y^*_{jk}\right] \geq 0, \forall x \in R_+^{mnL}, \forall y \in R_+^{no}. \tag{1.46}$$

Note that (1.46) provides a means for recovering the top-tiered prices, ρ_1^*.

Equilibrium Conditions for Consumers at the Demand Markets

Also, the equilibrium conditions for consumers at demand market k, thus, take the form: for all intermediaries: $j; j = 1, \ldots, n$:

$$\rho^*_{2j} + \hat{c}_{jk}(y^*)\begin{cases} = \rho^*_{3k}, & \text{if} \quad y^*_{jk} > 0 \\ \geq \rho^*_{3k}, & \text{if} \quad y^*_{jk} = 0, \end{cases} \tag{1.47}$$

with (1.47) holding for all demand markets k, which is equivalent to $y^* \in R_+^{no}$ satisfying:

$$\sum_{j=1}^{n}\sum_{k=1}^{o}(\rho_{2j}^* + \hat{c}_{jk}(y^*) - \rho_{3k}^*)) \times (y_{jk} - y_{jk}^*) \geq 0, \quad \forall y \in R_+^{no}. \tag{1.48}$$

Conditions (1.47) state that consumers at demand market k will purchase the product from intermediary j, if the price charged by the intermediary for the product plus the transaction cost (from the perspective of the consumers) does not exceed the price that the consumers are willing to pay for the product, that is, ρ_{3k}^*.

In Nagurney and Ke (2001) a variational inequality of the form (1.42) was derived in a manner distinct from that given above for a static financial network model with intermediation, but with a slightly different feasible set where it was assumed that the constraints (1.25) had to be tight, that is, to hold as an equality. Nagurney and Ke (2003), in turn, demonstrated how electronic transactions could be introduced into financial networks with intermediation by adding additional links to the network in Figure 1.5 and by including additional transaction costs and prices and expanding the objective functions of the decision-makers accordingly. We discuss electronic financial transactions subsequently, when we describe the financial engineering of integrated social and financial networks with intermediation.

1.4.1 The Discrete-Time Algorithm (Adjustment Process)

Note that the projected dynamical system (1.40) is a continuous time adjustment process. However, in order to further fix ideas and to provide a means of "tracking" the trajectory of (1.40), we present a discrete-time adjustment process, in the form of the Euler method, which is induced by the general iterative scheme of Dupuis and Nagurney (1993).

The statement of the Euler method is as follows:

Step 0: Initialization:

Start with a $Z^0 \in \mathcal{K}$. Set $\tau := 1$.

Step 1: Computation

Compute Z^τ by solving the variational inequality problem:

$$Z^\tau = P_{\mathcal{K}}(Z^{\tau-1} - \alpha_\tau F(Z^{\tau-1})), \tag{1.49}$$

where $\{\alpha_\tau; \tau = 1, 2, \ldots\}$ is a sequence of positive scalars such that $\sum_{\tau=1}^{\infty} \alpha_\tau = \infty$, $\alpha_\tau \to 0$, as $\tau \to \infty$ (which is required for convergence).

Step 2: Convergence Verification

If $|Z_b^\tau - Z_b^{\tau-1}| \leq \epsilon$, for some $\epsilon > 0$, a prespecific tolerance, then stop: otherwise, set $\tau := \tau + 1$, and go to Step 1.

The statement of this method in the context of the dynamic financial model takes the form:

The Euler Method

Step 0: Initialization Step

Set $(x^0, y^0, \rho_2^0, \rho_3^0) \in \mathcal{K}$. Let $\tau = 1$, where τ is the iteration counter, and set the sequence $\{\alpha_\tau\}$ so that $\sum_{\tau=1}^{\infty} \alpha_\tau = \infty$, $\alpha_\tau > 0$, $\alpha_\tau \to 0$, as $\tau \to \infty$.

Step 1: Computation Step

Compute $(x^\tau, y^\tau, \rho_2^\tau, \rho_3^\tau) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{L}\Bigg[x_{ijl}^{\tau} + \alpha_\tau(2Q_{z_{jl}}^{i} \cdot x_i^{\tau-1} + \frac{\partial c_{ijl}(x_{ijl}^{\tau-1})}{\partial x_{ijl}} + \frac{\partial c_j(x^{\tau-1})}{\partial x_{ijl}}$$

$$+\frac{\partial \hat{c}_{ijl}(x_{ijl}^{\tau-1})}{\partial x_{ijl}} - \rho_{2j}^{\tau-1}) - x_{ijl}^{\tau-1}\Bigg] \times \left[x_{ijl} - x_{ijl}^{\tau}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\Bigg[y_{jk}^{\tau} + \alpha_\tau(2Q_k^i \cdot y_j^{\tau-1} + \hat{c}_{jk}(y^{\tau-1}) + \frac{\partial c_{jk}(y_{jk}^{\tau-1})}{\partial y_{jk}}$$

$$+\rho_{2j}^{\tau-1} - \rho_{3k}^{\tau-1}) - y_{jk}^{\tau-1}\Bigg] \times \left[y_{jk} - y_{jk}^{\tau}\right]$$

$$+\sum_{j=1}^{n}\left[\rho_{2j}^{\tau} + \alpha_\tau(\sum_{i=1}^{m}\sum_{l=1}^{L} x_{ijl}^{\tau-1} - \sum_{k=1}^{o} y_{jk}^{\tau-1}) - \rho_{2j}^{\tau-1}\right] \times \left[\rho_{2j} - \rho_{2j}^{\tau}\right]$$

$$+\sum_{k=1}^{o}\left[\bar{\rho}_{3k}^{\tau} + \alpha_\tau(\sum_{j=1}^{n} y_{jk}^{\tau-1} - d_k(\rho_3^{\tau-1})) - \rho_{3k}^{\tau-1}\right] \times \left[\rho_{3k} - \rho_{3k}^{\tau}\right] \geq 0,$$

$$\forall (x, y, \rho_2, \rho_3) \in \mathcal{K}.$$

Step 2: Convergence Verification

If $|x_{ijl}^{\tau} - x_{ijl}^{\tau-1}| \leq \epsilon$, $|y_{jk}^{\tau} - y_{jk}^{\tau-1}| \leq \epsilon$, $|\rho_{2j}^{\tau} - \rho_{2j}^{\tau-1}| \leq \epsilon$, $|\rho_{3k}^{\tau} - \rho_{3k}^{\tau-1}| \leq \epsilon$, for all $i = 1, \cdots, m$; $j = 1, \cdots, n$; $l = 1, \ldots, L$; $k = 1, \cdots, o$, with $\epsilon > 0$, a pre-specified tolerance, then stop; otherwise, set $\tau := \tau + 1$, and go to Step 1.

Note that the variational inequality subproblem encountered in the computation step at each iteration of the Euler method can be solved explicitly and in closed form since it is actually a quadratic programming problem and the feasible set is a Cartesian product consisting of the product of K, which has a simple network structure, and the nonnegative orthants, R_+^{no}, R_+^{n}, and R_+^{o}, corresponding to the variables x, y, ρ_2, and ρ_3, respectively.

Computation of Financial Flows and Products

In fact, the subproblem in the x variables can be solved using exact equilibration (see also Dafermos and Sparrow (1969)) noted in the discussion of the modified projection method, whereas the remainder of the variables can be obtained by explicit formulae, which are provided below for convenience.

In particular, compute, at iteration τ, the y^{τ}_{jk}s, according to:

$$y^{\tau}_{jk} = \max\{0, y^{\tau-1}_{jk} - \alpha_{\tau}(2Q^{i}_{k} \cdot y^{\tau-1}_{j} + \hat{c}_{jk}(y^{\tau-1}) + \frac{\partial c_{jk}(y^{\tau-1}_{jk})}{\partial y_{jk}} + \rho^{\tau-1}_{2j} - \rho^{\tau-1}_{3k})\}, \quad \forall j, k. \tag{1.50}$$

Computation of the Prices

At iteration τ, compute the ρ^{τ}_{2j}s according to:

$$\rho^{\tau}_{2j} = \max\{0, \rho^{\tau-1}_{2j} - \alpha_{\tau}(\sum_{i=1}^{m}\sum_{l=1}^{L} x^{\tau-1}_{ijl} - \sum_{k=1}^{o} y^{\tau-1}_{jk})\}, \quad \forall j, \tag{1.51}$$

whereas the ρ^{τ}_{3k}s are computed explicitly and in closed form according to:

$$\rho^{\tau}_{3k} = \max\{0, \rho^{\tau-1}_{3k} - \alpha_{\tau}(\sum_{j=1}^{n} y^{\tau-1}_{jk} - d_k(\rho^{\tau-1}_{3}))\}, \quad \forall k. \tag{1.52}$$

1.5 Numerical Examples

In this section, the Euler method is applied to several numerical examples. The algorithm was implemented in FORTRAN. For the solution of the induced network subproblems in x, we utilized the exact equilibration algoritm, which fully exploits the simplicity of the special network structure of the subproblems.

The convergence criterion used was that the absolute value of the flows and prices between two successive iterations differed by no more than 10^{-4}. For the examples, the sequence $\{\alpha_{\tau}\} = .1\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\}$, which is of the form given in the intialization step of the algorithm in the preceding section. The numerical examples had the network structure depicted in Figure 1.6 and consisted of two source agents, two intermediaries, and two demand markets, with a single financial instrument handled by each intermediary.

The algorithm as follows was initialized as follows: since there was a single financial instrument associated with each of the intermediaries, we set $x_{ij1} = \frac{S^i}{n}$ for each source agent i. All the other variables, that is, the initial vectors y, ρ_2, and ρ_3 were set to zero. Additional details are given in Nagurney and Dong (2002).

Example 2

The data for this example were constructed for easy interpretation purposes. The supplies of the two source agents were: $S^1 = 10$ and $S^2 = 10$. The variance-covariance matrices Q^i and Q^j were equal to the identity matrices for all source agents i and all intermediaries j.

The transaction cost functions faced by the source agents associated with transacting with the intermediaries were given by:

$$c_{111}(x_{111}) = .5x^2_{111} + 3.5x_{111}, \quad c_{121}(x_{121}) = .5x^2_{121} + 3.5x_{121},$$

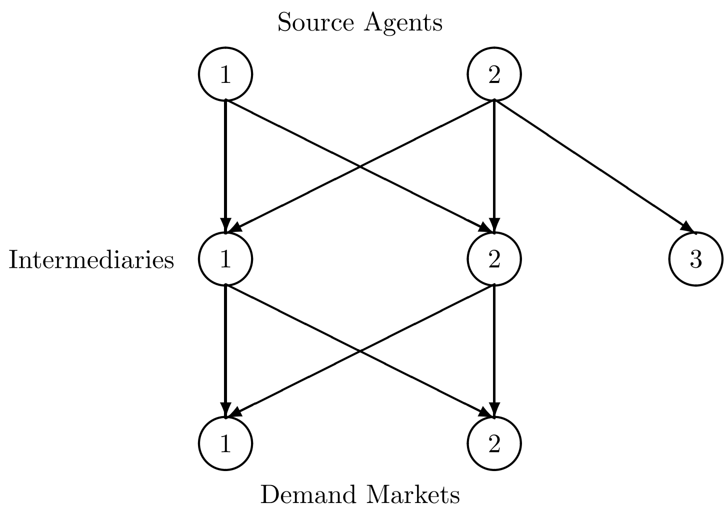

Figure 1.6: The Financial Network Structure of the Numerical Examples

$$c_{211}(x_{211}) = .5x_{211}^2 + 3.5x_{211}, \quad c_{221}(x_{221}) = .5x_{221}^2 + 3.5x_{221}.$$

The handling costs of the intermediaries, in turn, were given by:

$$c_1(x) = .5(\sum_{i=1}^{2} .5x_{i11})^2, \quad c_2(x) = .5(\sum_{i=1}^{2} x_{i21})^2.$$

The transaction costs of the intermediaries associated with transacting with the source agents were, respectively, given by:

$$\hat{c}_{111}(x_{111}) = 1.5x_{111}^2 + 3x_{111}, \quad \hat{c}_{121}(x_{121}) = 1.5x_{121}^2 + 3x_{121},$$

$$\hat{c}_{211}(x_{211}) = 1.5x_{211}^2 + 3x_{211}, \quad \hat{c}_{221}(x_{221}) = 1.5x_{221}^2 + 3x_{221}.$$

The demand functions at the demand markets were:

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1000, \quad d_2(\rho_3) = -2\rho_{32} - 1.5\rho_{31} + 1000,$$

and the transaction costs between the intermediaries and the consumers at the demand markets were given by:

$$\hat{c}_{11}(y) = y_{11} + 5, \quad \hat{c}_{12}(y) = y_{12} + 5, \quad \hat{c}_{21}(y) = y_{21} + 5, \quad \hat{c}_{22}(y) = y_{22} + 5.$$

It was assumed for this and the subsequent examples that the transaction costs as perceived by the intermediaries and associated with transacting with the demand markets were all zero, that is, $c_{jk}(y_{jk}) = 0$, for all j, k.

The Euler method converged and yielded the following equilibrium pattern:

$$x^*_{111} = x^*_{121} = x^*_{211} = x^*_{221} = 5.000,$$

$$y^*_{11} = y^*_{12} = y^*_{21} = y^*_{22} = 5.000.$$

The vector ρ^*_2 had components: $\rho^*_{21} = \rho^*_{22} = 262.6664$, and the computed demand prices at the demand markets were: $\rho^*_{31} = \rho^*_{32} = 282.8106$.

The optimality/equilibrium conditions were satisfied with good accuracy. Note that in this example, the budget constraint was tight for both source agents, that is, $s^*_1 = s^*_2 = 0$, where $s^*_i = S^i - \sum_{j=1}^{n} \sum_{l=1}^{L} x^*_{ijl}$, and, hence, there was zero flow on the links connecting node 3 with top tier nodes 1 and 2. Thus, it was optimal for both source agents to invest their entire financial holdings in each instrument made available by each of the two intermediaries.

Example 3

The following variant of Example 2 was then constructed to create Example 3. The data were identical to that in Example 2 except that the supply for each source sector was increased so that $S^1 = S^2 = 50$.

The Euler method converged and yielded the following new equilibrium pattern:

$$x^*_{111} = x^*_{121} = x^*_{211} = x^*_{221} = 23.6832,$$

$$y^*_{11} = y^*_{12} = y^*_{21} = y^*_{22} = 23.7247.$$

The vector ρ^*_2 had components: $\rho^*_{21} = \rho^*_{22} = 196.0174$, and the demand prices at the demand markets were: $\rho^*_{31} = \rho^*_{32} = 272.1509$.

It is easy to verify that the optimality/equilibrium conditions, again, were satisfied with good accuracy. Note, however, that unlike the solution for Example 2, both source agent 1 and source agent 2 did not invest their entire financial holdings. Indeed, each opted to not invest the amount 23.7209 and this was the volume of flow on each of the two links ending in node 3 in Figure 1.6.

Since the supply of financial funds increased, the price for the instruments charged by the intermediaries decreased from 262.6664 to 196.1074. The demand prices at the demand markets also decreased, from 282.8106 to 272.1509.

Example 4

Example 3 was then modified as follows: the data were identical to that in Example 3 except that the first diagonal term in the variance-covariance matrix Q^1 was changed from 1 to 2.

The Euler method again converged, yielding the following new equilibrium pattern:

$$x^*_{111} = 18.8676, \quad x^*_{121} = 23.7285, \quad x^*_{211} = 25.1543, \quad x^*_{221} = 23.7267,$$

$$y^*_{11} = y^*_{12} = 22.0501, \quad y^*_{21} = y^*_{22} = 23.7592.$$

The vector ρ^*_2 had components: $\rho^*_{21} = 201.4985$, $\rho^*_{22} = 196.3633$, and the demand prices at the demand markets were: $\rho^*_{31} = \rho^*_{32} = 272.6178$.

1.5.1 The Integration of Social Networks with Financial Networks

As noted by Nagurney, Cruz, and Wakolbinger (2004), globalization and technological advances have made major impacts on financial services in recent years and have allowed for the emergence of electronic finance. The financial landscape has been transformed through increased financial integration, increased cross border mergers, and lower barriers between markets. Moreover, as noted by several authors, boundaries between different financial intermediaries have become less clear (cf. Claessens and Jansen (2000), Claessens et al. (2003), G-10 (2001)).

For example, during the period 1980-1990, global capital transactions tripled with telecommunication networks and financial instrument innovation being two of the empirically identified major causes of globalization with regards to international financial markets (Kim (1999)). The growing importance of networks in financial services and their effects on competition have been also addressed by Claessens et al. (2003). Kim (1999) has argued for the necessity of integrating various theories, including portfolio theory with risk management, and flow theory in order to capture the underlying complexity of the financial flows over space and time.

At the same time that globalization and technological advances have transformed financial services, researchers have identified the importance of social networks in a plethora of financial transactions (cf. Nagurney, Cruz, and Wakolbinger (2004) and the references therein)), notably, in the context of personal relationships. The relevance of social networks within an international financial context needs to be examined both theoretically and empirically. It is clear that the existence of appropriate social networks can affect not only the risk associated with financial transactions but also transaction costs.

Given the prevalence of networks in the discussions of globalization and international financial flows, it seems natural that any theory for the illumination of the behavior of the decision-makers involved in this context as well as the impacts of their decisions on the financial product flows, prices, appreciation rates, etc., should be network-based. Recently, Nagurney, Cruz, and Wakolbinger (2004) took on a network perspective for the theoretical modeling, analysis, and computation of solutions to international financial networks with intermediation in which they explicitly integrated the social network component. They also captured electronic transactions within the framework since that aspect is critical in the modeling of international financial flows today.

Here, that model is highlighted. This model generalizes the model of Nagurney and Cruz (2003) to explicitly include social networks.

As in the model of Nagurney and Cruz (2003), the model consists of L countries, with a typical country denoted by l or $\hat{l}$; I "source" agents in each country with sources of funds, with a typical source agent denoted by i, and J financial intermediaries with a typical financial intermediary denoted by j. As noted earlier, examples of source agents are households and businesses, whereas examples of financial intermediaries include banks, insurance companies, investment companies, and brokers, where now we include electronic brokers, etc. Intermediaries in the framework need not be country-specific but, rather, may be virtual.

Assume that each source agent can transact directly electronically with the consumers through the Internet and can also conduct his financial transactions with the intermediaries either physically or electronically in different currencies. There are H currencies in the international economy, with a typical currency being denoted by h. Also, assume that there are K financial products which can be in distinct currencies and in different countries with a typical financial product (and associated with a demand market) being denoted by k. Hence, the financial intermediaries in the model, in addition to transacting with the source agents, also determine how to allocate the incoming financial resources among distinct uses, which are represented by the demand markets with a demand market corresponding to, for example, the market for real estate loans, household loans, or business loans, etc., which, as mentioned, can be associated with a distinct country and a distinct currency combination. Let m refer to a mode of transaction with $m = 1$ denoting a physical transaction and $m = 2$ denoting an electronic transaction via the Internet.

The depiction of the *supernetwork* (see also, e.g., Nagurney and Dong (2002)) is given in Figure 1.7. As this figure illustrates, the supernetwork is comprised of the social network, which is the bottom level network, and the international financial network, which is the top level network. Internet links to denote the possibility of electronic financial transactions are denoted in the figure by dotted arcs. In addition, dotted arcs/links are used to depict the integration of the two networks into a supernetwork.

The supernetwork in Figure 1.7 consists of a social and an international financial network with intermediation. Both networks consist of three tiers of decision-makers. The top tier of nodes consists of the agents in the different countries with sources of funds, with agent i in country l being referred to as agent il and associated with node il. There are, hence, IL top-tiered nodes in the network. The middle tier of nodes in each of the two networks consists of the intermediaries (which need not be country-specific), with a typical intermediary j associated with node j in this (second) tier of nodes in the networks. The bottom tier of nodes in both the social network and in the financial network consists of the demand markets, with a typical demand market for product k in currency h and country $\hat{l}$ associated with node $kh\hat{l}$. There are, as depicted in Figure 1.7, J middle (or second) tiered nodes corresponding to the intermediaries and KHL bottom (or third) tiered nodes in the international financial network. In addition, we add a node $J + 1$ to the middle tier of nodes in the financial network only in order to represent the possible non-investment (of a portion or all of the funds) by one or more of the source agents, as also done in the model in the previous section.

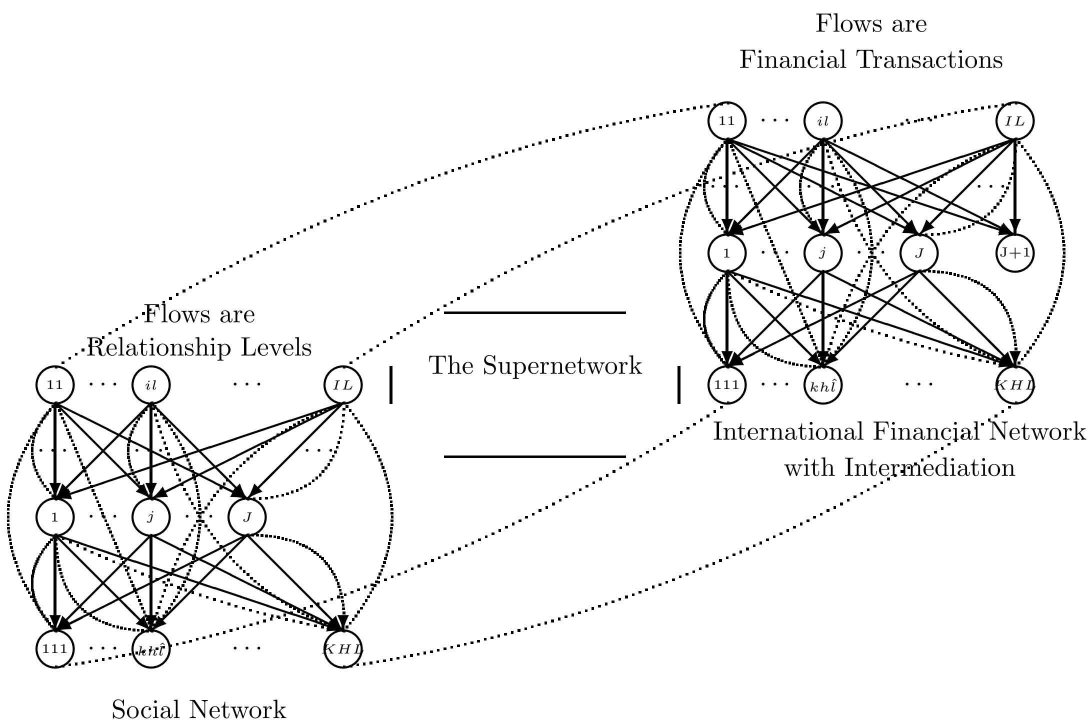

Figure 1.7: The Multilevel Supernetwork Structure of the Integrated International Financial Network / Social Network System

Note that the network in Figure 1.7 includes classical physical links as well as Internet links to allow for electronic financial transactions. Electronic transactions are possible between the source agents and the intermediaries, the source agents and the demand markets as well as the intermediaries and the demand markets. Physical transactions can occur between the source agents and the intermediaries and between the intermediaries and the demand markets.

Nagurney, Cruz, and Wakolbinger (2004) describe the behavior of the decision-makers in the model, and allow for multicriteria decision-making, which consists of profit maximization, risk minimization (with general risk functions), as well as the maximization of the value of relationships. Each decision-maker is allowed to weight the criteria individually. The dynamics of the interactions are discussed and the projected dynamical system derived. The Euler method is then used to track the dynamic trajectories of the financial flows (transacted either physically or electronically), the prices, as well as the relationship levels until the equilibrium state is reached.

Acknowledgments
The writing of this chapter was supported, in part, by NSF Grant No.: IIS 0002647 under the MKIDS Program. This support is gratefully acknowledged.

References

Ahuja, R. K., Magnanti, T. L., and Orlin, J. B. (1993), *Network Flows*, Prentice Hall, Upper Saddle River, New Jersey.

Arrow, K. J. (1951), "An Extension of the Basic Theorems of Classical Welfare Economics," *Econometrica* **51**, 1305-1323.

Barr, R. S. (1972), "The Multinational Cash Management Problem: A Generalized Network Approach," Working Paper, University of Texas, Austin, Texas.

Board of Governors (1980), *Introduction to Flow of Funds*, Flow of Funds Section, Division of Research and Statistics, Federal Reserve System, Washington, DC, June.

Boginski, V., Butenko, S., and Pardalos, P. M. (2003), "On Structural Properties of the Market Graph," in *Innovations in Financial and Economic Networks*, A. Nagurney, Editor, Edward Elgar Publishing, Cheltenham, England.

Charnes, A., and Cooper, W. W. (1958), "Nonlinear Network Flows and Convex Programming over Incidence Matrices," *Naval Research Logistics Quarterly* **5**, 231-240.

Charnes, A., and Cooper, W. W. (1961), *Management Models and Industrial Applications of Linear Programming*, John Wiley & Sons, New York.

Charnes, A., and Cooper, W. W. (1967), "Some Network Characterizations for Mathematical Programming and Accounting Approaches to Planning and Control," *The Accounting Review* **42**, 24-52.

Charnes, A., and Miller, M. (1957), "Programming and Financial Budgeting," Symposium on Techniques of Industrial Operations Research, Chicago, Illinois, June.

Christofides, N., Hewins, R. D., and, Salkin, G. R. (1979), "Graph Theoretic Approaches to Foreign Exchange Operations," *Journal of Financial and Quantitative Analysis* **14**, 481-500.

Claessens, S. and Jansen, M., Editors (2000), *Internationalization of Financial Services*, Kluwer Academic Publishers, Boston, Massachusetts.

Claessens, S., Dobos, G., Klingebiel, D., and Laeven, L. (2003), "The Growing Importance of networks in Finance and Their Effects on Competition," in *Innovations in Financial and Economic Networks*, A. Nagurney, Editor, Edward Elgar Publishing, Cheltenham, England.

Cohen, J. (1987), *The Flow of Funds in Theory and Practice*, *Financial and Monetary Studies* **15**, Kluwer Academic Publishers, Dordrecht, The Netherlands.

Copeland, M. A. (1952), *A Study of Moneyflows in the United States*, National Bureau of Economic Research, New York.

Cournot, A. A. (1838), *Researches into the Mathematical Principles of the Theory of Wealth*, English Translation, Macmillan, London, England, 1897.

Crum, R. L. (1976), "Cash Management in the Multinational Firm: A Constrained Generalized Network Approach," Working Paper, University of Florida, Gainesville, Florida.

Crum, R. L., Klingman, D. D., and Tavis, L. A. (1979), "Implementation of Large-Scale Financial Planning Models: Solution Efficient Transformations," *Journal of Financial and Quantitative Analysis* **14**, 137-152.

Crum, R. L., Klingman, D. D., and Tavis, L. A. (1983), "An Operational Approach to Integrated Working Capital Planning," *Journal of Economics and Business* **35**, 343-378.

Crum, R. L., and Nye, D. J. (1981), "A Network Model of Insurance Company Cash Flow Management," *Mathematical Programming Study* **15**, 86-101.

Dafermos, S. C., and Sparrow, F. T. (1969), "The Traffic Assignment Problem for a General Network," *Journal of Research of the National Bureau of Standards* **73B**, 91-118.

Dantzig, G. B., and Madansky, A. (1961), "On the Solution of Two-Stage Linear Programs under Uncertainty," in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability* **1**, pp. 165-176, University of California Press, Berkeley, California.

Debreu G. (1951), "The Coefficient of Resource Utilization," *Econometrica* **19**, 273-292.

Dong, J., and Nagurney, A. (2001), "Bicriteria Decision-Making and Financial Equilibrium: A Variational Inequality Perspective," *Computational Economics* **17**, 29-42.

Dong, J., Zhang, D., and Nagurney, A. (1996), "A Projected Dynamical Systems Model of General Financial Equilibrium with Stability Analysis," *Mathematical and Computer Modelling* **24**, 35-44.

Doumpos, M., Zopounidiz, C., and Pardalos, P. M. (2000), "Multicriteria Sorting Methodology: Application to Financial Decision Problems," *Parallel Algorithms and Applications* **15**, 113-129.

Dupuis, P., and Nagurney, A. (1993), "Dynamical Systems and Variational Inequalities," *Annals of Operations Research* **44**, 9-42.

Enke, S. (1951) "Equilibrium Among Spatially Separated Markets," *Econometrica* **10**, 40-47.

Euler, L. (1736), "Solutio Problematis ad Geometriam Situs Pertinentis," *Commetarii Academiae Scientiarum Imperialis Petropolitanae* **8**, 128-140.

Fei, J. C. H. (1960), "The Study of the Credit System by the Method of Linear Graph," *The Review of Economics and Statistics* **42**, 417-428.

Ferguson, A. R., and Dantzig, G. B. (1956), "The Allocation of Aircraft to Routes," *Management Science* **2**, 45-73.

Ford, L. R., and Fulkerson, D. R. (1962), *Flows in Networks*, Princeton University Press, Princeton, New Jersey.

Francis, J. C., and Archer, S. H. (1979), *Portfolio Analysis*, Prentice Hall, Englewood Cliffs, New Jersey.

G-10 (2001), "Report on Consolidation in Financial Services," Bank for International Settlements, Switzerland.

Guenes, J., and Pardalos, P. M. (2003), "Network Optimization in Supply Chain Management and Financial Engineering: An Annotated Bibliography," *Networks* **42**, 66-84.

Hughes, M., and Nagurney, A. (1992), "A Network Model and Algorithm for the Estimation and Analysis of Financial Flow of Funds," *Computer Science in Economics and Management* **5**, 23-39.

Korpelevich, G. M. (1977), "The Extragradient Method for Finding Saddle Points and Other Problems," *Matekon* **13**, 35-49.

Kim, H. M. (1999), *Globalization of International Financial Markets*, Ashgate, Hants, England.

Markowitz, H. M. (1952), "Portfolio Selection," *The Journal of Finance* **7**, 77-91.

Markowitz, H. M. (1959), *Portfolio Selection: Efficient Diversification of Investments*, John Wiley & Sons, Inc., New York.

Mulvey, J. M. (1987), "Nonlinear Networks in Finance," *Advances in Mathematical Programming and Financial Planning* **1**, 253-271.

Mulvey, J. M., Simsek, K. D., and Pauling, B. (2003), "A Stochastic Network Optimization Approach for Integrated Pension and Corporate Financial Planning," in *Innovations in Financial and Economic Networks*, A. Nagurney, Editor, Edward Elgar Publishing, Cheltenham, England.

Mulvey, J. M., and Vladimirou, H. (1989), "Stochastic Network Optimization Models for Investment Planning," *Annals of Operations Research* **20**, 187-217.

Mulvey, J. M., and Vladimirou, H. (1991), "Solving Multistage Stochastic Networks: An Application of Scenario Aggregation," *Networks* **21**, 619-643.

Nagurney, A. (1994), "Variational Inequalities in the Analysis and Computation of Multi-Sector, Multi-Instrument Financial Equilibria," *Journal of Economic Dynamics and Control* **18**, 161-184.

Nagurney, A. (1999), *Network Economics: A Variational Inequality Approach*, Second and Revised Edition, Kluwer Academic Publishers, Dordrecht, The Netherlands.

Nagurney, A. (2001), "Finance and Variational Inequalities," *Quantitative Finance* **1**, 309-317.

Nagurney, A., Editor (2003), *Innovations in Financial and Economic Networks*, Edward Elgar Publishing, Cheltenham, England.

Nagurney, A., and Cruz, J. (2003), "International Financial Networks with Electronic Transactions," in *Innovations in Financial and Economic Networks*, A. Nagurney, Editor, Edward Elgar Publishing, Cheltenham, England, pp. 135-167.

Nagurney, A., Cruz, J. M., and Wakolbinger, T. (2004), "The Co-Evolution and Emergence of Integrated International Financial Networks and Social Networks: Theory, Analysis, and Computations," to appear in *Globalization and Regional Economic Modeling*, R. J. Cooper, K. P. Donaghy, and G. J. D. Hewings, Editors, Springer, Berlin, Germany.

Nagurney, A., and Dong, J. (1996a), "Network Decomposition of General Financial Equilibria with Transaction Costs," *Networks* **28**, 107-116.

Nagurney, A., and Dong, J. (1996b), "General Financial Equilibrium Modelling with Policy Interventions and Transaction Costs," *Computational Economics* **9**, 363-384.

Nagurney, A., and Dong, J. (2002), *Supernetworks: Decision-Making for the Information Age*, Edward Elgar Publishers, Cheltenham, England.

Nagurney, A., Dong, J., and Hughes, M. (1992), "Formulation and Computation of General Financial Equilibrium," *Optimization* **26**, 339-354.

Nagurney, A., and Hughes, M. (1992), "Financial Flow of Funds Networks," *Networks* **22**, 145-161.

Nagurney, A., and Ke, K. (2001), "Financial Networks with Intermediation," *Quantitative Finance* **1**, 441-451.

Nagurney, A., and Ke, K. (2003), "Financial Networks with Electronic Transactions: Modeling, Analysis, and Computations," *Quantitative Finance* **3**, 71-87.

Nagurney, A., and Siokos, S. (1997), "Variational Inequalities for International General Financial Equilibrium Modeling and Computation," *Mathematical and Computer Modelling* **25**, 31-49.

Nagurney, A., and Zhang, D. (1996), *Projected Dynamical Systems and Variational Inequalities with Applications*, Kluwer Academic Publishers, Boston, Massachusetts.

Pigou, A. C. (1920), *The Economics of Welfare*, Macmillan, London, England.

Quesnay, F. (1758), *Tableau Economique*, 1758, reproduced in facsimile with an introduction by H. Higgs by the British Economic Society, 1895.

Rockafellar, R. T., and Wets, R. J.-B. (1991), "Scenarios and Policy in Optimization under Uncertainty," *Mathematics of Operations Research* **16**, 1-29.

Rudd, A., and Rosenberg, B. (1979),"Realistic Portfolio Optimization," *TIMS Studies in the Management Sciences* **11**, 21-46.

Rutenberg, D. P. (1970), "Maneuvering Liquid Assets in a Multi-National Company: Formulation and Deterministic Solution Procedures," *Management Science* **16**, 671-684.

Samuelson, P. A. (1952), "Spatial Price Equilibrium and Linear Programming," *American Economic Review* **42**, 283-303.

Shapiro, A. C., and Rutenberg, D. P. (1976), "Managing Exchange Risks in a Floating World," *Financial Management* **16**, 48-58.

Soenen, L. A. (1979), *Foreign Exchange Exposure Management: A Portfolio Approach*, Sijthoff and Noordhoff, Germantown, Maryland.

Srinivasan, V. (1974), "A Transshipment Model for Cash Management Decisions," *Management Science* **20**, 1350-1363.

Storoy, S., Thore, S., and Boyer, M. (1975), "Equilibrium in Linear Capital Market Networks," *The Journal of Finance* **30**, 1197-1211.

Takayama, T., and Judge, G. G. (1971), *Spatial and Temporal Price and Allocation Models*, North-Holland, Inc., Amsterdam, The Netherlands.

Thore, S. (1969), "Credit Networks," *Economica* **36**, 42-57.

Thore, S. (1970), "Programming a Credit Network under Uncertainty," *Journal of Money, Banking, and Finance* **2**, 219-246.

Thore, S. (1980), *Programming the Network of Financial Intermediation*, Universitetsforlaget, Oslo, Norway.

Thore, S. (1984), "Spatial Models of the Eurodollar Market," *Journal of Banking and Finance* **8**, 51-65.

Thore, S., and Kydland, F. (1972), "Dynamic for Flow-of-Funds Networks," in *Applications of Management Science in Banking and Finance*, pp. 259-276, S. Eilon and T. R. Fowkes, editors, Gower Press, England.

Wallace, S. (1986), "Solving Stochastic Programs with Network Recourse," *Networks* **16**, 295-317.