Projected Dynamical Systems and Evolutionary Variational Inequalities via Hilbert Spaces with Applications[1]

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Abstract. In this paper, we make explicit the connection between projected dynamical systems on Hilbert spaces and evolutionary variational inequalities. We give a novel formulation that unifies the underlying constraint sets for such inequalities, which arise in time-dependent traffic network, spatial price, and a variety of financial equilibrium problems. We emphasize the importance of the results in applications and provide a traffic network numerical example in which we compute the curve of equilibria.

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1. Introduction

In this paper we describe how the theory of projected dynamical systems (PDS) and that of evolutionary variational inequalities (EVI) can be intertwined for the theoretical analysis and computation of solutions to applied problems. The two theories have developed in parallel and have been advanced by the need to formulate, analyze, and solve a spectrum of dynamic problems in such disciplines as operations research/management science, engineering, notably, transportation science, economics, and finance.

The outline of the paper is as follows: we first provide the theoretical foundations and the historical developments of these two theories in Sections 2 and 3, respectively, along with the primary references. We then construct a unified definition of the constraint set that arises in the applications of concern here, that is, time-dependent traffic network, spatial price equilibrium, and financial equilibrium problems that have been formulated and studied as evolutionary variational inequalities.

In Section 4 we detail the links between PDS and EVI, and we provide a concrete

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numerical traffic network example. We close with a summary of the contributions.

The results in this paper enable the richer modeling of a variety of dynamic applications and also make possible the adaptation of existing algorithms for the computation of solutions to projected dynamical systems for the solution of evolutionary variational inequalities. In addition, the linkages established in this paper between PDS and EVI (that have developed in parallel) allow for additional synergies between these methodologies to be exploited. Moreover, we can expect new applications to arise because of the connections made in this paper.

2. Projected Dynamical Systems

In this section we first present the theoretical foundations of projected dynamical systems. We then identify numerous applications in a spectrum of disciplines in which this methodology has been utilized for the formulation, analysis, and solution of dynamic problems.

2.1 Theoretical Foundations and Historical Developments

In the early 90s, Dupuis and Nagurney (Ref. 1) introduced a class of dynamics given by solutions to a differential equation with a discontinuous right-hand side, namely

$$\frac{dx(t)}{dt} = \Pi_{\mathcal{K}}(x(t), -F(x(t))). \tag{1}$$

In this formulation, $\mathcal{K}$ is a convex polyhedral set in R^n, $F : \mathcal{K} \to R^n$ is a Lipschitz continuous function with linear growth and $\Pi_{\mathcal{K}} : R \times \mathcal{K} \to R^n$ is the Gateaux directional derivative

$$\Pi_{\mathcal{K}}(x, -F(x)) = \lim_{\delta \to 0^+} \frac{P_{\mathcal{K}}(x - \delta F(x)) - x}{\delta}$$

of the projection operator $P_{\mathcal{K}} : R^n \to \mathcal{K}$, given by

$$||P_{\mathcal{K}}(z) - z|| = \inf_{y \in \mathcal{K}} ||y - z||$$

(see Ref. 2 and Ref. 1).

Theorem 5.1 in Dupuis and Ishii (Ref. 2) proves the existence of local solutions (on an interval $[0, l] \subset R$) for the initial value problem $\frac{dx(t)}{dt} = \Pi_{\mathcal{K}}(x(t), -F(x(t)))$, $x(0) \in \mathcal{K}$. Dupuis and Nagurney (Ref. 1) came back to the class of differential equations

(1), extended the existence of solutions to the real axis, and introduced the notion

of **projected dynamical systems (PDS)** defined by these solutions, together with

several examples, and applications of such dynamics.

Although the papers above were the first to introduce PDS, a similar idea appears

in the literature earlier in the papers by Henry (Ref. 3), Cornet (Ref. 4), and in the

book by Aubin and Cellina (Ref. 5), where (1) is a particular case of the differential

inclusion

$$\frac{dx(t)}{dt} \in -F(x(t)) - N_{\mathcal{K}}(x(t)). \tag{2}$$

In (2), $\mathcal{K}$ is a non-empty, closed, and convex subset of R^n or of a Hilbert space X and

$F : \mathcal{K} \to 2^X$ is a closed, convex valued upper semicontinuous set-valued mapping and

$N_{\mathcal{K}}(x(t))$ is the normal, closed, convex cone to $\mathcal{K}$. In Ref. 5, Ref. 4, and Ref. 3, there

are results regarding the existence of solutions to (1), but these are distinct from the

one in Ref. 1. In Hipfel (Ref. 6), there appears yet another existence result for the

solutions to equation (1), for the particular case of $\mathcal{K} := R^n_+$, but with no relation to

projected dynamics.

Isac and Cojocaru (Ref. 7 and Ref. 8) started the systematic study of PDS on

infinite-dimensional Hilbert spaces in 2002. Cojocaru (Ref. 9) (see also Cojocaru and

Jonker (Ref. 10)) completely answered the question of existence of solutions to an

equation of type (1), known as projected differential equations (PrDE), on a Hilbert

space X with respect to any non-empty, closed and convex subset $\mathcal{K} \subset X$ and any

Lipschitz continuous vector field $F : \mathcal{K} \to X$. The solutions belong to the class

of absolutely continuous functions from $[0, \infty)$ to $\mathcal{K}$. Moreover, the linear growth

condition present in Ref. 1 was removed. For completeness, we include main result

below (cf. Ref. 9 and Ref. 10):

Theorem 2.1

Let X be a Hilbert space of arbitrary dimension and let $\mathcal{K} \subset X$ be a non-empty,

closed, and convex subset. Let $F : \mathcal{K} \to X$ be a Lipschitz continuous vector field and

let $x_0 \in \mathcal{K}$.

Then the initial value problem

$$\frac{dx(t)}{dt} = \Pi_{\mathcal{K}}(x(t), -F(x(t))), \ x(0) = x_0$$

has a unique solution on the interval $[0, \infty)$.

The solutions were shown to be unique through each initial point $x(0) \in \mathcal{K}$. As in the finite-dimensional case, Isac and Cojocaru (Ref. 7 and Ref. 8) and Cojocaru (Ref. 9), defined a projected dynamical system PDS given by the solutions to such PrDE. It is easily seen that a projected flow evolves only in the interior of the set $\mathcal{K}$ or on its boundary.

The key trait of a projected dynamical system, which we use in this paper, was first noted in Dupuis and Nagurney (Ref. 1). The authors showed that: the critical points of equation (1) are the same as the solutions to a variational inequality problem, that is, a problem of the type: given $F : \mathcal{K} \to R^n$, find the points $x \in \mathcal{K}$ such that $\langle F(x), y - x \rangle \geq 0$, for any $y \in \mathcal{K}$, where by $\langle \cdot, \cdot \rangle$ we denote the inner product on R^n. In Ref. 9, Cojocaru showed that the same result holds on a Hilbert space of any dimension, for any closed and convex subset $\mathcal{K}$ and a Lipschitz field F. Moreover, if F is strictly monotone or strictly pseudo-monotone, than there is a unique critical point of (1).

2.2 Applications

As noted in the Introduction, the development of PDS was motivated, in part, by the need to rigorously formulate and compute solutions to a variety of **dynamic** counterparts of problems arising in operations research/management science, engineering, including transportation science, economics as well as regional science, and in finance. Such problems had been studied principally at the equilibrium state using finite-dimensional variational inequality formulations as given above (see also, e.g., Nagurney (Ref. 11)). In applications from such disciplines, it was essential to be able to handle constraints (represented by the set $\mathcal{K}$) such as, for example, nonnegativity constraints on the variables, budget constraints, conservation of flow equations in network-based problems, etc. At the same time, the dynamics represented needed to be meaningful in the context of the applications.

Dupuis and Nagurney (Ref. 1) presented several applications of PDS, notably, dynamic models of oligopolistic market equilibrium, spatial price equilibrium, as well as traffic network equilibrium in the case of elastic travel demands. The oligopolistic market equilibrium problem is notable in economics and dates to Cournot (Ref.

12), whereas spatial price equilibrium problems were first formally studied by Enke (Ref. 13) and Samuelson (Ref. 14) (see also Takayama and Judge (Ref. 15) and the references therein). The traffic network equilibrium problem in the case of elastic demands was first formulated rigorously (under suitable assumptions) as an optimization problem by Beckmann, McGuire, and Winsten (Ref. 16) and governed by Wardrop's (Ref. 17) first principle in which all utilized paths on the network connecting an origin/destination pair in equilibrium have equal and minimal travel costs. The contributions of Smith (Ref. 18) and Dafermos (Ref. 19) established, in turn, the foundations for the use of finite-dimensional variational inequality theory for the development of more general (and realistic) models (cf. Ref. 20) in this domain as well as many other ones. The above noted applications from different disciplines (and many of their extensions) were subsequently studied as PDS both qualitatively in terms of, for example, stability analysis, as well as numerically from the perspective of the computation of solutions (see, e.g., Refs. 21-25).

Since the book by Nagurney and Zhang (Ref. 22), there have been many additional applications that have benefited from the PDS framework in terms of dynamic

problem formulation, analysis, as well as computations. In particular, finance has been a rich area of application of PDS, beginning with the work of Nagurney, Dong, and Zhang (Ref. 25) and Nagurney and Siokos (Ref. 26) who formulated a variety of general dynamic financial multi-sector, multi-instrument models. More recently, Nagurney and Ke (Ref. 27) demonstrated how variational inequality theory could be used to formulate and solve financial network problems with intermediation. This work has been extended to the case of dynamic international financial modeling with intermediation (and risk management) by Nagurney and Cruz (Ref. 28).

Recently, supply chain networks in which there are distinct tiers of decision-makers consisting of, for example, manufacturers, retailers, and consumers at the demand markets, who compete within a tier (in a Nash equilibrium context) but cooperate between tiers, have been studied as PDS (see Ref. 29). The first variational inequality formulation of supply chain network equilibrium is due to Nagurney, Dong, and Zhang (Ref. 30).

Cojocaru (Ref. 9 and Ref. 31), in turn, developed a dynamic multiclass migration network model and formulated it as a projected dynamical system based on a static

model developed by Isac, Bulavski, and Kalashnikov (Ref. 32). She also established stability analysis results. Until then, such migration network models had been studied primarily in a static framework (cf. Ref. 11 and the references therein).

Finally, Cojocaru (Ref. 9) also constructed an infinite-dimensional PDS for the classical one-dimensional obstacle problem as appears in Kinderlehrer and Stampacchia (Ref. 33). This example is the first to use projected dynamics along with the accompanying stability theory for the unique solution of this problem. Moreover, it is the first application to utilize PDS and to illustrate their advantages in the context of infinite dimensions.

3. Evolutionary Variational Inequalities

In this section we turn to evolutionary (time-dependent) variational inequalities and we first present the theoretical foundation along with the historical developments. We then provide a novel unified definition of the constraint set $\mathcal{K}$ proposed for the EVI arising in a variety of applications. We subsequently, for definiteness, further elaborate upon these applications which range from traffic network equilibrium problems to

financial equilibrium problems.

3.1 Theoretical Foundations and Historical Developments

The evolutionary variational inequalities (EVI) were originally introduced by Lions and Stampacchia (Ref. 34) and by Brezis (Ref. 35) to solve problems arising principally from mechanics. They also provided a theory for existence and uniqueness of the solution of such problems.

Steinbach (Ref. 36), on the other hand, studied an obstacle problem with a memory term by means of a variational inequality. In particular, under suitable assumptions on the time–dependent conductivity, he established existence and uniqueness results.

In this paper, we are interested in studying an evolutionary variational inequality in the form proposed by Daniele, Maugeri, and Oettli (Ref. 37 and Ref. 38). They modeled and studied the traffic network problem with feasible path flows which have to satisfy time–dependent capacity constraints and demands. They proved that the equilibrium conditions (in the form of generalized Wardrop (Ref. 17) (1952) con-

ditions) can be expressed by means of an EVI, for which existence theorems and computational procedures were given. The algorithm proposed was based on the subgradient method. In addition, EVI for spatial price equilibrium problems (see Daniele and Maugeri (Ref. 39) and Daniele (Ref. 40 and Ref. 41)) and for financial equilibria (see Daniele (Ref. 42)) have been derived.

The same framework has been used also by Scrimali in Ref. 43, who studied a special convex set $\mathcal{K}$ which depends on the solution of the evolutionary variational inequality, and gives rise to an evolutionary quasi–variational inequality. See also the recent work of Bliemer and Bovy (Ref. 44) in multiclass traffic networks. For an overview of dynamic traffic network problems, see Ran and Boyce (Ref. 45). For additional background on variational inequalities and quasi-variational inequalities, see Baiocchi and Capelo (Ref. 46).

In Gwinner (Ref. 47), the author presents a survey of several classes of time–dependent variational inequalities. Moreover, he deals with projected dynamical systems in a Hilbert space framework. Raciti (Ref. 48 and Ref. 49) applied these ideas to the dynamic traffic network problem. Both Gwinner and Raciti used known results

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in Aubin and Cellina (Ref. 5) for establishing the existence of infinite-dimensional

PDS.

3.2 General Formulation of the Constraint Set $\mathcal{K}$

We provide here a novel unified definition of the constraint set $\mathcal{K}$, proposed in

Refs. 37-42, for the EVI arising in time-dependent traffic network problems, spatial

equilibrium problems with either quantity or price formulations, and a variety of

financial equilibrium problems.

We consider a nonempty, convex, closed, bounded subset of the reflexive Banach

space $L^p([0, T], R^q)$ given by:

$$\mathcal{K} = \bigcup_{t \in [0,T]} \left\{ u \in L^p([0,T], R^q) \mid \lambda(t) \le u(t) \le \mu(t) \text{ a.e. in } [0,T]; \right.$$

$$\left. \sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \xi_{ji} \in \{0,1\}, i \in \{1,..,q\}, j \in \{1,\ldots,l\} \right\}.$$

$$(3)$$

We let $\lambda, \mu \in L^p([0,T], R^q)$, $\rho \in L^p([0,T], R^l)$ be convex functions in the above

definition. For chosen values of the scalars ξ_{ji}, of the dimensions q and l, and of

the boundaries λ, μ, we obtain each of the previous above-cited model constraint set

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formulations as follows:

- for the traffic network problem (see Ref. 37 and Ref. 38) we let $\xi_{ji} \in \{0, 1\}$, $i \in \{1, .., q\}$, $j \in \{1, \ldots, l\}$, and $\lambda(t) \geq 0$ for all $t \in [0, T]$;

- for the quantity formulation of spatial price equilibrium (see Ref. 41) we let $q = n + m + nm$, $l = n + m$, $\xi_{ji} \in \{0, 1\}$, $i \in \{1, .., q\}$, $j \in \{1, \ldots, l\}$; $\mu(t)$ large and $\lambda(t) = 0$, for any $t \in [0, T]$;

- for the price formulation of spatial price equilibrium (see Ref. 40 and Ref. 39) we let $q = n + m + mn$, $l = 1$, $\xi_{ji} = 0$, $i \in \{1, .., q\}$, $j \in \{1, \ldots, l\}$, and $\lambda(t) \geq 0$ for all $t \in [0, T]$;

- for the financial equilibrium problem (cf. Ref. 42) we let $q = 2mn + n$, $l = 2m$, $\xi_{ji} = \{0, 1\}$ for $i \in \{1, .., q\}$, $j \in \{1, \ldots, l\}$; $\mu(t)$ large and $\lambda(t) = 0$, for any $t \in [0, T]$.

Recall that

$$\ll \phi, u \gg := \int_0^T \langle \phi(t), u(t) \rangle dt,$$

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where $\phi \in (L^p([0,T], R^q))^*$ and $u \in L^p([0,T], R^q)$. Let $F : \mathcal{K} \to (L^p([0,T], R^q))^*$.

In this framework, we propose the following **standard form** of the evolutionary

variational inequality (EVI):

$$\text{find } v \in \mathcal{K} \text{ such that } \ll F(v), u - v \gg \geq 0, \ \forall u \in \mathcal{K}. \tag{4}$$

It was shown in Ref. 38 (see Theorem 5.1 and Corollary 5.1) that if F satisfies either

of the following conditions:

- F is hemicontinuous with respect to the strong topology on $\mathcal{K}$, and there exist

  $A \subseteq \mathcal{K}$ nonempty, compact, and $B \subseteq \mathcal{K}$ compact such that, for every $v \in \mathcal{K} \setminus A$,

  there exists $u \in B$ with $\ll F(v), u - v \gg \geq 0$;

- F is hemicontinuous with respect to the weak topology on $\mathcal{K}$;

- F is pseudomonotone and hemicontinuous along line segments,

then the EVI problem (4) admits a solution over the constraint set $\mathcal{K}$. In Stampacchia

(Ref. 50)), it is shown that if F is in addition strictly monotone, then the solution to

the EVI is unique.

3.3 Applications

As in the case of the development of PDS, it has been applications that have motivated the development of EVI. Below, we expand on the applications that can now be cast into standard form using (3) and (4).

Daniele, Maugeri, and Oettli (Ref. 37 and Ref. 38) were apparently the first to formulate time-dependent traffic network equilibria as evolutionary variational inequalities and to establish the existence as well as the calculation of such equilibria. They considered traffic networks in which the demand varied over the time horizon as well as the capacities on the flows on the paths connecting the origins to the destinations. The results therein demonstrated how traffic network equilibria evolve in the presence of such variations. Subsequently, Raciti (Ref. 49) applied the results of Ref. 38 to construct a concrete numerical traffic network example in which the demand was a function of time and the equilibrium at each time instant could be computed exactly and in closed form. Scrimali (Ref. 43) developed an elastic demand time-dependent traffic network model with delays and formulated the equilibrium conditions as a quasi-variational inequality problem. She then established existence

results and also provided a numerical example.

Daniele and Maugeri (Ref. 39) developed a time-dependent spatial equilibrium model (price formulation) in which there were imposed bounds over time on the supply market prices, the demand market prices, and the commodity shipments between the supply and demand market pairs. Moreover, they presented existence results.

Static spatial price equilibrium problems of this form had been studied by numerous researchers (cf. Ref. 11 and the references therein) as well as through (as noted above) using projected dynamical systems (see also Nagurney and Zhang (Ref. 22)). The contribution of Daniele and Maugeri (Ref. 39) allowed for the price and commodity shipment bounds to vary over time. Furthermore, the solution of the formulated EVI traces the curve(s) of the resulting equilibrium price and commodity shipment patterns.

Daniele (Ref. 41) then addressed the time-dependent spatial price equilibrium problem in which the variables were commodity shipments. Not only did she provide existence results, but also she performed stability analysis of the model based on

Dafermos and Nagurney (Ref. 51) (see also Nagurney (Ref. 11)).

In terms of evolutionary variational inequalities and financial equilibria, Daniele (Ref. 42) introduced a time-dependent financial network model consisting of multiple sectors, each of which seeks to determine its optimal portfolio given time-dependent supplies of the financial holdings. The work was motivated, in part, by the contributions of Nagurney and Siokos (Ref. 26) (see also the references therein) in the modeling of static and dynamic general financial equilibrium problems using, respectively, finite-dimensional variational inequality theory and projected dynamical system theory.

4. On the Relationship of EVI and PDS

Joining the previous two sections, we present here how the theory of EVI and that of PDS can be intertwined in the theoretical analysis of applied problems.

We consider the PDS defined on the closed and convex set $\mathcal{K}$, given as in (3), where we take $p = 2$. We note that the elements in the set $\mathcal{K}$ vary with time, but $\mathcal{K}$ is fixed in the space of functions $L^2([0, T], R^q)$, $T > 0$, fixed, as can be readily seen

considering, for example, $q := 2, T := 2, \rho_1(t) = t$ and $\xi_{ji} := 1$ for $i \in \{1, 2\}, j \in \{1\}$:

$$\mathcal{K} = \bigcup_{t \in [0,2]} \left\{ u \in L^2([0,T], R^2) \mid (0,0) \le (u_1(t), u_2(t)) \le \left(t, \frac{3}{2}t\right) \text{ a.e. in } [0,2] \,;\right.$$

$$\left.\sum_{i=1}^{2} u_i(t) = t, \text{ a.e. in } [0,2] \right\}. \tag{5}$$

We consider a vector field $F : \mathcal{K} \to L^2([0,T], R^q)$, assuming that F satisfies the

conditions from Theorem 2.1 and Theorem 5.1/Corollary 5.1 (see Section 3 or Ref.

38). The PrDE can be written as:

$$\frac{du(\cdot, \tau)}{d\tau} = \Pi_{\mathcal{K}}(u(\cdot, \tau), -F(u(\cdot, \tau))), \ u(\cdot, 0) = u(\cdot) \in \mathcal{K}, \tag{6}$$

where the time τ in this formulation is different than the time t in (3) and (4). This

fact deserves more comments. At each moment $t \in [0,T]$, the solution of the EVI

represents a static state of the underlying system. As t varies over the interval $[0,T]$,

the static states describe one (or more) curve(s) of equilibria. In contrast, $\tau \in [0, \infty)$

is the time that describes the dynamics of the system until it reaches one of the

equilibria on the curve(s).

Based on Theorem 2.1, (6) has solutions in the class of absolutely continuous

functions with respect to τ, from $[0, \infty)$ to $\mathcal{K}$. Moreover, we see that under the

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condition of strict monotonicity (present in both EVI and PDS theories) we obtain

a unique curve of equilibria. The following is an immediate consequence of Ref. 10,

Theorem 2.2.

Corollary 4.1

The solutions to the EVI problem (4) are the same as the critical points of (6) and

viceversa.

This result is crucial in merging the two theories and in computing and interpreting

problems ranging from spatial price (quantity and price formulations), traffic network

equilibrium problems, and general financial equilibrium problems as presented in

Section 3.2.

The method we employ is the following: EVI theory gives the existence of one (or

more) curve(s) of equilibria on the interval $[0, T]$; according to the above Corollary,

any point of such a curve is a critical point of an infinite-dimensional PDS. To use

this information in applications we proceed by discretizing the time-interval $[0, T]$.

We thus obtain a sequence of PDS, on distinct, finite-dimensional, closed, convex sets

$\mathcal{K}_t$. We compute the critical points of each such PDS, and thus find the equilibria of the system at some fixed moments $t \in [0, T]$. An interpolation of the sequence of critical points then gives an approximation of the curve(s) of equilibria.

4.1 Traffic Network Numerical Example

For an idea as to how this procedure works, we present here a simple traffic network equilibrium example (cf. Ref. 37 and Ref. 38). We consider a network consisting of a single origin/destination pair of nodes and two paths connecting these nodes of a single link each. The feasible set is given in (5) where $u(t)$ denotes the vector of path flows at t. The cost functions on the paths are defined as: $2u_1(t) - 1.5$ for the first path and $u_2(t) - 1$ for the second path. We consider a vector field F given by

$$F : \mathcal{K} \to L^2([0,1], R^2), \ (F_1(u(t)), F_2(u(t))) = (2u_1(t) - 1.5, u_2(t) - 1).$$

The theory of EVI (as described in Section 3.2 above) states that the system has a unique equilibrium, since F is strictly monotone, for any arbitrarily fixed point $t \in [0, 2]$. One can easily see that

$$\langle F(u_1, u_2) - F(v_1, v_2), (u_1 - v_1, u_2 - v_2) \rangle = 2(u_1 - v_1)^2 + (u_2 - v_2)^2 > 0,$$

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for any

$$u \neq v \in L^2\left([0, 2], R^2\right).$$

With the help of PDS theory, we can compute an approximate curve of equilibria, by choosing $t_0 \in \left\{\frac{k}{4} | k \in \{0, \ldots, 8\}\right\}$. Therefore, we obtain a sequence of PDS defined by the vector field

$$-F(u_1(t_0), u_2(t_0)) = (-2u_1(t_0) + 1.5, -u_2(t_0) + 1)$$

on nonempty, closed, convex, 1-dimensional subsets

$$\mathcal{K}_{t_0} := \left\{\left\{[0, t_0] \times \left[0, \frac{3}{2}t_0\right]\right\} \cap \{x + y = t_0\}\right\}.$$

For each we can compute the unique equilibrium of the system at t_0, i.e., the point

$$(u_1(t_0), u_2(t_0)) \in R^2 \text{ such that } -F(u_1(t_0), u_2(t_0)) \in N_{\mathcal{K}_{t_0}}(u_1(t_0), u_2(t_0)).$$

Using a simple MAPLE computation, we obtain that the equilibria are the points:

$$\left\{(0, 0), \left(\frac{1}{4}, 0\right), \left(\frac{1}{3}, \frac{1}{6}\right), \left(\frac{5}{12}, \frac{1}{3}\right), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{7}{12}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{5}{6}\right), \left(\frac{3}{4}, 1\right), \left(\frac{5}{6}, \frac{7}{6}\right)\right\}.$$

Interpolating these points we obtain the approximate curve of traffic network equilibria as displayed in Figure 1.

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5. Summary

In this paper, we have demonstrated how the two theories of projected dynamical systems and evolutionary variational inequalities that have been developed in parallel can be connected to enrich the modeling, analysis, and computation of solutions to a spectrum of time-dependent equilibrium problems that arise in such disciplines as operations research/management science, engineering, economics, and finance.

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