Evolution Variational Inequalities and Projected Dynamical Systems with Application to Human Migration

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Asbtract: In this paper, we explore the relationship between projected dynamical systems and evolution variational inequalities (also sometimes referred to as parabolic variational inequalities). The methodology of evolution variational inequalities is then utilized for the first time to model the dynamic adjustment of a social-economic process in the context of human migration. The questions of dynamics and convergence of algorithms in this framework are addressed and answered. In particular, we provide existence and uniqueness results for the solution path without assuming Lipschitz continuity and propose a finite-difference scheme for the solution of the human migration problem. The algorithm, an ordinary implicit scheme, is a discrete-time version of the model. Its convergence estimate is also established.

Key words: Evolution variational inequalities, parabolic variational inequalities, pro-

jected dynamical systems, human migration.

* It is with great sorrow that Jie Pan passed away before this and other papers that we had planned to write were completed. He was an excellent young researcher whose untimely death is a great loss. This paper is dedicated to his memory.

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1. Introduction

The dynamical modeling of competitive systems has long been an important focus in economics (cf. Arrow and Hurwicz [1, 2], Smale [3]) as well as a a subject of growing interest in operations research/management science and in engineering with applications ranging from congested urban transportation systems to problems of human migration. In this scenario, one needs not only to settle the fundamental question of existence of equilibria, but also to understand the dynamic process by which an equilibrium is arrived at. Once these issues are sufficiently understood, one then turns to the basic question of stability of the equilibrium pattern. It is clear that dynamics adds a greater validity to the theory of equilibria not only arising in economics and in engineering but in any branch of applied mathematics concerned with the formulation and analysis of competitive systems with interacting agents.

Dupuis and Nagurney [4] developed the basic theory of the existence, uniqueness, and algorithmic procedures for "projected" dynamical systems, and established the equivalence between the set of equilibria or stationary points of a projected dynamical system and the set of solutions to the corresponding finite-dimensional variational inequality problem. In this new form, an equilibrium becomes the end-product of a dynamic process, rather than an isolated solution of a variational inequality problem or any other static model. Hence, a plethora of applications that have been formulated and studied to-date as (finite-dimensional) variational inequality problems (cf. Nagurney [5]) can now be cast into a richer dynamical framework.

The need for a "projected" dynamical system is due to the polyhedral constraints found in many applications of competitive systems. For example, such constraints may be due to limited natural resources or budgets (as in the case of a variety of financial problems), to conservation of flow equations and nonnegativity constraints (as in the setting of congested urban transportation networks). In a projected dynamical system, the right-hand side is a projection operator, and, hence, it is discontinuous, in contrast to classical dynamical systems (cf. [6]).

In particular, under some conditions on the underlying function, Dupuis and Nagurney [4] established the unique existence of the solution path of the projected dynamical system.

One of the conditions utilized is the linear growth condition, which is implied by Lipschitz continuity, a condition that one may wish to further relax in equilibrium models.

Subsequently, Zhang and Nagurney [7] studied the stability of the equilibria of projected dynamical systems, providing a complete answer to Smale's [3] proposal and challenge. In particular, they proposed two distinct approaches, termed the "monotonicity" approach and the "regularity" approach for stability analysis of projected dynamical systems. This research was then applied to spatial price equilibrium problems and to oligopolistic market equilibrium problems by the same authors (cf. [8] and [9]).

One goal is to obtain better existence and uniqueness results for some of the dynamic equilibrium models. Specifically, we would like to remove the linear growth condition which Dupuis and Nagurney [4] required for the existence and uniqueness solution of a projected dynamical system. For this purpose, we recast a projected dynamical system into a different form. We then utilize the theory of evolution variational inequalities to answer the uniqueness and existence of a solution path in the dynamic adjustment process, under a much weaker condition than the linear growth condition. This is a direction that deserves further attention.

The motivation for this work, stems in part, from our earlier work (cf. [10]) where we began to explore the dynamic dimension of equilibrium modeling in the context of human migration problems. There, we developed a multi-stage migration model by establishing a connection between a sequence of variational inequalities and a non-homogeneous Markov chain. A stable population distribution was shown to exist under certain assumptions. The multi-stage model was in a discrete time framework, which allowed for straightforward conservation laws for the population flows.

Nevertheless, the discrete time model did not take advantage of results in continuous-time dynamical systems and the conditions for the existence of a stable population are not easily verifiable, as they deal with the transitional probabilities at future stages. On the other hand, Beckmann and Puu [11], in their book, studied a variety of equilibrium problems in the continuous setting, many of whose fundamental relationships are formulated as time-continuous dynamical systems.

In this paper, we demonstrate the use of *evolution* variational inequalities for the modeling of the dynamic socio-economic adjustment process associated with human migration. Since human population is comparable to durable goods in the sense that no consumption is incurred in the process, it is quite important to "put stocks (populations) and flows (migration rate) into the same model" (cf. Smale [3]). The human migration problem, however, is more complicated than the transport of goods because (rational) people migrate in response to the change in their utility values between their destination and origin locations. The migration model in this paper provides a good example as to how one might handle this complexity. Moreover, since the topic of human migration is one of increasing global concern, methodologies to formulate and study such a problem are of great practical as well as potentially humanitarian interest.

Evolution variational inequalities are time-dependent and have been used to-date principally in mathematical physics, where they are sometimes referred to as parabolic equations. Their usefulness outside of that application domain, however, has yet to be fully explored especially from an algorithmic perspective. For some recent applications that utilize variational inequalities of evolution with a focus on projected dynamical systems on Hilbert spaces in which trajectories of equilibria are of concern, along with additional references, see Cojocaru, Daniele, and Nagurney [12]. Cojocaru [13] and Cojocaru and Jonker [14], in turn, also demonstrated that the linear growth condition was not needed but the proof is distinct from that constructed in this paper. Moreover, here we outline an algorithmic scheme applied specifically to problems of human migration.

This paper is organized as follows. In Section 2, we review some general results in projected dynamical systems theory and in evolution variational inequalities. We also explore the relationship between the two formalisms.

In Section 3, we formulate a continuous-time dynamic migration process (without population growth) as an evolution variational inequality problem. We then establish the existence and uniqueness of the solution path of the evolution variational inequality.

In Section 4, we present an algorithm which is a discretized version of the model, along with a convergence estimate of the algorithm.

Finally, Section 5 concludes with a summary and some suggestions for future research.

2. Projected Dynamical Systems and Evolution Variational Inequalities

Dupuis and Nagurney [4] considered projected dynamical systems whose set of equilibria or stationary points coincide with the set of solutions of the corresponding finite-dimensional and time-independent variational inequality problem. This connection is of substantial significance since it provides a dynamical underpinning to an otherwise static framework. The projected dynamical system is the solution path of an ordinary differential equation (ODE) with a discontinuous right-hand side. Since the dynamical system in question evolves within convex polyhedral constraints, the vector field that drives the dynamical system must be projected onto the convex polyhedral constraints.

The mathematical foundations will now be recalled. Let b be a vector in R^n, and let K be a convex polyhedron in R^n. Also, let the set of inward normals at $x \in K$ be defined as: $n(x) = \{\gamma : \|\gamma\| = 1, \langle \gamma, x - y \rangle \leq 0, \forall y \in K\}$. Then, the projection of b at $x \in K$ to K, $\Pi_K(x, b)$, can be written as the following:

$$\Pi_K(x, b) = b + \max(\langle b, -n \rangle, 0) \cdot n, \tag{1}$$

where n is the vector in $n(x)$, such that: $\langle b, -n \rangle = \max_{\gamma \in n(x)} \langle b, -\gamma \rangle$.

Definition 1: A Projected Dynamical System

The ODE *for the projected dynamical system is given by:*

$$\dot{x} = \Pi_K(x, b(x)), \quad x(0) = x_0 \in K, \tag{2}$$

where $b(x) : K \mapsto R^n$ is a given continuous vector field on K.

We now recall the finite-dimensional variational inequality problem. Assuming that $F(x) : K \mapsto R^n$ is given, the time-independent and finite-dimensional variational inequality problem, VI(F, K), is defined as:

Definition 2: The Finite-Dimensional Variational Inequality Problem

Determine a vector $x^* \in K$, such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K. \tag{3}$$

Dupuis and Nagurney [4] established the connection between the stationary points of the projected dynamical system (2) and the solutions of the variational inequality problem (3) in the following theorem.

Theorem 1

Let $b(x)$=$-F(x)$*. Then the set of stationary points of the* ODE *(2) coincide with the set of solutions of the* VI(F, K) *(3).*

Of course, the ODE (2) is realistic only if there is a unique solution path from a given initial point. In that same paper, Dupuis and Nagurney established the existence and uniqueness for the solution path of the ODE. They approached the problem by rewriting the ODE as a pair of two equations, the first equation being the associated ODE without the projection, the second being a mapping that restricts the solution of the first equation to K. Such a mapping, found in the Skorokhod Problem (SP) (Skorokhod [15]), is Lipschitz continuous.

For completeness, we recall the definition of Lipschitz continuity.

Definition 3: Lipschitz Continuity

F *is said to be Lipschitz continuous if there is an* $L > 0$*, such that*

$$\|F(x_1) - F(x_2)\| \leq L\|x_1 - x_2\|, \quad \forall x_1, x_2 \in K. \tag{4}$$

Although their approach takes the advantage of the results of the SP, the assumptions made were rather strong. In particular, the linear growth condition on $F(x)$ defined as:

Definition 4: Linear Growth Condition

There exists a $B < \infty$*, such that the vector field* $b : R^n \mapsto R^n$ *satisfies the linear growth condition* $\|b(x)\| \leq B(1 + \|x\|)$ *for all* $x \in K$*, and also*

$$\langle b(x) - b(y), x - y\rangle \leq B\|x - y\|^2, \quad \forall x, y \in K, \tag{5}$$

is a sufficient condition for the linear growth condition.

It may be desirable to relax this condition in order to expand the set of functions that can be handled, realizing, of course, that it does not mean that the linear growth condition is necessary for the existence and uniqueness of the solution. In particular, by recasting the ODE problem as a different formulation might shed some light on how to weaken the condition.

Towards this end, we now switch our attention to evolution variational inequalities (EVI).

Unlike the VI(F, K), the EVI is a time-dependent (and infinite-dimensional) variational inequality problem. It has been used, heretofore, in many problems of mathematical physics (see, e.g., Kinderlehrer and Stampacchia [16]).

For simplicity, assume that V is a Banach space. Let $a : V \times V \mapsto R$ be continuous, bilinear, and elliptic in the sense that $a(x, x) \geq \alpha \|x\|^2, \forall x \in V$ for some $\alpha > 0$. On a finite time interval [0, T], $f(t) \in L^2(0, T; V)$. Also, $\phi : V \mapsto \overline{R}$ is convex, proper, and lower semi-continuous. An EVI, sometimes referred to as a parabolic variational inequality, is defined as follows:

Definition 5: An Evolution Variational Inequality Problem

Find $x(t) \in V$, a.e. $t \in (0, T)$, such that $x(0) = x_0$, and

$$\langle \frac{\partial x}{\partial t}, y - x \rangle + a(x, y - x) + \phi(y) - \phi(x) \geq \langle f, y - x \rangle, \quad \forall y \in K, \quad a.e.\ t \in (0, T). \tag{6}$$

The above EVI can be further generalized by replacing the bilinear form $a(x, y)$ with $\langle F(x), y \rangle$, where $F(x)$ is nonlinear and maximal monotone.

The existence and the uniqueness for a solution path $u(t) \in W_2^1(0, T; V)$ of an EVI is found in many nonlinear analysis texts, such as, for example, Zeidler [17]. It is interesting to observe that the condition of Lipschitz continuity, and, hence, the condition of linear growth, are also ultimately used therein. However, by making use of an ingenious tool called a "Yosida approximation" (cf. [17]), which produces a Lipschitz continuous approximation for any maximal monotone operator, the existence and uniqueness results for Lipschitz continuous operators can be extended to maximal monotone operators. Of course, in this process, the notion of solutions is understood in a generalized space $W_2^1(0,T;V)$, which contains $C^1(0,T;V)$ as a dense subspace.

Our focus is on a special case that is most relevant to the discussion of the paper, constructed as follows.

Definition 6: A Special Case of an EVI

Assume that K is a convex polyhedron in R^n, ϕ is the characteristic function of K, with $a(x,y)=\langle F(x),y\rangle$ for an $F(x)$ defined on K, $f=0$, where V is as outlined above, but is restricted to be a Hilbert space.

Then the EVI *then takes the following form: find $x(t)\in K$, a.e. $t\in(0,T)$, such that $x(0)=x_0$, and*

$$\langle\frac{\partial x}{\partial t},y-x\rangle+\langle F(x),y-x\rangle\geq 0,\quad\forall y\in K,\quad a.e.\ t\in(0,T). \tag{7}$$

Theorem 2

Assume the assumptions in Definition 6. Let $b(x)$=$-F(x)$. Then a solution path of the ODE *(2) is also a solution path of the* EVI *(7).*

Proof: Recall that the normal cone is defined as the following:

$$N_K(x)=\begin{cases}\{z\in R^n:\langle z,y-x\rangle\leq 0,\forall y\in K\}, & \text{if } x\in K\\ \emptyset, & \text{if } x\notin K.\end{cases} \tag{8}$$

Using the normal cone concept, we can write the EVI as the multi-valued equation:

$$\frac{\partial x}{\partial t}+F(x(t))\in -N_K(x(t)). \tag{9}$$

Similarly, if we use the earlier formula for calculating the projection of a vector to K at $x\in K$ (cf. also Cojocaru, Daniele, and Nagurney [18] and the references therein) we should be able to write the ODE for the projected dynamical system using the normal cone. We only need to note the fact that the set of inward normals $n(x)$ consists of all the unit vectors in $-N_K(x)$, i.e., $\frac{\partial x}{\partial t} + F(x(t)) = \max(\langle -F(x), -n \rangle, 0) \cdot n$. Hence, the solution path of the ODE also solves

$$\frac{\partial x}{\partial t} + F(x(t)) \in -N_K(x(t)). \tag{10}$$

The proof is complete.

We would like to point out that not all the solutions of the EVI need be solutions of the ODE for the PDS. Therefore, the EVI is more general than the ODE for the PDS in its current form.

3. An EVI Formulation of a Time-Continuous Migration Model

Human migration has been studied in the past by a diversity of researchers across disciplines. Both discrete models and continuous models can be traced to their sources of more than two decades ago. For citations to discrete models, see [5] and [10], and the references therein as well as [19] and [20]. For many continuous models, we refer the reader to the book by Beckmann and Puu [11], and the references therein. The continuous model takes a more compact form, and, hence, renders more readily the mathematical analysis. Furthermore, it often contains the discrete model as a special case, following discretization in space or time.

We select the topic of human migration in which to explore the potential of evolution variational inequalities for the modeling and computation of equilibria. The framework we find ourselves in is a mixed one — discrete in the space variables, since we assume that locations in which the populations reside and ultimately migrate to, are discrete in space, and yet continuous in the time dimension. This strategy allows us to avoid spatial discretization, and, at the same time, to make use of the results for EVI from nonlinear analysis.

Unlike in discrete-time models, the flows of population here are assumed to be continuous. Hence, the flows should be understood as the rate of migration, rather than the amount of migration as in discrete models, such as those in [10]. This rate of migration depends not only on the population distribution, but also on the utilities and the migration or transaction costs associated with moving between locations. Hence, we assume that associated with each location in space is a utility function that measures the attractiveness of that location.

We now develop the model. Let there be N discrete locations in space. Let $f(t) = (f_{12}(t), \ldots, f_{N-1,N}(t))$ denote the vector of flows of the population between locations, where $f_{ij}(t)$ denotes the flow of population from location i to location j at time t. Let $p(t) = (p_1(t), p_2(t), \ldots, p_N(t))$ denote the vector of population distributions at t, where $p_i(t)$ denotes the population at location i at time t.

We assume that there is no population growth, and, consequently,

$$\sum_{i=1}^{N} p_i(t) = C, \quad \forall t, \tag{11}$$

where $C > 0$ is a constant.

Also, let $u(p(t)) = (u_1(p(t)), u_2(p(t)), \ldots, u_N(p(t)))$ denote the vector of utility functions, where $u_i(t)$ denotes the utility of locating in location i at time t. Finally, let $c(f(t)) = (c_{ij}(f(t);\ i, j = 1, 2, \ldots, N, i \neq j)$ denote the vector of migration or transaction costs associated with migrating between locations, with $c_{ij}(t)$ denoting the cost associated with migrating between locations i and j at time t. We emphasize that this cost includes not only the cost of transportation but also "psychic" costs associated with dislocation.

We assume that the *rate of flow* is directly related to the difference between the utility values minus the migration cost. In other words, the migrants view the attractiveness of a potential destination for settlement relative to the attractiveness of their present location discounted by the cost of migration. More specifically, mathematically, the rate of change of migration flows between locations may be expressed as:

$$\frac{df}{dt} = \Pi_K(f(t), -F(p(t), f(t))), \tag{12}$$

where $\Pi_K(f(t), -F(p(t), f(t)))$ is the projection of the net gains in utility $-F(p(t), f(t)))$ on $K = \{f(t) \geq 0\}$ at $f(t) \in K$, with component F_{ij} defined as:

$$-F_{ij}(p(t), f(t)) = u_j(p(t)) - u_i(p(t)) - c_{ij}(f(t)). \tag{13}$$

As discussed in Section 2, ODE (12) can be written as the following EVI: Determine $\overline{f}(t) \geq 0$, such that

$$\langle \frac{d\overline{f}}{dt} + F(\overline{p}(t), \overline{f}(t)), f(t) - \overline{f}(t) \rangle \geq 0,\ \forall f(t) \geq 0. \tag{14}$$

Next, we must determine the relationship between the population distributions, $p(t)$, and the migration flows, $f(t)$. In the absence of any restrictions on $p(t)$, we would have a straightforward conservation of flow equation:

$$\frac{dp_i(t)}{dt} = \sum_{j \neq i} (-f_{ij}(t) + f_{ji}(t)),\ \forall i. \tag{15}$$

In other words, the rate of change of the population at a location i must be equal to the difference between the inflow and outflow of that location. However, the vector of population distributions, $p(t)$, must be nonnegative and, moreover, bounded by the no growth condition (11). Therefore, the vector force field should be projected onto the convex polyhedral constraint set, $K_1 = \{p(t) = (p_1(t), p_2(t), \ldots, p_N(t)) : p_i(t) \geq 0, \sum_{i=1}^N p_i(t) = C\}$, that is,

$$\frac{dp(t)}{dt} = \Pi_{K_1}(p(t), -G(f(t))), \tag{16}$$

where $-G(f(t)) = (\sum_{j,j\neq i}(-f_{ij}(t) + f_{ji}(t));\ i = 1, 2, \ldots, N.)$ The above equation determines the state of the migration system. Essentially, it is a constraint on the state variable $p(t)$ after the control variable $f(t)$ is determined by (14).

Similarly, we can write this ODE as an EVI: Determine $\bar{p}(t) \in K_1$ such that

$$\langle \frac{d\overline{p}(t)}{dt} + G(\overline{f}(t)), p(t) - \overline{p}(t)\rangle \geq 0,\ \forall p(t) \in K_1. \tag{17}$$

In order to make the model more compact, we group the two EVI conditions (14) and (17) for $\overline{p}(t), \overline{f}(t)$ together to form the EVI formulation of the migration model:

EVI Formulation

Determine $\overline{f}(t) \geq 0, \overline{p}(t) \geq 0, \sum_{i=1}^N \overline{p}_i(t) = C$, such that

$$\langle \frac{d\overline{f}}{dt} + F(\overline{p}(t), \overline{f}(t)), f(t) - \overline{f}(t)\rangle + \langle \frac{d\overline{p}(t)}{dt} + G(\overline{f}(t)), p(t) - \overline{p}(t)\rangle \geq 0, \tag{18}$$
$$\forall f(t) \geq 0, p(t) \in K_1.$$

The initial condition for the problem can be specified by the initial population distribution and the rate of migration.

It is clear that the unique solution of (12) and (16) would be equivalent to the unique solution of (18). However, the new formulation (18) enables us to prove the existence and uniqueness of the solution path without requiring Lipschitz continuity. As we will see in Section 4, it also gives a new discrete version which can be used to solve the problem numerically.

We now state one of the main results in this paper – that of existence and uniqueness of a solution path of the above EVI. Although we state the theorem in terms of utility functions and transaction cost functions, it should be clear that the same proof and results apply to a more general class of EVIs, where $F(p, f)$ is any maximal monotone operator with respect to f for any fixed p, and $G(f)$ is any Lipschitz continuous operator.

Of course, one may use the more familiar projected dynamical system form for the dynamic migration model. However, as indicated in the introduction, by using the EVI form, it becomes possible for us to establish the existence and uniqueness condition without the linear growth condition. Mathematically, the two forms are equivalent.

Theorem 3

If $-u(p(t))$ and $c(f(t))$ are maximal monotone operators, then EVI *(18) has a unique solution path in $W_2^1(0, T; V)$.*

Proof: First of all, for any given $\overline{p}(t)$ in the first half of the EVI, there must be a unique solution $\overline{f}(t) \geq 0$, such that

$$\langle \frac{d\overline{f}}{dt} + F(\overline{p}(t), \overline{f}(t)), f(t) - \overline{f}(t) \rangle \geq 0, \ \forall f(t) \geq 0. \tag{19}$$

The main theorem for the EVI problem states that there is a unique solution path if the operator in the EVI problem is maximal monotone (see Zeidler [17], Chapter 55). Hence, all we have to do is to verify that $F(\overline{p}(t), f(t))$ is maximal monotone, which is apparent from the same property of $c(f(t))$.

Next we show that the solution $\overline{f}(t)$ is continuously dependent on $\overline{p}(t)$. For this purpose, we assume that $f_1(t)$ is the solution path associated with $p_1(t)$, and $f_2(t)$ is the solution path associated with $p_2(t)$. Let $f(t) = f_1(t) - f_2(t), p(t) = p_1(t) - p_2(t)$. Then we have that

$$g(p_1(t)) - g(p_2(t)) \in f'(t) + c(f_1(t)) - c(f_2(t)) + \partial\chi_K(f_1(t)) - \partial\chi_K(f_2(t)), \tag{20}$$

where $g(p) = -F(p, f) + c(f)$, or $g_{ij}(p) = -u_i(p) + u_j(p)$.

Applying the Yosida approximation (cf. Zeidler [17] Chapter 55) to both sides of (20) yields

$$g(p_1(t)) - g(p_2(t)) = f'_\mu(t) + (c + \partial\chi_K)_\mu((f_1)_\mu(t)) - (c + \partial\chi_K)_\mu((f_2)_\mu(t)). \tag{21}$$

Now by taking inner products with $f_\mu(t)$ on both sides of (21), we obtain

$$\begin{aligned} &\langle g(p_1(t)) - g(p_2(t)), f_\mu(t)\rangle = \\ &\langle f'_\mu(t), f_\mu(t)\rangle + \langle (c + \partial\chi_K)_\mu(f_1)_\mu(t) - (c + \partial\chi_K)_\mu(f_2)_\mu(t), f_\mu(t)\rangle \\ &\geq \langle f'_\mu(t), f_\mu(t)\rangle. \end{aligned} \tag{22}$$

Hence,

$$\langle f'_\mu(t), f_\mu(t)\rangle \leq \|g(p_1(t)) - g(p_2(t))\|\|f_\mu(t)\| \leq (\|g(p_1(t)) - g(p_2(t))\|^2 + \|f_\mu(t)\|^2)/2. \tag{23}$$

Integration by parts gives:

$$\|f_\mu(t)\|^2 = 2\int_0^t \langle f'_\mu(s), f_\mu(s)\rangle ds \leq \int_0^t \|g(p_1(s)) - g(p_2(s))\|^2 ds + \int_0^t \|f_\mu(s)\|^2 ds. \tag{24}$$

By Gronwall's Lemma (cf. Perko [21]), we have, by noticing that $\|f_\mu(0)\|^2 = 0$, that

$$\|f_\mu(t)\|^2 \leq \text{Constant } \int_0^t \|g(p_1(s)) - g(p_2(s))\|^2 ds. \tag{25}$$

Hence, by taking the limit $\mu \to 0$, the continuous dependency of $\overline{f}(t)$ on $\overline{p}(t)$ is established. We denote this dependency by writing $\overline{f}(t) = h(\overline{p}(t))$. In fact, (25) implies that $h(\overline{p}(t))$ is Lipschitz continuous.

Finally, by substituting $\overline{f}(t)$ by $h(\overline{p}(t))$ in the second part of the EVI, we obtain an EVI of a continuous function on a compact set K_1. The existence of $\overline{p}(t)$ is then a consequence of a fixed-point theorem. Notice that $G(f)$ is linear, and, consequently, Lipschitz continuous. The fact that the solution is also unique is guaranteed by the Lipschitz continuity of $G(f(t)) = G(h(p(t)))$ as a function of $p(t)$.

4. A Finite-Difference Scheme and its Convergence Estimate

Although there are standard numerical algorithms for the solution of evolution (parabolic) variational inequalities available in the literature (see, e.g., [17], and the references therein), in the scenario of the socio-economic application under consideration here there is no need for spatial discretization, since the spatial variables are finite-dimensional.

We now present a finite-difference scheme (cf. [17], and the references therein), in particular, an ordinary implicit scheme, for the approximation of the solution to EVI (18), along with a convergence estimate. This methodology, hence, also provides one with an alternative approach to that of projected dynamical systems (cf. [4]) for the solution of dynamic versions of competitive equilibrium problems.

In this section, we assume that both the migration cost functions, $c(f)$, and the minus utility functions, $-u(p)$, are strongly monotone.

We now introduce a time step, Δt, and mark the time horizon at times: $t_n = n\Delta t$ ($n = 0, 1, 2, \ldots$). The EVI for the migration problem can then be approximated by the classical finite-difference schemes.

In the following, we use the notation $\overline{f}^n$ for the solution $\overline{f}(t^n)$ and $\overline{f}_a^n$ for the approximation of $\overline{f}^n$. Similarly, $\overline{p}^n$ is for $\overline{p}(t^n)$, and $\overline{p}_a^n$ is for the approximation of $\overline{p}^n$. The ordinary implicit scheme for the migration model is now stated:

The Finite-Difference Scheme

Given $\overline{f}_a^n \in K, \overline{p}_a^n \in K_1$, find $\overline{f}_a^{n+1} \in K, \overline{p}_a^{n+1} \in K_1$, such that, for all $f \in K$,

$$\langle \frac{\overline{f}_a^{n+1} - \overline{f}_a^n}{\Delta t} + F(\overline{p}_a^{n+1}, \overline{f}_a^{n+1}), f - \overline{f}_a^{n+1} \rangle \geq 0, \tag{26}$$

where $\overline{p}_a^{n+1}$ satisfies

$$\frac{\overline{p}_a^{n+1} - \overline{p}_a^n}{\Delta t} + G(\overline{f}_a^{n+1}) \in -N_{K_1}(\overline{p}_a^{n+1}). \tag{27}$$

We now turn to the convergence analysis. Note that, by definition, we have that, for any $f \geq 0$,

$$\langle \frac{d\overline{f}}{dt}(t^{n+1}) + F(\overline{p}^{n+1}, \overline{f}^{n+1}), f - \overline{f}^{n+1} \rangle \geq 0, \tag{28}$$

where $\overline{p}^{n+1}$ satisfies

$$\frac{d\overline{p}}{dt}(t^{n+1}) + G(\overline{f}^{n+1}) \in -N_{K_1}(\overline{p}^{n+1}). \tag{29}$$

Let $f = \overline{f}^{n+1}$ in (26), and $f = \overline{f}_a^{n+1}$ in (28). The we obtain

$$\langle \frac{\overline{f}_a^{n+1} - \overline{f}_a^n}{\Delta t} + F(\overline{p}_a^{n+1}, \overline{f}_a^{n+1}), e^{n+1} \rangle \geq 0, \tag{30}$$

$$\langle \frac{d\overline{f}}{dt}(t^{n+1}) + F(\overline{p}^{n+1}, \overline{f}^{n+1}), -e^{n+1} \rangle \geq 0, \tag{31}$$

where $e^{n+1} = \overline{f}^{n+1} - \overline{f}_a^{n+1}$.

Combining (30) and (31), we have that

$$\langle \frac{d\overline{f}}{dt}(t^{n+1}) - \frac{\overline{f}_a^{n+1} - \overline{f}_a^n}{\Delta t} + F(\overline{p}^{n+1}, \overline{f}^{n+1}) - F(\overline{p}_a^{n+1}, \overline{f}_a^{n+1}), e^{n+1} \rangle \leq 0.$$

Hence,

$$\langle \partial e^n + \frac{d\overline{f}}{dt}(t^{n+1}) - \frac{\overline{f}^{n+1} - \overline{f}^n}{\Delta t} + F(\overline{p}^{n+1}, \overline{f}^{n+1}) - F(\overline{p}_a^{n+1}, \overline{f}_a^{n+1}), e^{n+1} \rangle \leq 0,$$

where

$$\partial e^n = \frac{e^{n+1} - e^n}{\Delta t} = \frac{\overline{f}^{n+1} - \overline{f}^n}{\Delta t} - \frac{\overline{f}_a^{n+1} - \overline{f}_a^n}{\Delta t}.$$

Thus,

$$\begin{aligned}
&\langle \partial e^n + F(\overline{p}^{n+1}, \overline{f}^{n+1}) - F(\overline{p}_a^{n+1}, \overline{f}_a^{n+1}), e^{n+1} \rangle \\
&\leq \langle -\tfrac{d\overline{f}}{dt}(t^{n+1}) + \tfrac{\overline{f}^{n+1} - \overline{f}^n}{\Delta t}, e^{n+1} \rangle \\
&= \tfrac{1}{\Delta t} \textstyle\int_{t^n}^{t^{n+1}} \langle \tfrac{d\overline{f}}{dt}(t) - \tfrac{d\overline{f}}{dt}(t^{n+1}), e^{n+1} \rangle dt \\
&= \tfrac{1}{\Delta t} \textstyle\int_{t^n}^{t^{n+1}} \langle \max(0, -F(\overline{p}(t), \overline{f}(t))) - \max(0, F(\overline{p}^{n+1}, \overline{f}^{n+1})), e^{n+1} \rangle dt,
\end{aligned}$$

and, hence, it follows that

$$\begin{aligned}
&|\langle \partial e^n + F(\overline{p}^{n+1}, \overline{f}^{n+1}) - F(\overline{p}_a^{n+1}, \overline{f}_a^{n+1}), e^{n+1} \rangle| \\
&\leq \tfrac{1}{\Delta t} \textstyle\int_{t^n}^{t^{n+1}} \langle \| \max(0, -F(\overline{p}(t), \overline{f}(t))) - \max(0, -F(\overline{p}^{n+1}, \overline{f}^{n+1})) \|_0 \| e^{n+1} \|_0 \rangle dt \\
&\leq \tfrac{1}{\Delta t} \textstyle\int_{t^n}^{t^{n+1}} \langle \| - F(\overline{p}(t), \overline{f}(t)) + F(\overline{p}^{n+1}, \overline{f}^{n+1}) \|_0 \| e^{n+1} \|_0 \rangle dt \\
&= \| - F(\overline{p}(t^*), \overline{f}(t^*)) + F(\overline{p}^{n+1}, \overline{f}^{n+1}) \|_0 \| e^{n+1} \|_0,
\end{aligned}$$

where $t^* \in [t^n, t^{n+1}]$.

Multiplying the two ends by Δt, we obtain

$$
\begin{aligned}
&|\langle e^{n+1} - e^n + \Delta t(F(\overline{p}^{n+1}, \overline{f}^{n+1}) - F(\overline{p}_a^{n+1}, \overline{f}_a^{n+1})), e^{n+1}\rangle| \\
&\leq \Delta t\| - F(\overline{p}(t^*), \overline{f}(t^*)) + F(\overline{p}^{n+1}, \overline{f}^{n+1})\|_0 \|e^{n+1}\|_0.
\end{aligned}
$$

Since

$$
\begin{aligned}
&2\textstyle\sum_{n=0}^{N-1} \langle e^{n+1} - e^n, e^{n+1}\rangle = \sum_{n=0}^{N-1} \|e^{n+1} - e^n\|_0^2 + \|e^N\|_0^2 - \|e^0\|_0^2 \\
&\geq \max_{0\leq n\leq N} \|e^n\|_0^2 - \|e^0\|_0^2,
\end{aligned} \tag{32}
$$

summation over n, from 0 to $N-1$, yields

$$
\begin{aligned}
&\max_{0\leq n\leq N} \|e^n\|_0^2 + 2\Delta t \textstyle\sum_{n=0}^{N-1} \langle F(\overline{p}^{n+1}, \overline{f}^{n+1}) - F(\overline{p}_a^{n+1}, \overline{f}_a^{n+1}), e^{n+1}\rangle \\
&\leq \|e^0\|_0^2 + 2\Delta t\|F(\overline{p}^{n+1}, \overline{f}^{n+1}) - F(\overline{p}^{n+1}(t^*), \overline{f}^{n+1}(t^*)\|_0 \|e^{n+1}\|_0.
\end{aligned} \tag{33}
$$

Lastly, we provide a lower bound on $2\Delta t \sum_{n=0}^{N-1} \langle F(\overline{p}^{n+1}, \overline{f}^{n+1}) - F(\overline{p}_a^{n+1}, \overline{f}_a^{n+1}), e^{n+1}\rangle$ by using the definitions of F, $\overline{p}^{n+1}$, and $\overline{p}_a^{n+1}$, and the assumption that $c(f)$ and $-u(p)$ are strongly monotone. Note that

$$
\begin{aligned}
&2\Delta t \textstyle\sum_{n=0}^{N-1} \langle F(\overline{p}^{n+1}, \overline{f}^{n+1}) - F(\overline{p}_a^{n+1}, \overline{f}_a^{n+1}), e^{n+1}\rangle \\
&= 2\Delta t \textstyle\sum_{n=0}^{N-1} (\langle c(\overline{f}^{n+1}) - c(\overline{f}_a^{n+1}), e^{n+1}\rangle + \langle u(\overline{p}^{n+1}) - u(\overline{p}_a^{n+1}), G(\overline{f}^{n+1}) - G(\overline{f}_a^{n+1})\rangle) \\
&\geq 2\Delta t \textstyle\sum_{n=0}^{N-1} (\alpha \|e^{n+1}\|_0^2 + \langle -u(\overline{p}^{n+1}) + u(\overline{p}_a^{n+1}), -G(\overline{f}^{n+1}) + G(\overline{f}_a^{n+1})\rangle) \\
&= 2\Delta t \textstyle\sum_{n=0}^{N-1} (\alpha \|e^{n+1}\|_0^2 - \Delta u(p_n^*) \langle \overline{p}^{n+1} - \overline{p}_a^{n+1}, \frac{d\overline{p}}{dt}(t^{n+1}) - v^{n+1} - (\frac{\overline{p}_a^{n+1} - \overline{p}_a^n}{\Delta t} - v_a^{n+1})\rangle) \\
&(\text{where } -v^{n+1} \in N_{K_1}(\overline{p}^{n+1}), -v_a^{n+1} \in N_{K_1}(\overline{p}_a^{n+1})) \\
&= 2\Delta t \textstyle\sum_{n=0}^{N-1} \alpha \|e^{n+1}\|_0^2 \\
&\ + 2\Delta t \textstyle\sum_{n=0}^{N-1} (-\Delta u(p_n^*)) \langle \overline{p}^{n+1} - \overline{p}_a^{n+1}, \frac{\overline{p}^{n+1} - \overline{p}^n}{\Delta t} + O(\Delta t) - v^{n+1} - (\frac{\overline{p}_a^{n+1} - \overline{p}_a^n}{\Delta t} - v_a^{n+1})\rangle \\
&(\text{Note that } \langle -v^{n+1} + v_a^{n+1}, \overline{p}^{n+1} - \overline{p}_a^{n+1}\rangle \geq 0) \\
&\geq 2\Delta t \textstyle\sum_{n=0}^{N-1} \alpha \|e^{n+1}\|_0^2 + 2\Delta t \sum_{n=0}^{N-1} \beta \langle \tilde{e}^{n+1}, \frac{\tilde{e}^{n+1} - \tilde{e}^n}{\Delta t} + O(\Delta t)\rangle \\
&(\text{where } \tilde{e}^{n+1} = \overline{p}^{n+1} - \overline{p}_a^{n+1}) \\
&\geq 2\Delta t \textstyle\sum_{n=0}^{N-1} \alpha \|e^{n+1}\|_0^2 + \beta(\max_{0\leq n\leq N} \|\tilde{e}^n\|_0^2 - \|\tilde{e}^0\|_0^2) + 2\Delta t \sum_{n=0}^{N-1} \beta \langle \tilde{e}^{n+1}, O(\Delta t)\rangle.
\end{aligned} \tag{34}
$$

Furthermore, note that in the last inequality, we again used (32). Hence, by combining (33) and (34), we have that:

$$
\begin{aligned}
&\max_{0\leq n\leq N} \|e^n\|_0^2 + \beta \max_{0\leq n\leq N} \|\tilde{e}^n\|_0^2 + 2\Delta t\alpha \textstyle\sum_{n=0}^{N-1} \|e^{n+1}\|_0^2 + 2\Delta t\beta \sum_{n=0}^{N-1} \langle \tilde{e}^{n+1}, O(\Delta t)\rangle \\
&\leq \|e^0\|_0^2 + \beta \|\tilde{e}^0\|_0^2 + 2\Delta t\|F(\overline{p}^{n+1}, \overline{f}^{n+1}) - F(\overline{p}^{n+1}(t^*), \overline{f}^{n+1}(t^*))\|_0 \|e^{n+1}\|_0.
\end{aligned}
$$

Assuming now that $e^0 = 0$ and that $\tilde{e}^0 = 0$, and using Young's inequality, we obtain:

$$\begin{aligned}
&\max_{0\le n\le N} \|e^n\|_0^2 + 2\Delta t\alpha \sum_{n=0}^{N-1} \|e^{n+1}\|_0^2 + \beta \max_{0\le n\le N} \|\tilde{e}^n\|_0^2 \\
&\le o(\Delta t)\|e^{n+1}\|_0 + \sum_{n=0}^{N-1} o(\Delta t)\|\tilde{e}^{n+1}\|_0 \\
&\le \frac{1}{2\epsilon}(o(\Delta t))^2 + \frac{\epsilon}{2}\|e^{n+1}\|_0^2 + \sum_{n=0}^{N-1}(\frac{1}{2\tilde{\epsilon}}(o(\Delta t))^2 + \frac{\tilde{\epsilon}}{2}\|\tilde{e}^{n+1}\|_0^2).
\end{aligned}$$

Letting $\epsilon = 1, \tilde{\epsilon} = \beta/N$, yields

$$\begin{aligned}
&\frac{1}{2}\max_{0\le n\le N} \|e^n\|_0^2 + \frac{\beta}{2}\max_{0\le n\le N} \|\tilde{e}^n\|_0^2 \le \frac{1}{2}(o(\Delta t))^2 + \sum_{n=0}^{N-1} \frac{N}{2\beta}(o(\Delta t))^2 \\
&= \frac{1}{2}(o(\Delta t))^2 + \frac{1}{2\beta}(o(\Delta t))^2 N^2 = \frac{1}{2}(o(\Delta t))^2 + \frac{1}{2\beta}(O(\Delta t))^2.
\end{aligned}$$

We have, thus, established the following result:

Theorem 4

The error for the finite-difference algorithm defined by (26) and (27) is of order $O(\Delta t)$, provided that $c(f)$ and $-u(p)$ are both strongly monotone.

5. Summary and Conclusions

In this paper, we have applied, for the first time, the theory of evolution (parabolic) variational inequalities for the formulation, analysis, and solution of a dynamic socio-economic process, arising from the problem of human migration. Evolution variational inequalities, in contrast to finite-dimensional variational inequalities, are time-dependent and infinite-dimensional. Nevertheless, their utilization has focused on problems of mathematical physics and their utility in economics, operations research/management science, and engineering has been left unexplored.

Although the finite-dimensional variational inequality problem has been used to-date for the study of a plethora of competitive equilibrium problems, ranging from oligopolistic market equilibrium problems to congested urban transportation systems, that framework has been inherently static, with a focus on the equilibrium points of the system in question. Recent research on projected dynamical systems, in contrast, (cf. the book by Nagurney and Zhang [22]) has provided a dynamic dimension to such problems through the connection that the set of solutions of a projected dynamical system corresponds to the set of solutions to the corresponding finite-dimensional variational inequality problem.

In this paper, we explore another connection between two methodologies that can be used for the study of competitive equilibrium problems, in particular, projected dynamical systems, and evolution variational inequalities. Specifically, we provide conditions that are less restrictive than those imposed on a solution to a projected dynamical system in order to establish the existence and uniqueness of a solution path to an evolution variational inequality problem. Our theoretical results are in the framework of a new model of dynamic human migration.

We then propose a finite-difference scheme for the solution of the human migration model, along with a convergence estimate. Future research should include the computation of human migration problems using this new methodology, comparisons with algorithms based on projected dynamical systems theory, as well as the development of new dynamic models in other applications in economics, operations research/management science, and engineering. Moreover, we can expect additional linkages between projected dynamical systems and time-dependent variational inequalities such as those established recently by Cojocaru, Daniele, and Nagurney [12, 18, 23] which we expect will enhance both the theoretical foundations as well as the modeling paradigms for a variety of dynamic problems in different disciplines.

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