

# Supply Chain Networks, Electronic Commerce, and Supply Side and Demand Side Risk

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**Abstract:** In this paper, we develop a supply chain network model in which both physical and electronic transactions are allowed and in which supply side risk as well as demand side risk are included in the formulation. The model consists of three tiers of decision-makers: the manufacturers, the distributors, and the retailers, with the demands associated with the retail outlets being random. We model the optimizing behavior of the various decision-makers, with the manufacturers and the distributors being multicriteria decision-makers and concerned with both profit maximization and risk minimization. We derive the equilibrium conditions and establish the finite-dimensional variational inequality formulation. We provide qualitative properties of the equilibrium pattern in terms of existence and uniqueness results and also establish conditions under which the proposed computational procedure is guaranteed to converge. We illustrate the supply chain network model through several numerical examples for which the equilibrium prices and product shipments are computed. This is the first multitiered supply chain network equilibrium model with electronic commerce and with supply side and demand side risk for which modeling, qualitative analysis, and computational results have been obtained.

**Key Words:** Supply chains; Networks; Electronic commerce; Risk management; Multicriteria optimization; Network equilibrium; Variational inequalities

## 1. Introduction

Supply chain networks have evolved to become critical structures in the production and dissemination of goods in today's modern economies. Since they involve manufacturers, distributors, retailers, as well as consumers, which are spatially dispersed, they must respond to the realities of world events, which, in the given age, are characterized by heightened risks and uncertainty. Furthermore, the management of supply chain networks must consider the complexity of interactions among the various decision-makers, coupled with appropriate decision-making criteria, in this new world order.

Given the importance of supply chain networks in practice, there has been increased attention focused on disruptions to supply chains from various directions. Threats to the optimal functioning of supply chains have included both human disruptions (cf. Sheffi (2001)) as well as biological ones, including, for example, the SARS virus (see Engardio et al. (2003)). Such realities have brought even greater pressure and need for the rigorous formulation and analysis of supply chain networks.

On the positive side, advances in electronic commerce have unveiled new opportunities for the management of supply chain networks (cf. Nagurney and Dong (2002) and the references therein). Notably, electronic commerce (e-commerce) has had an immense effect on the manner in which businesses order goods and have them transported with the major portion of e-commerce transactions being in the form of business-to-business (B2B). Estimates of B2B electronic commerce range from approximately .1 trillion dollars to 1 trillion dollars in 1998 and with forecasts reaching as high as \$4.8 trillion dollars in 2003 in the United States (see Federal Highway Administration (2000), Southworth (2000)). Moreover, according to the National Research Council (2000), the principal effect of business-to-business (B2B) commerce, estimated to be 90% of all electronic commerce by value and volume, is in the creation of new and more profitable supply chain networks.

Furthermore, the availability of electronic commerce in which the physical ordering of goods (and supplies) (and, in some cases, even delivery) is replaced by electronic orders, offers the potential of reducing risks associated with physical transportation due to potential threats and disruptions as mentioned above.

In this paper, we take upon the challenge of formalizing decision-making in multitiered supply chain networks in a scenario of risk and uncertainty. Our motivation comes from practice but the approach is technical and rigorous. Our framework builds upon the recent advances in supply chain network equilibrium modeling in the case of random demands without and with electronic commerce, respectively (cf. Dong, Zhang, and Nagurney (2002a, 2003)) and with multicriteria decision-makers (cf. Dong, Zhang, and Nagurney (2002b) and Dong et al. (2003)). Here, however, and unlike the preceding papers, we are explicitly concerned with risk management on the supply side in terms of the manufacturers and the distributors. In addition, we retain the measurement of demand side risk. The need to incorporate both supply and demand side risks is well-documented in the literature (see, e.g., Smeltzer and Siferd (1998), Agrawal and Seshadri (2000), Johnson (2001), Zsidisin (2003)). For additional recent research on supply chains including electronic commerce, see the edited volumes by Geunes, Pardalos, and Romeijn (2002) and by Pardalos and Tsitsiringos (2002).

Our focus in this paper is to also provide the foundations upon which a global supply chain network model can be built that captures both supply side and demand side risk. Some preliminary results can be found in Nagurney, Cruz, and Matsypura (2003). Frameworks for risk management in a global supply chain context with a focus on centralized decision-making and optimization can be found in Huchzermeier and Cohen (1996), Cohen and Mallik (1997), and the references therein.

The paper is organized as follows. In Section 2, we develop the supply chain network model with decentralized decision-makers and with supply side and demand side risk. We derive the optimality conditions of the various decision-makers, and establish that the governing equilibrium conditions can be formulated as a finite-dimensional variational inequality problem. We emphasize here that the concept of equilibrium, first explored in a general setting for supply chains by Nagurney, Dong, and Zhang (2002) (see also Nagurney et al. (2002)), provides a valuable benchmark against which prices of the product at the various tiers of the network as well as product flows between tiers can be compared.

In Section 3, we study qualitative properties of the equilibrium pattern, and, under reasonable conditions, establish existence and uniqueness results. We also provide properties of the function that enters the variational inequality that allows us to establish convergence

of the proposed algorithmic scheme in Section 4. In Section 5, we apply the algorithm to several supply chain network examples for the computation of the equilibrium prices and shipments. The paper concludes with Section 6, in which we summarize our results and present suggestions for future research.

## 2. The Supply Chain Network Equilibrium Model with Supply Side and Demand Side Risk

In this section, we develop the supply chain network model. The multicriteria decision-makers on the supply side in the form of manufacturers and distributors are concerned not only with profit maximization but also with risk minimization. The demand side risk, in turn, is represented by the uncertainty surrounding the random demands at the retailers. The model allows for not only physical transactions but also electronic transactions.

In particular, we consider  $m$  manufacturers involved in the production of a homogeneous product which is then shipped to  $n$  distributors, who, in turn, ship the product to  $o$  retailers. The retailers can transact either physically (in the standard manner) with the distributors or directly, in an electronic manner, with the manufacturers (see also Nagurney et al. (2002)).

We denote a typical manufacturer by  $i$ , a typical distributor by  $j$ , and a typical retailer by  $k$ . Note that (cf. Figure 1) the manufacturers are located at the top tier of nodes of the network, the distributors at the middle tier, and the retailers at the third or bottom tier of nodes. The links in the supply chain *supernetwork* (cf. Nagurney and Dong (2002)) in Figure 1 include classical physical links as well as Internet links to allow for e-commerce. The introduction of e-commerce allows for “connections” that were, heretofore, not possible, such as, for example, those enabling retailers to purchase a product directly from the manufacturers.

The behavior of the various network decision-makers represented by the three tiers of nodes in Figure 1 is now described. We first focus on the manufacturers. We then turn to the distributors, and, subsequently, to the retailers. Both the manufacturers and the distributors are assumed to be multicriteria decision-makers and are concerned not only with profit maximization but also with risk minimization.

### The Behavior of the Manufacturers and their Optimality Conditions

Let  $q_i$  denote the nonnegative production output of manufacturer  $i$ . Group the production outputs of all manufacturers into the column vector  $q \in R_+^m$ . Here it is assumed that each manufacturer  $i$  is faced with a production cost function  $f_i$ , which can depend, in general, on

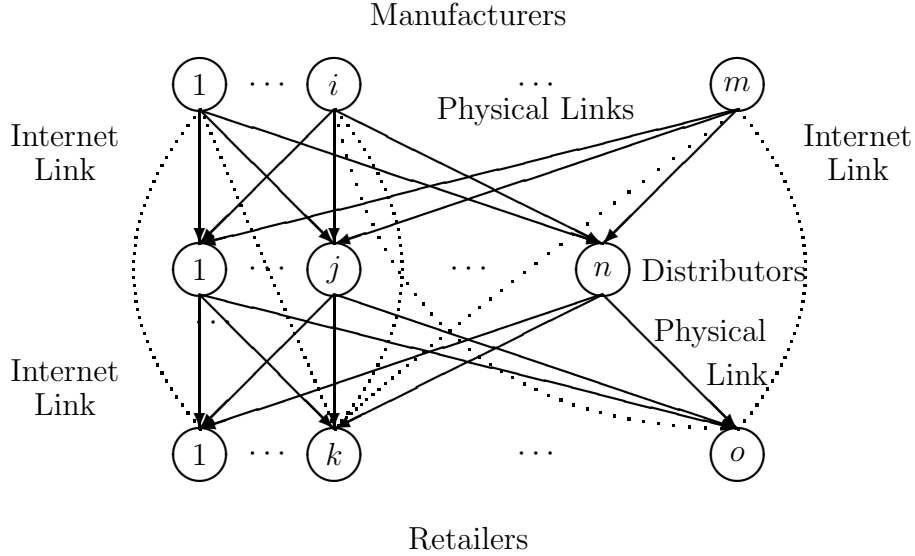


Figure 1: The *Supernetwork* Structure of the Supply Chain

the entire vector of production outputs, that is,

$$f_i = f_i(q), \quad \forall i. \quad (1)$$

Hence, the production cost of a particular manufacturer can depend not only on his production output but also on those of the other manufacturers. This allows one to model competition.

The transaction cost associated with manufacturer  $i$  transacting with distributor  $j$  is denoted by  $c_{ij}$ . The product shipment between manufacturer  $i$  and distributor  $j$  is denoted by  $q_{ij}$ . The product shipments between all pairs of manufacturers and distributors are grouped into the column vector  $Q^1 \in R_+^{mn}$ . In addition, a manufacturer  $i$  may transact directly with the retailer  $k$  with this transaction cost associated with the Internet transaction begin denoted by  $c_{ik}$  and the associated product shipment from manufacturer  $i$  to retailer  $k$  by  $q_{ik}$ . We group these product shipments into the column vector  $Q^2 \in R_+^{mo}$ .

The transaction cost between a manufacturer and distributor pair and the transaction cost between a manufacturer and retailer may depend upon the volume of transactions between

each such pair, and are given, respectively, by:

$$c_{ij} = c_{ij}(q_{ij}), \quad \forall i, j, \quad (2a)$$

and

$$c_{ik} = c_{ik}(q_{ik}), \quad \forall i, k. \quad (2b)$$

The quantity produced by manufacturer  $i$  must satisfy the following conservation of flow equation:

$$q_i = \sum_{j=1}^n q_{ij} + \sum_{k=1}^o q_{ik}, \quad (3)$$

which states that the quantity produced by manufacturer  $i$  is equal to the sum of the quantities shipped from the manufacturer to all distributors and to all retailers.

The total costs incurred by a manufacturer  $i$ , thus, are equal to the sum of the manufacturer's production cost plus the total transaction costs. His revenue, in turn, is equal to the price that the manufacturer charges for the product times the total quantity obtained/purchased of the product from the manufacturer by all the distributors and all the retailers. Let  $\rho_{1ij}^*$  denote the price charged for the product by manufacturer  $i$  to distributor  $j$  who has transacted, and let  $\rho_{1ik}^*$  denote the price charged by manufacturer  $i$  for the product to the retailer  $k$ . Hence, manufacturers can price according to their locations, as to whether the product is sold to the distributor or to the retailers directly, and according to whether the transaction was conducted via the Internet or not. How these prices are arrived at is discussed later in this section.

Noting the conservation of flow equations (3) and the production cost functions (1), one can express the criterion of profit maximization for manufacturer  $i$  as:

$$\text{Maximize } \sum_{j=1}^n \rho_{1ij}^* q_{ij} + \sum_{k=1}^o \rho_{1ik}^* q_{ik} - f_i(Q^1, Q^2) - \sum_{j=1}^n c_{ij}(q_{ij}) - \sum_{k=1}^o c_{ik}(q_{ik}), \quad (4)$$

subject to  $q_{ij} \geq 0$ , for all  $j$ , and  $q_{ik} \geq 0$ , for all  $k$ .

Note that the first two terms in (4) correspond to the revenue whereas the subsequent three terms denote the production cost and the transaction costs, respectively.

In addition to the criterion of profit maximization we also assume that each manufacturer is concerned with risk minimization. Here, for the sake of generality, we assume, as given, a risk function  $r_i$ , for manufacturer  $i$ , which is assumed to be continuous and convex and a function of not only the shipments associated with the manufacturer but also of the shipments of other manufacturers. Hence, we assume that

$$r_i = r_i(Q^1, Q^2), \quad \forall i. \quad (5)$$

Note that according to (5) the risk as perceived by a manufacturer is dependent not only upon the flows that he controls but also on those controlled by others. Hence, the second criterion of manufacturer  $i$  can be expressed as:

$$\text{Minimize } r_i(Q^1, Q^2), \quad (6)$$

subject to:  $q_{ij} \geq 0, \forall j$  and  $q_{ik} \geq 0, \forall k$ .

Each manufacturer  $i$  associates a nonnegative weight  $\alpha_i$  with the risk minimization criterion (6), with the weight associated with the profit maximization criterion (4) serving as the numeraire and being set equal to 1. Hence, we can construct a value function for each manufacturer (cf. Fishburn (1970), Chankong and Haimes (1983), Yu (1985), Keeney and Raiffa (1993)) using a constant additive weight value function. Consequently, the multicriteria decision-making problem for manufacturer  $i$  is transformed into:

$$\text{Maximize } \sum_{j=1}^n \rho_{1ij}^* q_{ij} + \sum_{k=1}^o \rho_{1ik}^* q_{ik} - f_i(Q^1, Q^2) - \sum_{j=1}^n c_{ij}(q_{ij}) - \sum_{k=1}^o c_{ij}(q_{ik}) - \alpha_i r_i(Q^1, Q^2), \quad (7)$$

subject to:  $q_{ij} \geq 0, \forall j; q_{ik} \geq 0, \forall k$ .

The manufacturers are assumed to compete in a noncooperative fashion. Also, it is assumed that the production cost functions and the transaction cost functions for each manufacturer are continuous and convex. The governing optimization/equilibrium concept underlying noncooperative behavior is that of Nash (1950, 1951), which states, in this context, that each manufacturer will determine his optimal production quantity and shipments, given the optimal ones of the competitors. Hence, the optimality conditions for all manufacturers *simultaneously* can be expressed as the following inequality (see also Gabay and



Moulin (1980), Bazaraa, Sherali, and Shetty (1993), Nagurney (1999), Nagurney, Dong, and Zhang (2002)): determine the solution  $(Q^{1*}, Q^{2*}) \in R_+^{mn+mo}$ , which satisfies:

$$\begin{aligned} & \sum_{i=1}^m \sum_{j=1}^n \left[ \frac{\partial f_i(Q^{1*}, Q^{2*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} + \alpha_i \frac{\partial r_i(Q^{1*}, Q^{2*})}{\partial q_{ij}} - \rho_{1ij}^* \right] \times [q_{ij} - q_{ij}^*] \\ & + \sum_{i=1}^m \sum_{k=1}^o \left[ \frac{\partial f_i(Q^{1*}, Q^{2*})}{\partial q_{ik}} + \frac{\partial c_{ik}(q_{ik}^*)}{\partial q_{ik}} + \alpha_i \frac{\partial r_i(Q^{1*}, Q^{2*})}{\partial q_{ik}} - \rho_{1ik}^* \right] \times [q_{ik} - q_{ik}^*] \geq 0, \quad (8) \\ & \forall (Q^1, Q^2) \in R_+^{mn+mo}. \end{aligned}$$

The inequality (8), which is a *variational inequality* has a nice economic interpretation. In particular, from the first term one can infer that, if there is a positive shipment of the product transacted between manufacturer and a distributor, then the marginal cost of production plus the marginal cost of transacting plus the weighted marginal risk associated with that transaction must be equal to the price that the distributor is willing to pay for the product. If the marginal cost of production plus the marginal cost and weighted marginal risk of transacting exceeds that price, then there will be zero volume of flow of the product between the two. The second term in (8) has a similar interpretation; in particular, there will be a positive volume of flow of the product from a manufacturer to a retailer if the marginal cost of production of the manufacturer plus the marginal cost of transacting with the retailer via the Internet and the weighted marginal risk is equal to the price the retailers are willing to pay for the product.

Note that in the above framework we explicitly allow each of the decision-makers, in the form of manufacturers, to not only have distinct risk functions but also distinct weights associated with their respective weight functions. We emphasize that the study of risk has had a long and prominent history in the field of finance, dating to the work of Markowitz (1952, 1959), whereas the incorporation of risk into supply chain management has only been addressed fairly recently. Furthermore, we emphasize that by allowing the risk function to be general one can then construct an appropriate function pertaining to the specific situation and decision-maker. For example, in the field of finance, measurement of risk has included the use of variance-covariance matrices, yielding quadratic expressions for the risk (see also, e.g., Nagurney and Siokos (1997)). In addition, in finance, the bicriteria optimization problem of

net revenue maximization and risk minimization is fairly standard (see also, e.g., Dong and Nagurney (2001)).

### The Behavior of the Distributors and their Optimality Conditions

The distributors, in turn, are involved in transactions both with the manufacturers as well as with the retailers. Let  $q_{jk}$  denote the amount of the product purchased by retailer  $k$  from distributor  $j$ . Group these shipment quantities into the column vector  $Q^3 \in R_+^{no}$ .

A distributor  $j$  is faced with what is termed a *handling* cost, which may include, for example, the loading/unloading and storage costs associated with the product. Denote this cost by  $c_j$  and, in the simplest case, one would have that  $c_j$  is a function of  $\sum_{i=1}^m q_{ij}$  and  $\sum_{k=1}^o q_{jk}$  that is, the inventory cost of a distributor is a function of how much of the product he has obtained from the various manufacturers and how much of the product he has shipped out to the various retailers. However, for the sake of generality, and to enhance the modeling of competition, allow the function to, in general, depend also on the amounts of the product held by other distributors. Therefore, one may write:

$$c_j = c_j(Q^1, Q^3), \quad \forall j. \quad (9)$$

Distributor  $j$  associates a price with the product, which is denoted by  $\gamma_j^*$ . This price, as will be shown, will also be endogenously determined in the model and will be, given a positive volume of flow between a distributor and any retailer, related to a clearing-type price. Assuming that the distributors are also profit-maximizers, the optimization problem of a distributor  $j$  is given by:

$$\text{Maximize } \gamma_j^* \sum_{k=1}^o q_{jk} - c_j(Q^1, Q^3) - \sum_{i=1}^m \rho_{ij}^* q_{ij} \quad (10)$$

subject to:

$$\sum_{k=1}^o q_{jk} \leq \sum_{i=1}^m q_{ij}, \quad (11)$$

and the nonnegativity constraints:  $q_{ij} \geq 0$ , and  $q_{jk} \geq 0$ , for all  $i$ , and  $k$ . Objective function (10) expresses that the difference between the revenues and the handling cost and the payout to the manufacturers should be maximized. Constraint (11) expresses that the retailers cannot purchase more from a distributor than is held in stock.

In addition, each distributor seeks to also minimize his risk associated with obtaining and shipping the product to the various retailers. We assume that each distributor  $j$  is faced with his own individual risk denoted by  $r_j$  with the function being assumed to be continuous and convex and dependent on the shipments to and from all the distributors, that is,

$$r_j = r_j(Q^1, Q^3), \quad \forall j. \quad (12)$$

The model assumes that each distributor associates a weight of 1 with the profit criterion (10) and a weight of  $\beta_j$  to his risk level. Therefore, the multicriteria decision-making problem for distributor  $j$ ;  $j = 1, \dots, n$ , can be transformed directly into the optimization problem:

$$\text{Maximize} \quad \gamma_j^* \sum_{k=1}^o q_{jk} - c_j(Q^1, Q^3) - \sum_{i=1}^m \rho_{1ij}^* q_{ij} - \beta_j r_j(Q^1, Q^3) \quad (13)$$

subject to:

$$\sum_{k=1}^o q_{jk} \leq \sum_{i=1}^m q_{ij}, \quad (14)$$

and the nonnegativity constraints:  $q_{ij} \geq 0$ , and  $q_{jk} \geq 0$ , for all  $i$  and  $k$ .

Objective function (13) represents a value function for distributor  $j$  with  $\beta_j$  having the interpretation as a conversion rate in dollar value.

The optimality conditions of the distributors are now obtained, assuming that each distributor is faced with the optimization problem (13), subject to (14), and the nonnegativity assumption on the variables.

Here it is also assumed that the distributors compete in a noncooperative manner, given the actions of the other distributors. Note that, at this point, we consider that distributors seek to determine not only the optimal amounts purchased by the retailers, but, also, the amount that they wish to obtain from the manufacturers. In equilibrium, all the shipments between the tiers of network decision-makers will have to coincide.

Assuming that the handling cost for each distributor is continuous and convex as are the transaction costs, the optimal  $(Q^{1*}, Q^{3*}, \rho_2^*) \in R_+^{mn+no+n}$  satisfy the optimality conditions for all the distributors or, equivalently, the variational inequality:

$$\sum_{i=1}^m \sum_{j=1}^n \left[ \frac{\partial c_j(Q^{1*}, Q^{3*})}{\partial q_{ij}} + \rho_{1ij}^* + \beta_j \frac{\partial r_j(Q^{1*}, Q^{3*})}{\partial q_{ij}} - \rho_{2j}^* \right] \times [q_{ij} - q_{ij}^*]$$

$$\begin{aligned}
& + \sum_{j=1}^n \sum_{k=1}^o \left[ -\gamma_j^* + \frac{\partial c_j(Q^{1*}, Q^{3*})}{\partial q_{jk}} + \rho_{2j}^* + \beta_j \frac{\partial r_j(Q^{1*}, Q^{3*})}{\partial q_{jk}} \right] \times [q_{jk} - q_{jk}^*] \\
& + \sum_{j=1}^n \left[ \sum_{i=1}^m q_{ij}^* - \sum_{k=1}^o q_{jk}^* \right] \times [\rho_{2j} - \rho_{2j}^*] \geq 0, \quad \forall Q^1 \in R_+^{mn}, \forall Q^3 \in R_+^{no}, \forall \rho_2 \in R_+^n, \quad (15)
\end{aligned}$$

where  $\rho_{2j}$  is the Lagrange multiplier associated with constraint (14) for distributor  $j$  and  $\rho_2$  is the column vector of all the distributors' multipliers. In this derivation, as in the derivation of inequality (8), the prices charged were not variables. The  $\gamma^* = (\gamma_1^*, \dots, \gamma_n^*)$  is the vector of endogenous equilibrium prices in the complete model.

The economic interpretation of the distributors' optimality conditions is now highlighted. From the second term in inequality (15), one has that, if retailer  $k$  purchases the product from a distributor  $j$ , that is, if the  $q_{jk}^*$  is positive, then the price charged by retailer  $j$ ,  $\gamma_j^*$ , is equal to the marginal handling cost and the weighted marginal risk, plus the shadow price term  $\rho_{2j}^*$ , which, from the third term in the inequality, serves as the price to clear the market from distributor  $j$ . Furthermore, from the first term in inequality (15), one can infer that, if a manufacturer transacts with a distributor resulting in a positive flow of the product between the two, then the price  $\rho_{2j}^*$  is precisely equal to distributor  $j$ 's payment to the manufacturer,  $\rho_{1ij}^*$ , plus his marginal cost of handling the product associated with transacting with the particular manufacturer plus the weighted marginal risk associated with the transaction.

In the above derivations, we have considered supply side risk from the perspective of the manufacturers as well as the distributors. We now turn to the retailers and consider demand side risk which is modeled in a stochastic manner to represent the uncertainty. Clearly, if the demands were known with certainty there would be no risk associated with them and the retailers and manufacturers could make their production and distribution decisions accordingly.

## The Retailers and their Optimality Conditions

The retailers, in turn, must decide how much to order from the distributors and from the manufacturers in order to cope with the random demand while still seeking to maximize their profits. A retailer  $k$  is also faced with what we term a *handling* cost, which may include, for example, the display and storage cost associated with the product. We denote this cost by

$c_k$  and, in the simplest case, we would have that  $c_k$  is a function of  $s_k = \sum_{i=1}^m q_{ik} + \sum_{j=1}^n q_{jk}$ , that is, the holding cost of a retailer is a function of how much of the product he has obtained from the various manufacturers directly through orders via the Internet and from the various distributors. However, for the sake of generality, and to enhance the modeling of competition, we allow the function to, in general, depend also on the amounts of the product held by other retailers and, therefore, we may write:

$$c_k = c_k(Q^2, Q^3), \quad \forall k. \quad (16)$$

Let  $\rho_{3k}$  denote the demand price of the product associated with retailer  $k$ . We assume that  $\hat{d}_k(\rho_{3k})$  is the demand for the product at the demand price of  $\rho_{3k}$  at retail outlet  $k$ , where  $\hat{d}_k(\rho_{3k})$  is a random variable with a density function of  $\mathcal{F}_k(x, \rho_{3k})$ , with  $\rho_{3k}$  serving as a parameter. Hence, we assume that the density function may vary with the demand price. Let  $P_k$  be the probability distribution function of  $\hat{d}_k(\rho_{3k})$ , that is,  $P_k(x, \rho_{3k}) = P_k(\hat{d}_k \leq x) = \int_0^x \mathcal{F}_k(x, \rho_{3k}) dx$ .

Retailer  $k$  can sell to the consumers no more than the minimum of his supply or his demand, that is, the actual sale of  $k$  cannot exceed  $\min\{s_k, \hat{d}_k\}$ . Let

$$\Delta_k^+ \equiv \max\{0, s_k - \hat{d}_k\} \quad (17)$$

and

$$\Delta_k^- \equiv \max\{0, \hat{d}_k - s_k\}, \quad (18)$$

where  $\Delta_k^+$  is a random variable representing the excess supply (inventory), whereas  $\Delta_k^-$  is a random variable representing the excess demand (shortage).

Note that the expected values of excess supply and excess demand of retailer  $k$  are scalar functions of  $s_k$  and  $\rho_{3k}$ . In particular, let  $e_k^+$  and  $e_k^-$  denote, respectively, the expected values:  $E(\Delta_k^+)$  and  $E(\Delta_k^-)$ , that is,

$$e_k^+(s_k, \rho_{3k}) \equiv E(\Delta_k^+) = \int_0^{s_k} (s_k - x) \mathcal{F}_k(x, \rho_{3k}) dx, \quad (19)$$

$$e_k^-(s_k, \rho_{3k}) \equiv E(\Delta_k^-) = \int_{s_k}^{\infty} (x - s_k) \mathcal{F}_k(x, \rho_{3k}) dx. \quad (20)$$

Assume that the unit penalty of having excess supply at retail outlet  $k$  is  $\lambda_k^+$  and that the unit penalty of having excess demand is  $\lambda_k^-$ , where the  $\lambda_k^+$  and the  $\lambda_k^-$  are assumed to be nonnegative. Then, the expected total penalty of retailer  $k$  is given by

$$E(\lambda_k^+ \Delta_k^+ + \lambda_k^- \Delta_k^-) = \lambda_k^+ e_k^+(s_k, \rho_{3k}) + \lambda_k^- e_k^-(s_k, \rho_{3k}).$$

Assuming, as already mentioned, that the retailers are also profit-maximizers, the expected revenue of retailer  $k$  is  $E(\rho_{3k} \min\{s_k, \hat{d}_k\})$ . Hence, the optimization problem of a retailer  $k$  can be expressed as:

$$\text{Maximize } E(\rho_{3k} \min\{s_k, \hat{d}_k\}) - E(\lambda_k^+ \Delta_k^+ + \lambda_k^- \Delta_k^-) - c_k(Q^2, Q^3) - \sum_{i=1}^m \rho_{1ik}^* q_{ik} - \sum_{j=1}^n \gamma_j^* q_{jk} \quad (21)$$

subject to:

$$q_{ik} \geq 0, \quad q_{jk} \geq 0, \text{ for all } i \text{ and } j. \quad (22)$$

Objective function (21) expresses that the expected profit of retailer  $k$ , which is the difference between the expected revenues and the sum of the expected penalty, the handling cost, and the payouts to the manufacturers and to the distributors, should be maximized.

Applying now the definitions of  $\Delta_k^+$ , and  $\Delta_k^-$ , we know that  $\min\{s_k, \hat{d}_k\} = \hat{d}_k - \Delta_k^-$ . Therefore, the objective function (21) can be expressed as

$$\text{Maximize } \rho_{3k} d_k(\rho_{3k}) - (\rho_{3k} + \lambda_k^-) e_k^-(s_k, \rho_{3k}) - \lambda_k^+ e_k^+(s_k, \rho_{3k}) - c_k(Q^2, Q^3) - \sum_{i=1}^m \rho_{1ik}^* q_{ik} - \sum_{j=1}^n \gamma_j^* q_{jk} \quad (23)$$

where  $d_j(\rho_{3k}) \equiv E(\hat{d}_k)$  is a scalar function of  $\rho_{3k}$ .

We now consider the optimality conditions of the retailers assuming that each retailer is faced with the optimization problem (21), subject to (22), which represents the nonnegativity assumption on the variables. Here, we also assume that the retailers compete in a noncooperative manner so that each maximizes his profits, given the actions of the other retailers. Note that, at this point, we consider that retailers seek to determine the amount that they wish to obtain from the manufacturers and from the distributors. First, however, we make the following derivation and introduce the necessary notation:

$$\frac{\partial e_k^+(s_k, \rho_{3k})}{\partial q_{ik}} = \frac{\partial e_k^+(s_k, \rho_{3k})}{\partial q_{jk}} = P_k(s_k, \rho_{3k}) = P_k\left(\sum_{i=1}^m q_{ik} + \sum_{j=1}^n q_{jk}, \rho_{3k}\right) \quad (24)$$

$$\frac{\partial e_k^-(s_k, \rho_{3k})}{\partial q_{ik}} = \frac{\partial e_k^-(s_k, \rho_{3k})}{\partial q_{jk}} = P_k(s_k, \rho_{3k}) - 1 = P_k\left(\sum_{i=1}^m q_{ij} + \sum_{j=1}^n q_{jk}, \rho_{3k}\right) - 1. \quad (25)$$

Assuming that the handling cost for each retailer is continuous and convex, then the optimality conditions for all the retailers satisfy the variational inequality: determine  $(Q^{2*}, Q^{3*}) \in R_+^{mo+no}$ , satisfying:

$$\begin{aligned} & \sum_{i=1}^m \sum_{k=1}^o \left[ \lambda_k^+ P_k(s_k^*, \rho_{3k}) - (\lambda_k^- + \rho_{3k})(1 - P_k(s_k^*, \rho_{3k})) + \frac{\partial c_k(Q^{2*}, Q^{3*})}{\partial q_{ik}} + \rho_{1ik}^* \right] \times [q_{ik} - q_{ik}^*] \\ & + \sum_{j=1}^n \sum_{k=1}^o \left[ \lambda_k^+ P_k(s_k^*, \rho_{3k}) - (\lambda_k^- + \rho_{3k})(1 - P_k(s_k^*, \rho_{3k})) + \frac{\partial c_k(Q^{2*}, Q^{3*})}{\partial q_{jk}} + \gamma_j^* \right] \times [q_{jk} - q_{jk}^*] \geq 0, \\ & \forall (Q^2, Q^3) \in R_+^{mo+no}. \end{aligned} \quad (26)$$

In this derivation, as in the derivation of inequalities (8) and (15), we have not had the prices charged be variables. They become endogenous variables in the complete supply chain network model. A similar derivation but in the absence of electronic commerce (and in the case of only a two-tiered rather than a three-tiered supply chain network) was obtained in Dong, Zhang, and Nagurney (2002a).

We now highlight the economic interpretation of the retailers' optimality conditions. In inequality (26), we can infer that, if a manufacturer  $i$  transacts with a retailer  $k$  resulting in a positive flow of the product between the two, then the selling price at retail outlet  $k$ ,  $\rho_{3k}$ , with the probability of  $(1 - P_k(\sum_{i=1}^m q_{ik}^* + \sum_{j=1}^n q_{jk}^*, \rho_{3k}))$ , that is, when the demand is not less than the total order quantity, is precisely equal to the retailer  $k$ 's payment to the manufacturer,  $\rho_{1ik}^*$ , plus his marginal cost of handling the product and the penalty of having excess demand with probability of  $P_k(\sum_{i=1}^m q_{ik}^* + \sum_{j=1}^o q_{jk}^*, \rho_{3k})$ , (which is the probability when actual demand is less than the order quantity), subtracted by the penalty of having shortage with probability of  $(1 - P_k(\sum_{i=1}^m q_{ik}^* + \sum_{j=1}^o q_{jk}^*, \rho_{3k}))$  (when the actual demand is greater than the order quantity).

Similarly, if a distributor  $j$  transacts with a retailer  $k$  resulting in a positive flow of the product between the two, then the selling price at retail outlet  $k$ ,  $\rho_{3k}$ , with the probability of  $(1 - P_k(\sum_{i=1}^m q_{ik}^* + \sum_{j=1}^n q_{jk}^*, \rho_{3k}))$ , that is, when the demand is not less than the total

order quantity, is precisely equal to the retailer  $k$ 's payment to the manufacturer,  $\gamma_j^*$ , plus his marginal cost of handling the product and the penalty of having excess demand with probability of  $P_k(\sum_{i=1}^m q_{ik}^* + \sum_{j=1}^o q_{jk}^*, \rho_{3k})$ , (which is the probability when actual demand is less than the order quantity), subtracted by the penalty of having shortage with probability of  $(1 - P_k(\sum_{i=1}^m q_{ik}^* + \sum_{j=1}^o q_{jk}^*, \rho_{3k}))$  (when the actual demand is greater than the order quantity).

### The Equilibrium Conditions

We now turn to a discussion of the market equilibrium conditions. Subsequently, we construct the equilibrium conditions for the entire supply chain network.

The equilibrium conditions associated with the transactions that take place between the retailers and the consumers are the stochastic economic equilibrium conditions, which, mathematically, take on the following form: For any retailer  $k$ ;  $k = 1, \dots, o$ :

$$\hat{d}_k(\rho_{3k}^*) \begin{cases} \leq \sum_{i=1}^m q_{ik}^* + \sum_{j=1}^o q_{jk}^* & \mathbf{a.e.}, \text{ if } \rho_{3k}^* = 0 \\ = \sum_{i=1}^m q_{ik}^* + \sum_{j=1}^o q_{jk}^* & \mathbf{a.e.}, \text{ if } \rho_{3k}^* > 0, \end{cases} \quad (27)$$

where **a.e.** means that the corresponding equality or inequality holds almost everywhere.

Conditions (27) state that, if the demand price at outlet  $k$  is positive, then the quantities purchased by the retailer from the manufacturers and from the distributors in the aggregate is equal to the demand, with exceptions of zero probability. These conditions correspond to the well-known economic equilibrium conditions (cf. Nagurney (1999) and the references therein). Related equilibrium conditions, but without electronic transactions allowed, were proposed in Dong, Zhang, and Nagurney (2002a).

Equilibrium conditions (27) are equivalent to the following variational inequality problem, after taking the expected value and summing over all retailers  $k$ : determine  $\rho_3^* \in R_+^o$  satisfying

$$\sum_{k=1}^o \left( \sum_{i=1}^m q_{ik}^* + \sum_{j=1}^n q_{jk}^* - d_k(\rho_{3k}^*) \right) \times [\rho_{3k} - \rho_{3k}^*] \geq 0, \quad \forall \rho_3 \in R_+^o, \quad (28)$$

where  $\rho_3$  is the  $o$ -dimensional column vector with components:  $\{\rho_{31}, \dots, \rho_{3o}\}$ .



## The Equilibrium Conditions of the Supply Chain

In equilibrium, we must have that the sum of the optimality conditions for all manufacturers, as expressed by inequality (8), the optimality conditions of the distributors, as expressed by condition (15), the optimality conditions for all retailers, as expressed by inequality (26), and the market equilibrium conditions, as expressed by inequality (28) must be satisfied. Hence, the shipments that the manufacturers ship to the retailers must be equal to the shipments that the retailers accept from the manufacturers. In addition, the shipments shipped from the manufacturers to the distributors, must be equal to those accepted by the distributors, and, finally, the shipments from the distributors to the retailers must coincide with those accepted by the retailers. We state this explicitly in the following definition:

### Definition 1: Supply Chain Network Equilibrium with Supply Side and Demand Side Risk

*The equilibrium state of the supply chain network with supply and demand side risk is one where the product flows between the tiers of the decision-makers coincide and the product shipments and prices satisfy the sum of the optimality conditions (8), (15), and (26), and the conditions (28).*

The summation of inequalities (8), (15), (26), and (28) (with the prices at the manufacturers, the distributors, and at the retailers denoted, respectively, by their values at the equilibrium, and denoted by  $\rho_1^*$ ,  $\rho_2^*$ , and  $\rho_3^*$ ), after algebraic simplification, yields the following result:

### Theorem 1: Variational Inequality Formulation

*A product shipment and price pattern  $(Q^{1*}, Q^{2*}, Q^{3*}, \rho_2^*, \rho_3^*) \in \mathcal{K}$  is an equilibrium pattern of the supply chain model according to Definition 1 if and only if it satisfies the variational inequality problem:*

$$\begin{aligned} & \sum_{i=1}^m \sum_{j=1}^n \left[ \frac{\partial f_i(Q^{1*}, Q^{2*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} + \frac{\partial c_j(Q^{1*}, Q^{3*})}{\partial q_{ij}} + \alpha_i \frac{\partial r_i(Q^{1*}, Q^{2*})}{\partial q_{ij}} + \beta_j \frac{\partial r_j(Q^{1*}, Q^{3*})}{\partial q_{ij}} - \rho_{2j}^* \right] \\ & \times [q_{ij} - q_{ij}^*] + \sum_{i=1}^m \sum_{k=1}^o \left[ \frac{\partial f_i(Q^{1*}, Q^{2*})}{\partial q_{ik}} + \frac{\partial c_{ik}(q_{ik}^*)}{\partial q_{ik}} + \frac{\partial c_k(Q^{2*}, Q^{3*})}{\partial q_{ik}} + \alpha_i \frac{\partial r_i(Q^{1*}, Q^{2*})}{\partial q_{ik}} \right] \end{aligned}$$

$$\begin{aligned}
& + \lambda_k^+ P_k(s_k^*, \rho_{3k}^*) - (\lambda_k^- + \rho_{3k}^*)(1 - P_k(s_k^*, \rho_{3k}^*)) \Big] \times [q_{ik} - q_{ik}^*] \\
& + \sum_{j=1}^n \sum_{k=1}^o \left[ \lambda_k^+ P_k(s_k^*, \rho_{3k}^*) - (\lambda_k^- + \rho_{3k}^*)(1 - P_k(s_k^*, \rho_{3k}^*)) + \frac{\partial c_j(Q^{1*}, Q^{3*})}{\partial q_{jk}} \right. \\
& + \left. \frac{\partial c_k(Q^{2*}, Q^{3*})}{\partial q_{jk}} + \beta_j \frac{\partial r_j(Q^{1*}, Q^{3*})}{\partial q_{jk}} + \rho_{2j}^* \right] \times [q_{jk} - q_{jk}^*] + \sum_{j=1}^n \left[ \sum_{i=1}^m q_{ij}^* - \sum_{k=1}^o q_{jk}^* \right] \times [\rho_{2j} - \rho_{2j}^*] \\
& + \sum_{k=1}^o \left[ \sum_{j=1}^n q_{jk}^* + \sum_{i=1}^m q_{ik}^* - d_k(\rho_3^*) \right] \times [\rho_{3k} - \rho_{3k}^*] \geq 0, \quad \forall (Q^1, Q^2, Q^3, \rho_2, \rho_3) \in \mathcal{K}, \quad (29)
\end{aligned}$$

where  $\mathcal{K} \equiv \{(Q^1, Q^2, Q^3, \rho_2, \rho_3) | (Q^1, Q^2, Q^3, \rho_2, \rho_3) \in R_+^{mn+mo+no+n+o}\}$ .

For easy reference in the subsequent sections, variational inequality problem (29) can be rewritten in standard variational inequality form (cf. Nagurney (1999)) as follows: Determine  $X^* \in \mathcal{K}$  satisfying:

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K} \equiv R_+^{mn+mo+no+n+o}, \quad (30)$$

where  $X \equiv (Q^1, Q^2, Q^3, \rho_2, \rho_3)$ ,  $F(X) \equiv (F_{ij}, F_{ik}, F_{jk}, F_j, F_k)_{i=1, \dots, m; j=1, \dots, n; k=1, \dots, o}$ , and the specific components of  $F$  are given by the functional terms preceding the multiplication signs in (29). The term  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $N$ -dimensional Euclidean space.

Note that the variables in the model (and which can be determined from the solution of either variational inequality (29) or (30)) are: the equilibrium product shipments between manufacturers and the distributors given by  $Q^{1*}$ , the equilibrium product shipments transacted electronically between the manufacturers and the retailers denoted by  $Q^{2*}$ , and the equilibrium product shipments between the distributors and the retailers given by  $Q^{3*}$ , as well as the equilibrium demand prices  $\rho_3^*$  and the equilibrium prices  $\rho_2^*$ . We now discuss how to recover the prices  $\rho_1^*$  associated with the top tier of nodes of the supply chain supernetwork and the prices  $\gamma^*$  associated with the middle tier.

First note that from (8), we have that (as already discussed briefly) if  $q_{ij}^* > 0$ , then the price  $\rho_{1ij}^* = \frac{\partial f_i(Q^{1*}, Q^{2*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} + \alpha_i \frac{\partial r_i(Q^{1*}, Q^{2*})}{\partial q_{ij}}$ . Also, from (8) it follows that if  $q_{ik}^* > 0$ , then the price  $\rho_{1ik}^* = \frac{\partial f_i(Q^{1*}, Q^{2*})}{\partial q_{ik}} + \frac{\partial c_{ik}(q_{ik}^*)}{\partial q_{ik}} + \alpha_i \frac{\partial r_i(Q^{1*}, Q^{2*})}{\partial q_{ik}}$ . Hence, the product is priced at the manufacturer's level according to whether it has been transacted physically or electronically;

and also according to the distributor or retailer with which the transaction has taken place. On the other hand, from (15) it follows that if  $q_{jk}^* > 0$ , then  $\gamma_j^* = \rho_{2j}^* + \frac{\partial c_j(Q^{1*}, Q^{3*})}{\partial q_{jk}} + \beta_j \frac{\partial r_j(Q^{1*}, Q^{3*})}{\partial q_{jk}}$ .

### 3. Qualitative Properties

In this section, we provide some qualitative properties of the solution to variational inequality (29) (equivalently, variational inequality (30)). In particular, we derive existence and uniqueness results. We also investigate properties of the function  $F$  (cf. (30)) that enters the variational inequality of interest here.

Since the feasible set is not compact we cannot derive existence simply from the assumption of continuity of the functions. Nevertheless, we can impose a rather weak condition to guarantee existence of a solution pattern.

Let

$$\mathcal{K}_b = \{(Q^1, Q^2, Q^3, \rho_2, \rho_3) | 0 \leq Q^l \leq b_l, l = 1, 2, 3; 0 \leq \rho_2 \leq b_4; 0 \leq \rho_3 \leq b_5\}, \quad (31)$$

where  $b = (b_1, \dots, b_5) \geq 0$  and  $Q^l \leq b_l; \rho_2 \leq b_4; \rho_3 \leq b_5$  means that  $q_{ij} \leq b_1$ ,  $q_{ik} \leq b_2$ ,  $q_{jk} \leq b_3$ , and  $\rho_{2j} \leq b_4$ ,  $\rho_{3k} \leq b_5$  for all  $i, j, k$ . Then  $\mathcal{K}_b$  is a bounded closed convex subset of  $R^{mn+mo+no+n+o}$ . Thus, the following variational inequality

$$\langle F(X^b), X - X^b \rangle \geq 0, \quad \forall X^b \in \mathcal{K}_b, \quad (32)$$

admits at least one solution  $X^b \in \mathcal{K}_b$ , from the standard theory of variational inequalities, since  $\mathcal{K}_b$  is compact and  $F$  is continuous. Following Kinderlehrer and Stampacchia (1980) (see also Theorem 1.5 in Nagurney (1999)), we then have:

#### Theorem 2

*Variational inequality (29) admits a solution if and only if there exists a  $b > 0$ , such that variational inequality (32) admits a solution in  $\mathcal{K}_b$  with*

$$Q^{1b} < b_1, \quad Q^{2b} < b_2, \quad Q^{3b} < b_3, \quad \rho_2^b < b_4, \quad \rho_3 < b_5. \quad (33)$$

#### Theorem 3: Existence

*Suppose that there exist positive constants  $M, N, R$  with  $R > 0$ , such that:*

$$\frac{\partial f_i(Q^1, Q^2)}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij})}{\partial q_{ij}} + \frac{\partial c_j(Q^1, Q^3)}{\partial q_{ij}} + \alpha_i \frac{\partial r_i(Q^1, Q^2)}{\partial q_{ij}} + \beta_j \frac{\partial r_j(Q^1, Q^3)}{\partial q_{ij}} \geq M,$$

$$\begin{aligned} & \forall Q^1 \text{ with } q_{ij} \geq N, \quad \forall i, j, & (34a) \\ & \frac{\partial f_i(Q^1, Q^2)}{\partial q_{ik}} + \frac{\partial c_{ik}(q_{ik})}{\partial q_{ik}} + \frac{\partial c_k(Q^2, Q^3)}{\partial q_{jk}} + \alpha_i \frac{\partial r_i(Q^1, Q^2)}{\partial q_{ik}} \\ & + \lambda_k^+ P_k(s_k, \rho_{3k}) - (\lambda_k^- + \rho_{3k})(1 - P_k(s_k, \rho_{3k})) + \beta_j \frac{\partial r_j(Q^1, Q^3)}{\partial q_{jk}} \geq M, \end{aligned}$$

$$\begin{aligned} & \forall Q^2 \text{ with } q_{ik} \geq N, \quad \forall i, k, & (34b) \\ & \lambda_k^+ P_k(s_k, \rho_{3k}) - (\lambda_k^- + \rho_{3k})(1 - P_k(s_k, \rho_{3k})) + \frac{\partial c_j(Q^1, Q^3)}{\partial q_{jk}} + \frac{\partial c_k(Q^2, Q^3)}{\partial q_{jk}} \geq M, \end{aligned}$$

$$\forall Q^3 \text{ with } q_{jk} \geq N, \quad \forall j, k, \quad (34c)$$

and

$$d_k(\rho_{3k}) \leq N, \quad \forall \rho_3 \text{ with } \rho_{3k} \geq R, \quad \forall k. \quad (35)$$

Then, variational inequality (30) admits at least one solution.

**Proof:** Follows using analogous arguments as the proof of existence for Proposition 1 in Nagurney and Zhao (1993) (see also existence proof in Nagurney, Dong, and Zhang (2002)).

□

Assumptions (34a), (34b), (34c) and (35) can be economically justified as follows. In particular, when the product shipment,  $q_{ij}$ , between manufacturer  $i$  and distributor  $j$ , and the shipment,  $q_{ik}$ , between manufacturer  $i$  and retailer  $k$ , are large, one can expect the corresponding sum of the marginal costs associated with the production, transaction, and holding plus the weighted marginal risk to exceed a positive lower bound, say  $M$ . At the same time, the large  $q_{ij}$  and  $q_{ik}$  causes a greater  $s_k$ , which in turn causes the probability distribution  $P_k(s_k, \rho_{3k})$  to be close to 1. Consequently, the sum of the middle two terms on the left-hand side of (34b),  $\lambda_k^+ P_k(s_k, \rho_{3k}) - (\lambda_k^- + \rho_{3k})(1 - P_k(s_k, \rho_{3k}))$  is seen to be positive. Therefore, the left-hand sides of (34b) and (34c), respectively, are greater than or equal to the lower bound  $M$ . On the other hand, a high price  $\rho_{3k}$  at retailer  $k$  will drive the demand at that retailer down, in line with the decreasing nature of any demand function, which ensures (35).

We now recall the concept of additive production cost, which was introduced by Zhang and Nagurney (1996) in the stability analysis of dynamic spatial oligopolies, and has also

been employed in the qualitative analysis of supply chains by Nagurney, Dong, and Zhang (2002).

**Definition 2: Additive Production Cost**

Suppose that for each manufacturer  $i$ , the production cost  $f_i$  is additive, that is

$$f_i(q) = f_i^1(q_i) + f_i^2(\bar{q}_i), \tag{36}$$

where  $f_i^1(q_i)$  is the internal production cost that depends solely on the manufacturer's own output level  $q_i$ , which may include the production operation and the facility maintenance, etc., and  $f_i^2(\bar{q}_i)$  is the interdependent part of the production cost that is a function of all the other manufacturers' output levels  $\bar{q}_i = (q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_m)$  and reflects the impact of the other manufacturers' production patterns on manufacturer  $i$ 's cost. This interdependent part of the production cost may describe the competition for the resources, consumption of the homogeneous raw materials, etc.

We now explore additional qualitative properties of the vector function  $F$  that enters the variational inequality problem. Specifically, we show that  $F$  is monotone as well as Lipschitz continuous. These properties are fundamental in establishing the convergence of the algorithmic scheme in the subsequent section.

**Lemma 1**

Let  $g_k(s_k, \rho_{3k})^T = (P_k(s_k, \rho_{3k}) - \rho_{3k}(1 - P_k(s_k, \rho_{3k})), s_k - \rho_{3k})$ , where  $P_k$  is a probability distribution with the density function of  $\mathcal{F}_k(x, \rho_{3k})$ . Then  $g_k(s_k, \rho_{3k})$  is monotone, that is,

$$\begin{aligned} & [-\rho'_{3k}(1 - P_k(s'_k, \rho'_{3k})) + \rho''_{3k}(1 - P_k(s''_k, \rho''_{3k}))] \times [q'_{jk} - q''_{jk}] \\ & + [s'_k - d_k(\rho'_{3k}) - s''_k + d_k(\rho''_{3k})] \times [\rho'_{3k} - \rho''_{3k}] \geq 0, \quad \forall (s'_k, \rho'_{3k}), (s''_k, \rho''_{3k}) \in R^2_+ \end{aligned} \tag{37}$$

if and only if  $d'_k(\rho_{3k}) \leq -(4\rho_{3k}\mathcal{F}_k)^{-1}(P_k + \rho_{3k}\frac{\partial P_k}{\partial \rho_{3k}})^2$ .

**Proof:** In order to prove that  $g_k(s_k, \rho_{3k})$  is monotone with respect to  $s_k$  and  $\rho_{3k}$ , we only need to show that its Jacobian matrix is positive semidefinite, which will be the case if all eigenvalues of the symmetric part of the Jacobian matrix are nonnegative real numbers.

The Jacobian matrix of  $g_k$  is

$$\nabla g_k(s, \rho_{3k}) = \begin{bmatrix} \rho_{3k} \mathcal{F}_k(s_k, \rho_{3k}) & -1 + P_k(s_k, \rho_{3k}) + \rho_{3k} \frac{\partial P_k(s_k, \rho_{3k})}{\partial \rho_{3k}} \\ 1 & -d'_k(\rho_{3k}) \end{bmatrix}, \quad (38)$$

and its symmetric part is

$$\frac{1}{2}[\nabla g_k(s_k, \rho_{3k}) + \nabla^T g_k(s_k, \rho_{3k})] = \begin{bmatrix} \rho_{3k} \mathcal{F}_k(s_k, \rho_{3k}), & \frac{1}{2} \left( \rho_{3k} \frac{\partial P_k}{\partial \rho_{3k}} + P_k(s_k, \rho_{3k}) \right) \\ \frac{1}{2} \left( \rho_{3k} \frac{\partial P_k}{\partial \rho_{3k}} + P_k(s_k, \rho_{3k}) \right), & -d'_k(\rho_{3k}) \end{bmatrix}. \quad (39)$$

The two eigenvalues of (39) are

$$\gamma_{min}(s_k, \rho_{3k}) = \frac{1}{2} \left[ (\rho_{3k} \mathcal{F}_k - d'_k) - \sqrt{(\rho_{3k} \mathcal{F}_k - d'_k)^2 + \left( \rho_{3k} \frac{\partial P_k}{\partial \rho_{3k}} + P_k \right)^2 + 4\rho_{3k} \mathcal{F}_k d'_k} \right], \quad (40)$$

$$\gamma_{max}(s_k, \rho_{3k}) = \frac{1}{2} \left[ (\rho_{3k} \mathcal{F}_k - d'_k) + \sqrt{(\rho_{3k} \mathcal{F}_k - d'_k)^2 + \left( \rho_{3k} \frac{\partial P_k}{\partial \rho_{3k}} + P_k \right)^2 + 4\rho_{3k} \mathcal{F}_k d'_k} \right]. \quad (41)$$

Moreover, since what is inside the square root in both (40) and (41) can be rewritten as

$$(\rho_{3k} \mathcal{F}_k + d'_k)^2 + \left( \rho_{3k} \frac{\partial P_k}{\partial \rho_{3k}} + P_k \right)^2$$

and can be seen as being nonnegative, both eigenvalues are real. Furthermore, under the condition of the lemma,  $d'_k$  is non-positive, so the first item in (40) and in (41) is nonnegative. The condition further implies that the second item in (40) and in (41), the square root part, is not greater than the first item, which guarantees that both eigenvalues are nonnegative real numbers.  $\square$

The condition of Lemma 1 states that the expected demand function of a retailer is a nonincreasing function with respect to the demand price and its first order derivative has an upper bound.

#### Theorem 4: Monotonicity

*The function that enters the variational inequality problem (30) is monotone, if the condition assumed in Lemma 1 is satisfied for each  $k$ ;  $k = 1, \dots, o$ , and if the following conditions are also satisfied.*

Suppose that the production cost functions  $f_i; i = 1, \dots, m$ , are additive, as defined in Definition 2, and that the  $f_i^1; i = 1, \dots, m$ , are convex functions. If the  $c_{ij}, c_{ik}, c_k, c_j, r_i$  and  $r_j$  functions are convex, for all  $i, j, k$ , then the vector function  $F$  that enters the variational inequality (30) is monotone, that is,

$$\langle F(X') - F(X''), X' - X'' \rangle \geq 0, \quad \forall X', X'' \in \mathcal{K}. \quad (42)$$

**Proof:** Let  $X' = (Q^{1'}, Q^{2'}, Q^{3'}, \rho'_2, \rho'_3)$ ,  $X'' = (Q^{1''}, Q^{2''}, Q^{3''}, \rho''_2, \rho''_3)$ . Then, inequality (42) can be seen in the following deduction:

$$\begin{aligned} & \langle F(X') - F(X''), X' - X'' \rangle \\ &= \sum_{i=1}^m \sum_{j=1}^n \left[ \frac{\partial f_i(Q^{1'}, Q^{2'})}{\partial q_{ij}} - \frac{\partial f_i(Q^{1''}, Q^{2''})}{\partial q_{ij}} \right] \times [q'_{ij} - q''_{ij}] \\ &+ \sum_{i=1}^m \sum_{j=1}^n \left[ \frac{\partial c_j(Q^{1'}, Q^{3'})}{\partial q_{ij}} - \frac{\partial c_j(Q^{1''}, Q^{3''})}{\partial q_{ij}} \right] \times [q'_{ij} - q''_{ij}] \\ &\quad + \sum_{i=1}^m \sum_{j=1}^n \left[ \frac{\partial c_{ij}(q'_{ij})}{\partial q_{ij}} - \frac{\partial c_{ij}(q''_{ij})}{\partial q_{ij}} \right] \times [q'_{ij} - q''_{ij}] \\ &+ \sum_{i=1}^m \sum_{j=1}^n \alpha_i \left[ \frac{\partial r_i(Q^{1'}, Q^{2'})}{\partial q_{ij}} - \frac{\partial r_i(Q^{1''}, Q^{2''})}{\partial q_{ij}} \right] \times [q'_{ij} - q''_{ij}] \\ &+ \sum_{i=1}^m \sum_{j=1}^n \beta_j \left[ \frac{\partial r_j(Q^{1'}, Q^{3'})}{\partial q_{ij}} - \frac{\partial r_j(Q^{1''}, Q^{3''})}{\partial q_{ij}} \right] \times [q'_{ij} - q''_{ij}] \\ &+ \sum_{i=1}^m \sum_{k=1}^o \left[ \frac{\partial f_i(Q^{1'}, Q^{2'})}{\partial q_{ik}} - \frac{\partial f_i(Q^{1''}, Q^{2''})}{\partial q_{ik}} \right] \times [q'_{ik} - q''_{ik}] \\ &+ \sum_{i=1}^m \sum_{k=1}^o \left[ \frac{\partial c_k(Q^{2'}, Q^{3'})}{\partial q_{ik}} - \frac{\partial c_k(Q^{2''}, Q^{3''})}{\partial q_{ik}} \right] \times [q'_{ik} - q''_{ik}] \\ &\quad + \sum_{i=1}^m \sum_{k=1}^o \left[ \frac{\partial c_{ik}(q'_{ik})}{\partial q_{ik}} - \frac{\partial c_{ik}(q''_{ik})}{\partial q_{ik}} \right] \times [q'_{ik} - q''_{ik}] \\ &+ \sum_{i=1}^m \sum_{k=1}^o [\lambda_k^+ P_k(s'_k, \rho'_{3k}) - \lambda_k^+ P_k(s''_k, \rho''_{3k})] \times [q'_{ik} - q''_{ik}] \end{aligned}$$



$$\begin{aligned}
& + \sum_{i=1}^m \sum_{k=1}^o \alpha_i \left[ \frac{\partial r_i(Q^{1'}, Q^{2'})}{\partial q_{ik}} - \frac{\partial r_i(Q^{1''}, Q^{2''})}{\partial q_{ik}} \right] \times [q'_{ik} - q''_{ik}] \\
& + \sum_{i=1}^m \sum_{k=1}^o [-\lambda_k^-(1 - P_k(s'_k, \rho'_{3k})) + \lambda_k^-(1 - P_k(s''_k, \rho''_{3k}))] \times [q'_{ik} - q''_{ik}] \\
& + \sum_{i=1}^m \sum_{k=1}^o [-\rho'_{3k}(1 - P_k(s'_k, \rho'_{3k})) + \rho''_{3k}(1 - P_k(s''_k, \rho''_{3k}))] \times [q'_{ik} - q''_{ik}] \\
& + \sum_{j=1}^n \sum_{k=1}^o \left[ \frac{\partial c_j(Q^{1'}, Q^{3'})}{\partial q_{jk}} - \frac{\partial c_j(Q^{1''}, Q^{3''})}{\partial q_{jk}} \right] \times [q'_{jk} - q''_{jk}] \\
& + \sum_{j=1}^n \sum_{k=1}^o \left[ \frac{\partial c_k(Q^{2'}, Q^{3'})}{\partial q_{jk}} - \frac{\partial c_k(Q^{2''}, Q^{3''})}{\partial q_{jk}} \right] \times [q'_{jk} - q''_{jk}] \\
& + \sum_{j=1}^n \sum_{k=1}^o \beta_j \left[ \frac{\partial r_j(Q^{1'}, Q^{3'})}{\partial q_{jk}} - \frac{\partial r_j(Q^{1''}, Q^{3''})}{\partial q_{jk}} \right] \times [q'_{jk} - q''_{jk}] \\
& + \sum_{j=1}^n \sum_{k=1}^o [\lambda_k^+ P_k(s'_k, \rho'_{3k}) - \lambda_k^+ P_k(s''_k, \rho''_{3k})] \times [q'_{jk} - q''_{jk}] \\
& + \sum_{j=1}^m \sum_{k=1}^o [-\lambda_k^-(1 - P_k(s'_k, \rho'_{3k})) + \lambda_k^-(1 - P_k(s''_k, \rho''_{3k}))] \times [q'_{jk} - q''_{jk}] \\
& + \sum_{j=1}^n \sum_{k=1}^o [-\rho'_{3k}(1 - P_k(s'_k, \rho'_{3k})) + \rho''_{3k}(1 - P_k(s''_k, \rho''_{3k}))] \times [q'_{jk} - q''_{jk}] \\
& + \sum_{k=1}^o [s'_k - d_k(\rho'_{3k}) - s''_k + d_k(\rho''_{3k})] \times [\rho'_{3k} - \rho''_{3k}] \\
& = (I) + (II) + (III) + \cdots + (XV) + \cdots + (XIX). \tag{43}
\end{aligned}$$

Since the  $f_i$ ;  $i = 1, \dots, m$ , are additive, and the  $f_i^1$ ;  $i = 1, \dots, m$ , are convex functions, one has

$$\begin{aligned}
(I) + (VI) & = \sum_{i=1}^m \sum_{j=1}^n \left[ \frac{\partial f_i^1(Q^{1'}, Q^{2'})}{\partial q_{ij}} - \frac{\partial f_i^1(Q^{1''}, Q^{2''})}{\partial q_{ij}} \right] \times [q'_{ij} - q''_{ij}] \\
& + \sum_{k=1}^o \left[ \frac{\partial f_i(Q^{1'}, Q^{2'})}{\partial q_{ik}} - \frac{\partial f_i(Q^{1''}, Q^{2''})}{\partial q_{ik}} \right] \times [q'_{ik} - q''_{ik}] \geq 0. \tag{44}
\end{aligned}$$

The convexity of  $c_j, \forall j$ ;  $c_{ij}, \forall i, j$ ;  $c_k, \forall k$ ;  $c_{ik}, \forall i, k$ ;  $r_j, \forall j$ ; and  $r_i, \forall i$ , gives, respectively,

$$(II) + (XIII) = \sum_{j=1}^n \sum_{i=1}^m \left[ \frac{\partial c_j(Q^{1'}, Q^{3'})}{\partial q_{ij}} - \frac{\partial c_j(Q^{1''}, Q^{3''})}{\partial q_{ij}} \right] \times [q'_{ij} - q''_{ij}]$$

$$+ \sum_{k=1}^o \left[ \frac{\partial c_j(Q^{1'}, Q^{3'})}{\partial q_{jk}} - \frac{\partial c_j(Q^{1''}, Q^{3''})}{\partial q_{jk}} \right] \times [q'_{jk} - q''_{jk}] \geq 0 \quad (45)$$

$$(III) = \sum_{i=1}^m \sum_{j=1}^n \left[ \frac{\partial c_{ij}(q'_{ij})}{\partial q_{ij}} - \frac{\partial c_{ij}(q''_{ij})}{\partial q_{ij}} \right] \times [q'_{ij} - q''_{ij}] \geq 0 \quad (46)$$

$$(VII) + (XIV) = \sum_{k=1}^o \left\{ \sum_{i=1}^m \left[ \frac{\partial c_k(Q^{2'}, Q^{3'})}{\partial q_{ik}} - \frac{\partial c_k(Q^{2''}, Q^{3''})}{\partial q_{ik}} \right] \times [q'_{ik} - q''_{ik}] \right. \\ \left. + \sum_{j=1}^n \left[ \frac{\partial c_k(Q^{2'}, Q^{3'})}{\partial q_{jk}} - \frac{\partial c_k(Q^{2''}, Q^{3''})}{\partial q_{jk}} \right] \times [q'_{jk} - q''_{jk}] \right\} \geq 0 \quad (47)$$

$$(VIII) = \sum_{i=1}^m \sum_{k=1}^o \left[ \frac{\partial c_{ik}(q'_{ik})}{\partial q_{ik}} - \frac{\partial c_{ik}(q''_{ik})}{\partial q_{ik}} \right] \times [q'_{ik} - q''_{ik}] \geq 0 \quad (48)$$

$$(IV) + (IX) = \sum_{i=1}^m \alpha_i \left\{ \sum_{j=1}^n \left[ \frac{\partial r_i(Q^{1'}, Q^{2'})}{\partial q_{ij}} - \frac{\partial r_i(Q^{1''}, Q^{2''})}{\partial q_{ij}} \right] \times [q'_{ij} - q''_{ij}] \right. \\ \left. + \sum_{k=1}^o \left[ \frac{\partial r_i(Q^{1'}, Q^{2'})}{\partial q_{ik}} - \frac{\partial r_i(Q^{1''}, Q^{2''})}{\partial q_{ik}} \right] \times [q'_{ik} - q''_{ik}] \right\} \geq 0 \quad (49)$$

$$(V) + (XV) = \sum_{j=1}^n \beta_j \left\{ \sum_{i=1}^m \left[ \frac{\partial r_j(Q^{1'}, Q^{3'})}{\partial q_{ij}} - \frac{\partial r_j(Q^{1''}, Q^{3''})}{\partial q_{ij}} \right] \times [q'_{ij} - q''_{ij}] \right. \\ \left. + \sum_{k=1}^o \left[ \frac{\partial r_j(Q^{1'}, Q^{3'})}{\partial q_{jk}} - \frac{\partial r_j(Q^{1''}, Q^{3''})}{\partial q_{jk}} \right] \times [q'_{jk} - q''_{jk}] \right\} \geq 0. \quad (50)$$

Since the probability function  $P_k$  is an increasing function w.r.t.  $s_k$ , for all  $k$ , and  $s_k = \sum_{i=1}^m q_{ik} + \sum_{j=1}^n q_{jk}$ , hence, we have the followings.

$$(X) + (XVI) = \sum_{k=1}^o [\lambda_k^+ P_k(s'_k, \rho'_{3k}) - \lambda_k^+ P_k(s''_k, \rho''_{3k})] \times [s'_k - s''_k] \geq 0 \quad (51)$$

$$(XI) + (XVII) = \sum_{k=1}^o [-\lambda_k^- (1 - P_k(s'_k, \rho'_{3k})) + \lambda_k^- (1 - P_k(s''_k, \rho''_{3k}))] \times [s'_k - s''_k] \geq 0. \quad (52)$$

Since for each  $k$ , applying Lemma 1, we can see that  $g_k(s_k, \rho_{3k})$  is monotone, hence, we have:

$$(XII) + (XVIII) + (XIX) = \sum_{k=1}^o [-\rho'_{3k} (1 - P_k(s'_k, \rho'_{3k})) + \rho''_{3k} (1 - P_k(s''_k, \rho''_{3k}))] \times [s'_k - s''_k]$$

$$+ \sum_{k=1}^o [s'_k - d_k(\rho'_{3k}) - s''_k + d_k(\rho''_{3k})] \times [\rho'_{3k} - \rho''_{3k}] \geq 0. \quad (53)$$

Therefore, we conclude that (42) is nonnegative in  $\mathcal{K}$ . The proof is complete.  $\square$

### Theorem 5: Strict Monotonicity

The function that enters the variational inequality problem (30) is strictly monotone, if the conditions mentioned in Lemma 1 for  $g_k(s_k, \rho_{3k})$  are satisfied strictly for all  $k$  and if the following conditions are also satisfied.

Suppose that the production cost functions  $f_i; i = 1, \dots, m$ , are additive, as defined in Definition 2, and that the  $f_i^1; i = 1, \dots, m$ , are strictly convex functions. If the  $c_{ij}, c_{ik}, c_k, c_j, r_i$  and  $r_j$  functions are strictly convex, for all  $i, j, k$ , then the vector function  $F$  that enters the variational inequality (30) is strictly monotone, that is,

$$\langle F(X') - F(X''), X' - X'' \rangle > 0, \quad \forall X', X'' \in \mathcal{K}. \quad (54)$$

### Theorem 6: Uniqueness

Under the conditions indicated in Theorem 5, the function that enters the variational inequality (30) has a unique solution in  $\mathcal{K}$ .

From Theorem 6 it follows that, under the above conditions, the equilibrium product shipment pattern between the manufacturers and the retailers, as well as the equilibrium price pattern at the retailers, is unique.

### Theorem 7: Lipschitz Continuity

The function that enters the variational inequality problem (30) is Lipschitz continuous, that is,

$$\|F(X') - F(X'')\| \leq L\|X' - X''\|, \quad \forall X', X'' \in \mathcal{K}, \text{ with } L > 0, \quad (55)$$

under the following conditions:

(i). Each  $f_i; i = 1, \dots, m$ , is additive and has a bounded second order derivative;

(ii). The  $c_{ij}$ ,  $c_{ik}$ ,  $c_k$ ,  $c_j$ ,  $r_i$  and  $r_j$  have bounded second order derivatives, for all  $i, j, k$ ;

**Proof:** Since the probability function  $P_k$  is always less than or equal to 1, for each retailer  $k$ , the result is direct by applying a mid-value theorem from calculus to the vector function  $F$  that enters the variational inequality problem (30).  $\square$

## 4. The Algorithm

In this section, an algorithm is presented which can be applied to solve any variational inequality problem in standard form (see (30)), that is:

Determine  $X^* \in \mathcal{K}$ , satisfying:

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}. \quad (56)$$

The algorithm is guaranteed to converge provided that the function  $F$  that enters the variational inequality is monotone and Lipschitz continuous (and that a solution exists). The algorithm is the modified projection method of Korpelevich (1977).

The statement of the modified projection method is as follows, where  $\mathcal{T}$  denotes an iteration counter:

### Modified Projection Method

#### Step 0: Initialization

Set  $X^0 \in \mathcal{K}$ . Let  $\mathcal{T} = 1$  and let  $a$  be a scalar such that  $0 < a \leq \frac{1}{L}$ , where  $L$  is the Lipschitz continuity constant (cf. Korpelevich (1977)) (see (55)).

#### Step 1: Computation

Compute  $\bar{X}^{\mathcal{T}}$  by solving the variational inequality subproblem:

$$\langle \bar{X}^{\mathcal{T}} + aF(X^{\mathcal{T}-1}) - X^{\mathcal{T}-1}, X - \bar{X}^{\mathcal{T}} \rangle \geq 0, \quad \forall X \in \mathcal{K}. \quad (57)$$

#### Step 2: Adaptation

Compute  $X^{\mathcal{T}}$  by solving the variational inequality subproblem:

$$\langle X^{\mathcal{T}} + aF(\bar{X}^{\mathcal{T}}) - X^{\mathcal{T}-1}, X - X^{\mathcal{T}} \rangle \geq 0, \quad \forall X \in \mathcal{K}. \quad (58)$$

### Step 3: Convergence Verification

If  $\max |X_l^{\mathcal{T}} - X_l^{\mathcal{T}-1}| \leq \epsilon$ , for all  $l$ , with  $\epsilon > 0$ , a prespecified tolerance, then stop; else, set  $\mathcal{T} =: \mathcal{T} + 1$ , and go to Step 1.

We now state the convergence result for the modified projection method for this model.

### Theorem 8: Convergence

*Assume that the function that enters the variational inequality (29) (or (30)) has at least one solution and satisfies the conditions in Theorem 4 and in Theorem 7. Then the modified projection method described above converges to the solution of the variational inequality (29) or (30).*

**Proof:** According to Korpelevich (1977), the modified projection method converges to the solution of the variational inequality problem of the form (30), provided that the function  $F$  that enters the variational inequality is monotone and Lipschitz continuous and that a solution exists. Existence of a solution follows from Theorem 3. Monotonicity follows Theorem 5. Lipschitz continuity, in turn, follows from Theorem 7.  $\square$

We emphasize that, in view of the fact that the feasible set  $\mathcal{K}$  underlying the supply chain network model with supply and demand side risk is the nonnegative orthant, the projection operation encountered in (57) and (58) takes on a very simple form for computational purposes. Indeed, the product shipments as well as the product prices at a given iteration in both (57) and in (58) can be exactly and computed in closed form.

## 5. Numerical Examples

In this section, we apply the modified projection method to six numerical examples. The algorithm was implemented in FORTRAN and the computer system used was a DEC Alpha system located at the University of Massachusetts at Amherst. The convergence criterion used was that the absolute value of the product shipments and prices between two successive iterations differed by no more than  $10^{-4}$ . The parameter  $a$  in the modified projection method (see (57) and (58)) was set to .01 for all the examples.

In all the examples, we assumed that the demands associated with the retail outlets followed a uniform distribution. Hence, we assumed that the random demand,  $\hat{d}_k(\rho_{3k})$ , of retailer  $k$ , is uniformly distributed in  $[0, \frac{b_k}{\rho_{3k}}]$ ,  $b_k > 0$ ;  $k = 1, \dots, o$ . Therefore,

$$P_k(x, \rho_{3k}) = \frac{x\rho_{3k}}{b_k}, \quad (59)$$

$$\mathcal{F}_k(x, \rho_{3k}) = \frac{\rho_{3k}}{b_k}, \quad (60)$$

$$d_k(\rho_{3k}) = E(\hat{d}_k) = \frac{1}{2} \frac{b_k}{\rho_{3k}}; \quad k = 1, \dots, o. \quad (61)$$

It is easy to verify that the expected demand function  $d_k(\rho_{3k})$  associated with retailer  $k$  is a decreasing function of the price at the demand market.

The modified projection method was initialized as follows: all variables were set to zero, except for the initial retail prices  $\rho_{3k}$  which were set to 1 for all retailers  $k$ .

The six examples solved consisted of two manufacturers, two distributors, and two retailers, as depicted in Figure 2.

### Example 1a

The data for this example were constructed for easy interpretation purposes. The production cost functions for the manufacturers were given by:

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 2q_1, \quad f_2(q) = 2.5q_2^2 + q_1q_2 + 2q_2.$$

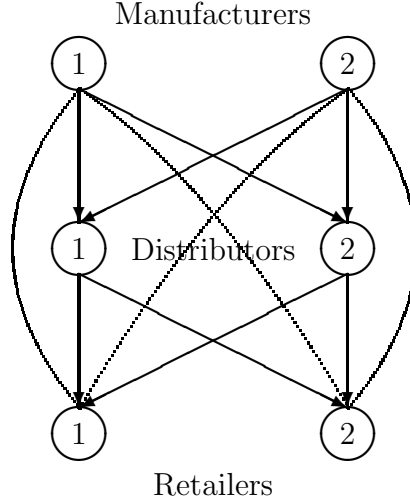


Figure 2: Supply Chain Network Structure for the Numerical Examples

The transaction cost functions faced by the manufacturers and associated with transacting with the distributors were given by:

$$c_{ij}(q_{ij}) = .5q_{ij}^2 + 3.5q_{ij}, \quad \text{for } i = 1, 2; j = 1, 2.$$

The transaction cost functions faced by the manufacturers but associated with transacting electronically with the retailers were given by:

$$c_{ik}(q_{ik}) = .5q_{ik}^2 + 5q_{ik}, \quad \text{for } i = 1, 2; k = 1, 2.$$

The handling costs of the distributors, in turn, were given by:

$$c_j(Q^1, Q^3) = .5\left(\sum_{i=1}^2 q_{ij}\right)^2, \quad \text{for } j = 1, 2,$$

whereas the handling costs of the retailers were given by:

$$c_k(Q^2, Q^3) = .5\left(\sum_{j=1}^2 q_{jk}\right)^2, \quad \text{for } k = 1, 2.$$



The  $b_k$ s were set to 100 for both retailers yielding probability distribution functions as in (59) and the expected demand functions as in (61). The weights associated with the excess supply and excess demand at the retailers were:  $\lambda_k^+ = \lambda_k^- = 1$  for  $k = 1, 2$ . Hence, we assigned equal weights for each retailer for excess supply and for excess demand.

Hence, in this example, the manufacturers and the distributors were only concerned about profit maximization and not with risk minimization. In terms of the model, this would correspond to (cf. (29)) the weights  $\alpha_i$  and  $\beta_j$  for  $i = 1, 2$ ;  $j = 1, 2$  equal to zero.

The modified projection method converged and yielded the following equilibrium pattern: the product shipments between the two manufacturers and the two distributors were:  $q_{ij}^* = .3697$  for  $i = 1, 2$ ;  $j = 1, 2$ , whereas the product shipments transacted electronically between the manufacturers and the retailers were:  $q_{ik}^* = .3487$  for  $i = 1, 2$ ;  $k = 1, 2$ , and, finally, the product shipments between the distributors and the retailers were:  $q_{jk}^* = .3697$  for  $j = 1, 2$ ;  $k = 1, 2$ . The computed equilibrium prices, in turn, were:  $\rho_{2j}^* = 15.2301$  for  $j = 1, 2$  and  $\rho_{3k}^* = 34.5573$  for  $k = 1, 2$ . The expected demands (see (61)) were:  $d_1(\rho_{31}^*) = d_2(\rho_{32}^*) = 1.4469$ .

### Example 1b

Example 1b was constructed from Example 1a as follows. We kept the data as in Example 1a but we assumed now that the first manufacturer was a multicriteria decision-maker and also concerned with risk minimization with his risk function being given by:

$$r_1 = \left( \sum_{k=1}^2 q_{1k} - 2 \right)^2$$

and with the weight  $\alpha_1 = 2$ . This risk measure can be explained as follows. The first manufacturer is concerned with physical transactions associated with the retailers and wishes to increase directly the volume of transactions with the consumers by having the related shipments lie as close as possible to a target (of 2).

The modified projection method yielded the following new solution: the product shipments between the two manufacturers and the two distributors were:  $q_{1j}^* = 0.0000$ , for  $j = 1, 2$  and  $q_{2j}^* = .5442$  for  $j = 1, 2$ . The product shipments transacted electronically between manufacturers and the retailers were:  $q_{1k}^* = .7969$  for  $k = 1, 2$  and  $q_{2k}^* = .1327$  for

$k = 1, 2$ . Finally, the product shipments between the distributors and the retailers were:  $q_{jk}^* = .2721$  for  $j = 1, 2$  and  $k = 1, 2$ . The computed equilibrium prices were:  $\rho_{2j}^* = 14.9516$  for  $j = 1, 2$  and  $\rho_{3k}^* = 33.9690$  for  $k = 1, 2$ .

Hence, the volume of product transacted electronically between the first manufacturer and the consumer markets almost doubled. An increase in the Internet transactions was expected given that the manufacturer was concerned with the physical transactions and his risk function reflected that. The prices decreased at the distributor level since the distributors handled a reduced volume of product shipments.

### Example 2a

Example 2a was constructed from Example 1a as follows. We retained all the data as in Example 1, except that we increased  $b_1$  and  $b_2$  from 100 to 1000. This has the interpretation that the expected demand at both retailers increased.

The modified projection method converged and yielded the following equilibrium pattern: the product shipments between the two manufacturers and the two distributors were now:  $q_{ij}^* = .6974$  for  $i = 1, 2; j = 1, 2$ , whereas the product shipments transacted electronically between the manufacturers and the retailers were:  $q_{ik}^* = 1.9870$  for  $i = 1, 2; k = 1, 2$ , and, finally, the product shipments between the distributors and the retailers were now:  $q_{jk}^* = .6973$  for  $j = 1, 2; k = 1, 2$ . The computed equilibrium prices, in turn, were:  $\rho_{2j}^* = 39.8051$  for  $j = 1, 2$  and  $\rho_{3k}^* = 92.9553$  for  $k = 1, 2$ . The expected demands increased (as expected) relative to those obtained in Example 1a with  $d_1(\rho_{31}^*) = d_2(\rho_{32}^*) = 5.3789$ .

### Example 2b

This example was constructed from Example 2a as follows. The data were identical except now both manufacturers  $i = 1, 2$  were assumed to also be risk minimizers. The risk functions of the two manufacturers were:

$$r_i = \left( \sum_{k=1}^2 q_{ik} - 4 \right)^2,$$

with weights given by  $\alpha_i = 2$  for  $i = 1, 2$ . Hence, unlike Example 1b, in which just the first manufacturer was a multicriteria decision-maker, in this example, all the manufacturers were.

The modified projection method converged to the following solution. The product shipments between the two manufacturers and the two distributors were:  $q_{ij}^* = .6907$  for  $i = 1, 2$  and  $j = 1, 2$ . The product shipments transacted electronically were:  $q_{ik}^* = 1.9948$  for  $i = 1, 2$  and  $k = 1, 2$ , whereas the product shipments between the distributors and the retailers were:  $q_{jk}^* = .6906$  for  $j = 1, 2$  and  $k = 1, 2$ .

The equilibrium prices were:  $\rho_{2j}^* = 39.7978$  for  $j = 1, 2$  and  $\rho_{3k}^* = 92.9181$  for  $k = 1, 2$ . Note that the volume of products transacted electronically increased from those reported in Example 2a.

### Example 3a

Example 3a was constructed from Example 2a as follows. We retained all the data as in Example 2a, except that now we decreased the transaction costs associated with transacting electronically, where now  $c_{ik}(q_{ik}) = q_{ik} + 1$ ,  $i = 1, 2; k = 1, 2$ .

The modified projection method converged and yielded the following equilibrium pattern: the product shipments between the two manufacturers and the two distributors were now:  $q_{ij}^* = .0484$  for  $i = 1, 2; j = 1, 2$ , whereas the product shipments transacted electronically between the manufacturers and the retailers were:  $q_{ik}^* = 2.7418$  for  $i = 1, 2; k = 1, 2$ , and, finally, the product shipments between the distributors and the retailers were now:  $q_{jk}^* = .0483$  for  $j = 1, 2; k = 1, 2$ . The computed equilibrium prices, in turn, were:  $\rho_{2j}^* = 39.1269$  for  $j = 1, 2$  and  $\rho_{3k}^* = 89.4390$  for  $k = 1, 2$ . Hence, the product shipments between the manufacturers and the retailers increased and the prices at the retailers decreased (relative to those obtained in Example 2a).

### Example 3b

Example 3b had identical to that of Example 3a except that we added the following data. In this example the first distributor as a multicriteria decision-maker and concerned with both profit maximization and risk minimization. His risk function was given by

$$r_1 = \left( \sum_{k=1}^2 q_{1k} - 2 \right)^2$$

with a weight  $\beta_1 = 1$ .

The modified projection method yielded the following equilibrium solution. The product shipments between the manufacturers and the distributors were:  $q_{i1}^* = .4516$  for  $i = 1, 2$  and  $q_{i2}^* = .0128$  for  $i = 1, 2$ . The product shipments between the manufacturers and the retailers were:  $q_{ik}^* = 2.5639$  for  $i = 1, 2$  and  $k = 1, 2$ . Finally, the product shipments between the distributors and the retailers were:  $q_{11}^* = .4512, q_{12}^* = .4518$  and  $q_{21}^* = .0130, q_{22}^* = .0124$ . The equilibrium prices at the distributors' level were now:  $\rho_{21}^* = 40.4076$  and  $\rho_{22}^* = 39.0912$  whereas those at the retailers' level were:  $\rho_{3k}^* = 89.2497$  for  $k = 1, 2$ .

The above numerical results illustrate the variety of problems that can be solved.

## 6. Summary and Conclusions

This paper has developed a three-tiered supply chain network equilibrium model consisting of manufacturers, distributors, and retailers with electronic commerce. The model allows for physical transactions between the different tiers of decision-makers as well as electronic transactions in the form of B2B commerce between manufacturers and the retailers. In addition, the demands for the product associated with the retailers are no longer assumed to be known with certainty but rather, are random. Furthermore, the manufacturers as well as the distributors are assumed to be multicriteria decision-makers and concerned not only with profit maximization but also with risk minimization.

The model generalized previous supply chain network equilibrium models to include electronic commerce, multiple tiers of decision-makers as well as supply side and demand side risk within the same framework.

Finite-dimensional variational inequality theory was used to formulate the derived equilibrium conditions, to study the model qualitatively, and also to obtain convergence results for the proposed algorithmic scheme. Finally, numerical examples were presented to illustrate the model and computational procedure.

Future research will include empirical work as well as extensions of the model to global supply chain networks.

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