Supply Chain Supernetworks
with
Random Demands

June Dong and Ding Zhang
Department of Marketing and Management
School of Business
State University of New York at Oswego
Oswego, New York 13126

Anna Nagurney
Department of Finance and Operations Management
Isenberg School of Management
University of Massachusetts
Amherst, Massachusetts 01003

Abstract: In this paper, we develop a supply chain network model in the form of a supernetwork, in which both physical and electronic transactions are allowed. The model consists of three tiers of decision-makers: the manufacturers, the distributors, and the retailers, with the demands associated with the retail outlets being random. We model the optimizing behavior of the various decision-makers, derive the equilibrium conditions, and establish the finite-dimensional variational inequality formulation. We provide qualitative properties of the equilibrium pattern in terms of existence and uniqueness results and also establish conditions under which the proposed computational procedure is guaranteed to converge. We illustrate the supernetwork model through several numerical examples for which the equilibrium prices and product shipments are computed. This is the first multitiered supply chain network equilibrium model with electronic commerce and with random demands for which modeling, qualitative analysis, and computational results have been obtained.

Key Words: supply chain management, supernetworks, electronic commerce, random demands, variational inequalities

1. Introduction

The study of supply chain network problems through modeling, analysis, and computation has been an active area of research due to the complexity of the relationships among the various decision-makers, such as suppliers, manufacturers, distributors, and retailers as well as the practical importance of the topic for the efficient movement of products. The topic is multidisciplinary by nature since it involves particulars of manufacturing, transportation and logistics, retailing/marketing, as well as economics.

Recently, the introduction of electronic commerce has unveiled new opportunities in terms of research and practice in supply chain analysis and management (see, e.g., Kuglin and Rosenbaum (2001)). Notably, electronic commerce (e-commerce) has had a huge effect on the manner in which businesses order goods and have them transported with the major portion of e-commerce transactions being in the form of business-to-business (B2B). Estimates of B2B electronic commerce range from approximately .1 trillion dollars to 1 trillion dollars in 1998 and with forecasts reaching as high as $4.8 trillion dollars in 2003 in the United States (see Federal Highway Administration (2000), Southworth (2000)). Moreover, it has been emphasized by Handfield and Nichols (1999) and by the National Research Council (2000) that the principal effect of business-to-business (B2B) commerce, estimated to be 90% of all e-commerce by value and volume, is in the creation of new and more profitable supply chain networks.

In this paper, we introduce the first supernetwork supply chain model with random demands. The term *supernetwork* here refers to a network in which decision-making regarding transportation and telecommunications tradeoffs (such as those that arise in electronic commerce) are modeled in a unified fashion. This concept has been explored to-date in supply chains, in financial networks with intermediation, in transportation and location decision-making, as well as in other applications relevant to the Information Age. For an introduction to the subject, as well as numerous citations, see the book by Nagurney and Dong (2002).

In particular, in this paper, we build upon the recent work of Nagurney, Loo, Dong, and Zhang (2001) who modeled supply chain networks with electronic commerce but who assumed that all the underlying functions were known with certainty. Here, in contrast, we consider the more realistic situation in which the demands associated with the product at the retail outlets are now random. This paper also generalizes the results of Dong, Zhang, and Nagurney (2002), who considered two-tiered supply chains, consisting of manufacturers and retailers, in which the demand at the retail outlets are random. In this paper, however, we extend that earlier framework to include another tier of decision-makers consisting of distributors of the product and, significantly, we introduce e-commerce, in the form of B2B transactions, between manufacturers and retailers. Nagurney (2002), et al., in turn, considered dynamic supply chains viewed as multilevel networks, but did not consider electronic commerce nor random demands.

We emphasize that the interplay of transportation networks with telecommunication networks is a subject that has been studied in the context of other applications, notably, Intelligent Transportation Systems (see Boyce (1988a, b), Boyce, Kirson, and Schofer (1994), Ran and Boyce (1994, 1996), and the references therein). In addition, the role of such networks as the foundation of our modern economies and societies has also received attention from the regional science communities (cf. Batten, Casti, and Thord (1995) and Beckmann, et al. (1998)). In this paper, we take the perspective of Network Economics (cf. Nagurney (1999)) in order to formalize the interactions among distinct decision-makers on multitiered networks in the form of supply chains who can compete within a tier of nodes but must cooperate between tiers of nodes.

The paper is organized as follows. In Section 2, we develop the supernetwork supply chain model with random demands, derive the optimality conditions of the various decision-makers, and establish that the governing equilibrium conditions can be formulated as a finite-dimensional variational inequality problem. We emphasize here that the concept of equilibrium, first explored in a general setting for supply chains by Nagurney, Dong, and Zhang (2002), provides a valuable benchmark against which prices of the product at the various tiers of the network as well as product flows between tiers can be compared.

In Section 3, we study qualitative properties of the equilibrium pattern, and, under reasonable conditions, establish existence and uniqueness results. We also provide properties of the function that enters the variational inequality that allows us to establish convergence of the proposed algorithmic scheme in Section 4. In Section 5, we apply the algorithm to several supply chain examples for the computation of the equilibrium prices and shipments. We conclude the paper with Section 6, in which we summarize our results and present suggestions for future research.

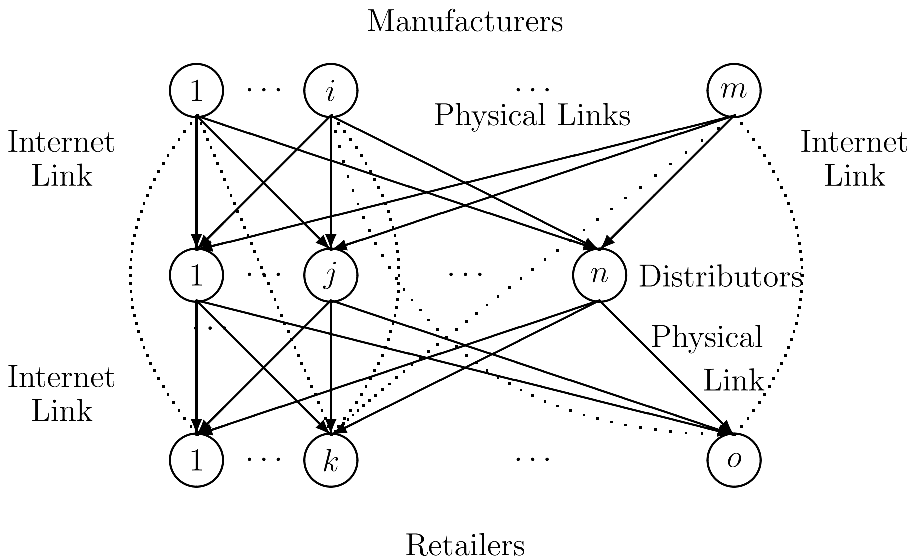

Figure 1: The Supernetwork Structure of the Supply Chain

2. The Supernetwork Supply Chain Model

In this Section, we present a supply chain supernetwork model with random demands. Specifically, we consider m manufacturers involved in the production of a homogeneous product which is then shipped to n distributors, who, in turn, ship the product to o retailers. The retailers can transact either physically (in the standard manner) with the distributors or directly, in an electronic manner, with the manufacturers.

We denote a typical manufacturer by i, a typical distributor by j, and a typical retailer by k. Note that (cf. Figure 1) the manufacturers are located at the top tier of nodes of the network, the distributors at the middle tier, and the retailers at the third or bottom tier of nodes. The links in the supply chain supernetwork in Figure 1 include classical physical links as well as Internet links to allow for e-commerce, with the Internet links denoted by dotted arcs. The introduction of e-commerce allows for "connections" that were, heretofore, not possible, such as, for example, those enabling retailers to purchase a product directly from the manufacturers.

The behavior of the various network decision-makers represented by the three tiers of nodes in Figure 1 is now described. We first focus on the manufacturers. We then turn to the distributors, and, subsequently, to the retailers.

The Behavior of the Manufacturers and their Optimality Conditions

Let q_i denote the nonnegative production output of manufacturer i. Group the production outputs of all manufacturers into the column vector $q \in R^m_+$. Here it is assumed that each manufacturer i is faced with a production cost function f_i, which can depend, in general, on the entire vector of production outputs, that is,

$$f_i = f_i(q), \quad \forall i. \tag{1}$$

Hence, the production cost of a particular manufacturer can depend not only on his production output but also on those of the other manufacturers. This allows one to model competition.

The transaction cost associated with manufacturer i transacting with distributor j is denoted by c_{ij}. The product shipment between manufacturer i and distributor j is denoted by q_{ij}. The product shipments between all pairs of manufacturers and distributors are grouped into the column vector $Q^1 \in R^{mn}_+$. In addition, a manufacturer i may transact directly with the retailer k with this transaction cost associated with the Internet transaction begin denoted by c_{ik} and the associated product shipment from manufacturer i to retailer k by q_{ik}. We group these product shipments into the column vector $Q^2 \in R^{mo}_+$.

The transaction cost (which we assume includes the cost of transportation) between a manufacturer and distributor pair and the transaction cost (which also includes the transportation cost) between a manufacturer and retailer may depend upon the volume of transactions between each such pair, and are given, respectively, by:

$$c_{ij} = c_{ij}(q_{ij}), \quad \forall i, j, \tag{2a}$$

and

$$c_{ik} = c_{ik}(q_{ik}), \quad \forall i, k. \tag{2b}$$

The quantity produced by manufacturer i must satisfy the following conservation of flow equation:

$$q_i = \sum_{j=1}^{n} q_{ij} + \sum_{k=1}^{o} q_{ik}, \tag{3}$$

which states that the quantity produced by manufacturer i is equal to the sum of the quantities shipped from the manufacturer to all distributors and to all retailers.

The total costs incurred by a manufacturer i, thus, are equal to the sum of the manufacturer's production cost plus the total transaction costs. His revenue, in turn, is equal to the price that the manufacturer charges for the product times the total quantity obtained/purchased of the product from the manufacturer by all the distributors and all the retailers. Let ρ_{1ij}^* denote the price charged for the product by manufacturer i to distributor j who has transacted, and let ρ_{1ik}^* denote the price charged by manufacturer i for the product to the retailer k. Hence, manufacturers can price according to their locations, as to whether the product is sold to the distributor or to the retailers directly, and according to whether the transaction was conducted via the Internet or not. How these prices are arrived at is discussed later in this section.

Noting the conservation of flow equations (3) and the production cost functions (1), one can express the criterion of profit maximization for manufacturer i as:

$$\text{Maximize} \sum_{j=1}^{n} \rho_{1ij}^* q_{ij} + \sum_{k=1}^{o} \rho_{1ik}^* q_{ik} - f_i(Q^1, Q^2) - \sum_{j=1}^{n} c_{ij}(q_{ij}) - \sum_{k=1}^{o} c_{ik}(q_{ik}), \tag{4}$$

subject to $q_{ij} \geq 0$, for all j, and $q_{ik} \geq 0$, for all k.

The manufacturers are assumed to compete in a noncooperative fashion. Also, it is assumed that the production cost functions and the transaction cost functions for each manufacturer are continuous and convex. The governing optimization/equilibrium concept underlying noncooperative behavior is that of Nash (1950, 1951), which states, in this context, that each manufacturer will determine his optimal production quantity and shipments, given the optimal ones of the competitors. Hence, the optimality conditions for all manufacturers *simultaneously* can be expressed as the following inequality (see also Gabay and Moulin (1980), Dafermos and Nagurney (1987), Bazarra, Sherali, and Shetty (1993), and

Nagurney (1999)): determine the solution $(Q^{1^*}, Q^{2^*}) \in R_+^{mn+mo}$, which satisfies:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(Q^{1^*}, Q^{2^*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} - \rho_{1ij}^*\right] \times \left[q_{ij} - q_{ij}^*\right]$$

$$+\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\frac{\partial f_i(Q^{1^*}, Q^{2^*})}{\partial q_{ik}} + \frac{\partial c_{ik}(q_{ik}^*)}{\partial q_{ik}} - \rho_{1ik}^*\right] \times [q_{ik} - q_{ik}^*] \geq 0, \quad \forall (Q^1, Q^2) \in R_+^{mn+mo}. \quad (5)$$

The inequality (5), which is a *variational inequality* has a nice economic interpretation. In particular, from the first term one can infer that, if there is a positive shipment of the product transacted between manufacturer and a distributor, then the marginal cost of production plus the marginal cost of transacting must be equal to the price that the distributor is willing to pay for the product. If the marginal cost of production plus the marginal cost of transacting exceeds that price, then there will be zero volume of flow of the product between the two. The second term in (5) has a similar interpretation; in particular, there will be a positive volume of flow of the product from a manufacturer to a retailer if the marginal cost of production of the manufacturer plus the marginal cost of transacting with the retailer via the Internet is equal to the price the retailers are willing to pay for the product.

The Behavior of the Distributors and their Optimality Conditions

The distributors, in turn, are involved in transactions both with the manufacturers since they wish to obtain the product for their inventory, as well as with the retailers. Thus, a distributor conducts transactions both with the manufacturers as well as with the retailers.

Let q_{jk} denote the amount of the product purchased by retailer k from distributor j. Group these shipment quantities into the column vector $Q^3 \in R_+^{no}$.

A distributor j is faced with what is termed a *handling* cost, which may include, for example, the loading/unloading and storage costs associated with the product. Denote this cost by c_j and, in the simplest case, one would have that c_j is a function of $\sum_{i=1}^{m} q_{ij}$ and $\sum_{k=1}^{o} q_{jk}$ that is, the inventory cost of a distributor is a function of how much of the product he has obtained from the various manufacturers and how much of the product he has shipped out to the various retailers. However, for the sake of generality, and to enhance the modeling of competition, allow the function to, in general, depend also on the amounts of the product held by other distributors. Therefore, one may write:

$$c_j = c_j(Q^1, Q^3), \quad \forall j. \tag{6}$$

Distributor j associates a price with the product, which is denoted by γ_j^*. This price, as will be shown, will also be endogenously determined in the model and will be, given a positive volume of flow between a distributor and any retailer, equal to a clearing-type price. Assuming, that the distributors are also profit-maximizers, the optimization problem of a distributor j is given by:

$$\text{Maximize} \quad \gamma_j^* \sum_{k=1}^{o} q_{jk} - c_j(Q^1, Q^3) - \sum_{i=1}^{m} \rho_{1ij}^* q_{ij} \tag{7}$$

subject to:

$$\sum_{k=1}^{o} q_{jk} \leq \sum_{i=1}^{m} q_{ij}, \tag{8}$$

and the nonnegativity constraints: $q_{ij} \geq 0$, and $q_{jk} \geq 0$, for all i, and k. Objective function (7) expresses that the difference between the revenues and the handling cost and the payout to the manufacturers should be maximized. Constraint (8) expresses that the retailers cannot purchase more from a distributor than is held in stock.

The optimality conditions of the distributors are now obtained, assuming that each distributor is faced with the optimization problem (7), subject to (8), and the nonnegativity assumption on the variables. Here it is also assumed that the distributors compete in a noncooperative manner so that each maximizes his profits, given the actions of the other distributors. Note that, at this point, we consider that distributors seek to determine not only the optimal amounts purchased by the retailers, but, also, the amount that they wish to obtain from the manufacturers. In equilibrium, all the shipments between the tiers of network decision-makers will have to coincide.

Assuming that the handling cost for each distributor is continuous and convex as are the transaction costs, the optimal $(Q^{1*}, Q^{3*}, \rho_2^*) \in R_+^{mn+no+n}$ satisfy the optimality conditions for all the distributors or, equivalently, the variational inequality:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial c_j(Q^{1*}, Q^{3*})}{\partial q_{ij}} + \rho_{1ij}^* - \rho_{2j}^* \right] \times \left[q_{ij} - q_{ij}^* \right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[-\gamma_j^*+\frac{\partial c_j(Q^{1*},Q^{3*})}{\partial q_{jk}}+\rho_{2j}^*\right]\times\left[q_{jk}-q_{jk}^*\right]+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}q_{ij}^*-\sum_{k=1}^{o}q_{jk}^*\right]\times\left[\rho_{2j}-\rho_{2j}^*\right]\geq 0,$$

$$\forall Q^1\in R_+^{mn},\forall Q^3\in R_+^{no},\forall\rho_2\in R_+^n, \tag{9}$$

where ρ_{2j} is the Lagrange multiplier associated with constraint (8) for distributor j and ρ_2 is the column vector of all the distributors' multipliers. In this derivation, as in the derivation of inequality (5), the prices charged were not variables. The $\gamma^*=(\gamma_1^*,\ldots,\gamma_n^*)$ is the vector of endogenous equilibrium prices in the complete model.

The economic interpretation of the distributors' optimality conditions is now highlighted. From the second term in inequality (9), one has that, if retailer k purchases the product from a distributor j, that is, if the q_{jk}^* is positive, then the price charged by retailer j, γ_j^*, plus the marginal handling cost, is precisely equal to ρ_{2j}^*, which, from the third term in the inequality, serves as the price to clear the market from distributor j. Also, note that, from the second term, one sees that if no product is sold by a particular distributor, then the price associated with holding the product can exceed the price charged to the retailers. Furthermore, from the first term in inequality (9), one can infer that, if a manufacturer transacts with a distributor resulting in a positive flow of the product between the two, then the price ρ_{2j}^* is precisely equal to distributor j's payment to the manufacturer, ρ_{1ij}^*, plus his marginal cost of handling the product associated with transacting with the particular manufacturer.

The Retailers and their Optimality Conditions

The retailers, in turn, must decide how much to order from the distributors and from the manufacturers in order to cope with the random demand while still seeking to maximize their profits. A retailer k is also faced with what we term a *handling* cost, which may include, for example, the display and storage cost associated with the product. We denote this cost by c_k and, in the simplest case, we would have that c_k is a function of $s_k=\sum_{i=1}^m q_{ik}+\sum_{j=1}^n q_{jk}$, that is, the holding cost of a retailer is a function of how much of the product he has obtained from the various manufacturers directly and from the various distributors. However, for the sake of generality, and to enhance the modeling of competition, we allow the function to, in general, depend also on the amounts of the product held by other retailers and, therefore, we may write:

$$c_k = c_k(Q^2, Q^3), \quad \forall k. \tag{10}$$

Let ρ_{3k} denote the demand price of the product associated with retailer k. We assume that $\hat{d}_k(\rho_{3k})$ is the demand for the product at the demand price of ρ_{3k} at retail outlet k, where $\hat{d}_k(\rho_{3k})$ is a random variable with a density function of $\mathcal{F}_k(x, \rho_{3k})$, with ρ_{3k} serving as a parameter. Hence, we assume that the density function may vary with the demand price. Let P_k be the probability distribution function of $\hat{d}_k(\rho_{3k})$, that is, $P_k(x, \rho_{3k}) = P_k(\hat{d}_k \leq x) = \int_0^x \mathcal{F}_k(x, \rho_{3k})dx$.

Retailer k can sell to the consumers no more than the minimum of his supply or his demand, that is, the actual sale of k cannot exceed $\min\{s_k, \hat{d}_k\}$. Let

$$\Delta_k^+ \equiv \max\{0, s_k - \hat{d}_k\} \tag{11}$$

and

$$\Delta_k^- \equiv \max\{0, \hat{d}_k - s_k\}, \tag{12}$$

where Δ_k^+ is a random variable representing the excess supply (inventory), whereas Δ_k^- is a random variable representing the excess demand (shortage).

Note that the expected values of excess supply and excess demand of retailer k are scalar functions of s_k and ρ_{3k}. In particular, let e_k^+ and e_k^- denote, respectively, the expected values: $E(\Delta_k^+)$ and $E(\Delta_k^-)$, that is,

$$e_k^+(s_k, \rho_{3k}) \equiv E(\Delta_k^+) = \int_0^{s_k} (s_k - x)\mathcal{F}_k(x, \rho_{3k})dx, \tag{13}$$

$$e_k^-(s_k, \rho_{3k}) \equiv E(\Delta_k^-) = \int_{s_k}^{\infty} (x - s_k)\mathcal{F}_k(x, \rho_{3k})dx. \tag{14}$$

Assume that the unit penalty of having excess supply at retail outlet k is λ_k^+ and that the unit penalty of having excess demand is λ_k^-, where the λ_k^+ and the λ_k^- are assumed to be nonnegative. Then, the expected total penalty of retailer k is given by

$$E(\lambda_k^+ \Delta_k^+ + \lambda_k^- \Delta_k^-) = \lambda_k^+ e_k^+(s_k, \rho_{3k}) + \lambda_k^- e_k^-(s_k, \rho_{3k}).$$

Assuming, as already mentioned, that the retailers are also profit-maximizers, the expected revenue of retailer k is $E(\rho_{3k}\min\{s_k,\hat{d}_k\})$. Hence, the optimization problem of a retailer k can be expressed as:

$$\text{Maximize}\, E(\rho_{3k}\min\{s_k,\hat{d}_k\}) - E(\lambda_k^+\Delta_k^+ + \lambda_k^-\Delta_k^-) - c_k(Q^2,Q^3) - \sum_{i=1}^{m}\rho_{1ik}^* q_{ik} - \sum_{j=1}^{n}\gamma_j^* q_{jk} \quad (15)$$

subject to:

$$q_{ik} \geq 0, \quad q_{jk} \geq 0, \text{ for all } i \text{ and } j. \quad (16)$$

Objective function (15) expresses that the expected profit of retailer k, which is the difference between the expected revenues and the sum of the expected penalty, the handling cost, and the payouts to the manufacturers and to the distributors, should be maximized.

Applying now the definitions of Δ_k^+, and Δ_k^-, we know that $\min\{s_k,\hat{d}_k\} = \hat{d}_k - \Delta_k^-$. Therefore, the objective function (15) can be expressed as

$$\text{Maximize} \quad \rho_{3k}d_k(\rho_{3k}) - (\rho_{3k}+\lambda_k^-)e_k^-(s_k,\rho_{3k}) - \lambda_k^+e_k^+(s_k,\rho_{3k}) - c_k(Q^2,Q^3) - \sum_{i=1}^{m}\rho_{1ik}^* q_{ik} - \sum_{j=1}^{n}\gamma_j^* q_{jk} \quad (17)$$

where $d_j(\rho_{3k}) \equiv E(\hat{d}_k)$ is a scalar function of ρ_{3k}.

We now consider the optimality conditions of the retailers assuming that each retailer is faced with the optimization problem (15), subject to (16), which represents the nonnegativity assumption on the variables. Here, we also assume that the retailers compete in a noncooperative manner so that each maximizes his profits, given the actions of the other retailers. Note that, at this point, we consider that retailers seek to determine the amount that they wish to obtain from the manufacturers and from the distributors. First, however, we make the following derivation and introduce the necessary notation:

$$\frac{\partial e_k^+(s_k,\rho_{3k})}{\partial q_{ik}} = \frac{\partial e_k^+(s_k,\rho_{3k})}{\partial q_{jk}} = P_k(s_k,\rho_{3k}) = P_k(\sum_{i=1}^{m} q_{ik} + \sum_{j=1}^{n} q_{jk}, \rho_{3k}) \quad (18)$$

$$\frac{\partial e_k^-(s_k,\rho_{3k})}{\partial q_{ik}} = \frac{\partial e_k^-(s_k,\rho_{3k})}{\partial q_{jk}} = P_k(s_k,\rho_{3k}) - 1 = P_k(\sum_{i=1}^{m} q_{ij} + \sum_{j=1}^{n} q_{jk}, \rho_{3k}) - 1. \quad (19)$$

Assuming that the handling cost for each retailer is continuous and convex, then the optimality conditions for all the retailers satisfy the variational inequality: determine $(Q^{2*}, Q^{3*}) \in R_+^{mo+no}$, satisfying:

$$\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\lambda_k^+ P_k(s_k^*, \rho_{3k}) - (\lambda_k^- + \rho_{3k})(1 - P_k(s_k^*, \rho_{3k})) + \frac{\partial c_k(Q^{2*}, Q^{3*})}{\partial q_{ik}} + \rho_{1ik}^*\right] \times [q_{ik} - q_{ik}^*]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\lambda_k^+ P_k(s_k^*, \rho_{3k}) - (\lambda_k^- + \rho_{3k})(1 - P_k(s_k^*, \rho_{3k})) + \frac{\partial c_k(Q^{2*}, Q^{3*})}{\partial q_{jk}} + \gamma_j^*\right] \times \left[q_{jk} - q_{jk}^*\right] \geq 0,$$

$$\forall (Q^2, Q^3) \in R_+^{mo+no}. \tag{20}$$

In this derivation, as in the derivation of inequality (5), we have not had the prices charged be variables. They become endogenous variables in the complete supply chain supernetwork model.

We now highlight the economic interpretation of the retailers' optimality conditions. In inequality (20), we can infer that, if a manufacturer i transacts with a retailer k resulting in a positive flow of the product between the two, then the selling price at retail outlet k, ρ_{3k}, with the probability of $(1 - P_k(\sum_{i=1}^m q_{ik}^* + \sum_{j=1}^n q_{jk}^*, \rho_{3k}))$, that is, when the demand is not less then the total order quantity, is precisely equal to the retailer k's payment to the manufacturer, ρ_{1ik}^*, plus his marginal cost of handling the product and the penalty of having excess demand with probability of $P_k(\sum_{i=1}^m q_{ik}^* + \sum_{j=1}^o q_{jk}^*, \rho_{3k})$, (which is the probability when actual demand is less than the order quantity), subtracted by the penalty of having shortage with probability of $(1 - P_k(\sum_{i=1}^m q_{ik}^* + \sum_{j=1}^o q_{jk}^*, \rho_{3k}))$ (when the actual demand is greater than the order quantity).

Similarly, a distributor j transacts with a retailer k resulting in a positive flow of the product between the two, then the selling price at retail outlet k, ρ_{3k}, with the probability of $(1 - P_k(\sum_{i=1}^m q_{ik}^* + \sum_{j=1}^n q_{jk}^*, \rho_{3k}))$, that is, when the demand is not less then the total order quantity, is precisely equal to the retailer k's payment to the manufacturer, γ_j^*, plus his marginal cost of handling the product and the penalty of having excess demand with probability of $P_k(\sum_{i=1}^m q_{ik}^* + \sum_{j=1}^o q_{jk}^*, \rho_{3k})$, (which is the probability when actual demand is less than the order quantity), subtracted by the penalty of having shortage with probability of $(1 - P_k(\sum_{i=1}^m q_{ik}^* + \sum_{j=1}^o q_{jk}^*, \rho_{3k}))$ (when the actual demand is greater than the order quantity).

The Equilibrium Conditions

We now turn to a discussion of the market equilibrium conditions. Subsequently, we construct the equilibrium conditions for the entire supply chain.

The equilibrium conditions associated with the transactions that take place between the retailers and the consumers are the *stochastic economic equilibrium conditions*, which, mathematically, take on the following form: For any retailer k; $k = 1, \ldots, o$:

$$\hat{d}_k(\rho_{3k}^*) \begin{cases} \leq \sum_{i=1}^m q_{ik}^* + \sum_{j=1}^o q_{jk}^* & \textbf{a.e.}, \quad \text{if} \quad \rho_{3k}^* = 0 \\ = \sum_{i=1}^m q_{ik}^* + \sum_{j=1}^o q_{jk}^* & \textbf{a.e.}, \quad \text{if} \quad \rho_{3k}^* > 0, \end{cases} \tag{21}$$

where **a.e.** means that the corresponding equality or inequality holds almost everywhere.

Conditions (21) state that, if the demand price at outlet k is positive, then the quantities purchased by the retailer from the manufacturers and from the distributors in the aggregate is equal to the demand, with exceptions of zero probability. These conditions correspond to the well-known economic equilibrium conditions (cf. Nagurney (1999) and the references therein). Related equilibrium conditions, but in without electronic transactions allowed, were proposed in Dong, Zhang, and Nagurney (2002).

Equilibrium conditions (21) are equivalent to the following variational inequality problem, after taking the expected value and summing over all retailers k: determine $\rho_3^* \in R_+^o$ satisfying

$$\sum_{k=1}^o (\sum_{i=1}^m q_{ik}^* + \sum_{j=1}^n q_{jk}^* - d_j(\rho_{3k}^*)) \times [\rho_{3k} - \rho_{3k}^*] \geq 0, \qquad \forall \rho_3 \in R_+^o, \tag{22}$$

where ρ_3 is the o-dimensional column vector with components: $\{\rho_{31}, \ldots, \rho_{3o}\}$.

The Equilibrium Conditions of the Supply Chain

In equilibrium, we must have that the sum of the optimality conditions for all manufacturers, as expressed by inequality (5), the optimality conditions of the distributors, as expressed by condition (9), the optimality conditions for all retailers, as expressed by inequality (20), and the market equilibrium conditions, as expressed by inequality (22) must be satisfied. Hence, the shipments that the manufacturers ship to the retailers must be equal to the shipments that the retailers accept from the manufacturers. In addition, the shipments shipped from the manufacturers to the distributors, must be equal to those accepted by the distributors, and, finally, the shipments from the distributors to the retailers must coincide with those accepted by the retailers. We state this explicitly in the following definition:

Definition 1: Supply Chain Network Equilibrium with Random Demands

The equilibrium state of the supply chain with random demands is one where the product flows between the tiers of the decision-makers coincide and the product shipments and prices satisfy the sum of the optimality conditions (5), (9), and (20), and the conditions (22).

The summation of inequalities (5), (9), (20), and (22) (with the prices at the manufacturers, the distributors, and at the retailers denoted, respectively, by their values at the equilibrium, and denoted by ρ_1^*, ρ_2^*, and ρ_3^*), after algebraic simplification, yields the following result:

Theorem 1: Variational Inequality Formulation

A product shipment and price pattern $(Q^{1^*}, Q^{2^*}, Q^{3^*}, \rho_2^*, \rho_3^*) \in \mathcal{K}$ is an equilibrium pattern of the supply chain model according to Definition 1 if and only if it satisfies the variational inequality problem:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(Q^{1^*},Q^{2^*})}{\partial q_{ij}}+\frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}}+\frac{\partial c_j(Q^{1^*},Q^{3^*})}{\partial q_{ij}}-\rho_{2j}^*\right]\times\left[q_{ij}-q_{ij}^*\right]$$

$$\sum_{i=1}^{m}\sum_{k=1}^{o}\left[\frac{\partial f_i(Q^{1^*},Q^{2^*})}{\partial q_{ik}}+\frac{\partial c_{ik}(q_{ik}^*)}{\partial q_{ik}}+\frac{\partial c_k(Q^{2^*},Q^{3^*})}{\partial q_{ik}}\right.$$

$$\left.+\lambda_k^+P_k(s_k^*,\rho_{3k}^*)-(\lambda_k^-+\rho_{3k}^*)(1-P_k(s_k^*,\rho_{3k}^*))\right]\times[q_{ik}-q_{ik}^*]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\lambda_k^+P_k(s_k^*,\rho_{3k}^*)-(\lambda_k^-+\rho_{3k}^*)(1-P_k(s_k^*,\rho_{3k}^*))+\frac{\partial c_j(Q^{1^*},Q^{3^*})}{\partial q_{jk}}\right.$$

$$\left.+\frac{\partial c_k(Q^{2^*},Q^{3^*})}{\partial q_{jk}}+\rho_{2j}^*\right]\times\left[q_{jk}-q_{jk}^*\right]+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}q_{ij}^*-\sum_{k=1}^{o}q_{jk}^*\right]\times\left[\rho_{2j}-\rho_{2j}^*\right]$$

$$+\sum_{k=1}^{o}\left[\sum_{j=1}^{n}q_{jk}^*+\sum_{i=1}^{m}q_{ik}^*-d_k(\rho_3^*)\right]\times[\rho_{3k}-\rho_{3k}^*]\geq 0,\quad\forall(Q^1,Q^2,Q^3,\rho_2,\rho_3)\in\mathcal{K},\tag{23}$$

where $\mathcal{K} \equiv \{(Q^1, Q^2, Q^3, \rho_2, \rho_3) | (Q^1, Q^2, Q^3, \rho_2, \rho_3) \in R_+^{mn+mo+no+n+o}\}$.

For easy reference in the subsequent sections, variational inequality problem (23) can be rewritten in standard variational inequality form (cf. Nagurney (1999)) as follows: Determine $X^* \in \mathcal{K}$ satisfying:

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K} \equiv R_+^{mn+mo+no+n+o}, \tag{24}$$

where $X \equiv (Q^1, Q^2, Q^3, \rho_2, \rho_3)$, $F(X) \equiv (F_{ij}, F_{ik}, F_{jk}, F_j, F_k)_{i=1,\ldots,m; j=1,\ldots,n; k=1,\ldots,o}$, and the specific components of F are given by the functional terms preceding the multiplication signs in (24). The term $\langle \cdot, \cdot \rangle$ denotes the inner product in N-dimensional Euclidean space.

Note that the variables in the model (and which can be determined from the solution of either variational inequality (23) or (24)) are: the equilibrium product shipments between manufacturers and the distributors given by Q^{1*}, the equilibrium product shipments transacted electronically between the manufacturers and the retailers denoted by Q^{2*}, and the equilibrium product shipments between the distributors and the retailers given by Q^{3*}, as well as the equilibrium demand prices ρ_3^* and the equilibrium distributor prices γ^*. We now discuss how to recover the prices ρ_1^* associated with the top tier of nodes of the supply chain supernetwork and the prices ρ_2^* associated with the middle tier.

First note that from (5), we have that (as already discussed briefly) that if $q_{ij}^* > 0$, then the price $\rho_{1ij}^* = \frac{\partial f_i(Q^{1*}, Q^{2*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}}$. Also, from (5) it follows that if $q_{ik}^* > 0$, then the price $\rho_{1ik}^* = \frac{\partial f_i(Q^{1*}, Q^{2*})}{\partial q_{ik}} + \frac{\partial c_{ik}(q_{ik}^*)}{\partial q_{ik}}$. Hence, the product is priced at the manufacturer's level according to whether it has been transacted physically or electronically; and also according to the distributor or retailer with which the transaction has taken place. On the other hand, from (9) it follows that if $q_{jk}^* > 0$, then $\gamma_j^* = \rho_{2j}^* + \frac{\partial c_j(Q^{1*}, Q^{2*})}{\partial q_{jk}}$.

3. Qualitative Properties

In this Section, we provide some qualitative properties of the solution to variational inequality (23) (equivalently, variational inequality (24)). In particular, we derive existence and uniqueness results. We also investigate properties of the function F (cf. (24)) that enters the variational inequality of interest here.

Since the feasible set is not compact we cannot derive existence simply from the assumption of continuity of the functions. Nevertheless, we can impose a rather weak condition to guarantee existence of a solution pattern.

Let

$$\mathcal{K}_b = \{(Q^1, Q^2, Q^3, \rho_2, \rho_3) | 0 \leq Q^l \leq b_l, l = 1, 2, 3;\ 0 \leq \rho_2 \leq b_4;\ 0 \leq \rho_3 \leq b_5\}, \tag{25}$$

where $b = (b_1, \cdots, b_5) \geq 0$ and $Q^l \leq b_l; \rho_2 \leq b_4; \rho_3 \leq b_5$ means that $q_{ij} \leq b_1$, $q_{ik} \leq b_2$, $q_{jk} \leq b_3$, and $\rho_{2j} \leq b_4$, $\rho_{3k} \leq b_5$ for all i, j, k. Then $\mathcal{K}_b$ is a bounded closed convex subset of $R^{mn+mo+no+n+o}$. Thus, the following variational inequality

$$\langle F(X^b), X - X^b \rangle \geq 0, \quad \forall X^b \in \mathcal{K}_b, \tag{26}$$

admits at least one solution $X^b \in \mathcal{K}_b$, from the standard theory of variational inequalities, since $\mathcal{K}_b$ is compact and F is continuous. Following Kinderlehrer and Stampacchia (1980) (see also Theorem 1.5 in Nagurney (1999)), we then have:

Theorem 2

Variational inequality (23) admits a solution if and only if there exists a $b > 0$, such that variational inequality (26) admits a solution in $\mathcal{K}_b$ with

$$Q^{1b} < b_1, \quad Q^{2b} < b_2, \quad Q^{3b} < b_3, \quad \rho_2^b < b_4, \quad \rho_3 < b_5. \tag{27}$$

Theorem 3: Existence

Suppose that there exist positive constants M, N, R with $R > 0$, such that:

$$\frac{\partial f_i(Q^1, Q^2)}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij})}{\partial q_{ij}} + \frac{\partial c_j(Q^1, Q^3)}{\partial q_{ij}} \geq M, \quad \forall Q^1 \text{ with } q_{ij} \geq N, \quad \forall i, j \tag{28a}$$

$$\frac{\partial f_i(Q^1, Q^2)}{\partial q_{ik}} + \frac{\partial c_{ik}(q_{ik})}{\partial q_{ik}} + \frac{\partial c_k(Q^2, Q^3)}{\partial q_{jk}} + \lambda_k^+ P_k(s_k, \rho_{3k}) - (\lambda_k^- + \rho_{3k})(1 - P_k(s_k, \rho_{3k})) \geq M,$$

$$\forall Q^2 \ \textit{with} \ q_{ik} \geq N, \quad \forall i, k, \tag{28b}$$

$$\lambda_k^+ P_k(s_k, \rho_{3k}) - (\lambda_k^- + \rho_{3k})(1 - P_k(s_k, \rho_{3k})) + \frac{\partial c_j(Q^1, Q^3)}{\partial q_{jk}} + \frac{\partial c_k(Q^2, Q^3)}{\partial q_{jk}} \geq M,$$

$$\forall Q^3 \ \textit{with} \ q_{jk} \geq N, \quad \forall j, k, \tag{28c}$$

and

$$d_k(\rho_{3k}) \leq N, \quad \forall \rho_3 \quad \textit{with} \quad \rho_{3k} \geq R, \quad \forall k. \tag{29}$$

Then, variational inequality (24) admits at least one solution.

Proof: Follows using analogous arguments as the proof of existence for Proposition 1 in Nagurney and Zhao (1993) (see also existence proof in Nagurney, Dong, and Zhang (2002)). □

Assumptions (28a), (28b), (28c) and (29) can be economically justified as follows. In particular, when the product shipment, q_{ij}, between manufacturer i and distributor j, and the shipment, q_{ik}, between manufacturer i and retailer k, are large, one can expect the corresponding sum of the marginal costs associated with the production, transaction, and holding to exceed a positive lower bound, say M. At same time, the large q_{ij} and q_{ik} causes a greater s_k, which in turn causes the probability distribution $P_k(s_k, \rho_{3k})$ to be close to 1. Consequently, the sum of the last two terms on the left-hand side of (28b), $\lambda_k^+ P_k(s_k, \rho_{3k}) - (\lambda_k^- + \rho_{3k})(1 - P_k(s_k, \rho_{3k}))$ is seen to be positive. Therefore, the left-hand sides of (28b) and (28c), respectively, are greater than or equal to the lower bound M. On the other hand, a high price ρ_{3k} at retailer k will drive the demand at that retailer down, in line with the decreasing nature of any demand function, which ensures (29).

We now recall the concept of additive production cost, which was introduced by Zhang and Nagurney (1996) in the stability analysis of dynamic spatial oligopolies, and has also been employed in the qualitative analysis by Nagurney, Dong, and Zhang (2000) for the study of spatial economic networks with multicriteria producers and consumers and in the case of supply chains by Nagurney, Dong, and Zhang (2002).

Definition 2: Additive Production Cost

Suppose that for each manufacturer i, the production cost f_i is additive, that is

$$f_i(q) = f_i^1(q_i) + f_i^2(\bar{q}_i), \tag{30}$$

where $f_i^1(q_i)$ is the internal production cost that depends solely on the manufacturer's own output level q_i, which may include the production operation and the facility maintenance, etc., and $f_i^2(\bar{q}_i)$ is the interdependent part of the production cost that is a function of all the other manufacturers' output levels $\bar{q}_i = (q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_m)$ and reflects the impact of the other manufacturers' production patterns on manufacturer i's cost. This interdependent part of the production cost may describe the competition for the resources, consumption of the homogeneous raw materials, etc.

We now explore additional qualitative properties of the vector function F that enters the variational inequality problem. Specifically, we show that F is monotone as well as Lipschitz continuous. These properties are fundamental in establishing the convergence of the algorithmic scheme in the subsequent section.

Lemma 1

Let $g_k(s_k, \rho_{3k})^T = (P_k(s_k, \rho_{3k}) - \rho_{3k}(1 - P_k(s_k, \rho_{3k})),\ s_k - \rho_{3k}))$, where P_k is a probability distribution with the density function of $\mathcal{F}_k(x, \rho_{3k})$. Then $g_k(s_k, \rho_{3k})$ is monotone, that is,

$$[-\rho'_{3k}(1 - P_k(s'_k, \rho'_{3k})) + \rho''_{3k}(1 - P_k(s''_k, \rho''_{3k}))] \times [q'_{jk} - q''_{jk}]$$
$$+[s'_k - d_k(\rho'_{3k}) - s''_k + d_k(\rho''_{3k})] \times [\rho'_{3k} - \rho''_{3k}] \geq 0, \quad \forall (s'_k, \rho'_{3k}), (s''_k, \rho''_{3k}) \in R^2_+ \tag{31}$$

if and only if $d'_k(\rho_{3k}) \leq -(4\rho_{3k}\mathcal{F}_k)^{-1}(P_k + \rho_{3k}\frac{\partial P_k}{\partial \rho_{3k}})^2$.

Proof: In order to prove that $g_k(s_k, \rho_{3k})$ is monotone with respect to s_k and ρ_{3k}, we only need to show that its Jacobian matrix is positive semidefinite, which will be the case if all eigenvalues of the symmetric part of the Jacobian matrix are nonnegative real numbers.

The Jacobian matrix of g_k is

$$\nabla g_k(s_, \rho_{3k}) = \begin{bmatrix} \rho_{3k}\mathcal{F}_k(s_k, \rho_{3k}) & -1 + P_k(s_k, \rho_{3k}) + \rho_{3k}\frac{\partial P_k(s_k, \rho_{3k})}{\partial \rho_{3k}} \\ 1 & -d'_k(\rho_{3k}) \end{bmatrix}, \tag{32}$$

and its symmetric part is

$$\frac{1}{2}[\nabla g_k(s_k,\rho_{3k})+\nabla^T g_k(s_k,\rho_{3k})]=\left[\begin{array}{cc}\rho_{3k}\mathcal{F}_k(s_k,\rho_{3k}), & \frac{1}{2}\left(\rho_{3k}\frac{\partial P_k}{\partial \rho_{3k}}+P_k(s_k,\rho_{3k})\right)\\ \frac{1}{2}\left(\rho_{3k}\frac{\partial P_k}{\partial \rho_{3k}}+P_k(s_k,\rho_{3k})\right), & -d'_k(\rho_{3k})\end{array}\right]. \tag{33}$$

The two eigenvalues of (33) are

$$\gamma_{min}(s_k,\rho_{3k})=\frac{1}{2}\left[(\rho_{3k}\mathcal{F}_k-d'_k)-\sqrt{(\alpha_k\mathcal{F}_k-d'_k)^2+(\rho_{3k}\frac{\partial P_k}{\partial \rho_{3k}}+P_k)^2+4\rho_{3k}\mathcal{F}_k d'_k}\right], \tag{34}$$

$$\gamma_{max}(s_k,\rho_{3k})=\frac{1}{2}\left[(\rho_{3k}\mathcal{F}_k-d'_k)+\sqrt{(\rho_{3k}\mathcal{F}_k-d'_k)^2+(\rho_{3k}\frac{\partial P_k}{\partial \rho_{3k}}+P_k)^2+4\rho_{3k}\mathcal{F}_k d'_k}\right]. \tag{35}$$

Moreover, since what is inside the square root in both (34) and (35) can be rewritten as

$$(\rho_{3k}\mathcal{F}_k+d'_k)^2+\left(\rho_{3k}\frac{\partial P_k}{\partial \rho_{3k}}+P_k\right)^2$$

and can be seen as being nonnegative, both eigenvalues are real. Furthermore, under the condition of the lemma, d'_k is non-positive, so the first item in (34) and in (35) is nonnegative. The condition further implies that the second item in (34) and in (35), the square root part, is not greater than the first item, which guarantees that both eigenvalues are nonnegative real numbers. □

The condition of Lemma 1 states that the expected demand function of a retailer is a nonincreasing function with respect to the demand price and its first order derivative has an upper bound.

Theorem 4: Monotonicity

The function that enters the variational inequality problem (24) is monotone, if the condition assumed in Lemma 1 is satisfied for each k; $k=1,\cdots,o$, and if the following conditions are also satisfied.

Suppose that the production cost functions $f_i; i=1,...,m$, are additive, as defined in Definition 2, and that the $f_i^1; i=1,...,m$, are convex functions. If the c_{ij}, c_{ik}, c_k and c_j

functions are convex, for all i, j, k, then the vector function F that enters the variational inequality (24) is monotone, that is,

$$\langle F(X') - F(X''), X' - X'' \rangle \geq 0, \quad \forall X', X'' \in \mathcal{K}. \tag{36}$$

Proof: Let $X' = (Q^{1'}, Q^{2'}, Q^{3'}, \rho_2', \rho_3')$, $X'' = (Q^{1''}, Q^{2''}, Q^{3''}, \rho_2'', \rho_3'')$. Then, inequality (36) can been seen in the following deduction:

$$\langle F(X') - F(X''), X' - X'' \rangle$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial f_i(Q^{1'}, Q^{2'})}{\partial q_{ij}} - \frac{\partial f_i(Q^{1''}, Q^{2''})}{\partial q_{ij}} \right] \times \left[q_{ij}' - q_{ij}'' \right]$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial c_j(Q^{1'}, Q^{3'})}{\partial q_{ij}} - \frac{\partial c_j(Q^{1''}, Q^{3''})}{\partial q_{ij}} \right] \times \left[q_{ij}' - q_{ij}'' \right]$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial c_{ij}(q_{ij}')}{\partial q_{ij}} - \frac{\partial c_{ij}(q_{ij}'')}{\partial q_{ij}} \right] \times \left[q_{ij}' - q_{ij}'' \right]$$

$$+ \sum_{i=1}^{m} \sum_{k=1}^{o} \left[\frac{\partial f_i(Q^{1'}, Q^{2'})}{\partial q_{ik}} - \frac{\partial f_i(Q^{1''}, Q^{2''})}{\partial q_{ik}} \right] \times [q_{ik}' - q_{ik}'']$$

$$+ \sum_{i=1}^{m} \sum_{k=1}^{o} \left[\frac{\partial c_k(Q^{2'}, Q^{3'})}{\partial q_{ik}} - \frac{\partial c_k(Q^{2''}, Q^{3''})}{\partial q_{ik}} \right] \times [q_{ik}' - q_{ik}'']$$

$$+ \sum_{i=1}^{m} \sum_{k=1}^{o} \left[\frac{\partial c_{ik}(q_{ik}')}{\partial q_{ik}} - \frac{\partial c_{ik}(q_{ik}'')}{\partial q_{ik}} \right] \times [q_{ik}' - q_{ik}'']$$

$$+ \sum_{i=1}^{m} \sum_{k=1}^{o} [\lambda_k^+ P_k(s_k', \rho_{3k}') - \lambda_k^+ P_k(s_k'', \rho_{3k}'')] \times [q_{ik}' - q_{ik}'']$$

$$+ \sum_{i=1}^{m} \sum_{k=1}^{o} [-\lambda_k^-(1 - P_k(s_k', \rho_{3k}')) + \lambda_k^-(1 - P_k(s_k'', \rho_{3k}''))] \times [q_{ik}' - q_{ik}'']$$

$$+ \sum_{i=1}^{m} \sum_{k=1}^{o} [-\rho_{3k}'(1 - P_k(s_k', \rho_{3k}')) + \rho_{3k}''(1 - P_k(s_k'', \rho_{3k}''))] \times [q_{ik}' - q_{ik}'']$$

$$+ \sum_{j=1}^{n} \sum_{k=1}^{o} \left[\frac{\partial c_j(Q^{1'}, Q^{3'})}{\partial q_{jk}} - \frac{\partial c_j(Q^{1''}, Q^{3''})}{\partial q_{jk}} \right] \times \left[q_{jk}' - q_{jk}'' \right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\frac{\partial c_k(Q^{2'},Q^{3'})}{\partial q_{jk}}-\frac{\partial c_k(Q^{2''},Q^{3''})}{\partial q_{jk}}\right]\times\left[q'_{jk}-q''_{jk}\right]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}[\lambda_k^+P_k(s'_k,\rho'_{3k})-\lambda_k^+P_k(s''_k,\rho''_{3k})]\times[q'_{jk}-q''_{jk}]$$

$$+\sum_{j=1}^{m}\sum_{k=1}^{o}[-\lambda_k^-(1-P_k(s'_k,\rho'_{3k}))+\lambda_k^-(1-P_k(s''_k,\rho''_{3k}))]\times[q'_{jk}-q''_{jk}]$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{o}[-\rho'_{3k}(1-P_k(s'_k,\rho'_{3k}))+\rho''_{3k}(1-P_k(s''_k,\rho''_{3k}))]\times[q'_{jk}-q''_{jk}]$$

$$+\sum_{k=1}^{o}[s'_k-d_k(\rho'_{3k})-s''_k+d_k(\rho''_{3k})]\times[\rho'_{3k}-\rho''_{3k}]$$

$$=(I)+(II)+(III)+\cdots+(XV). \tag{37}$$

Since the f_i; $i=1,...,m$, are additive, and the f_i^1; $i=1,...,m$, are convex functions, one has

$$(I)+(IV)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i^1(Q^{1\prime},Q^{2\prime})}{\partial q_{ij}}-\frac{\partial f_i^1(Q^{1\prime\prime},Q^{2\prime\prime})}{\partial q_{ij}}\right]\times\left[q'_{ij}-q''_{ij}\right]$$

$$+\sum_{k=1}^{o}\left[\frac{\partial f_i(Q^{1\prime},Q^{2\prime})}{\partial q_{ik}}-\frac{\partial f_i(Q^{1\prime\prime},Q^{2\prime\prime})}{\partial q_{ik}}\right]\times[q'_{ik}-q''_{ik}]\geq 0. \tag{38}$$

The convexity of $c_j, \forall j$, $c_{ij}, \forall i, j$, $c_k, \forall k$, and $c_{ik}, \forall i, k$ gives, respectively,

$$(II)+(X)=\sum_{j=1}^{n}\sum_{i=1}^{m}\left[\frac{\partial c_j(Q^{1'},Q^{3'})}{\partial q_{ij}}-\frac{\partial c_j(Q^{1''},Q^{3''})}{\partial q_{ij}}\right]\times\left[q'_{ij}-q''_{ij}\right]$$

$$\sum_{k=1}^{o}\left[\frac{\partial c_j(Q^{1'},Q^{3'})}{\partial q_{jk}}-\frac{\partial c_j(Q^{1''},Q^{3''})}{\partial q_{jk}}\right]\times\left[q'_{jk}-q''_{jk}\right]\geq 0 \tag{39}$$

$$(III)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial c_{ij}(q'_{ij})}{\partial q_{ij}}-\frac{\partial c_{ij}(q''_{ij})}{\partial q_{ij}}\right]\times\left[q'_{ij}-q''_{ij}\right]\geq 0 \tag{40}$$

$$(V)+(XI)=\sum_{k=1}^{o}\left\{\sum_{i=1}^{m}\left[\frac{\partial c_k(Q^{2'},Q^{3'})}{\partial q_{ik}}-\frac{\partial c_k(Q^{2''},Q^{3''})}{\partial q_{ik}}\right]\times[q'_{ik}-q''_{ik}]\right.$$

$$\left.+\sum_{j=1}^{n}\left[\frac{\partial c_k(Q^{2'},Q^{3'})}{\partial q_{jk}}-\frac{\partial c_k(Q^{2''},Q^{3''})}{\partial q_{jk}}\right]\times\left[q'_{jk}-q''_{jk}\right]\right\}\geq 0 \tag{41}$$

$$(VI) = \sum_{i=1}^{m}\sum_{k=1}^{o}\left[\frac{\partial c_{ik}(q'_{ik})}{\partial q_{ik}} - \frac{\partial c_{ik}(q''_{ik})}{\partial q_{ik}}\right] \times [q'_{ik} - q''_{ik}] \geq 0 \tag{42}$$

Since the probability function P_k is an increasing function w.r.t. s_k, for all k, and $s_k = \sum_{i=1}^{m} q_{ik} + \sum_{j=1}^{n} q_{jk}$, hence, we have the followings.

$$(VII) + (XII) = \sum_{k=1}^{o}[\lambda_k^+ P_k(s'_k, \rho'_{3k}) - a_k^+ P_k(s''_k, \rho''_{3k})] \times [s'_k - s''_k] \geq 0 \tag{43}$$

$$(VIII) + (XIII) = \sum_{k=1}^{o}[-\lambda_k^-(1 - P_k(s'_k, \rho'_{3k})) + \lambda_k^-(1 - P_k(s''_k, \rho''_{3k}))] \times [s'_k - s''_k] \geq 0 \tag{44}$$

Since for each k, applying Lemma 1, we can see that $g_k(s_k, \rho_{3k})$ is monotone, hence, we have:

$$(IX) + (XIV) + (XV) = \sum_{k=1}^{o}[-\rho'_{3k}(1 - P_k(s'_k, \rho'_{3k})) + \rho''_{3k}(1 - P_k(s''_k, \rho''_{3k}))] \times [s'_k - s''_k]$$

$$+ \sum_{k=1}^{o}[s'_k - d_k(\rho'_{3k}) - s''_k + d_k(\rho''_{3k})] \times [\rho'_{3k} - \rho''_{3k}] \geq 0 \tag{45}$$

Therefore, we conclude that (37) is nonnegative in $\mathcal{K}$. The proof is complete. □

Theorem 5: Strict Monotonicity

The function that enters the variational inequality problem (24) is strictly monotone, if the conditions mentioned in Lemma 1 for $g_k(s_k, \rho_{3k})$ are satisfied strictly for all k and if the following conditions are also satisfied.

Suppose that the production cost functions $f_i; i = 1, ..., m$, are additive, as defined in Definition 2, and that the $f_i^1; i = 1, ..., m$, are strictly convex functions. If the c_{ij}, c_{ik},c_k and c_j functions are strictly convex, for all i, j, k, then the vector function F that enters the variational inequality (24) is strictly monotone, that is,

$$\langle F(X') - F(X''), X' - X'' \rangle > 0, \quad \forall X', X'' \in \mathcal{K}. \tag{46}$$

Theorem 6: Uniqueness

Under the conditions indicated in Theorem 5, the function that enters the variational inequality (24) has a unique solution in $\mathcal{K}$.

From Theorem 6 it follows that, under the above conditions, the equilibrium product shipment pattern between the manufacturers and the retailers, as well as the equilibrium price pattern at the retailers, is unique.

Theorem 7: Lipschitz Continuity

The function that enters the variational inequality problem (24) is Lipschitz continuous, that is,

$$\|F(X') - F(X'')\| \leq L\|X' - X''\|, \quad \forall X', X'' \in \mathcal{K}, \text{ with } L > 0, \tag{47}$$

under the following conditions:

(i). Each f_i; $i = 1, ..., m$, is additive and has a bounded second order derivative;

(ii). The c_{ij}, c_{ik}, c_k, and c_j have bounded second order derivatives, for all i, j, k;

Proof: Since the probability function P_k is always less than or equal to 1, for each retailer k, the result is direct by applying a mid-value theorem from calculus to the vector function F that enters the variational inequality problem (24). □

4. The Algorithm

In this Section, an algorithm is presented which can be applied to solve any variational inequality problem in standard form (see (24)), that is:

Determine $X^* \in \mathcal{K}$, satisfying:

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{48}$$

The algorithm is guaranteed to converge provided that the function F that enters the variational inequality is monotone and Lipschitz continuous (and that a solution exists). The algorithm is the modified projection method of Korpelevich (1977).

The statement of the modified projection method is as follows, where $\mathcal{T}$ denotes an iteration counter:

Modified Projection Method

Step 0: Initialization

Set $X^0 \in \mathcal{K}$. Let $\mathcal{T} = 1$ and let α be a scalar such that $0 < a \leq \frac{1}{L}$, where L is the Lipschitz continuity constant (cf. Korpelevich (1977)) (see (47)).

Step 1: Computation

Compute $\bar{X}^{\mathcal{T}}$ by solving the variational inequality subproblem:

$$\langle \bar{X}^{\mathcal{T}} + aF(X^{\mathcal{T}-1}) - X^{\mathcal{T}-1}, X - \bar{X}^{\mathcal{T}} \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{49}$$

Step 2: Adaptation

Compute $X^{\mathcal{T}}$ by solving the variational inequality subproblem:

$$\langle X^{\mathcal{T}} + aF(\bar{X}^{\mathcal{T}}) - X^{\mathcal{T}-1}, X - X^{\mathcal{T}} \rangle \geq 0, \quad \forall X \in \mathcal{K}. \tag{50}$$

Step 3: Convergence Verification

If max $|X_l^{\mathcal{T}} - X_l^{\mathcal{T}-1}| \leq \epsilon$, for all l, with $\epsilon > 0$, a prespecified tolerance, then stop; else, set $\mathcal{T} =: \mathcal{T} + 1$, and go to Step 1.

We now state the convergence result for the modified projection method for this model.

Theorem 8: Convergence

Assume that the function that enters the variational inequality (23) (or (24)) has at least one solution and satisfies the conditions in Theorem 4 and in Theorem 7. Then the modified projection method described above converges to the solution of the variational inequality (23) or (24).

Proof: According to Korpelevich (1977), the modified projection method converges to the solution of the variational inequality problem of the form (24), provided that the function F that enters the variational inequality is monotone and Lipschitz continuous and that a solution exists. Existence of a solution follows from Theorem 3. Monotonicity follows Theorem 5. Lipschitz continuity, in turn, follows from Theorem 7. □

We emphasize that, in view of the fact that the feasible set $\mathcal{K}$ underlying the supply chain supernetwork model with random demands is the nonnegative orthant, the projection operation encountered in (49) and (50) takes on a very simple form for computational purposes. Indeed, the product shipments as well as the product prices at a given iteration in both (49) and in (50) can be exactly and computed in closed form.

5. Numerical Examples

In this Section, we apply the modified projection method to six numerical examples. The algorithm was implemented in FORTRAN and the computer system used was a DEC Alpha system located at the University of Massachusetts at Amherst. The convergence criterion used was that the absolute value of the product shipments and prices between two successive iterations differed by no more than 10^{-4}. The parameter a in the modified projection method (see (49) and (50)) was set to .01 for all the examples.

In all the examples, we assumed that the demands associated with the retail outlets followed a uniform distribution. Hence, we assumed that the random demand, $\hat{d}_k(\rho_{3k})$, of retailer k, is uniformly distributed in $[0, \frac{b_k}{\rho_{3k}}]$, $b_k > 0$; $k = 1, \ldots, o$. Therefore,

$$P_k(x, \rho_{3k}) = \frac{x\rho_{3k}}{b_k}, \tag{51}$$

$$\mathcal{F}_j(x, \rho_{3k}) = \frac{\rho_{3k}}{b_k}, \tag{52}$$

$$d_k(\rho_{3k}) = E(\hat{d}_k) = \frac{1}{2}\frac{b_k}{\rho_{3k}}; \quad k = 1, \ldots, o. \tag{53}$$

It is easy to verify that the expected demand function $d_k(\rho_{3k})$ associated with retailer k is a decreasing function of the price at the demand market.

The modified projection method was initialized as follows: all variables were set to zero, except for the initial retail prices ρ_{3k} which were set to 1 for all retailers k.

Example 1

The first and subsequent two numerical supply chain examples with electronic commerce consisted of two manufacturers, two distributors, and two retailers, as depicted in Figure 2.

The data for this example were constructed for easy interpretation purposes. The production cost functions for the manufacturers were given by:

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 2q_1, \quad f_2(q) = 2.5q_2^2 + q_1q_2 + 2q_2.$$

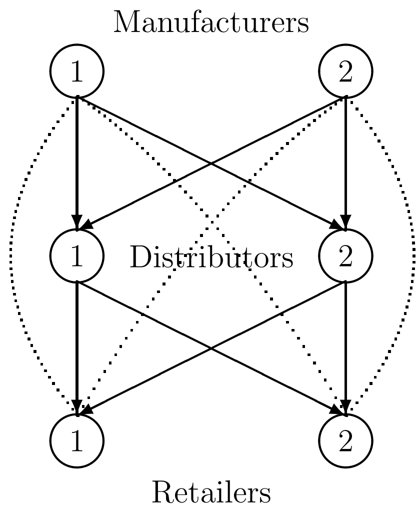

Figure 2: Supply Chain Supernetwork Structure for the Examples 1 through 3

The transaction cost functions faced by the manufacturers and associated with transacting with the distributors were given by:

$$c_{ij}(q_{ij}) = .5q_{ij}^2 + 3.5q_{ij}, \quad \text{for } i = 1,2; \; j = 1,2.$$

The transaction cost functions faced by the manufacturers but associated with transacting electronically with the retailers were given by:

$$c_{ik}(q_{ik}) = .5q_{ik}^2 + 5q_{ik}, \quad \text{for } i = 1,2; \; k = 1,2.$$

The handling costs of the distributors, in turn, were given by:

$$c_j(Q^1, Q^3) = .5(\sum_{i=1}^{2} q_{ij})^2, \quad \text{for } j = 1,2,$$

whereas the handling costs of the retailers were given by:

$$c_k(Q^2, Q^3) = .5(\sum_{j=1}^{2} q_{jk})^2, \quad \text{for } k = 1,2.$$

The b_ks were set to 100 for both both retailers yielding probability distribution functions as in (51) and the expected demand functions as in (53). The weights associated with the excess supply and excess demand at the retailers were: $\lambda_k^+ = \lambda_k^- = 1$ for $k = 1, 2$. Hence, we assigned equal weights for each retailer for excess supply and for excess demand.

The modified projection method converged and yielded the following equilibrium pattern: the product shipments between the two manufacturers and the two distributors were: $q_{ij}^* = .3697$ for $i = 1, 2$; $j = 1, 2$, whereas the product shipments transacted electronically between the manufacturers and the retailers were: $q_{ik}^* = .3487$ for $i = 1, 2$; $k = 1, 2$, and, finally, the product shipments between the distributors and the retailers were: $q_{jk}^* = .3697$ for $j = 1, 2$; $k = 1, 2$. The computed equilibrium prices, in turn, were: $\rho_{2j}^* = 15.2301$ for $j = 1, 2$ and $\rho_{3k}^* = 34.5573$ for $k = 1, 2$. The expected demands (see (53)) were: $d_1(\rho_{31}^*) = d_2(\rho_{32}^*) = 1.4469$.

Example 2

Example 2 was constructed from Example 1 as follows. We retained all the data as in Example 1, except that we increased b_1 and b_2 from 100 to 1000. This has the interpretation that the expected demand at both retailers increased.

The modified projection method converged and yielded the following equilibrium pattern: the product shipments between the two manufacturers and the two distributors were now: $q_{ij}^* = .6974$ for $i = 1, 2$; $j = 1, 2$, whereas the product shipments transacted electronically between the manufacturers and the retailers were: $q_{ik}^* = 1.9870$ for $i = 1, 2$; $k = 1, 2$, and, finally, the product shipments between the distributors and the retailers were now: $q_{jk}^* = .6973$ for $j = 1, 2$; $k = 1, 2$. The computed equilibrium prices, in turn, were: $\rho_{2j}^* = 39.8051$ for $j = 1, 2$ and $\rho_{3k}^* = 92.9553$ for $k = 1, 2$. The expected demands increased (as expected) relative to those obtained in Example 1 with $d_1(\rho_{31}^*) = d_2(\rho_{32}^*) = 5.3789$.

Example 3

Example 3 was constructed from Example 2 as follows. We retained all the data as in Example 2, except that now we decreased the transaction costs associated with transacting electronically, where now $c_{ik}(q_{ik}) = q_{ik} + 1$, $i = 1, 2; k = 1, 2$.

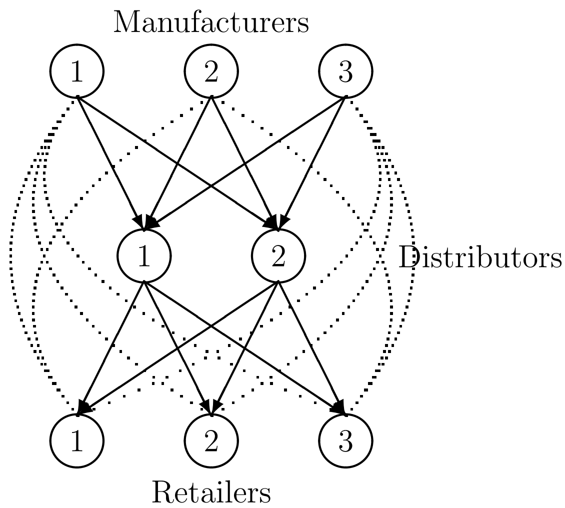

Figure 3: Supply Chain Supernetwork for Examples 4 through 6

The modified projection method converged and yielded the following equilibrium pattern: the product shipments between the two manufacturers and the two distributors were now: $q^*_{ij} = .0484$ for $i = 1, 2$; $j = 1, 2$, whereas the product shipments transacted electronically between the manufacturers and the retailers were: $q^*_{ik} = 2.7418$ for $i = 1, 2$; $k = 1, 2$, and, finally, the product shipments between the distributors and the retailers were now: $q^*_{jk} = .0483$ for $j = 1, 2$; $k = 1, 2$. The computed equilibrium prices, in turn, were: $\rho^*_{2j} = 39.1269$ for $j = 1, 2$ and $\rho^*_{3k} = 89.4390$ for $k = 1, 2$. Hence, the product shipments between the manufacturers and the retailers increased and the prices at the retailers decreased (relative to those obtained in Example 2).

Example 4

Examples 4 through 6 had the supernetwork structure depicted in Figure 3 and consisted of 3 manufacturers, 2 distributors, and 3 retailers.

The production cost functions for the manufacturers were given by:

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 2q_1, \quad f_2(q) = 2.5q_2^2 + q_1q_2 + 2q_2, \quad f_3(q) = .5q_3^2 + .5q_1q_3 + 2q_3.$$

The transaction cost functions faced by the manufacturers and associated with transacting with the distributors were given by:

$$c_{11}(q_{11}) = .5q_{11}^2 + 1.5, \quad c_{12}(q_{12}) = .5q_{12}^2 + 3.5,$$

$$c_{21}(q_{21}) = .5q_{21}^2 + 5.5, \quad c_{22}(q_{22}) = .5q_{22}^2 + 3.5,$$

$$c_{31}(q_{31}) = .5q_{31}^2 + 2, \quad c_{32}(q_{32}) = .5q_{32}^2 + 2.$$

The transaction cost functions faced by the manufacturers but associated with transacting electronically with the retailers were given by:

$$c_{ik}(q_{ik}) = .5q_{ik}^2 + 5q_{ik}, \quad \text{for } i = 1, 2;\ k = 1, 2, 3.$$

The handling costs of the distributors, in turn, were given by:

$$c_j(Q^1, Q^3) = .5(\sum_{i=1}^{2} q_{ij})^2, \quad \text{for } j = 1, 2,$$

whereas the handling costs of the retailers were given by:

$$c_k(Q^2, Q^3) = .5(\sum_{j=1}^{2} q_{jk})^2, \quad \text{for } k = 1, 2, 3.$$

The b_ks were set to 1000 for both all three retailers yielding probability distribution functions as in (51) and the expected demand functions as in (53). The weights associated with the excess supply and excess demand at the retailers were: $\lambda_k^+ = \lambda_k^- = 1$ for $k = 1, 2, 3$.

The modified projection method converged and yielded the following equilibrium pattern: the product shipments between the two manufacturers and the two distributors were:

$$Q^{1*} := q_{11}^* = 1.2656,\ q_{12}^* = 0.0000,\ q_{21}^* = 0.0000, q_{22}^* = .2543,\ q_{31}^* = .0790,\ q_{32}^* = .7564,$$

whereas the product shipments transacted electronically between the manufacturers and the retailers were:

$$Q^{2*} := q_{11}^* = q_{12}^* = q_{13}^* = .4596; q_{21}^* = q_{22}^* = q_{23}^* = .7709;$$

$$q_{31}^* = q_{32}^* = q_{33}^* = 4.7730,$$

and, finally, the product shipments between the distributors and the retailers were:

$$Q^{3*} := q_{11}^* = q_{12}^* = q_{13}^* = .4590; q_{21}^* = q_{22}^* = q_{23}^* = .3401.$$

The computed equilibrium prices, in turn, were: $\rho_{21}^* = 21.8951$ and $\rho_{22}^* = 22.2489$ and $\rho_{3k}^* = 73.6386$ for $k = 1, 2, 3$. The expected demands were: $d_1(\rho_{31}^*) = d_2(\rho_{32}^*) = d_3(\rho_{33}^*) = 6.7899$.

Example 5

The data for Example 5 were identical to the data in Example 4 except now we modified the weights associated with excess supply and excess demand as follows. We set $\lambda_k^+ = 10$ for $k = 1, 2, 3$ and $\lambda_k^- = 0$ for $k = 1, 2, 3$. Hence, only excess supply (or inventory was penalized).

The modified projection method converged and yielded the following equilibrium pattern: the product shipments between the two manufacturers and the two distributors were:

$$Q^{1*} := q_{11}^* = 1.2941,\ q_{12}^* = 0.0000,\ q_{21}^* = 0.0000, q_{22}^* = .3040,\ q_{31}^* = 0.0000,\ q_{32}^* = .6330,$$

whereas the product shipments transacted electronically between the manufacturers and the retailers were:

$$Q^{2*} := q_{11}^* = q_{12}^* = q_{13}^* = .3798; q_{21}^* = q_{22}^* = q_{23}^* = .6843;$$

$$q_{31}^* = q_{32}^* = q_{33}^* = 4.5133,$$

and, finally, the product shipments between the distributors and the retailers were:

$$Q^{3*} := q_{11}^* = q_{12}^* = q_{13}^* = .4347; q_{21}^* = q_{22}^* = q_{23}^* = .3097.$$

The computed equilibrium prices, in turn, were: $\rho_{21}^* = 20.6190$ and $\rho_{22}^* = 20.9572$ and $\rho_{3k}^* = 79.0529$ for $k = 1, 2, 3$. The expected demands were now: $d_1(\rho_{31}^*) = d_2(\rho_{32}^*) = d_3(\rho_{33}^*) = 6.3249$. Hence, the expected demand at the retailers decreased relative to the values obtained in the preceding example.

Since the penalty of having excess supply, λ_k^+, increased for all k (as compared to those values in Example 4) and the penalty of having excess demand, λ_k^-, decreased, as expected, the shipments between distributors and retailers, Q^{3*}, as well as the shipments between manufacturers and the retailers, Q^{2*}, decreased.

Example 6

The data for Example 6 were identical to the data in Example 5 except now we modified the weights associated with excess supply and excess demand as follows. We set $\lambda_k^+ = 0$ for $k = 1, 2, 3$ and $\lambda_k^- = 10$ for $k = 1, 2, 3$. Hence, only excess demand was penalized.

The modified projection method converged and yielded the following equilibrium pattern: the product shipments between the two manufacturers and the two distributors were:

$$Q^{1*} := q_{11}^* = 1.2130,\ q_{12}^* = 0.0000,\ q_{21}^* = 0.0000, q_{22}^* = .2017,\ q_{31}^* = 0.2043,\ q_{32}^* = .8816,$$

whereas the product shipments transacted electronically between the manufacturers and the retailers were:

$$Q^{2*} := q_{11}^* = q_{12}^* = q_{13}^* = .5522; q_{21}^* = q_{22}^* = q_{23}^* = .8636;$$

$$q_{31}^* = q_{32}^* = q_{33}^* = 5.0435,$$

and, finally, the product shipments between the distributors and the retailers were:

$$Q^{3*} := q_{11}^* = q_{12}^* = q_{13}^* = .4691; q_{21}^* = q_{22}^* = q_{23}^* = .3644.$$

The computed equilibrium prices, in turn, were: $\rho_{21}^* = 23.2677$ and $\rho_{22}^* = 23.6216$ and $\rho_{3k}^* = 68.5502$ for $k = 1, 2, 3$. The expected demands were now: $d_1(\rho_{31}^*) = d_2(\rho_{32}^*) = d_3(\rho_{33}^*) = 7.2939$.

In this example, because of the decrease in the penalties of having excess supply (inventory) and the increase in the penalties of having excess demand (shortage), the shipments between manufacturers and retailers and the shipments between distributors adn retailers increased, as opposed to Example 5.

6. Summary and Conclusions

This paper has developed a three-tiered supply chain network equilibrium model consisting of manufacturers, distributors, and retailers. The model allows for physical transactions between the different tiers of decision-makers as well as electronic transactions in the form of B2B commerce between manufacturers and the retailers. In addition, the demands for the product associated with the retailers are no longer assumed to be known with certainty but rather, are random.

Finite-dimensional variational inequality theory was used to formulate the derived equilibrium conditions, to study the model qualitatively, and also to obtain convergence results for the proposed algorithmic scheme. Finally, numerical examples were presented to illustrate the model and computational procedure.

Future research will include empirical work as well as allowing for other components of the model to also be random.

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