A LETTERS JOURNAL EXPLORING
THE FRONTIERS OF PHYSICS

OFFPRINT

Robustness of transportation networks subject to degradable links

A. NAGURNEY and Q. QIANG

EPL, **80** (2007) 68001

A Letters Journal Exploring the Frontiers of Physics

December 2007

EPL, **80** (2007) 68001 www.epljournal.org
doi: 10.1209/0295-5075/80/68001

Robustness of transportation networks subject to degradable links

A. Nagurney and Q. Qiang

University of Massachusetts - Amherst, MA 01003, USA

received 27 September 2007; accepted in final form 17 October 2007
published online 13 November 2007

PACS 87.23.Ge – Dynamics of social systems
PACS 89.40.-a – Transportation
PACS 89.90.+n – Other topics in areas of applied and interdisciplinary physics

Abstract – In this paper, we demonstrate how to capture the robustness of a transportation network in the case of degradable links represented by decreasing capacities. The analysis is conducted by utilizing Bureau of Public Road link travel cost functions and a recently proposed network efficiency measure for congested networks. For specific networks we are able to derive lower bounds for the robustness when percentage reductions in the link capacities take place.

Introduction. – Transportation networks play a major role as critical infrastructure networks underpinning our societies and economies. Although the rigorous modeling and analysis of transportation networks and the underlying behavior of travelers dates to the seminal book of Beckmann, McGuire, and Winsten [1], the research into the robustness of transportation networks in the presence of disruptions is relatively recent. This is particularly interesting, since the degradation of our transportation networks due to poor maintenance, natural disasters, deterioration over time, as well as unforeseen attacks now lead to estimates of $94 billion in the US in terms of needed repairs for roads alone (cf. [2]). Poor road conditions in the United States cost US motorists $54 billion in repairs and operating costs annually.

To the best of our knowledge, the works of Sakakibara *et al.* [3] and that of Scott *et al.* [4] stand as the first attempts to address the robustness of transportation networks. [3] proposed a topological index and considered a transportation network to be robust if it is "dispersed" in terms of the number of links connected to each node. [4], on the other hand, examined transportation network robustness by analyzing the increase in the total network cost after the removal of certain network components.

The concept of *system robustness*, in turn, has been studied in both computer science and in engineering. According to [5], robustness can be defined as "the degree to which a system or component can function correctly in the presence of invalid inputs or stressful environmental conditions." Gribble [6] defined system robustness as "the ability of a system to continue to operate correctly across a wide range of operational conditions, and to fail gracefully outside of that range." Ali *et al.* [7] considered an allocation mapping to be robust if it "guarantees the maintenance of certain desired system characteristics despite fluctuations in the behavior of its component parts or its environment." Schillo *et al.* [8] argued that robustness has to be studied "in relation to some definition of performance measure." According to Holmgren [9]: "Robustness signifies that the system will retain its system structure (function) intact (remain unchanged or nearly unchanged) when exposed to perturbations."

In addition, the physics research on complex networks has also examined network robustness according to different network measures and the accompanying degradation of network performance in the presence of attacks on the network; see, for example, [10]. However, the focus of that research has been on the impact of the removal of nodes on networks, whereas in this paper we focus on the degradation of links through reductions in their capacities and the effects on the induced travel costs in the presence of known travel demands and different functional forms for the links. Hence, we are not concerned directly with extreme events that may lead to the removal of nodes and links from the network but, rather, with the deterioration of the network infrastructure, such as roads, through changes in the link practical capacities. Finally, it is worth noting that there is a literature in robust optimization, which is a mathematical approach to deal with uncertainty and, in particular, when a problem's data may be known only within certain bounds. Robustness is a well-known concept in control, and the subject of robust optimization dates to the pioneering work of Soyster [11] and Ben-Tal and Nemirovsky [12,13]. Here, however, we deal with specific link functional forms, which are assumed to be known, and we vary the specific link capacities.

In this paper, we will utilize for our transportation network robustness analysis a recently proposed network efficiency measure for congested networks (see Nagurney and Qiang [14]). It is well-known that congestion is a fundamental problem in a variety of modern network systems, including urban transportation networks (cf. [15–17]). Moreover, we will consider transportation networks in which the user link cost functions are of the general Bureau of Public Roads form, which is widely used in practice. Such a functional form contains the practical capacity of a link implicitly and, hence, it will allow us to investigate the changes in performance of a transportation network when the link practical capacities are decreased.

This paper is organized as follows. We first briefly recall the well-known traffic network equilibrium model (see, *e.g.*, [1,15,16,18–22]). We then recall the network efficiency measure for congested networks. Subsequently, we propose the robustness measure and provide several examples. In addition, we derive some lower bounds for the robustness of transportation networks of special structure. We conclude the paper with a summary of the results.

Traffic network equilibrium model. – For completeness and easy reference, we recall the traffic network equilibrium model ([15–22]), which is widely used. Consider a network G with the set of directed links L with n_L elements and the set of origin/destination (O/D) pairs W with n_W elements. We denote the set of acyclic paths joining O/D pair w by P_w. The set of (acyclic) paths for all O/D pairs is denoted by P and there are n_P paths in the network. Links are denoted by a, b, etc; paths by p, q, etc., and O/D pairs by w_1, w_2, etc.

We assume that the demand d_w is known for all $w \in W$. We denote the nonnegative flow on path p by x_p and the flow on link a by f_a and we group the path flows into the vector $x \in R^{n_P}_+$ and the link flows into the vector $f \in R^{n_L}_+$.

The following conservation of flow equations must hold:

$$\sum_{p \in P_w} x_p = d_w, \qquad \forall w \in W, \tag{1}$$

which means that the sum of path flows on paths connecting each O/D pair must be equal to the demand for that O/D pair.

The link flows are related to the path flows, in turn, through the following conservation of flow equations:

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \qquad \forall a \in L, \tag{2}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and $\delta_{ap} = 0$, otherwise. Hence, the flow on a link is equal to the sum of the flows on paths that contain that link.

The user (travel) cost on a path p is denoted by C_p and the user (travel) cost on a link a by c_a. The user costs on paths are related to user costs on links through the following equations:

$$C_p = \sum_{a \in L} c_a \delta_{ap}, \qquad \forall p \in P, \tag{3}$$

that is, the user cost on a path is equal to the sum of user costs on links that make up the path. In engineering practice (see [21]), the travel time on a link is used as a proxy for the travel cost.

Since we are concerned with transportation networks, we allow the user link cost function on each link to depend upon the flow on that link, so that

$$c_a = c_a(f_a), \qquad \forall a \in L. \tag{4}$$

We assume that the link cost functions are continuous and monotonically increasing. In view of (1), (2), and (3), we may write

$$C_p = C_p(x), \qquad \forall p \in P. \tag{5}$$

A network equilibrium is defined as follows. A path flow pattern $x^* \in \mathcal{K}^1$, where $\mathcal{K}^1 \equiv \{x | x \in R^{n_P}_+ \text{ and (1) holds}\}$, is said to be a network equilibrium, if the following conditions hold for each O/D pair $w \in W$ and each path $p \in P_w$:

$$C_p(x^*) \begin{cases} = \lambda_w, & \text{if} \quad x^*_p > 0, \\ \geqslant \lambda_w, & \text{if} \quad x^*_p = 0. \end{cases} \tag{6}$$

The interpretation of conditions (6) is that all used paths connecting an O/D pair w have equal and minimal costs (with the minimal path costs equal to the equilibrium travel disutility, denoted by λ_w). These conditions are also referred to as the user-optimized conditions (cf. [18]). In this *classical* traffic network equilibrium problem, in which the cost on each link (cf. (4)) depends solely on the flow on that link, the traffic network equilibrium conditions (6) can be reformulated as the solution to an appropriately constructed optimization problem, as established in [1]. Indeed, the equilibrium link flow (and path flow pattern) can be obtained via the solution of the following optimization problem:

$$\text{Minimize}_{f \in \mathcal{K}^2} \quad \sum_{a \in L} \int_0^{f_a} c_a(y) \mathrm{d}y, \tag{7}$$

where $\mathcal{K}^2 \equiv \{f \in R^n_+ | \exists x \in R^{n_P}_+ \text{ satisfying (1), (2)}\}$. For additional background on this model, along with its impacts, see [23]. In particular, we know that if the user link cost functions are strictly monotone (cf. [22]) then the equilibrium link flow pattern is unique.

In this paper, we will consider user link cost functions known as Bureau of Public Road (BPR) functions, given by

$$c_a(f_a) = t^0_a \left[1 + k \left(\frac{f_a}{u_a} \right)^{\beta} \right], \qquad \forall a \in L, \tag{8}$$

where f_a is the flow on link a; u_a is the "practical" capacity on link a, which also has the interpretation of the level-of-service flow rate; t^0_a is the free-flow travel time or cost on link a; k and β are the model parameters and both take on positive values (see [24] and [21]). Often in applications $k = 0.15$ and $\beta = 4$.

The efficiency measure for congested networks recalled. – We now recall a network efficiency measure for congested networks proposed by [14]. The measure is defined in the context of network equilibrium, and it captures demands and costs, and the underlying behavior of "users" of the network. The formal definition is as follows. The network performance/efficiency measure, $\mathcal{E}(G, c, d)$, for a given network topology G, vector of user link cost functions c, and demand vector d, is defined as

$$\mathcal{E} = \mathcal{E}(G, c, d) = \frac{\sum_{w \in W} \frac{d_w}{\lambda_w}}{n_W}, \tag{9}$$

where recall that n_W is the number of O/D pairs in the network, and d_w and λ_w are, respectively, the demand and the equilibrium disutility for O/D pair w (*cf.* (6)).

Note that this measure has a nice, economic meaning in that it measures the average (O/D pair based) performance *vs.* cost or price, with the performance being measured by the demands and the cost or price by the travel disutility.

Robustness of a transportation network. – Based on the above network efficiency measure, we define the robustness of a transportation network as follows.

The robustness measure $\mathcal{R}^\gamma$ for a transportation network G with the vector of demands d, the vector of user link cost functions c, and the vector of link capacities u is defined as the relative performance retained under a given uniform capacity retention ratio γ with $\gamma \in (0, 1]$ so that the new capacities (cf. (8)) are given by γu. Its mathematical definition is given as

$$\mathcal{R}^\gamma = \mathcal{R}(G, c, d, \gamma, u) = \frac{\mathcal{E}^\gamma}{\mathcal{E}} \times 100\%, \tag{10}$$

where $\mathcal{E}$ and $\mathcal{E}^\gamma$ are the network performance measures with the original capacities and the remaining capacities, respectively. For example, if $\gamma = 0.9$ this means that the user link cost functions given by (8) now have the link capacities given by $0.9u_a$ for $a \in L$; if $\gamma = 0.7$ then the link capacities become $0.7u_a$ for all links $a \in L$, and so on.

From the above definition, a network, under a given level of capacity retention/deterioration, is considered to be robust if the network performance stays close to the original level.

Below we illustrate the above robustness concept with several network examples.

We then establish some theoretical results.

A simple network. Consider a simple network as depicted in fig. 1. There are two nodes: 1 and 2; two links: a and b; and a single O/D pair $w = (1, 2)$. Therefore, there are two paths connecting the single O/D pair, which are denoted, respectively, by: $p_1 = a$ and $p_2 = b$. The demand is given by $d_{w_1} = 10$.

We further assume that in the BPR link cost functions (cf. (8)) that $k = 1$ and $\beta = 4$; and $t^0_a = 10$ and $t^0_b = 1$. We evaluate the network robustness under two sets of capacities for links a and b; namely, capacity set A and set B. In capacity set A, $u_a = u_b = 50$ while in capacity set B, $u_a = 50$ and $u_b = 10$. Therefore, under capacity set A, the BPR link cost functions are given by

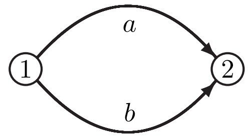

Fig. 1: A simple network.

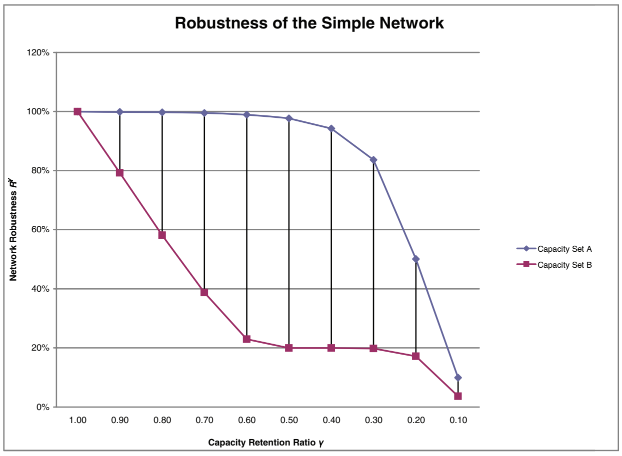

Fig. 2: Robustness *vs.* capacity retention ratio.

$$c_a(f_a) = 10\left(1 + \left(\frac{f_a}{50}\right)\right)^4 \quad c_b(f_b) = 10\left(1 + \left(\frac{f_b}{50}\right)\right)^4.$$

Under capacity set B, the BPR link cost functions are given by

$$c_a(f_a) = 10\left(1 + \left(\frac{f_a}{50}\right)\right)^4 \quad c_b(f_b) = 10\left(1 + \left(\frac{f_b}{10}\right)\right)^4.$$

In fig. 2, we show the relationship between network robustness and the capacity retention ratios for the two capacity sets.

We can see from fig. 2 that the network with capacity set A is more robust under different capacity retention ratios. This is due to the fact that with the given demand, capacity set A has more slack/redundant capacity that is available when links in the network are subject to partial degradation. Furthermore, the above analysis also has implication for policy-making and planning. Obviously, an effective policy should keep the network robustness above a certain critical value. For example, in the case of capacity set A, the network robustness drops significantly when the capacity retention ratio is below 0.30. Hence, appropriate maintenance measures need to be taken in order to maintain the capacity retention ratio above 0.30.

The Braess network. We now consider the Braess paradox network after the addition of a new link e and as depicted in fig. 3 (see also [25] and [26]). There are four nodes: 1, 2, 3, 4; five links: a, b, c, d, e; and a single O/D pair $w_1 = (1, 4)$. There are, hence, three paths connecting

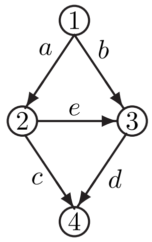

Fig. 3: The Braess network topology.

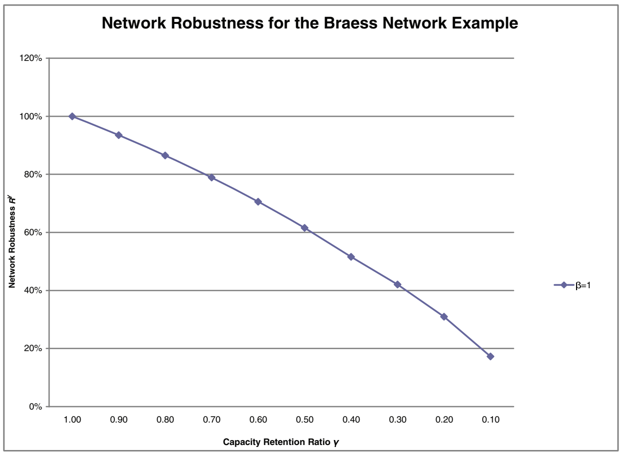

Fig. 4: Robustness *vs.* capacity retention ratio ($\beta = 1$).

the single O/D pair, which are denoted, respectively, by $p_1 = (a, c)$, $p_2 = (b, d)$, and $p_3 = (a, e, d)$.

Instead of using the original link cost functions, however, since they are not in BPR form, we construct a set of BPR functions under which the Braess paradox still occurs (without any capacity reduction). We assume that $k = 1$. Let $t_a^0 = t_d^0 = 1$, $t_b^0 = t_c^0 = 50$ and $t_e^0 = 10$. Furthermore, let $u_a = u_d = 20$, $u_b = u_c = 50$ and $u_e = 100$. The link cost functions are given by

$$c_a(f_a) = 1 + \left(\frac{f_a}{20}\right)^{\beta}, \ c_b(f_b) = 50\left(1 + \left(\frac{f_b}{50}\right)^{\beta}\right),$$

$$c_c(f_c) = 50\left(1 + \left(\frac{f_b}{50}\right)^{\beta}\right), \ c_d(f_d) = 1 + \left(\frac{f_d}{20}\right)^{\beta},$$

$$c_e(f_e) = 10\left(1 + \left(\frac{f_e}{100}\right)^{\beta}\right).$$

The demand is given by $d_{w_1} = 110$. Figures 4 through 7 present the network robustness for the Braess network under β values equal to 1, to 2, to 3, and to 4, respectively.

From the above example, we see that, for a given capacity retention ratio, when the value of β is small, the robustness of the network drops less severely than when β is large. This is due to the fact that β indicates, in part, the effect of congestion on links in a network. Therefore, for a certain capacity reduction, a "less congestion-sensitive" network can keep its efficiency closer to the original value.

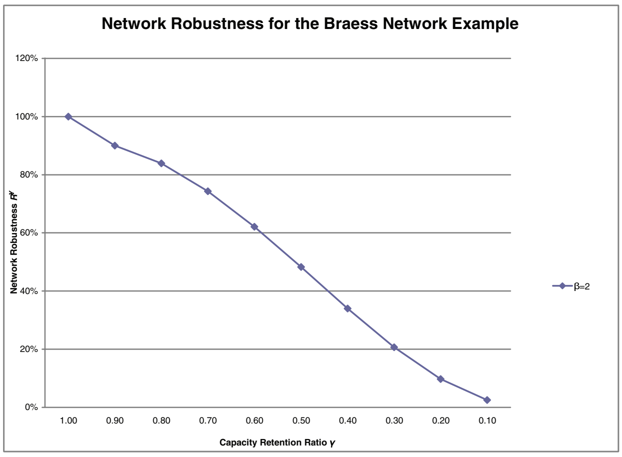

Fig. 5: Robustness *vs.* capacity retention ratio ($\beta = 2$).

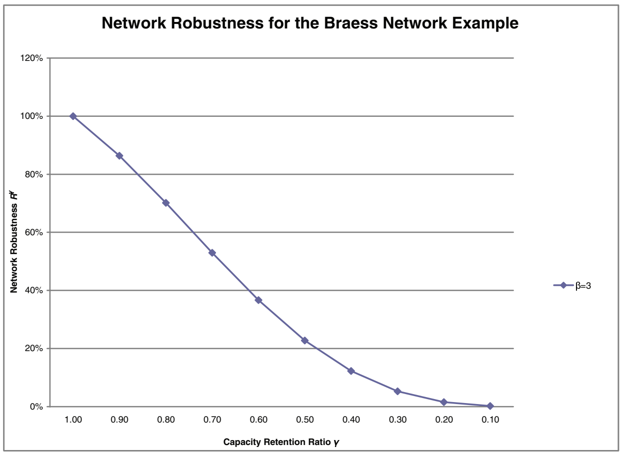

Fig. 6: Robustness *vs.* capacity retention ratio ($\beta = 3$).

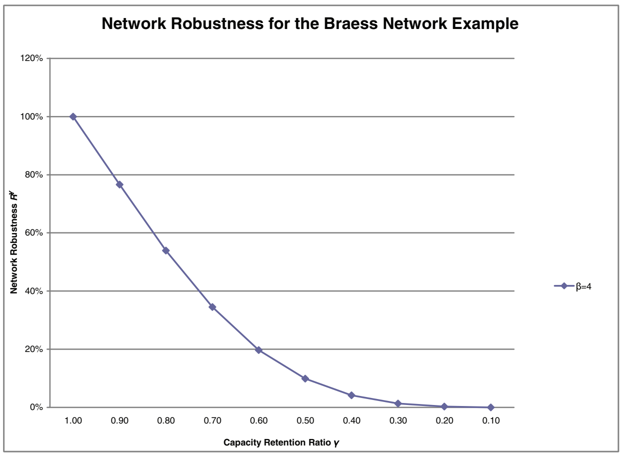

Fig. 7: Robustness *vs.* capacity retention ratio ($\beta = 4$).

Some theoretical results. – In this section, we consider transportation networks with special structure for which we can obtain some theoretical results in terms of robustness. In particular, we first consider a very simple network with BPR functions given by (8) for any β. We then consider networks of special topology consisting of a single O/D pair with parallel links for which the associated BPR link cost functions have $\beta=1$.

Theorem 1
Consider a network consisting of two nodes 1 and 2 as in fig. 1, which are connected by a single link a and with a single O/D pair $w_1=(1,2)$. Assume that the user link cost function associated with link a is of the BPR form given by (8). Then the network robustness given by the expression (cf. (10)) is given by the explicit formula:

$$\mathcal{R}^\gamma=\frac{\gamma^\beta[u_a^\beta+kd_{w_1}^\beta]}{[\gamma^\beta u_a^\beta+kd_{w_1}^\beta]}\times 100\%, \tag{11}$$

where d_{w_1} is the given demand for O/D pair $w_1=(1,2)$.

Moreover, the network robustness $\mathcal{R}$ is bounded from below by $\gamma^\beta\times 100\%$.

Proof: Clearly, since there is only a single O/D pair, and a single path, we have that

$$\mathcal{R}^\gamma=\frac{\mathcal{E}^\gamma}{\mathcal{E}}\times 100\%=\frac{\lambda_{w_1}}{\lambda_{w_1}^\gamma}\times 100\%,$$

where $\lambda_{w_1}^\gamma$ denotes the incurred travel disutility for travelers between O/D pair w_1 with the capacity γu_a on link a.

We can write out $\lambda_{w_1}^\gamma$ and λ_{w_1} explicitly for this simple network, which yields

$$\mathcal{R}^\gamma=\frac{t_a^0\left[1+k\left(\frac{d_{w_1}}{u_a}\right)^\beta\right]}{t_a^0\left[1+k\left(\frac{d_{w_1}}{\gamma u_a}\right)^\beta\right]}\times 100\%.$$

After simplification, we obtain

$$\mathcal{R}^\gamma=\frac{\gamma^\beta[u_a^\beta+kd_{w_1}^\beta]}{[\gamma^\beta u_a^\beta+kd_{w_1}^\beta]}\times 100\%,$$

which is exactly the form of (11).

To show the lower bound of $\mathcal{R}^\gamma$, we can rearrange (11) and get the following form:

$$\mathcal{R}^\gamma=\frac{\gamma^\beta\left[1+k\left(\frac{d_{w_1}}{u_a}\right)^\beta\right]}{\left[\gamma^\beta+k\left(\frac{d_{w_1}}{u_a}\right)^\beta\right]}\times 100\%.$$

Since $\gamma\in(0,1]$, we have that $\gamma^\beta\in(0,1]$, $\forall\beta>0$. Hence, we have the following:

$$\mathcal{R}^\gamma\geqslant\frac{\gamma^\beta\left[1+k\left(\frac{d_{w_1}}{u_a}\right)^\beta\right]}{\left[1+k\left(\frac{d_{w_1}}{u_a}\right)^\beta\right]}\times 100\%=\gamma^\beta\times 100\%,$$

which completes the proof.

Now let us consider a network with a special topology as depicted in fig. 8. The network consists of a single O/D pair which is connected by parallel links. In the following theorem, we give the general form of the network robustness as well as its lower bound for the above network.

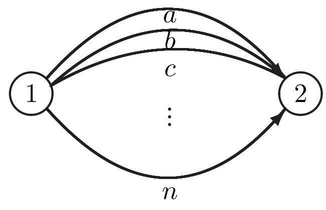

Fig. 8: A special network.

Theorem 2
Consider a network consisting of two nodes 1 and 2 as in fig. 8, which are connected by a set of parallel links. Assume that the associated BPR link cost functions have $\beta=1$ (cf. (8)). Furthermore, let us assume that there are positive flows on all the links at both the original and partially degraded capacity levels. Then the network robustness given by the expression (cf. (10)) is given by the explicit formula

$$\mathcal{R}^\gamma=\frac{\gamma U+k\gamma d_{w_1}}{\gamma U+kd_{w_1}}\times 100\%, \tag{12}$$

where d_{w_1} is the given demand for O/D pair $w_1=(1,2)$ and $U\equiv u_a+u_b+\ldots+u_n$.

Moreover, the network robustness $\mathcal{R}^\gamma$ is bounded from below by $\gamma\times 100\%$.

Proof: Clearly, since there is only a single O/D pair, we have that

$$\mathcal{R}^\gamma=\frac{\mathcal{E}^\gamma}{\mathcal{E}}\times 100\%=\frac{\lambda_{w_1}}{\lambda_{w_1}^\gamma}\times 100\%,$$

where $\lambda_{w_1}^\gamma$ denotes the incurred travel disutility for travelers between O/D pair w_1 under the capacity retention ratio γ.

Due to the special structure of the network as well as the assumption that there are positive flows on all the links before and after the capacity reduction, by referring to the traffic network equilibrium conditions (6), we can write λ_{w_1} and $\lambda_{w_1}^\gamma$ explicitly as follows:

$$\lambda_{w_1}=t_a^0\left(1+k\frac{f_a^*}{u_a}\right)=t_b^0\left(1+k\frac{f_b^*}{u_b}\right)=\ldots=t_n^0\left(1+k\frac{f_n^*}{u_n}\right),$$

where $f_a^*,\ f_b^*\ldots f_n^*$ are the equilibrium link flows under the link capacities: $u_a,\ u_b,\ldots,u_n$, respectively, and

$$\lambda_{w_1}^\gamma=t_a^0\left(1+k\frac{f_a^{**}}{\gamma u_a}\right)=t_b^0\left(1+k\frac{f_b^{**}}{\gamma u_b}\right)=\ldots=t_n^0\left(1+k\frac{f_n^{**}}{\gamma u_n}\right),$$

where $f_a^{**},\ f_b^{**},\ldots,f_n^{**}$ are the equilibrium link flows under the link capacities: $\gamma u_a,\ \gamma u_b,\ldots,\gamma u_n$, respectively.

Hence, we have

$$\mathcal{R}^\gamma = \frac{\lambda_{w_1}}{\lambda^\gamma_{w_1}} \times 100\% = \frac{t^0_a\left(1+k\frac{f^*_a}{u_a}\right)}{t^0_a\left(1+k\frac{f^{**}_a}{\gamma u_a}\right)} \times 100\% = \frac{t^0_b\left(1+k\frac{f^*_b}{u_b}\right)}{t^0_b\left(1+k\frac{f^{**}_b}{\gamma u_b}\right)} \times 100\% = \cdots = \frac{t^0_n\left(1+k\frac{f^*_n}{u_n}\right)}{t^0_n\left(1+k\frac{f^{**}_n}{\gamma u_n}\right)} \times 100\%,$$

which yields

$$\mathcal{R}^\gamma = \frac{\left(1+k\frac{f^*_a}{u_a}\right)+\left(1+k\frac{f^*_b}{u_b}\right)+\cdots+\left(1+k\frac{f^*_n}{u_n}\right)}{\left(1+k\frac{f^{**}_a}{\gamma u_a}\right)+\left(1+k\frac{f^{**}_b}{\gamma u_b}\right)+\cdots+\left(1+k\frac{f^{**}_n}{\gamma u_n}\right)} \times 100\%.$$

After some simplification and from the fact that $f^*_a + f^*_b + \ldots + f^*_n = f^{**}_a + f^{**}_b + \ldots + f^{**}_n = d_{w_1}$, we have

$$\mathcal{R}^\gamma = \frac{\gamma U + k\gamma d_{w_1}}{\gamma U + k d_{w_1}} \times 100\%,$$

which is exactly the form of (12).

To show the lower bound of the network robustness, we can rearrange (12) and get the following form:

$$\mathcal{R}^\gamma = \frac{\gamma\left(1+k\frac{d_{w_1}}{U}\right)}{\gamma + k\frac{d_{w_1}}{U}} \times 100\%.$$

Since $\gamma \in (0,1]$, we have the following:

$$\mathcal{R}^\gamma \geqslant \frac{\gamma\left(1+k\frac{d_{w_1}}{U}\right)}{\left(1+k\frac{d_{w_1}}{U}\right)} \times 100\% = \gamma \times 100\%,$$

which completes the proof.

Summary of the results. – In this paper, we presented a rigorous measure of transportation network robustness based on our earlier proposed network efficiency measure. We provided numerical examples and established some theoretical results. In the future, it would be very interesting to consider whether methods of robust optimization could be applied to assess transportation network robustness in the case of uncertain underlying data.

* * *

This research was supported by the John F. Smith Memorial Fund at the University of Massachusetts at Amherst. This support is gratefully appreciated. The authors acknowledge the helpful comments and suggestions of two anonymous reviewers on an earlier version of this paper.

REFERENCES

[1] Beckmann M. J., McGuire C. B. and Winsten C. B., *Studies in the Economics of Transportation* (Yale University Press, New Haven, Connecticut) 1956.

[2] American Society of Civil Engineers, *Report Card for America's Infrastructure* (American Society of Civil Engineers) 2005.

[3] Sakakibara H., Kajitani Y. and Okada N., *J. Transp. Engin.*, **130** (2004) 560.

[4] Scott D. M., Novak D., Aultman-Hall L. and Guo F., *J. Transp. Engin.*, **14** (2006) 215.

[5] Institute of Electrical and Electronics Engineers, *IEEE Standard Computer Dictionary: A Compilation of IEEE Standard Computer Glossaries* (Institute of Electrical and Electronics Engineers) 1990.

[6] Gribble S. D., *Proceedings of the 8th Workshop on Hot Topics in Operating Systems (HotOS-VIII)* (IEEE Computer Society) 2001, p. 21.

[7] Ali S., Maciejewski A. A., Siegel H. J. and Kim J., *Parallel and Distributed Processing Symposium* (IEEE Society) 2003.

[8] Schillo M., Bürchert H., Fischer K. and Klusch M., *Proceedings of 5th International Conference on Autonomous Agents* (ACM) 2001, p. 75.

[9] Holmgren Å. J., *A framework for vulnerability assessment of electric power systems*, in *Reliability and Vulnerability in Critical Infrastructure: A Quantitative Geographic Perspective*, edited by Murray A. and Grubesic T. (Springer-Verlag) 2007.

[10] Albert R., Jeong H. and Barabasi A-L., *Nature*, **406** (2000) 378.

[11] Soyster A. L., *Oper. Res.*, **21** (1973) 1154.

[12] Ben-Tal A. and Nemirovsky A., *Math. Oper. Res.*, **23** (1998) 769.

[13] Ben-Tal A. and Nemirovsky A., *Oper. Res. Lett.*, **25** (1999) 1.

[14] Nagurney A. and Qiang Q., *Europhys. Lett.*, **79** (2007) 38005.

[15] Nagurney A., *Sustainable Transportation Networks* (Edward Elgar Publishing, Cheltenham, England) 2000.

[16] Nagurney A., *Supply Chain Networks Economics: Dynamics of Prices Flows and Profits* (Edward Elgar Publishing, Cheltenham, England) 2006.

[17] Nagurney A., Liu Z., Cojocaru M.-G. and Daniele P., *Transp. Res. E*, **43** (2007) 624.

[18] Dafermos S. C. and Sparrow F. T., *J. Res. Natl. Bur. Stand., Sect. B*, **73** (1969) 91.

[19] Smith M., *Transp. Res. B*, **13** (1979) 259.

[20] Dafermos S. C., *Transp. Sci.*, **14** (1980) 42.

[21] Sheffi Y., *Urban Transportation Networks - Equilibrim Analysis with Mathematical Programming Methods* (Prentice-Hall, Englewood Cliffs, NJ) 1985.

[22] Nagurney A., *Network Economics: A Variational Inequality Approach* (Kluwer Academic Publishers, Dordrecht, The Netherlands) 1999.

[23] Boyce D. E., Mahmassani H. S. and Nagurney A., *Pap. Reg. Sci.*, **84** (2005) 85.

[24] Bureau of Public Roads, *Traffic Assignment Manual* (U.S. Department of Commerce, Urban Planning Division, Washington DC) 1964.

[25] Braess D., *Unternehmensforschung*, **12** (1968) 258.

[26] Braess D., Nagurney A. and Wakolbinger T., *Transp. Sci.*, **39** (2005) 446.