Dynamics of Supply Chains:
A Multilevel (Logistical/Informational/Financial)
Network Perspective

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Abstract: In this paper, we propose a multilevel network perspective for the conceptualization of the dynamics underlying supply chains in the presence of competition. The multilevel network consists of: the logistical network, the informational network, and the financial network. We describe the behavior of the network decision-makers, which are spatially separated, and which consist of the manufacturers/producing firms, the retailers, and the consumers located at the demand markets. We propose a projected dynamical system, along with stability analysis results, that captures the adjustments of the commodity shipments and the prices over space and time. A discrete-time adjustment process is described and implemented in order to illustrate the evolution of the commodity shipments and prices to the equilibrium solution in several numerical examples.

1. Introduction

Networks are fundamental to the functioning of today's societies and economies and come in many forms: transportation networks, logistical networks, telecommunication networks, as well as a variety of economic networks, including financial networks. Increasingly, these networks are interrelated. In the case of electronic commerce, orders over the Internet trigger shipments over logistical and transportation networks, and financial payments, in turn, over a financial network. Teleshopping as well as telecommuting are examples of applications in which economic activities are conducted on interrelated networks, in particular, transportation and telecommunication networks.

Recently, there has been growing interest in the development of theoretical frameworks for the unified treatment of such *complex network systems* (cf. Nagurney and Dong (2002) and Nagurney, Dong, and Mokhtarian (2001, 2002)) with the focus being principally the study of teleshopping versus shopping and telecommuting versus commuting decision-making. In this paper, in contrast, we tackle another problem entirely – that of supply chain analysis, a topic of interdisciplinary nature and one which has received great attention (cf. Bramel and Simchi-Levi (1997), Poirier (1996, 1999), Stadtler and Kilger (2000), Mentzer (2000), Bovet (2000), and Miller (2001), and the references therein). Our approach, in contrast to that of other authors, which tends to be an optimization approach, focuses on both the equilibrium (see also Nagurney, Dong, and Zhang (2002) and Nagurney, Loo, Dong, and Zhang (2001)), as well as the disequilibrium aspects of supply chains under competition. However, rather than formulating such a problem over a single network, as was done by Nagurney, Dong, and Zhang (2002) and Nagurney, Loo, Dong, and Zhang (2001), who proposed static models of supply chain networks under competition, we propose a *multilevel* network framework and address the dynamics.

The novelty of the proposed multilevel network framework allows one to capture distinct flows, in particular, logistical, informational, and financial within the same network system, while retaining the spatial nature of the network decision-makers. Moreover, since both the logistical and financial networks are hierarchical, we are able to observe, through a discrete-time process, how the prices as well as the commodity shipments are adjusted from iteration to iteration (time period to time period), until the equilibrium state is reached. Although we focus on a supply chain consisting of competing producers/manufacturers, retailers, and consumers, the framework is sufficiently general to include other levels of decision-makers in the network such as owners of distribution centers, for example.

We assume that the manufacturers, who are spatially separated, are involved in the production of a homogeneous commodity which is then shipped to the retailers, which are located at distinct locations. The manufacturers obtain a price for the product (which is endogenous) and seek to determine their optimal production and shipment quantities, given the production costs as well as the *transaction* costs associated with conducting business with the different retailers. Note that we consider a transaction cost to be sufficiently general, for example, to include transportation/shipping cost.

The retailers, in turn, must agree with the manufacturers as to the volume of shipments since they are faced with the handling cost associated with having the product in their retail outlet. In addition, they seek to maximize their profits with the price that the consumers are willing to pay for the product being endogenous.

Finally, in this supply chain, the consumers provide the "pull" in that, given the demand functions at the various demand markets, they determine their optimal consumption levels from the various retailers subject both to the prices charged for the product as well as the cost of conducting the transaction. The demand markets are "separated" spatially from the retailers through the inclusion of explicit transaction costs which also include the cost associated with transportation.

The paper is organized as follows. In Section 2, we present the multilevel network representing the supply chain system and consisting of the logistical, the informational, and the financial networks. We propose the disequilibrium dynamics underlying the supply chain problem, derive the projected dynamical system, and discuss the stationary/equilibrium point. In Section 3, we provide some qualitative properties of the dynamic trajectories, along with stability analysis results. In Section 4, we describe the discrete-time adjustment process, which is a time discretization of the continuous adjustment process given in Section 2.

In Section 5, we implement the discrete-time adjustment process and apply it to several numerical examples. A summary and suggestions for future research are given in Section 6.

2. The Dynamic Supply Chain Model

In this Section, we develop the dynamic supply chain model with manufacturers, retailers, and consumers using a multilevel network perspective. This multilevel network consists of a logistical network, an informational network, and a financial network. We first identify the multilevel network structure of the problem and the corresponding flows and prices. We then describe the underlying functions and the behavior of the various network decision-makers, who are spatially separated and consist of the manufacturers, the retailers, and the consumers located at the demand markets. Our perspective is an equilibrium one, since we believe that, in an environment of competition, an equilibrium state provides a valuable benchmark. We also provide a mechanism for describing and understanding the disequilibrium dynamics.

We now describe the structure of the supply chain as well as the distinct networks and associated flows that make up the entire system. We assume that there are m manufacturers involved in the production of a homogeneous commodity, which can then be sold and shipped to n retailers, and, finally, purchased (and consumed) by consumers at o demand markets. Both the manufacturers as well as the consumers at the demand markets can be located at distinct spatial locations. We denote a typical manufacturer by i, a typical retailer by j, and a typical demand market by k. Let q_{ij} denote the nonnegative volume of commodity shipment between manufacturer i and retailer j and let q_{jk} denote the nonnegative volume of the commodity between retailer j and consumers at demand market k. We group the commodity shipments between the manufacturers and the retailers into the column vector $Q^1 \in R^{mn}_+$ and the commodity shipments between the retailers and the demand markets into the column vector $Q^2 \in R^{no}_+$. Let q_i denote the amount of the commodity produced by manufacturer i and group the production outputs of all manufacturers into the column vector $q \in R^m_+$.

The logistical network (cf. Figure 1) is the bottom network of the multilevel network for the supply chain model. Specifically, the logistical network represents the commodity production outputs and the shipments between the network agents, that is, between the manufacturers and the retailers, and the retailers and the demand markets. As depicted in Figure 1, the top tier of nodes of the logistical network consists of the manufacturers, the middle tier consists of nodes of the retailers, and the bottom tier, of the demand markets.

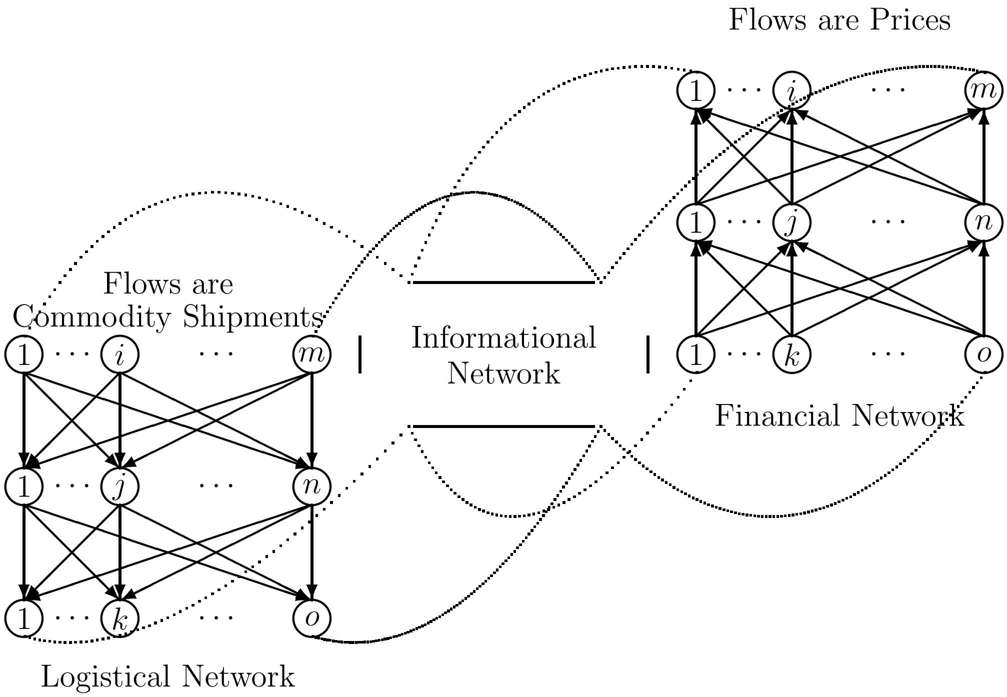

Figure 1: Multilevel Network Structure of the Supply Chain System

The links joining two tiers of nodes correspond to transactions between the nodes in the supply chain that take place. The flows on the links in the logistical network correspond to the commodity shipments with the flow on a link (i, j) joining a node i in the top tier with node j in the middle tier given by q_{ij} and the flow on a link (j, k) joining node j at the middle tier with node k at the bottom tier by q_{jk}.

The financial network, in turn, is the top network in the multilevel framework shown in Figure 1 and its flows are the prices associated with the commodity. This financial network also has a three-tiered nodal structure as does the logistical network with the nodes corresponding to distinct decision-makers as before. However, the links in the financial network go in the opposite direction from those in the logistical network. This reflects both the "bottom up" approach of our model, in that the consumers provide the "pull" through the prices they are willing to pay for the product, as well as representing the payments for the commodity, which move in an upward direction.

Specifically, we let ρ_{1ij} denote the price of the commodity of manufacturer i (located at tier 1 of the financial network) associated with retailer j, and group the manufacturer's prices for the product (at the beginning of the supply chain) into the column vector $\rho_{1i} \in R^n_+$. We then further group all the manufacturers' prices into the column vector $\rho_1 \in R^{mn}_+$. We denote the price charged by retailer j, located at tier 2, by ρ_{2j} and we group the retailers' prices into the vector $\rho_2 \in R^n_+$. Finally, we let ρ_{3k} denote the true price of the commodity as perceived by consumers located at demand market k at the third tier of nodes and group these prices into the column vector $\rho_3 \in R^o_+$.

Central to the multilevel network perspective of the supply chain model is the Informational Network depicted between the Logistical and Financial Networks in Figure 1. Note that the links in the Informational Network are bidirectional since the informational network, as we shall subsequently describe, stores and provides the commodity shipment and price information over time, which allows for the recomputation of the new commodity shipments and prices, until, ultimately, the equilibrium pattern is attained.

The Dynamics

We now turn to describing the dynamics by which the manufacturers adjust their commodity shipments over time, the consumers adjust their consumption amounts based on the prices of the product at the demand markets, and the retailers operate between the two. We also describe the dynamics by which the prices adjust over time. The commodity shipment flows evolve over time on the logistical network, whereas the prices do so over the financial network. The informational network stores and provides the commodity shipment and price information so that the new commodity shipments and prices can be computed. The dynamics are derived from the bottom tier of nodes on up since, as mentioned previously, we assume that it is the demand for the product (and the corresponding prices) that actually drives the supply chain dynamics.

The Demand Market Price Dynamics

We begin by describing the dynamics underlying the prices of the product associated with the demand markets (see the bottom-tiered nodes in the financial network). We assume, as given, a demand function d_k, which can depend, in general, upon the entire vector of prices

ρ_3, that is,

$$d_k = d_k(\rho_3), \quad \forall k. \tag{1}$$

Moreover, we assume that the rate of change of the price ρ_{3k}, denoted by $\dot{\rho}_{3k}$, is equal to the difference between the demand at the demand market k, as a function of the demand market prices, and the amount available from the retailers at the demand market. Hence, if the demand for the product at the demand market (at an instant in time) exceeds the amount available, the price at that demand market will increase; if the amount available exceeds the demand at the price, then the price at the demand market will decrease. Furthermore, we guarantee that the prices do not become negative. Hence, the dynamics of the price ρ_{3k} associated with the commodity at demand market k can be expressed as:

$$\dot{\rho}_{3k} = \begin{cases} d_k(\rho_3) - \sum_{j=1}^n q_{jk}, & \text{if} \quad \rho_{3k} > 0 \\ \max\{0, d_k(\rho_3) - \sum_{j=1}^n q_{jk}\}, & \text{if} \quad \rho_{3k} = 0. \end{cases} \tag{2}$$

The Dynamics of the Commodity Shipments Between the Retailers and the Demand Markets

We now describe the dynamics of the commodity shipments of the logistical network taking place over the links joining the retailers to the demand markets. We assume that retailer j has a unit transaction cost c_{jk} associated with transacting with consumers at demand market k where

$$c_{jk} = c_{jk}(Q^2), \quad \forall j, k, \tag{3}$$

that is, the unit cost associated with transacting can, in general, depend upon the vector Q^2. Note that this unit cost is assumed to also include the transportation cost associated with consumers at demand market k obtaining the commodity from retailer j.

We assume that the rate of change of the commodity shipment q_{jk} is equal to the difference between the price the consumers are willing to pay for the product at demand market k minus the unit transaction cost and the price charged for the product at the retail outlet. Of course, we also must guarantee that these commodity shipments do not become negative. Hence, we may write:

$$\dot{q}_{jk} = \begin{cases} \rho_{3k} - c_{jk}(Q^2) - \rho_{2j}, & \text{if} \quad q_{jk} > 0 \\ \max\{0, \rho_{3k} - c_{jk}(Q^2) - \rho_{2j}\}, & \text{if} \quad q_{jk} = 0. \end{cases} \tag{4}$$

Thus, according to (4), if the price the consumers are willing to pay for the product at a demand market exceeds the price the retailers charge for the product at the outlet plus the unit transaction cost (at an instant in time), then the volume of the commodity between that retail and demand market pair will increase; if the price charged by the retailer plus the transaction cost exceeds the price the consumers are will to pay, then the volume of flow of the commodity between that pair will decrease.

The Dynamics of the Prices at the Retail Outlets

The prices charged for the product at the retail outlets, in turn, must reflect supply and demand conditions as well. In particular, we assume that the price for the product associated with retail outlet j, ρ_{2j}, and computed at node j lying in the second tier of nodes of the financial network, evolves over time according to:

$$\dot{\rho}_{2j} = \begin{cases} \sum_{k=1}^{o} q_{jk} - \sum_{i=1}^{m} q_{ij}, & \text{if} \quad \rho_{2j} > 0 \\ \max\{0, \sum_{k=1}^{o} q_{jk} - \sum_{i=1}^{m} q_{ij}\}, & \text{if} \quad \rho_{2j} = 0. \end{cases} \tag{5}$$

Hence, if the amount of the commodity desired to be transacted by the consumers (at an instant in time) exceeds that available at the retail outlet, then the price charged at the retail outlet will increase; if the amount available is greater than that desired by the consumers, then the price charged at the retail outlet will decrease.

The Dynamics of Commodity Shipments Between Manufacturers and Retailers

We now describe the dynamics underlying the commodity shipments between the manfacturers and the retailers. We assume that each manufacturer i is faced with a production cost function f_i, which can depend, in general, on the entire vector of production outputs. Since, according to the conservation of flow equations:

$$q_i = \sum_{j=1}^{n} q_{ij}, \quad \forall i, \tag{6}$$

without any loss in generality, we may write the production cost functions as a function of the vector Q^1, that is, we have that

$$f_i = f_i(Q^1), \quad \forall i. \tag{7}$$

In addition, we associate with each manufacturer and retailer pair (i, j) a transaction cost, denoted by c_{ij}. The transaction cost includes the cost of shipping the commodity. We assume that the transaction costs are of the form:

$$c_{ij} = c_{ij}(q_{ij}), \quad \forall i, j. \tag{8}$$

The total costs incurred by a manufacturer i, thus, are equal to the sum of the manufacturer's production cost plus the total transaction costs. Its revenue, in turn, is equal to the price that the manufacturer charges for the product to the retailers (and the retailers are willing to pay) times the quantity of the commodity obtained/purchased the manufacturer by the retail outlets.

We assume that a fair price for the product for a given manufacturer is equal to its marginal costs of production and transacting, that is: $\frac{\partial f_i(Q^1)}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij})}{\partial q_{ij}}$.

A retailer j, in turn, is faced with what we term a *handling* cost, which may include, for example, the display and storage cost associated with the product. We denote this cost by c_j and, in the simplest case, we would have that c_j is a function of $\sum_{i=1}^{m} q_{ij}$, that is, the holding cost of a retailer is a function of how much of the product he has obtained from the various manufacturers. However, for the sake of generality, and to enhance the modeling of competition, we allow the function to, in general, depend also on the amounts of the product held by other retailers and, therefore, we may write:

$$c_j = c_j(Q^1), \quad \forall j. \tag{9}$$

The retailer j, on the other hand, ideally, would accept a commodity shipment from manufacturer i at a price that is equal to the price charged at the retail outlet for the commodity (and that the consumers are willing to pay) minus its marginal cost associated with handling the product. Now, since the commodity shipments sent from the manufacturers must be accepted by the retailers in order for the transactions to take place in the supply chain, we propose the following rate of change for the commodity shipments between the top tier of nodes and the middle tier in the logistical network:

$$\dot{q}_{ij} = \begin{cases} \rho_{2j} - \frac{\partial c_j(Q^1)}{\partial q_{ij}} - \frac{\partial f_i(Q^1)}{\partial q_{ij}} - \frac{\partial c_{ij}(q_{ij})}{\partial q_{ij}}, & \text{if} \quad q_{ij} > 0 \\ \max\{0, \rho_{2j} - \frac{\partial c_j(Q^1)}{\partial q_{ij}} - \frac{\partial f_i(Q^1)}{\partial q_{ij}} - \frac{\partial c_{ij}(q_{ij})}{\partial q_{ij}}\}, & \text{if} \quad q_{ij} = 0. \end{cases} \tag{10}$$

Following the above discussion, (10) states that the commodity shipment between a manufacturer/retailer pair evolves according to the difference between the price charged for the product by the retailer and its marginal cost, and the price charged by the manufacturer (which recall, assuming profit-maximizing behavior, was set to the marginal cost of production plus its marginal cost of transacting with the retailer). We also guarantee that the commodity shipments do not become negative as they evolve over time.

The Projected Dynamical System

Consider now the dynamic model in which the demand prices evolve according to (2) for all demand market prices k, the retail/demand market commodity shipments evolve according to (4) for all pairs of retailers/demand markets j, k, the retail prices evolve according to (5) for all retailers j, and the commodity shipments between the manufacturers and retailers evolve over time according to (10) for all manufacturer/retailer pairs i, j. Let now X denote the aggregate column vector $(Q^1, Q^2, \rho_2, \rho_3)$ in the feasible set $K \equiv R_+^{mn+no+n+o}$. Define the column vector $F(X) \equiv (F_{ij}, F_{jk}, F_j, F_k)_{i=1,\ldots,m;j=1,\ldots,n;k=1,\ldots,o}$, where $F_{ij} \equiv \frac{\partial f_i(Q^1)}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij})}{\partial q_{ij}} + \frac{\partial c_j(Q^1)}{\partial q_{ij}} - \rho_{2j}$; $F_{jk} \equiv \rho_{2j} + c_{jk}(Q^2) - \rho_{3k}$; $F_j \equiv \sum_{i=1}^m q_{ij} - \sum_{k=1}^o q_{jk}$, and $F_k \equiv \sum_{j=1}^n q_{jk} - d_k(\rho_3)$.

Then the dynamic model described by (2), (4), (5), and (10) for all k, j, i can be rewritten as the projected dynamical system (PDS) (cf. Nagurney and Zhang (1996)) defined by the following initial value problem:

$$\dot{X} = \Pi_K(X, -F(X)), \quad X(0) = X_0, \tag{11}$$

where Π_K is the projection operator of $-F(X)$ onto K at X and $X_0 = ({Q^1}^0, {Q^2}^0, \rho_2^0, \rho_3^0)$ is the initial point corresponding to the initial commodity shipments between the manufacturers and the retailers, and the retailers and the demand markets, and the initial retailers' prices and demand prices. The trajectory of (11) describes the dynamic evolution of and the dynamic interactions among the commodity shipments between the tiers of the logistical network and the prices on the financial network. The informational network, in turn, stores and provides the commodity shipment and price information over time as needed for the dynamic evolution of the supply chain transactions.

We emphasize that the dynamical system (11) is non-classical in that the right-hand side is discontinuous in order to guarantee that the constraints, which in the context of the above model are nonnegativity constraints on the variables, are not violated. Such dynamical systems were introduced by Dupuis and Nagurney (1993) and to-date have been used to model a variety of applications ranging from dynamic traffic network problems (cf. Nagurney and Zhang (1997)) and oligopoly problems (see Nagurney, Dupuis, and Zhang (1994)) and spatial price equilibrium problems (cf. Nagurney, Takayama, and Zhang (1995)).

A Stationary/Equilibrium Point

We now discuss the stationary point of the projected dynamical system (11). Recall that a stationary point is that point when $\dot{X} = 0$ and, hence, in the context of our model, when there is no change in the commodity shipments in the logistical network and no change in the prices in the financial network. This point is also an equilibrium point and, furthermore, has a variational inequality formulation (see Nagurney and Zhang (1996), Nagurney (1999)). We identify an equilibrium point, henceforth, with an "*".

Note that the stationary point such that $\dot{\rho}_{3k} = 0$ for all demand markets k and $\dot{q}_{jk} = 0$ for all retail/demand market pairs j, k (see (2) and (4), respectively) coincides with the following equilibrium conditions: For all retailers and demand markets: $j = 1, \ldots, n$; $k = 1, \ldots, o$, we must have that:

$$\rho_{2j}^* + c_{jk}(Q^{2*}) \begin{cases} = \rho_{3k}^*, & \text{if} \quad q_{jk}^* > 0 \\ \geq \rho_{3k}^*, & \text{if} \quad q_{jk}^* = 0, \end{cases} \tag{12}$$

and for all demand markets $k = 1, \ldots, o$, we must have that:

$$d_k(\rho_3^*) \begin{cases} = \sum_{j=1}^{n} q_{jk}^*, & \text{if} \quad \rho_{3k}^* > 0 \\ \leq \sum_{j=1}^{n} q_{jk}^*, & \text{if} \quad \rho_{3k}^* = 0. \end{cases} \tag{13}$$

Conditions (12) state that consumers at demand market k will purchase the product from retailer j, if the price charged by the retailer for the product plus the unit transaction cost does not exceed the price that the consumers are willing to pay for the product. Conditions (13) state, in turn, that if the price the consumers are willing to pay for the product at the demand market is positive, then the quantities purchased of the product from the retailers will be precisely equal to the demand for that product at the demand market. These conditions correspond to the well-known spatial price equilibrium conditions (cf. Samuelson (1952), Takayama and Judge (1971), Nagurney (1999) and the references therein).

In terms of the prices of the product at the retail outlets (cf. (5)), if $\dot{\rho}_{2j} = 0$ for all j, then the following equilibrium condition must be satisfied: For all retailers $j = 1, \ldots, n$, we must have that:

$$\sum_{i=1}^{m} q_{ij}^* - \sum_{k=1}^{o} q_{jk}^* \begin{cases} = 0, & \text{if} \quad \rho_{2j}^* > 0 \\ \geq 0, & \text{if} \quad \rho_{2j}^* = 0. \end{cases} \tag{14}$$

Conditions (14) state that the price of the product at a retail outlet j will be positive if the "supply" at the outlet, that is, $\sum_{i=1}^{m} q_{ij}^*$ of the product is equal to the "demand" at the outlet, that is, $\sum_{k=1}^{o} q_{jk}^*$; if the supply exceeds the demand, then the price will be zero at that outlet.

Finally, for all $\dot{q}_{ij}$ to be zero (cf. (10)), we must have: for all manufacturer/retailer pairs that: $i = 1, \ldots, m$; $j = 1, \ldots, n$:

$$\frac{\partial f_i(Q^{1*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} + \frac{\partial c_j(Q^{1*})}{\partial q_{ij}} - \rho_{2j}^* \begin{cases} = 0, & \text{if} \quad q_{ij}^* > 0 \\ \geq 0, & \text{if} \quad q_{ij}^* = 0. \end{cases} \tag{15}$$

Conditions (15) state that, in equilibrium, if there is a positive volume of commodity flow between a manufacturer/retailer pair, then the marginal cost of production plus the marginal cost of transacting and the marginal cost of handling the product must be equal to the price of the product at the retail outlet. If the marginal costs exceed the price, then there will be no commodity shipments between that pair.

In equilibrium, the conditions (12) – (15) are all satisfied simultaneously.

Furthermore, as established in Dupuis and Nagurney (1993), the set of stationary points of a projected dynamical system of the form given in (11) coincides with the set of solutions to the variational inequality problem given by: Determine $X^* \in K$, such that

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in K, \tag{16}$$

where, in our problem, $F(X)$ was defined prior to (11). Furthermore, the vector $X^* = (Q^{1*}, Q^{2*}, \rho_2^*, \rho_3^*)$ satisfying the equilibrium conditions (12) – (15) satisfies the variational inequality (16), which explicitly has the form;

$$
\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(Q^{1^*})}{\partial q_{ij}}+\frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}}+\frac{\partial c_j(Q^{1^*})}{\partial q_{ij}}-\rho_{2j}^*\right]\times\left[q_{ij}-q_{ij}^*\right]
$$

$$
+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\rho_{2j}^*+c_{jk}(Q^{2^*})-\rho_{3k}^*\right]\times\left[q_{jk}-q_{jk}^*\right]+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}q_{ij}^*-\sum_{k=1}^{o}q_{jk}^*\right]\times\left[\rho_{2j}-\rho_{2j}^*\right]
$$

$$
+\sum_{k=1}^{o}\left[\sum_{j=1}^{n}q_{jk}^*-d_k(\rho_3^*)\right]\times[\rho_{3k}-\rho_{3k}^*]\geq 0,\quad \forall(Q^1,Q^2,\rho_2,\rho_3)\in R_+^{mn+no+n+o}. \tag{17}
$$

Interestingly, Nagurney, Dong, and Zhang (2002) derived the same variational inequality formulation (using a slightly different notation for the price vector ρ_2) of their static supply chain model using, however, an entirely different approach from the one above. They also obtained existence and uniqueness results under reasonable conditions.

In the next Section, we provide some qualitative properties of the dynamic trajectories for (11). In particular, we establish the existence of a unique trajectory satisfying (11), as well as a global stability result.

3. Qualitative Properties

In this Section, we provide some qualitative properties. We first recall the definition of an additive production cost which was introduced in Zhang and Nagurney (1996) and also utilized by Nagurney, Dong, and Zhang (2002) and Nagurney, Loo, Dong, and Zhang (2001).

Definition 1: Additive Production Cost

Suppose that for each manufacturer i, the production cost f_i is additive, that is,

$$f_i(q) = f_i^1(q_i) + f_i^2(\bar{q}_i), \tag{18}$$

where $f_i^1(q_i)$ is the internal production cost that depends solely on the manufacturer's own output level q_i, which may include the production operation and the facility maintenance, etc., and $f_i^2(\bar{q}_i)$ is the interdependent part of the production cost that is a function of all the other manufacturers' output levels $\bar{q}_i = (q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_m)$ and reflects the impact of the other manufacturers' production patterns on manufacturer i's cost. This interdependent part of the production cost may describe the competition for the resources, consumption of the homogeneous raw materials, etc..

We now recall two results obtained by Nagurney, Dong, and Zhang (2002), which we will utilize to obtain qualitative properties of the dynamical system (11). In particular, we have:

Theorem 1: Monotonicity (Nagurney, Dong, and Zhang (2002))

Suppose that the production cost functions $f_i; i = 1, ..., m$, are additive, as defined in Definition 1, and $f_i^1; i = 1, ..., m$, are convex functions. If the c_{ij} and c_j functions are convex, the c_{jk} functions are monotone increasing, and the d_k functions are monotone decreasing functions of the demand market prices, for all i, j, k, then the vector function F that enters the variational inequality (16) is monotone, that is,

$$\langle F(X') - F(X''), X' - X'' \rangle \geq 0, \quad \forall X', X'' \in R_+^{mn+no+n+o}. \tag{19}$$

Theorem 2: Lipschitz Continuity (Nagurney, Dong, and Zhang (2002))

The function that enters the variational inequality problem (16) is Lipschitz continuous, that is,

$$\|F(X') - F(X'')\| \leq L\|X' - X''\|, \quad \forall X', X'' \in K, \tag{20}$$

under the following conditions:

(i). Each f_i; $i = 1, ..., m$, is additive and has a bounded second-order derivative;

(ii). c_{ij} and c_j have bounded second-order derivatives, for all i, j;

(iii). c_{jk} and d_k have bounded first-order derivatives.

We now state a fundamental property of the projected dynamical system (11).

Theorem 3: Existence and Uniqueness

Assume the conditions of Theorem 2. Then, for any $X_0 \in K$, there exists a unique solution $X_0(t)$ to the initial value problem (11).

Proof: Follows from Theorem 2.5 in Nagurney and Zhang (1996). □

We now turn to addressing the stability (see also Zhang and Nagurney (1995) and Nagurney and Zhang (2001)) of the supply chain network system through the initial value problem (11). We first recall the following:

Definition 2: Stability of the System

The system defined by (11) is stable if, for every X_0 and every equilibrium point X^*, the Euclidean distance $\|X^* - X_0(t)\|$ is a monotone nonincreasing function of time t.

We state a global stability result in the next theorem.

Theorem 4: Stability of the System

Assume the conditions of Theorem 1. Then the dynamical system (11) underlying the supply chain is stable.

Proof: Under the assumptions of Theorem 1, $F(X)$ is monotone and, hence, the conclusion follows directly from Theorem 4.1 of Zhang and Nagurney (1995). $\square$

4. The Discrete-Time Adjustment Process

Note that the projected dynamical system (11) is a continuous time adjustment process. However, in order to further fix ideas and to provide a means of "tracking" the trajectory, we propose a discrete-time adjustment process. The discrete-time adjustment process is a special case of the general iterative scheme of Dupuis and Nagurney (1993) and is, in fact, an Euler method, where at iteration τ the process takes the form:

$$X^{\tau} = P_K(X^{\tau-1} - \alpha_{\tau-1} F(X^{\tau-1})), \tag{21}$$

where P_K denotes the operator of projection (in the sense of the least Euclidean distance (cf. Nagurney (1999)) onto the closed convex set K and $F(X)$ is as defined preceding (11). Specifically, the complete statement of this method in the context of our model takes the form:

Step 0: Initialization Step

Set $(Q^{10}, Q^{20}, \rho_2^0, \rho_3^0) \in K$. Let $\tau = 1$ and set the sequence $\{\alpha_\tau\}$ so that $\sum_{\tau=1}^{\infty} \alpha_\tau = \infty$, $\alpha_\tau > 0$, $\alpha_\tau \to 0$, as $\tau \to \infty$. We note that the sequence $\{\alpha_\tau\}$ must satisfy the above-stated conditions (cf. Dupuis and Nagurney (1993)) for the scheme to converge.

Step 1: Computation Step

Compute $(Q^{1^\tau}, Q^{2^\tau}, \rho_2^\tau, \rho_3^\tau) \in K$ by solving the variational inequality

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left[q_{ij}^{\tau} + \alpha_\tau \left(\frac{\partial f_i(Q^{1^{\tau-1}})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^{\tau-1})}{\partial q_{ij}} + \frac{\partial c_j(Q^{1^{\tau-1}})}{\partial q_{ij}} - \rho_{2j}^{\tau-1} \right) - q_{ij}^{\tau-1} \right] \times \left[q_{ij} - q_{ij}^{\tau} \right]$$

$$+ \sum_{j=1}^{n} \sum_{k=1}^{o} \left[q_{jk}^{\tau} + \alpha_\tau (\rho_{2j}^{\tau-1} + c_{jk}(Q^{2^{\tau-1}}) - \rho_{3k}^{\tau-1}) - q_{jk}^{\tau-1} \right] \times \left[q_{jk} - q_{jk}^{\tau} \right]$$

$$+ \sum_{j=1}^{n} \left[\rho_{2j}^{\tau} + \alpha_\tau \left(\sum_{i=1}^{m} q_{ij}^{\tau-1} - \sum_{k=1}^{o} q_{jk}^{\tau-1} \right) - \rho_{2j}^{\tau-1} \right] \times \left[\rho_{2j} - \rho_{2j}^{\tau} \right]$$

$$+ \sum_{k=1}^{o} \left[\rho_{3k}^{\tau} + \alpha_\tau \left(\sum_{j=1}^{n} q_{jk}^{\tau-1} - d_k(\rho_3^{\tau-1}) \right) - \rho_{3k}^{\tau-1} \right] \times [\rho_{3k} - \rho_{3k}^{\tau}] \geq 0, \ \forall (Q^1, Q^2, \rho_2, \rho_3) \in K. \tag{22}$$

Step 2: Convergence Verification

If $|q_{ij}^\tau - q_{ij}^{\tau-1}| \leq \epsilon$, $|q_{jk}^\tau - q_{jk}^{\tau-1}| \leq \epsilon$, $|\rho_{2j}^\tau - \rho_{2j}^{\tau-1}| \leq \epsilon$, $|\rho_{3k}^\tau - \rho_{3k}^{\tau-1}| \leq \epsilon$, for all $i = 1, \cdots, m$; $j = 1, \cdots, n$; $k = 1, \cdots, o$, with $\epsilon > 0$, a pre-specified tolerance, then stop; otherwise, set $\tau := \tau + 1$, and go to Step 1.

The sequence $\{\alpha_\tau\}$ can be interpreted as a series of parameters that model a type of learning process in that, at the beginning of the discrete-time process, the terms are larger, whereas as time proceeds and the decision-makers have acquired more experience, the terms in the series decrease.

We note that since K is the nonnegative orthant the solution of (22) is accomplished exactly and in closed form as follows:

Computation of the Commodity Shipments:

At iteration τ compute the q_{ij}^τs according to:

$$q_{ij}^\tau = \max\{0, q_{ij}^{\tau-1} - \alpha_\tau(\frac{\partial f_i(Q^{1^{\tau-1}})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^{\tau-1})}{\partial q_{ij}} + \frac{\partial c_j(Q^{1^{\tau-1}})}{\partial q_{ij}} - \rho_{2j}^{\tau-1})\}, \quad \forall i, j. \tag{23}$$

Also, at iteration τ compute the q_{jk}^τs according to:

$$q_{jk}^\tau = \max\{0, q_{jk}^{\tau-1} - \alpha_\tau(\rho_{2j}^{\tau-1} + c_{jk}(Q^{2^{\tau-1}}) - \rho_{3k}^{\tau-1}\}, \quad \forall j, k. \tag{24}$$

Computation of the Prices:

The prices, ρ_{2j}^τ, in turn, are computed at iteration τ explicitly according to:

$$\rho_{2j}^\tau = \max\{0, \rho_{2j}^{\tau-1} - \alpha_\tau(\sum_{i=1}^m q_{ij}^{\tau-1} - \sum_{k=1}^o q_{jk}^{\tau-1})\}, \quad \forall j, \tag{25}$$

whereas the prices, ρ_{3k}, are computed according to:

$$\rho_{3k}^\tau = \max\{0, \rho_{3k}^{\tau-1} - \alpha_\tau(\sum_{j=1}^n q_{jk}^{\tau-1} - d_k(\rho_3^{\tau-1}))\}, \quad \forall k. \tag{26}$$

Discussion of Computations at the Logistical and Financial Network Levels and the Information Flows Through the Informational Network Level

We now discuss the above discrete-time adjustment process in the context of the multilevel network in Figure 1. The computation of the commodity shipments according to (23) and (24) takes place on the logistical network, whereas the computation of the prices at the tiers according to (25) and (26) takes place on the financial network.

Note that, according to (23), in order to compute the new commodity shipment between a manufacturer/retailer pair, the logistical network requires the price at the particular retailer from the preceding iteration as well as all the commodity shipments between the manufacturers and the retailers at the preceding iteration from the informational network. This type of computation can be done simultaneously for all pairs of manufacturers and retailers. Similarly, according to (24), in order to compute the new commodity shipment between a retailer and demand market pair, the logistical network requires from the informational network the prices at the particular retail outlet from the preceding iteration, and also the demand market price, as well as the commodity shipments between the retailers and the demand markets from the preceding iteration.

The financial network requires from the informational network, at a given iteration (or time period) for the computation of the retail prices (cf. (25)), the commodity shipments to and from the particular retailer at the preceding iteration, as well as the retail price at the retail outlet from the preceding iteration.

Finally, for the computation of the demand market price at a given iteration according to (26), the financial network requires from the informational network the demand prices for the product at all the demand markets from the preceding iteration and the commodity shipments at the preceding iteration from the retailers to the particular demand market. The computation of the demand market prices at all the demand markets are accomplished in a similar manner.

The informational network, thus, provides the information necessary for the computation of both the logistical flows and the financial flows at each iteration. Of course, different informational networks may provide information which is of different quality levels or timeliness but in this paper the focus is on the multilevel network concept, the dynamics, as well as what information is needed for the iterative process to converge.

Hence, according to the discrete-time adjustment process described above, the process is initialized with a vector of commodity shipments and prices. The informational network stores this information and then the logistical network and the financial network simultaneously compute the new commodity shipments between tiers of nodes and the new prices at the nodes, respectively. This information is then fed back to the informational network and transferred to the logistical and financial networks at the next iteration as needed. This process continues until convergence is reached, that is, until the absolute difference of the commodity shipments between two successive iterations and that of the prices between two successive iterations lies within an acceptable tolerance. Clearly, it is easy to see from (22) as well as from (23) – (26) that once the convergence tolerance has been reached (and, hence, these differences are approximately zero) then the equilibrium conditions (12) – (15) are satisfied; equivalently, a stationary point of the projected dynamical system (11) is attained, and, also a solution to variational inequality (16).

In Figure 2, the multilevel network with the equilibrium flows and prices is given. Note that in the financial network, we have explicitly labeled the prices ρ_1^* at the top tier of nodes in the financial network. Although we do not compute these prices explicitly through the discrete-time algorithm, they can, nevertheless, be recovered from the equilibrium solution, according to:

$$\rho_{1ij}^* = \frac{\partial f_i(Q^{1*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}} = \rho_{2j}^* - \frac{\partial c_j(Q^{1*})}{\partial q_{ij}}, \quad \forall i, j, \text{ such that } q_{ij}^* > 0;$$

otherwise, $\rho_{1ij}^* = \frac{\partial f_i(Q^{1*})}{\partial q_{ij}} + \frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}}$, since we have assumed profit-maximization behavior for both the manufacturers and the retailers.

Convergence conditions for this method can be found in Dupuis and Nagurney (1993) and studied in a variety of application contexts in Nagurney and Zhang (1996).

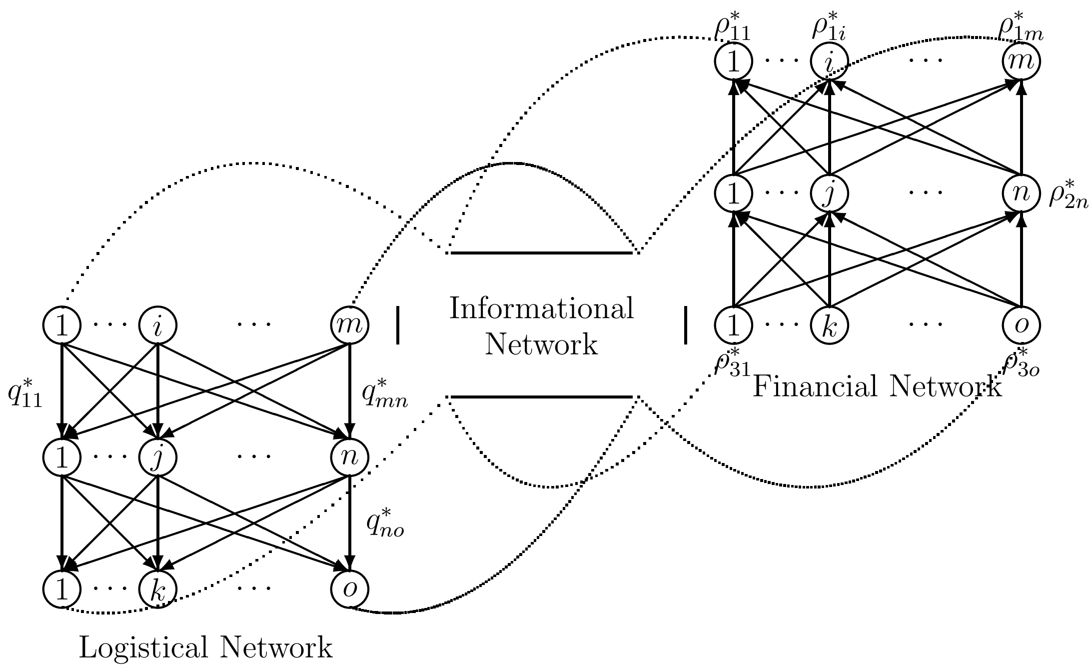

Figure 2: The Supply Chain Multilevel Network at Equilibrium

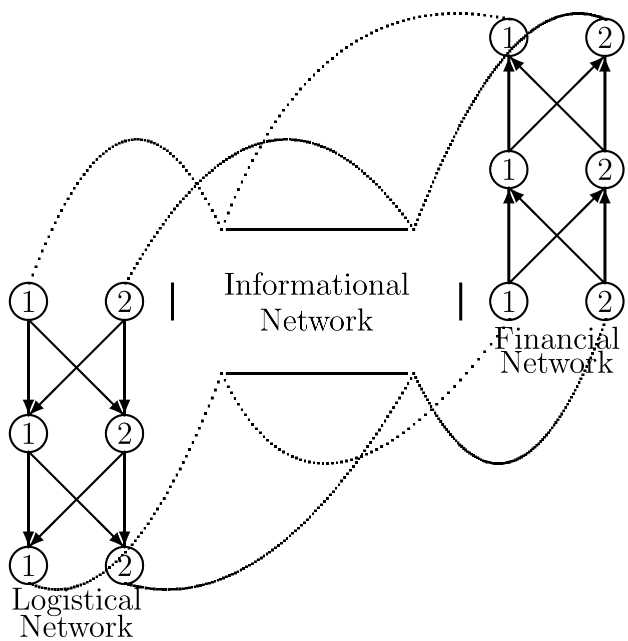

Figure 3: Multilevel Network for Numerical Example 1

5. Numerical Examples

In this Section, we apply the discrete-time adjustment process (algorithm) to five distinct numerical supply chain examples. The algorithm was implemented in FORTRAN and the computer system used was a DEC Alpha system located at the University of Massachusetts at Amherst. The convergence criterion used was that the absolute value of the flows and prices between two successive iterations differed by no more than 10^{-4}. The sequence $\{\alpha_\tau\}$ was set to $\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\}$ for all the examples. The initial commodity shipments and prices were all set to zero for each example.

Example 1

The first numerical example consisted of two manufacturers, two retailers, and two demand markets. Its multilevel network structure is given in Figure 3.

The data for this example were constructed in order to readily enable interpretation. The production cost functions for the manufacturers were given by:

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 2q_1, \quad f_2(q) = 2.5q_2^2 + q_1q_2 + 2q_2.$$

The transaction cost functions faced by the manufacturers and associated with transacting with the retailers were given by:

$$c_{11}(q_{11}) = .5q_{11}^2+3.5q_{11},\ c_{12}(q_{12}) = .5q_{12}^2+3.5q_{12},\ c_{21}(q_{21}) = .5q_{21}^2+3.5q_{21},\ c_{22}(q_{22}) = .5q_{22}^2+3.5q_{22}.$$

The handling costs of the retailers, in turn, were given by:

$$c_1(Q^1) = .5(\sum_{i=1}^{2} q_{i1})^2, \quad c_2(Q^1) = .5(\sum_{i=1}^{2} q_{i2})^2.$$

The demand functions at the demand markets were:

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1000, \quad d_2(\rho_3) = -2\rho_{32} - 1.5\rho_{31} + 1000,$$

and the transaction costs between the retailers and the consumers at the demand markets were given by:

$$c_{11}(Q^2) = q_{11} + 5, \quad c_{12}(Q^2) = q_{12} + 5, \quad c_{21}(Q^2) = q_{21} + 5, \quad c_{22}(Q^2) = q_{22} + 5.$$

The discrete-time adjustment process required 151 iterations for convergence and yielded the following equilibrium pattern: the commodity shipments between the two manufacturers and the two retailers were:

$$Q^{1^*} : q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 16.608,$$

the commodity shipments (consumption volumes) between the two retailers and the two demand markets were:

$$Q^{2^*} : q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 16.608,$$

the vector ρ_2^* had components:

$$\rho_{21}^* = \rho_{22}^* = 254.617,$$

and the demand prices at the demand markets were:

$$\rho_{31}^* = \rho_{32}^* = 276.224.$$

It is easy to verify that the optimality/equilibrium conditions were satisfied with good accuracy. We note that the same problem was solved in Nagurney, Dong, and Zhang (2002) but with the modified projection method, which, nevertheless, does not "track" the continuous adjustment process as the discrete-time algorithm thus. Moreover, the modified projection method required 257 iterations to converge to the same solution as above, whereas the discrete-time algorithm only required 151 iterations. Furthermore, each iteration of the discrete-time algorithm is simpler than that of the modified projection method.

Example 2

We then modified Example 1 as follows: The production cost function for manufacturer 1 was now given by:

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 12q_1,$$

whereas the transaction costs for manufacturer 1 were now given by:

$$c_{11}(q_{11}) = q_{11}^2 + 3.5q_{11}, \quad c_{12}(q_{12}) = q_{12}^2 + 3.5q_{12}.$$

The remainder of the data was as in Example 1. Hence, both the production costs and the transaction costs increased for manufacturer 1. The multilevel network is still as given in Figure 3.

The discrete-time algorithm converged in 154 iterations and yielded the following equilibrium pattern: the commodity shipments between the two manufacturers and the two retailers were now:

$$Q^{1^*} : q_{11}^* = q_{12}^* = 14.507, \quad q_{21}^* : q_{22}^* = 17.230,$$

the commodity shipments (consumption amounts) between the two retailers and the two demand markets were now:

$$Q^{2^*} : q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 15.869,$$

the vector ρ_2^*, was now equal to

$$\rho_{21}^* = \rho_{22}^* = 255.780,$$

and the demand prices at the demand markets were:

$$\rho_{31}^* = \rho_{32}^* = 276.646.$$

Thus, manufacturer 1 now produced less than it did in Example 1, whereas manufacturer 2 increased its production output. The price charged by the retailers to the consumers increased, as did the demand price at the demand markets, with a decrease in the incurred demand.

We note that in Nagurney, Dong, and Zhang (2002) this example was also solved, but with the modified projection method, which required 258 iterations for convergence, which is a higher number than required by the discrete-time algorithm, which, nevertheless, yielded the same solution as the modified projection method.

Example 3

The third supply chain problem consisted of two manufacturers, three retailers, and two demand markets. Its multilevel network structure is given in Figure 4.

The data for the third example were constructed from Example 2, but we added data for the manufacturers' transaction costs associated with the third retailer; handling cost data for the third retailer, and the transaction cost data between the new retailer and the demand markets. Hence, the complete data for this example were given by:

The production cost functions for the manufacturers were given by:

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 2q_1, \quad f_2(q) = 2.5q_2^2 + q_1q_2 + 12q_2.$$

The transaction cost functions faced by the two manufacturers and associated with transacting with the three retailers were given by:

$$c_{11}(q_{11}) = q_{11}^2 + 3.5q_{11}, \quad c_{12}(q_{12}) = q_{12}^2 + 3.5q_{12}, \quad c_{13}(q_{13}) = .5q_{13}^2 + 5q_{13},$$

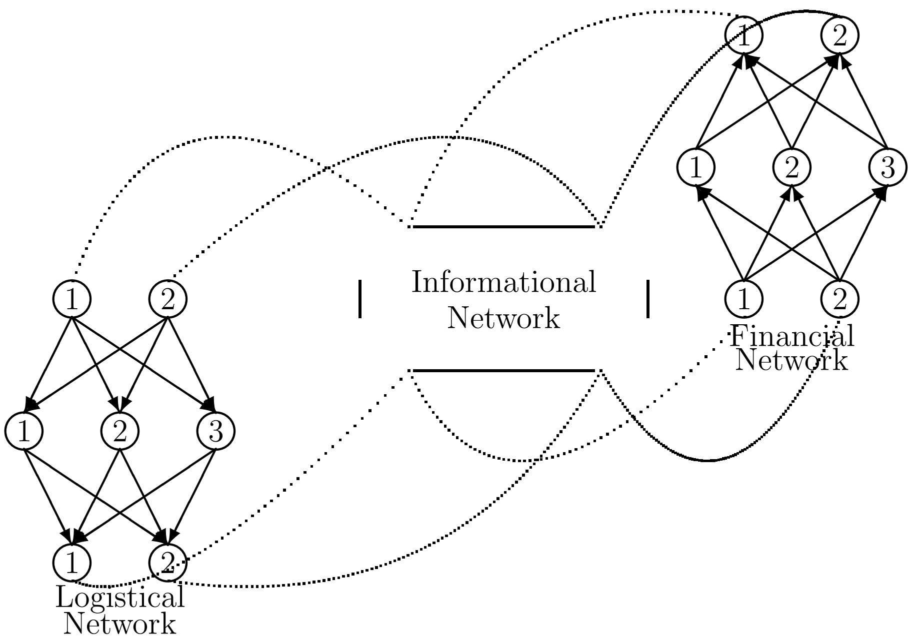

Figure 4: Multilevel Network for Example 3

$$c_{21}(q_{21}) = .5q_{21}^2 + 3.5q_{21}, \quad c_{22}(q_{22}) = .5q_{22}^2 + 3.5q_{22}, \quad c_{23}(q_{23}) = .5q_{23}^2 + 5q_{23}.$$

The handling costs of the retailers, in turn, were given by:

$$c_1(Q^1) = .5(\sum_{i=1}^{2} q_{i1})^2, \quad c_2(Q^1) = .5(\sum_{i=1}^{2} q_{i2})^2, \quad c_3(Q^1) = .5(\sum_{i=1}^{2} q_{i3})^2.$$

The demand functions at the demand markets, again, were:

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1000, \quad d_2(\rho_3) = -2\rho_{32} - 1.5\rho_{31} + 1000,$$

and the transaction costs between the retailers and the consumers at the demand markets were given by:

$$c_{11}(Q^2) = q_{11} + 5, \quad c_{12}(Q^2) = q_{12} + 5,$$
$$c_{21}(Q^2) = q_{21} + 5, \quad c_{22}(Q^2) = q_{22} + 5,$$
$$c_{31}(Q^2) = q_{31} + 5, \quad c_{32}(Q^2) = q_{32} + 5.$$

The discrete-time method converged in 215 iterations and yielded the following equilibrium pattern: the commodity shipments between the two manufacturers and the three retailers were:

$$Q^{1^*} : q_{11}^* = q_{12}^* = 9.243, \quad q_{13}^* = 14.645, \quad q_{21}^* = q_{22}^* = 13.567, \quad q_{23}^* = 9.726,$$

the commodity shipments between the three retailers and the two demand markets were:

$$Q^{2^*} : q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 11.404, \quad q_{31}^* = q_{32}^* = 12.184.$$

The vector ρ_2^* had components:

$$\rho_{21}^* = \rho_{22}^* = 259.310, \quad \rho_{23}^* = 258.530,$$

and the prices at the demand markets were:

$$\rho_{31}^* = \rho_{32}^* = 275.717.$$

Note that the prices at the demand markets were now lower than in Example 2, since there is now an additional retailer and, hence, increased competition. The incurred demand also increased at both demand markets, as did the production outputs of both manufacturers. Since the retailers now handled a greater volume of commodity flows, the prices charged for the product at the retail outlets, nevertheless, increased due to increased handling cost. The same problem was also solved in Nagurney, Dong, and Zhang (2002) with the modified projection method which required 361 iterations for convergence. As already noted, each iteration of the modified projection method is more complex than that of the discrete-time algorithm. In fact, it requires two steps with each step of the same order of complexity as the computation step in the discrete-time algorithm. Hence, these numerical results suggest that the discrete-time algorithm is a viable alternative to the modified projection method for the computation of the equilibrium pattern. Furthermore, as well shall see in Example 5, it also tracks the dynamic trajectories of the commodity shipments and the prices.

Example 4

The fourth numerical example consisted of three manufacturers, two retailers, and three demand markets. The multilevel network structure for this supply chain problem is given in Figure 5.

The data for the fourth example was constructed from the data for Example 1, with the addition of the necessary functions for the third manufacturer and the third demand market resulting in the following functions:

The production cost functions for the manufacturers were given by:

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 2q_1, \quad f_2(q) = 2.5q_2^2 + q_1q_2 + 2q_2, \quad f_3(q) = .5q_3^2 + .5q_1q_3 + 2q_3.$$

The transaction cost functions faced by the manufacturers and associated with transacting with the retailers were given by:

$$c_{11}(q_{11}) = .5q_{11}^2+3.5q_{11},\ c_{12}(q_{12}) = .5q_{12}^2+3.5q_{12},\ c_{21}(q_{21}) = .5q_{21}^2+3.5q_{21},\ c_{22}(q_{22}) = .5q_{22}^2+3.5q_{22},$$

$$c_{31}(q_{31}) = .5q_{31}^2 + 2q_{31}, \quad c_{32}(q_{32}) = .5q_{32}^2 + 2q_{32}.$$

The handling costs of the retailers, in turn, were given by:

$$c_1(Q^1) = .5(\sum_{i=1}^{2} q_{i1})^2, \quad c_2(Q^1) = .5(\sum_{i=1}^{2} q_{i2})^2.$$

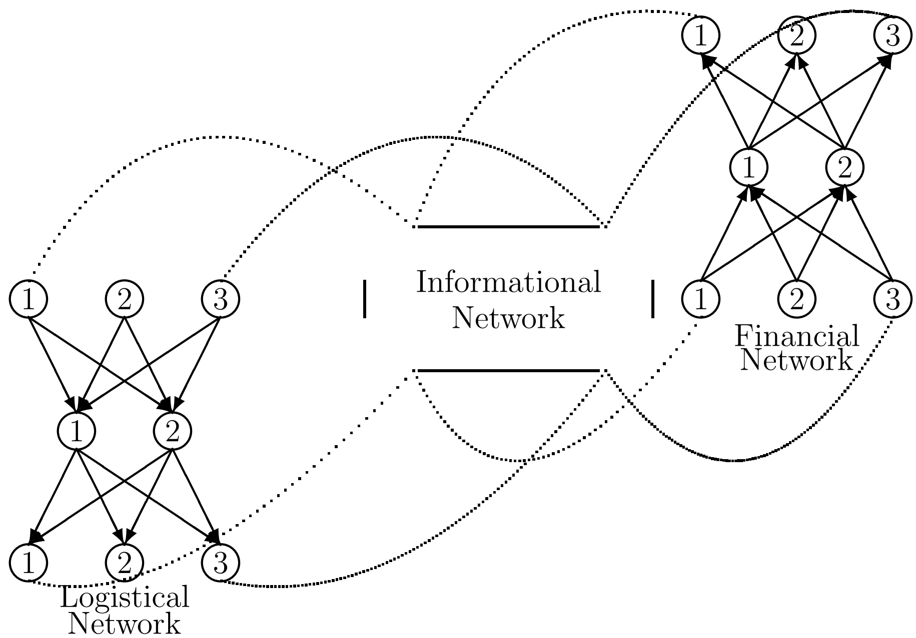

Figure 5: Multilevel Network for Example 4

The demand functions at the demand markets were:

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1000, \quad d_(\rho_3) = -2\rho_{32} - 1.5\rho_{31} + 1000,$$

$$d_3(\rho_3) = -2\rho_{33} - 1.5\rho_{31} + 1000,$$

and the transaction costs between the retailers and the consumers at the demand markets were given by:

$$c_{11}(Q^2) = q_{11} + 5, \quad c_{12}(Q^2) = q_{12} + 5, \quad c_{13}(Q^2) = q_{13} + 5,$$

$$c_{21}(Q^2) = q_{21} + 5, \quad c_{22}(Q^2) = q_{22} + 5, \quad c_{23}(Q^2) = q_{23} + 5.$$

The discrete-time algorithm converged in 175 iterations and yielded the following equilibrium pattern: the commodity shipments between the three manufacturers and the two retailers were:

$$Q^{1^*} : q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 12.395, \quad q_{31}^* = q_{32}^* = 50.078.$$

The commodity shipments (consumption levels) between the two retailers and the three demand markets were computed as:

$$Q^{2^*} : q_{11}^* = q_{12}^* = q_{13}^* = q_{21}^* = q_{22}^* = q_{23}^* = 24.956,$$

whereas the vector ρ_2^*, was now equal to:

$$\rho_{21}^* = \rho_{22}^* = 241.496,$$

and the demand prices at the three demand markets were:

$$\rho_{31}^* = \rho_{32}^* = \rho_{33}^* = 271.454.$$

Note that, in comparison to the results in Example 1, with the addition of a new manufacturer, the price charged at the retail outlets was now lower, due to the competition, and increased supply of the product. The consumers at the three demand markets gained, as well, with a decrease in the generalized demand market and an increased demand. The equilibrium solution was identical to that computed by the modified projection method for the same example in Nagurney, Dong, and Zhang (2002), but in 230 iterations.

Example 5

The fifth, and final, example consisted of two manufacturers, two retailers, and two demand markets, and its multilevel network structure was, hence, as depicted in Figure 3. For this example, we not only report the equilibrium solution computed by the discrete-time algorithm, which converged in 196 iterations, but we print out the iterates to show the tracking of the dynamic trajectories over time to the equilibrium solution.

The production cost functions for the manufacturers were given by:

$$f_1(q) = 2.5q_1^2 + q_1q_2 + 10q_1, \quad f_2(q) = 2.5q_2^2 + q_1q_2 + 2q_2.$$

The transaction cost functions faced by the manufacturers and associated with transacting with the retailers were given by:

$$c_{11}(q_{11}) = q_{11}^2+3.5q_{11},\ c_{12}(q_{12}) = .5q_{12}^2+3.5q_{12},\ c_{21}(q_{21}) = .5q_{21}^2+3.5q_{21},\ c_{22}(q_{22}) = .5q_{22}^2+3q_{22}.$$

The handling costs of the retailers, in turn, were given by:

$$c_1(Q^1) = .5(\sum_{i=1}^{2} q_{i1})^2, \quad c_2(Q^1) = .75(\sum_{i=1}^{2} q_{i2})^2.$$

The demand functions at the demand markets were:

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1200, \quad d_2(\rho_3) = -2.5\rho_{32} - 1\rho_{31} + 1000,$$

and the transaction costs between the retailers and the consumers at the demand markets were given by:

$$c_{11}(Q^2) = q_{11} + 5, \quad c_{12}(Q^2) = q_{12} + 5, \quad c_{21}(Q^2) = 3q_{21} + 5, \quad c_{22}(Q^2) = q_{22} + 5.$$

The discrete-time adjustment process yielded the following equilibrium pattern: the commodity shipments between the two manufacturers and the two retailers were:

$$Q^{1^*} : q_{11}^* = 19.002, \quad q_{12}^* = 16.920, \quad q_{21}^* = 30.225, \quad q_{22}^* = 9.6402,$$

the commodity shipments (consumption volumes) between the two retailers and the two demand markets were:

$$Q^{2^*} : q^*_{11} = 49.228, \quad q^*_{12} = 0.000, \quad q^*_{21} = 26.564, \quad q^*_{22} = 0.000,$$

the vector ρ^*_2 had components:

$$\rho^*_{21} = 320.2058, \quad \rho^*_{22} = 289.7407,$$

and the demand prices at the demand markets were:

$$\rho^*_{31} = 374.433, \quad \rho^*_{32} = 250.227.$$

Note that there were zero shipments of the commodity from both retailers to demand market 2, where the demand for the commodity was zero.

In the subsequent figures we provide the iterates from the algorithm (cf. (22)) in which we present the outputs of the logistical network (see (23) and (24)) and the financial network (see (25) and (26)).

Indeed, in Figure 6, we provide the generated commodity shipments between the manufacturers and the retailers, the q_{ij}s. In Figure 7, we show the commodity shipments between the retailers and the demand markets, the q_{jk}s over time. In Figure 8, in turn, we graph the retail prices, the ρ_2s, and in Figure 9, we present the demand market prices, the ρ_3s over time. Note that our convergence tolerance is quite tight with $\epsilon = 10^{-4}$. Nevertheless, the commodity shipments and the prices appear to have stabilized after the fiftieth iteration or so. Also, note, from Figure 7, that the commodity shipments to the second demand market remain at level zero after the fourth iteration.

6. Summary and Conclusions

In this paper, we have proposed a multilevel network framework, consisting of logistical, informational, and financial networks, for the conceptualization of supply chain problems. The manufacturers, as well as the retailers, and the consumers at the demand markets are spatially separated. We proposed the underlying dynamics of these network agents and showed that the dynamical system satisfied a projected dynamical system.

We studied the dynamical system qualitatively and established conditions for the existence of a unique trajectory as well as a global stability analysis result. We discussed the stationary/equilibrium point and gave an interpretation of the equilibrium conditions from an economic perspective.

We then proposed a discrete-time approximation to the continuous time adjustment process in which the computation of the commodity shipments takes place on the logistical network whereas the computation of the prices takes place on the financial network. The informational network stores and provides the necessary information from time period to time period for the computations/adjustments to take place. The computations that take place from time period to time period are very simple and yield closed form expressions for the new commodity shipments and the prices.

We then implemented the discrete-time adjustment process/algorithm and applied it to several supply chain examples. The method not only computed the equilibrium solutions in a timely manner but actually outperformed a previous algorithm. Furthermore, it tracks the dynamic trajectories.

Future research will include an actual application of this framework to a particular product. In addition, the association of logistical nodes with physical locations will be made and the incorporation of the physical transportation network for the actual shipment of the goods. Another potential research question involves allowing the informational network to vary with respect to the timeliness and quality of information provided and to allow the decision-makers on the network to select among different modes of information with different costs.

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References

Bovet D., 2000, **Value Nets: Breaking the Supply Chain to Unlock Hidden Profits**, John Wiley & Sons, New York.

Bramel J. and Simchi-Levi D., 1997, **The Logic of Logistics: Theory, Algorithms and Applications for Logistics Management**, Springer-Verlag, New York.

Dupuis P. and Nagurney A., 1993, "Dynamical systems and variational inequalities," *Annals of Operations Research* **44**, 9-42.

Mentzer J. T., editor, 2001, **Supply Chain Management**, Sage Publishers, Thousand Oaks, California.

Miller T. C., 2001, **Hierarchical Operations and Supply Chain Planning**, Springer-Verlag, London, England.

Nagurney A., 1999, **Network Economics: A Variational Inequality Approach**, second and revised edition, Kluwer Academic Publishers, Dordrecht, The Netherlands.

Nagurney A. and Dong J., 2002, "Urban location and transportation in the Information Age: A multiclass, multicriteria network equilibrium perspective," *Environment & Planning B* **29**, 53-74.

Nagurney A., Dong J. and Mokhtarian P. L., 2001, "Teleshopping versus shopping: A multicriteria network equilibrium framework," *Mathematical and Computer Modelling* **34**, 783-798.

Nagurney A., Dong J. and Mokhtarian P. L., 2002, "A space-time network for telecommuting versus commuting decision-making," to appear in *Papers in Regional Science*.

Nagurney A., Dong J. and Zhang D., 2002, "A supply chain network equilibrium model," *Transportation Research E*, in press.

Nagurney A., Loo J., Dong J. and Zhang D., 2001, "Supply chain networks and electronic commerce: A theoretical perspective," Isenberg School of Management, University of Massachusetts, Amherst, Massachusetts, 01003.

Nagurney A., Dupuis P. and Zhang D., 1994, "A dynamical systems approach for network oligopolies and variational inequalities," *Annals of Regional Science* **28**, 263-283.

Nagurney A., Takayama T. and Zhang D., 1995, "Massively parallel computation of spatial price equilibrium problems as dynamical systems," *Journal of Economic Dynamics and Control* **18**, 3-37.

Nagurney A. and Zhang D., 1996, **Projected Dynamical Systems and Variational Inequalities with Applications**, Kluwer Academic Publishers, Boston, Massachusetts.

Nagurney A. and Zhang D., 1997, "Projected dynamical systems in the formulation, stability analysis, and computation of fixed demand traffic network equilibria," *Transportation Science* **31**, 147-158.

Nagurney A. and Zhang D., 2001, "Dynamics of a transportation pollution permit system with stability analysis and computations," *Transportation Research D* **6**, 243-268.

Poirier C. C., 1996, **Supply Chain Optimization: Building a Total Business Network**, Berrett-Kochler Publishers, San Francisco, California.

Poirier C. C., 1999, **Advanced Supply Chain Management: How to Build a Sustained Competitive Advantage**, Berrett-Kochler Publishers, San Francisco, California.

Samuelson P. A., 1952, "Spatial price equilibrium and linear programming," *American Economic Review* **42**, 293-303.

Stadtler H. and Kilger C., editors, 2000, **Supply Chain Management and Advanced Planning**, Springer-Verlag, Berlin, Germany.

Takayama T. and Judge G. G., 1971, **Spatial and Temporal Price and Allocation Models**, North-Holland, Amsterdam, The Netherlands.

D. Zhang and A. Nagurney, "On the stability of projected dynamical systems," *Journal of*

Optimization Theory and Applications **85** (1995) 97-124.

D. Zhang and A. Nagurney, "Stability analysis of an adjustment process for oligopolistic market equilibrium modeled as a projected dynamical system," *Optimization* **36** (1996) 263-285.