A Supply Chain Network Economy: Modeling and Qualitative Analysis

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1. Introduction

This chapter proposes a modeling intitiative that is network-based to formalize both intra-supply chain cooperation and inter-supply chain competition. In particular, we propose a general network model for a supply chain economy since it is now recognized that when it comes to production, distribution, and consumption today, it is no longer simply firms that compete with one another, but, rather, it is *supply chains*, consisting of economic decision-makers and their spectrum of activities. The supply chain network economic model that we develop in this chapter aims to answer such fundamental questions as: How do economic decision-makers form coalitions in a competitive environment? How do supply chains compete with one another, and how does one identify the *winning* supply chain?

We note that, to-date, the major emphasis in the study of supply chains in the literature has been the optimization of a supply chain that is owned (and controlled) by a single entity with various issues explored such as the distribution network design, coordination, and inventory management (see, e.g., Chen (1998) and the reviews by Geoffrion and Power (1995) and Beamon (1998)). More recently, competition among supply chain agents has been addressed using game theoretic formulations as in the inventory-capacity game models of Cachon and Zipkin (1999) and Cachon and Lariviere (1999a, b) who studied two-stage serial supply chains. Corbett and Karmarkar (2001), on the other hand, investigated a supply chain network consisting of several tiers of decision-makers who compete within a tier. Under the assumption of identical linear production cost functions, they determined the number of entrants under Nash equilibrium.

Nagurney, Dong, and Zhang (2002), in turn, developed a three tiered supply chain network equilibrium model consisting of manufacturers, retailers, and consumers, who may compete within a tier but cooperate between tiers. Using a variational inequality formulation, the equilibrium production outputs, shipments, and consumption quantities, along with the prices of the product associated with the different tiers could then be determined with an iterative computational procedure. Nagurney et al. (2002a) further extended the basic framework to incorporate electronic commerce whereas Nagurney et al. (2002b) focused on the dynamics of supply chain networks. Subsequently, Dong, Zhang, and Nagurney (2002) developed a supply chain network equilibrium model with random demands associated with the product at different retail outlets and formulated and solved the model as a variational inequality problem. Our work is similar in spirit, although derived from an entirely different perspective and application setting, to that of Dafermos and Nagurney (1984), who constructed a general network model of production, along with its variational inequality formulation.

This chapter is organized as follows. In Section 2, we introduce the concept of a supply chain economy and develop the general network model. In particular, we consider an economy with multiple profit-driven decision-makers or "agents," who represent distinct supply chain entities such as raw material suppliers, manufacturers, logistic firms, wholesalers, and/or retailers. The supply chain economy is, hence, a network of agents and their activities, which can include, for example, the production, distribution, sale, and consumption of one or more products. Hence, a supply chain network economy may consist of several supply chains which interact in their procurement, production, logistic, and retailing activities.

The notable features of the network model include:

1. The model is not limited to a fixed number of tiers with similar functions/operations associated with a given tier as in the work of Cachon and Zipkin (1999), Cachon and Lariviere (1999a, 1999b), Nagurney, Dong, and Zhang (2002), Nagurney et al. (2002a, b), Dong, Zhang, and Nagurney (2002). For an example of a supply chain network with a fixed number of tiers and similar functions associated with a given tier, see Figure 1. The model developed in this chapter, in contrast, allows for an entirely *general structure* of the supply chains. In addition, the model allows for the study of *inter*-chain competition as well as coordination and integration of *intra*-chain activities with appropriately defined variables, network topology, and costs associated with the links.

2. The model addresses inter-supply chain competition at multiple markets and does not assume homogeneity of the product throughout the chain but, rather, captures also the costs associated with the subcomponents as the "product" proceeds down the chain.

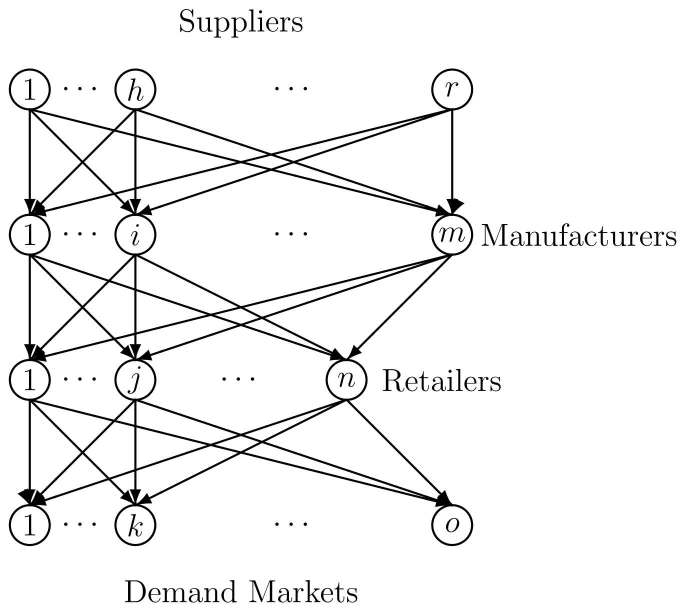

Figure 1: An example of a four-tiered supply chain network with suppliers, manufacturers, retailers, and demand markets and with similar functions associated with a given tier

3. The network model includes operation links as well as interface links with the former corresponding to such possible business operations as manufacturing, storage, and transportation and with the latter acting as a bridge between businesses.

The cost on a specific interface link, in turn, can measure the effectiveness of the coordination and the integration of the network agents. Moreover, we impose no artificial conditions on the cost functions associated with the various links.

In Section 3, we derive the equilibrium conditions, which are based on the identification of those chains in the network with minimal marginal chain cost and positive flow. The concept, which is based on the ideas of Wardrop (1952), and which has had wide application in traffic networks as well as other application settings (see Nagurney (1999), Nagurney and Dong (2002), and the references therein), is here generalized to apply to network chains, rather than paths, and to material flows, rather than identical types of network flows (such as travelers). We establish the variational inequality formulation of the equilibrium conditions governing the supply chain network economy. It is noteworthy that the model enables us to bring potential supply chain structures into our study. Moreover, the solution of the model identifies potential supply chain structures with minimum marginal costs and, hence, predicts what supply chain may prevail in the future. Consequently, the model can suggest effective supply chain formations.

In Section 4, we then turn to the examination of the qualitative properties of the equilibrium flow pattern and provide both existence and uniqueness results. Section 5 summarizes the results of this chapter and presents suggestions for future research.

2. A Network Model of the Supply Chain Economy

It is clear that, in today's business environment, multiple supply chains coexist and many of them compete in similar businesses such as Wal-Mart, Target, and Sears. The interaction among different supply chains, however, is not solely one of competition, but may involve other dimensions, such as cooperation. In this section, we propose a new concept – that of a *Supply Chain Network Economy* – to address the multi-dimensional interaction among supply chains. We first present the definition and then develop the model.

Definition 1: A Supply Chain Network Economy

A supply chain economy is a network of interrelated activities of procurement, production, distribution, sale, and consumption of one or more products, conducted by coalitions of business entities who act collectively within a coalition. Given the definition of a supply chain, we state that a supply chain network economy is a network of interrelated supply chains.

A supply chain network economy describes the environment (competitive and cooperative, in parts) of all the market-related and operation-related activities of business firms who belong to many supply chains and these supply chains compete in several related markets.

In this chapter, we consider a supply chain network economy associated with a particular product with multiple markets. We assume that there are n markets that are spatially separated, with a typical one denoted by j.

Let $G = [N, L]$ denote the network of the supply chain economy, where N denotes the set of nodes and L the set of oriented links in the network. The orientation of a link indicates the flow of material. The supply chain network economy may consist of several supply chains with each one being a subnetwork (connected subgraph) of G.

There are three different types of nodes in the network model: the origin nodes, the intermediate nodes, and the destination nodes. An origin node corresponds to the "beginning" or origin of a supply chain or supply chains and usually indicates the source of raw material or other resource. A destination node corresponds to the destination of a supply chain material flow and usually indicates a market for the end product. An intermediate node, in turn, lies in between the other nodes, including other intermediate ones, and serves as a connection for the supply chain links.

We consider two basic types of links in the network model: an operation link and an interface link. An operation link represents a business function performed by a firm in a supply chain network, and can reflect a manufacturing, transportation, storage, or, even, service operation. A firm may have several operation links in the supply chain network economy with distinct links corresponding to the firm's distinct business functions. An interface link, in turn, serves as a business-to-business bridge and lies, typically, between operation links in the network model. If we denote the set of operation links by A and the set of interface links by B, then we have that $L = A \cup B$.

We now distinguish between potential and active supply chains, with the latter playing an especially important role in terms of competitiveness of the supply chain network economy. A potential supply chain is represented by a connected subgraph of the network characterized by a destination node corresponding to its pertinent market. It is recognized in the model that a possible supply chain structure may not necessarily be prevailing at a particular time. However, a potential chain can evolve to prevail in the supply chain economy if it possesses a competitive structure. A potential supply chain becomes active in our model if it carries positive flow of material; hence, a potential supply chain is active if and only if its chain flow of material is positive. Later, we provide conditions that allow us to explicitly identify such chains and chain flows.

Let S_j denote the set of all potential supply chains pertaining to market j. S denotes the set of all potential supply chains in the network representing the supply chain economy.

The *chain flow* of a supply chain is defined to be the volume of the final or end product that the supply chain delivers to its pertinent market, that is, it is the output of the supply chain. Let X_s denote the chain flow of chain s and let X be the vector of all the chain flows of all the supply chains in the supply chain economy. The volume of the end product of a supply chain determines the amount of "work" processed at each stage of the material flow stream. For example, by tracking back the flow of material from the end product and exploding the series of bills of materials, one can determine the amount of work (and information) processed or resources (such as labor hours, machine hours, storage space, etc.) utilized in each participator link of a supply chain. To make such a measure rigorous, we define the process rate below.

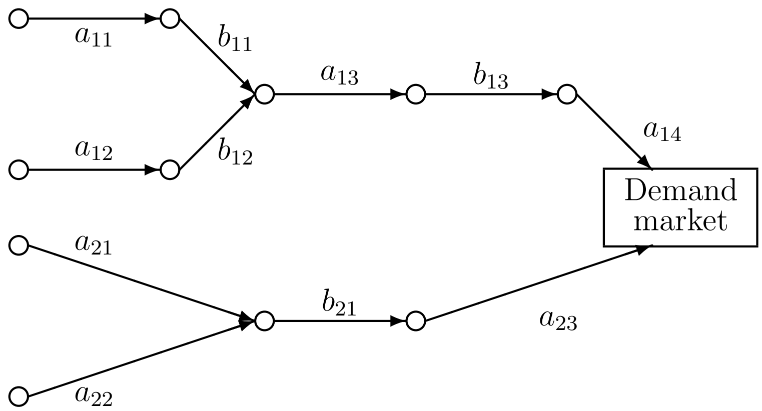

Figure 2: An example of two supply chains competing at a demand market

Definition 2: Process Rate

For a supply chain s and its participant operation link a, the process rate of a with respect to s, denoted by λ_{as}, is the amount of resource utilized (work conducted or information processed) by link a necessary to make one unit of end product of the supply chain s.

Consider, for example, an automobile supply chain, in which a participating tire manufacturer would have a process rate of four since each car requires four tires. Hence, an output of two hundred cars from such a chain would imply eight hundred tires to be manufactured by the participating tire manufacturer. Consequently, the actual material flow down the supply chain varies in substance as well as in quantity. It is distinct from a transportation network in that the actual amount of flow in a supply chain does not always equal its chain flow but is, in fact, λ_{as} *times* the chain flow on the link a. Here, for the sake of technical simplicity, we adopt the convention that the value of any particular λ_{as} is not less than one. In practice, a processing rate can always be scaled to an appropriate unit so that this assumption is satisfied.

We assume, for simplicity, that no link is shared by two different supply chains. However, this assumption may be ultimately relaxed. Therefore, the operation links and interface links can be partitioned into their supply chain families, A_s and B_s, that is,

$$A = \sum_{s \in S} A_s, \quad B = \sum_{s \in S} B_s. \tag{1}$$

In Figure 2, we depict an example of two supply chains competing at a demand market. Supply chain 1 is comprised of four operation links: a_{11}, a_{12}, a_{13}, and a_{14}, and three interface links: b_{11}, b_{12}, and b_{13}. Supply chain 2, on the other hand, is comprised of three operation links: a_{21}, a_{22}, and a_{23}, and one interface link b_{21}.

For any link $a \in L$, we let x_a denote the flow on the link. In the case of an operation link, x_a is the total amount of work processed on the link. Units for measuring the flow on an operation link can vary from application to application and should be appropriately selected. Examples of appropriate units, based on the nature of the operation, can correspond to the number of subassemblies, the parts determined by the bill of materials, the number of machine hours, the hours of labor of the production operations, the number of truckloads required for transportation, or even the square footage in the warehouse used for storage purposes.

Recall that X_s denotes the chain flow of chain s. Then, one has the following relationship between an operation link flow and a chain flow:

$$x_a = \lambda_{as} X_s, \quad \forall a \in A_s, \quad \forall s \in S. \tag{2}$$

For an interface link b, on the other hand, the link flow x_b indicates the amount of integration work, coordination effort, and/or information processed on this link. Therefore, the link flow of an interface link is determined by the chain flow of its governing supply chain, that is,

$$x_b = x_b(X_s), \quad \forall b \in B_s, \quad \forall s \in S. \tag{3a}$$

It is reasonable to assume that the chain-link flow relationship given in (3a) is a strictly monotone increasing function, although it can be nonlinear; that is, the greater the volume of the supply chain output, the greater the integration work required on all the interface links. Hence, the inverse function of (3a) exists, that is,

$$X_s = X_s^{-1}(x_b), \quad \forall b \in B_s, \quad \forall s \in S. \tag{3b}$$

The link cost of an operation link is defined to be the total cost incurred by the firm in performing the corresponding task function, which is in concert with the definition of an operation link being a business function performed by a firm. Therefore, the link cost $\bar{c}_a$ of link a is a function of its flow x_a, and can be expressed as

$$\bar{c}_a = \bar{c}_a(x_a), \quad \forall a \in A; \tag{4}$$

in other words, the cost of carrying out a task function depends on the amount of work processed in the task. Depending upon the nature of the operation corresponding to the link, the link cost function can be nonlinear, with consolidation and congestion being the two primary factors that determine the cost structure. For example, if a batch or lot size is required for a production operation or if a truckload or shipload is applicable, a consolidation effect can be expected to be present and, thus, the greater the volume, the cheaper the rate. On the other hand, if the link represents a bottleneck operation, then one can expect a congestion effect, with the function increasing as the volume of "traffic" increases.

The cost of an interface link, since such a link is an *inter-link*, reflects the effectiveness of coordination and integration of the two operation links that it bridges. The geographic distance, the experience of past cooperation, and/or the sophistication of the information system integrated in the two joined operations, as well as the compatibility of the two, represent several factors that could account for the cost of an interface link. Since effective coordination of two successive operation links of a supply chain may depend upon other joints of this chain as well, we assume that the cost of an interface link may, in general, depend on the flows of all the interface links belonging to the same supply chain. However, in light of ($3a$), it can be determined by the chain flow of its governing supply chain, and according to ($3b$), it also uniquely depends on the link flow of the interface link, that is,

$$\bar{c}_b = \bar{c}_b(X_s) = \bar{c}_b(x_b), \quad \forall b \in B_s, \quad \forall s \in S. \tag{5}$$

The cost of a supply chain s is then the sum of the link costs of its operation and interface links, which can be expressed as

$$\bar{C}_s = \sum_{a \in A_s} \bar{c}_a + \sum_{b \in B_s} \bar{c}_b. \tag{6}$$

We assume that the demand for the product at a marketplace is elastic and can be characterized by its market value. Let q_j and v_j denote, respectively, the supply and the market value of the product at market j. Further, let q, and v be, respectively, the corresponding n-dimensional vectors of supplies and prices. By definition, the supply at a market is equal to the sum of the chain flows of all the supply chains that are driven by that market, that is,

$$q_j = \sum_{s \in S_j} X_s, \quad j = 1, \ldots, n. \tag{7}$$

In general, we assume that the market value of the product at a market may depend on the vector of supplies of the products at all the markets, that is,

$$v_j = v_j(q), \quad j = 1, \ldots, n, \tag{8}$$

or, in vector form,

$$v = v(q) = (v_j(q); \quad j = 1, \ldots, n). \tag{9}$$

It is reasonable to assume, from an economic perspective, that, in (8) and (9), the market value is a decreasing function of the supply.

3. Equilibrium Conditions

In this section, we present the equilibrium conditions, the mathematical formulation of which can be utilized to determine the answers to the central questions of this chapter. These are: Who, among the multiple agents will enter this economy? Which potential chains will win through competition and become active chains? What outputs do the active chains deliver to the end markets?

We first, however, need to introduce the concepts of *link marginal cost* and *chain marginal cost*.

Definition 3: Link Marginal Cost

For any operation link, $a \in A_s$, the link marginal cost, c_a, is defined to be the unit link processing cost, if the flow on link a is no less than the processing rate, λ_{as}, for its governing supply chain, s; otherwise, it is defined to be the unit link cost for the processing rate, λ_{as}, namely,

$$c_a(x_a) = \begin{cases} \bar{c}_a(x_a)/x_a, & \text{if} \quad x_a \geq \lambda_{as} \\ \bar{c}_a(\lambda_{as})/\lambda_{as}, & \text{if} \quad x_a \leq 1. \end{cases} \tag{10}$$

For any interface link, $b \in B_s$, the link marginal cost, c_b, is defined to be the unit link cost, if the chain flow of its governing supply chain is no less than the one unit; otherwise, it is defined to be the link cost for one unit of the chain flow, namely,

$$c_b(X_s) = \begin{cases} \bar{c}_b(X_s)/X_s, & \text{if} \quad X_s \geq 1 \\ \bar{c}_b(1), & \text{if} \quad X_s \leq 1. \end{cases} \tag{11}$$

Since any active operation link in a supply chain s must process a minimum of one unit of the chain flow, and, consequently, the amount of λ_{as} link flow; in practice, the link marginal cost (10) is the unit processing cost for an active link. On the other hand, if the link flow turns out to be less than λ_{as}, then it would imply that the link is not on an active supply chain. In this case, the link marginal cost in (10) can be interpreted as the *triggering cost*, that is, the marginal cost necessary to activate this link. The link marginal cost for an interface link is similarly defined, except that it is based on the chain flow of the governing supply chain.

In line with the above definitions of operation and interface link marginal costs, we state the definition of the chain marginal cost as follows.

Definition 4: Chain Marginal Cost

The chain marginal cost of a supply chain s is defined to be the unit chain processing cost of chain s, if the chain flow X_s is no less than one unit; otherwise, it is the chain cost for one unit of flow.

$$C_s = \begin{cases} \bar{C}_s(X_s)/X_s, & \text{if} \quad X_s \geq 1 \\ \bar{C}(1), & \text{if} \quad X_s \leq 1. \end{cases} \tag{12}$$

It is clear that the above-defined link marginal costs and chain marginal cost are continuous when the link costs are continuous, and are differentiable when the link costs are differentiable. Furthermore, in view of (6), we can spell out the chain marginal cost in terms of link marginal costs in a uniform expression regardless of the value of X_s:

$$C_s = \sum_{a \in A_s} \lambda_{as} c_a(x_a) + \sum_{b \in B_s} c_b(X_s). \tag{13}$$

An end market can be viewed as the *destination* of the product, which, in turn, "pulls" the flow of materials through all its pertinent supply chains. The supply chains, as alternative "paths" to the destination are "racing" to deliver the product. In line with the perspective that the firms' goal in terms of cooperation in the supply chain is to make the final product overall at less cost than competing supply chain firms (see Cavinato (1992)), the winning chains in this race or competition are, in effect, the "shortest paths," that is, those that can deliver the product at the least marginal cost.

There is an interesting analogy between this interpretation and that occurring in the context of traffic networks in which travelers seek to determine the shortest paths from their origins to their destinations. In a congested urban transportation network, travelers are assumed to select shortest paths (minimal cost paths) from their origins to their destinations with Wardrop (1952) being credited with the following (first) principle:

Wardrop's First Principle

The travel costs of all paths actually used are equal and less than those which would be experienced by a single traveler on any unused path.

One should realize that the network structure of a supply chain can be any connected subgraph, with the simplest topology being that of a tree and, hence, it is more complex than a path in a transportation network from an origin to a destination. Also, unlike travelers (in vehicles), who arrive at their destinations in the same form as they leave their origins, the flow of material in a supply chain network economy evolves from raw materials, through fabricated parts, subassemblies, to finished products, compounded at various stages of a supply chain process.

Nevertheless, the economic rationale that an inferior or less cost-effective supply chain will lose to its rivals is in the same spirit as Wardrop's first principle.

We now formally state the definition of an equilibrium in this framework.

Definition 5: Equilibrium of a Supply Chain Network Economy

In a supply chain network equilibrium, the active supply chains are those whose marginal cost is equal to the market value of the product at the pertinent market, and the inactive supply chains are those whose marginal cost is greater than or equal to the market value of the product at the pertinent market. Mathematically, a feasible supply chain network economy with flows X^*, with induced q^* through (7), constitutes a supply chain network equilibrium if and only if the following system of equalities and inequalities holds true:

$$C_s(X) \begin{cases} = v_j(q^*), & \text{if} \quad X_s^* > 0 \\ \geq v_j(q^*), & \text{if} \quad X_s^* = 0, \end{cases} \quad \forall s \in S_j, \tag{14}$$

for all end markets j; $j = 1, \ldots, n$.

With this definition, we now have the answers to the aforementioned questions. The winning (active) chains are those supply chains with least marginal cost for their pertinent markets, and they are the only ones that carry positive flow of materials.

Before deriving the variational inequality formulation of the governing equilibrium conditions, we introduce link capacities. Recall that since an operation link may represent a manufacturing process, transport, and/or storage, it may face such constraints as limited machine hours, a fixed number of trucks available, and/or limited storage space in a warehouse.

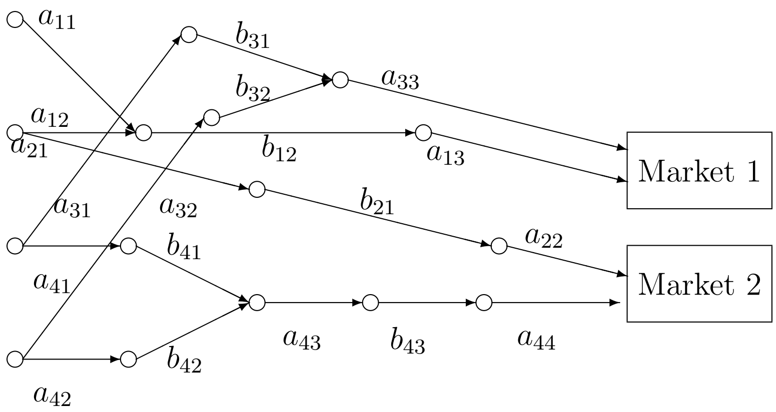

Figure 3: An example of a supply chain network economy

Denote the link capacity of an operation link by h_a. Then, we have that

$$\lambda_{as} X_s = x_a \leq h_a, \quad \forall a \in A_s, \quad \forall s \in S. \tag{15}$$

The feasible set underlying the supply chain economy in terms of the chain variables' accounting operational capacities is expressed as

$$\Omega_X = \{X \geq 0 : (15) \text{ holds for all } a \in A_s,\ s \in S\}. \tag{16}$$

This set is equivalent to the feasible set in chain variables and supply patterns, defined as

$$\Omega_{Xq} = \{X \geq 0,\ q \geq 0 : (7), (15) \text{ hold for all } a \in A_s,\ s \in S;\ j = 1, \ldots, n\}. \tag{17}$$

The following theorem gives a mathematical formulation for the supply chain network equilibrium in chain variables.

Theorem 1: Variational Inequality Formulation

X^* with induced q^* through (7) is a supply chain network equilibrium if and only if (X^*, q^*) solves the following variational inequality problem: determine $(X^*, q^*) \in \Omega_{Xq}$ such that

$$\sum_{j=1}^{n} \left[\sum_{s \in S_j} C_s(X^*)(X_s - X_s^*) - v_j(q^*)(q_j - q_j^*) \right] \geq 0, \quad (X, q) \in \Omega_{Xq}. \tag{18}$$

Proof: The proof is standard (see, e.g., Nagurney (1999)). □

3.1 An Example

In this subsection, we present a numerical example.

Example

Consider a supply chain network example as depicted in Figure 3. There are two demand markets, Market 1 and Market 2. There are four supply chains, denoted, respectively, by S_1, S_2, S_3, and S_4 and also referred to, without loss of generality, as Supply Chains 1, 2, 3, and 4. Supply chain S_1 is comprised of operation links a_{11}, a_{12}, a_{13} and interface link b_{12}. S_2 is comprised of operation links a_{21} and a_{22} and interface link b_{21}. S_3, in turn, is comprised of operation links a_{31}, a_{32}, and a_{33} and interface links b_{31} and b_{32}. Supply chain S_4 consists of operation links a_{41}, a_{42}, a_{43}, and a_{44} and interface links b_{41}, b_{42}, and b_{43}. Supply Chain 1 and Supply Chain 3 are competing for Market 1, while Supply Chain 2 and Supply Chain 4 are competing for Market 2.

The process rates for the operation links are given by:

$$\lambda_{11} = 2, \quad \lambda_{12} = 3, \quad \lambda_{13} = 1,$$

$$\lambda_{21} = 4, \quad \lambda_{22} = 1,$$

$$\lambda_{31} = 2, \quad \lambda_{32} = 2, \quad \lambda_{33} = 1,$$

$$\lambda_{41} = 2, \quad \lambda_{42} = 4, \quad \lambda_{43} = 2, \quad \lambda_{44} = 1.$$

The link marginal cost functions of the operation and the interface links, for each supply chain, are as given below.

$$c_{a11} = 0.5x_{a11} + 1, \quad c_{a12} = 0.1x_{a12},$$

$$c_{b11} = 2.1x_{b11} + 1, \quad c_{a13} = 5x_{a13} + 3,$$

$$c_{a21} = 0.5x_{a21} + 0.5, \quad c_{b21} = X_2, \quad c_{a22} = x_{a22} + 2,$$

$$c_{a31} = x_{a31} + 4, \quad c_{a32} = x_{a32} + 4, \quad c_{b31} = X_3 + 4,$$

$$c_{b32} = X_3 + 4, \quad c_{a33} = x_{a33} + 4,$$

$$c_{a41} = c_{a42} = c_{a43} = 0.5,$$

$$c_{a44} = 2x_{a44} + 1, \quad c_{b41} = c_{b42} = c_{b43} = X_4.$$

The chain marginal costs can be computed according to (13) by using the link marginal costs. They are as follows:

$$C_{S_1} = 10X_1 + 6, \quad C_{S_2} = 10X_2 + 4, \quad C_{S_3} = 11X_3 + 28, \quad C_{S_4} = 5X_4 + 5.$$

Suppose now that the two demand markets are interrelated and that their market value functions are:

$$v_1(q) = 40 - 5q_1 - q_2, \quad v_2(q) = 60 - q_1 - 10q_2.$$

The supply chain network equilibrium in this example is given by

$$X^* = (X_1^*, X_2^*, X_3^*, X_4^*) = (2, 1.4, 0, 2.6),$$

which generates the equilibrium supply pattern

$$q^* = (q_1^*, q_2^*) = (2, 4).$$

The market value at Market 1 at equilibrium is 26, whereas the market value at Market 2 at equilibrium is 18.

Examining the two competing supply chains for Market 1 at the equilibrium, one see that the marginal chain cost of Supply Chain 1 is 26 and the marginal chain cost of Supply Chain 3 is 28. This verifies the equilibrium condition that the market value is equal to the marginal chain cost for the active (winning) supply chain, Supply Chain 1, and that the inactive chain (the loser), Supply Chain 3, has a marginal cost that is higher than the market value. The equilibrium condition also holds for Market 2. In this case, the two competing supply chains, Supply Chain 2 and Supply Chain 4, are both active, and have the same marginal chain cost, 18, which is equal to the market value at Market 2 in equilibrium.

4. Qualitative Properties

This section provides the fundamental qualitative properties of the mathematical formulation of the equilibrium of the supply chain network economy presented in the previous section. First, we establish the existence of an equilibrium and then turn to its uniqueness.

Theorem 2: Existence

Suppose that the link cost functions for all the operation and interface links are continuous. Then, there exists an equilibrium of the supply chain network economy.

Proof: We claim that the feasible set Ω_{Xq} is compact. It is easy to see that Ω_{Xq} is closed. To show that it is bounded, one has, in view of (15), for every supply chain s

$$X_s \leq \lambda_{as}^{-1} h_a, \quad \forall a \in A_s, \tag{19}$$

which yields

$$X_s \leq \min_{a \in A_s} \{\lambda_{as}^{-1} h_a\}, \quad \forall s \in S. \tag{20}$$

(20) suggests that in (7),

$$q_j \leq \sum_{s \in S_j} \min_{a \in A_s} \{\lambda_{as}^{-1} h_a\}, \quad j = 1, \ldots, n. \tag{21}$$

Expressions (20) and (21) imply that all the variables are, indeed, bounded from above, and, hence, Ω_{Xq} is bounded.

As noted before, the continuity of the link cost functions guarantees the continuity of the link marginal cost functions, which further implies the continuity of the chain marginal cost functions through (13). Therefore, the variational inequality (18) is a continuous mapping over a compact set. It, thus, according to the standard theory of variational inequalities (see Kinderlehrer and Stampacchia (1980)) has at least one solution. □

The uniqueness of the supply chain network equilibrium pattern can, in general, be ensured under an assumption of strict monotonicity (cf. Nagurney (1999) and the references therein) of the marginal link cost functions and the market value functions as given by the following theorem.

Theorem 3: Uniqueness

Suppose that the marginal link cost functions are strictly monotone increasing, and the market value functions are strictly monotone decreasing, with respect to their arguments. Then, there exists a unique equilibrium of the supply chain network economy.

Proof: The monotonicity assumption on link marginal cost functions implies that the chain marginal cost function is monotone increasing, in view of (13). Therefore, under the condition of the theorem, the vector function of the variational inequality problem (18) is monotone. The conclusion of the theorem follows from Theorem 2 and the standard theory of variational inequality problem (see, e.g., Nagurney (1999)). □

Let $J_q v$ denote the Jacobian matrix of the market value function with respect to the supply vector q. It is well-known (see, e.g., Nagurney (1999)) that if $J_q v$ is symmetric, then there is a real-valued function $\bar{v}$ whose gradient is v, namely,

$$v = \nabla_q \bar{v}. \tag{22}$$

A special case of v being a gradient of a real-valued function is that the market value function is separable; namely, in this case, the market value of the product at market j depends only on the supply at market j. In this case, the Jacobian matrix of v is a diagonal matrix with the diagonal entries being the derivative of the market value with respect to their own supply.

Under certain conditions, the supply chain network equilibrium can also be formulated as the following optimization problem:

$$\min_{(X,q)\in\Omega_{Xq}} \sum_{j=1}^{n} \sum_{s\in S_j} \int_0^{X_s} C_s(u)du - \bar{v}(q). \tag{23}$$

Theorem 4: Sufficient Condition

Suppose that the Jacobian matrix of the market value function is symmetric. Then, a sufficient condition for (X^*, q^*) to be an equilibrium of a supply chain network economy is that it solves the optimization problem (23).

Proof: According to Theorem 1, it suffices to show that any solution to optimization problem (23) is a solution to (18). However, the variational inequality problem (18) is simply a restatement of the first-order necessary condition of the optimization problem (23) by noticing that the vector function entering into (18) is the gradient of the objective function in (23). □

Theorem 5: Necessary and Sufficient Condition

Suppose that the marginal link cost functions are monotone increasing, and the Jacobian matrix of the market value function is symmetric and negative semi-definite. Then, (X^*, q^*) is an equilibrium of the supply chain network economy if and only if it solves the optimization problem (23).

Proof: The vector function entering (18) is the gradient of the objective function in (23). Hence, the assumption of monotone marginal link costs, together with negative semi-definiteness of the Jacobian matrix of the market value function, implies that the objective function of (23) is convex. According to Bazaraa, Sherali, and Shetty (1993), the variational inequality problem (23) is the necessary and sufficient condition of the solution to the convex programming (23). □

5. Summary and Conclusions

In this chapter, we proposed a new framework, that is network-based, to formalize the modeling and analysis of supply chains. The framework allows for the simultaneous study of both intra-supply chain cooperation as well as inter-supply chain competition. Moreover, it has the notable feature that unlike many of the supply chain models in the literature, the underlying network topological structure is general. In addition, the model allows for the transformation of material flows as they proceed through the network.

We developed the supply chain network economic model, derived the governing equilibrium conditions, and then provided the variational inequality formulation. We established existence of an equilibrium pattern, and also gave conditions under which uniqueness is guaranteed.

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