

The Braess Paradox

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Abstract

The Braess Paradox is a counterintuitive phenomenon that may arise in congested urban transportation networks that was discovered by Dietrich Braess and described in his classic 1968 paper. In particular, the Braess Paradox occurs only in networks in which the users operate independently and noncooperatively, in a decentralized manner. The counterintuitive phenomenon is that the addition of a new road may result in the increase of travel cost/time for all travelers in the network! The Paradox remains relevant to this day and has continued to fascinate researchers in a wide range of scientific disciplines, inspiring further advances in both theory and practice.

Keywords: Braess paradox, transportation networks, user-optimization, system-optimization, congestion, networks, decentralized behavior, counterintuitive phenomena, complex systems

Introduction

Congested urban transportation networks are examples of complex systems in which users, that is, travelers, interact with infrastructure. The behavior of the users is, typically, delineated as being either that of user-optimization or system-optimization based, respectively, on Wardrop's (1952) two principles of travel behavior. In the case of the user-optimization, each user "selfishly" selects his own optimal route of travel from an origin to a destination with an equilibrium being achieved when no user has any incentive to alter his travel path. The governing equilibrium conditions state that all used paths, that is, those with positive flow, connecting each origin/destination pair of nodes of travel will have equal and minimal travel cost. In the case of system-optimization, there exists a central controller that routes the traffic flow from origin nodes to destination nodes so that the total cost in the transportation network is minimized. Importantly, in congested transportation networks, the cost (usually reflecting travel time although the cost may be generalized to include monetary cost, risk, etc.) on a link is an increasing function of the volume of the traffic flow on the link.

In 1968, Dietrich Braess in his classic paper, written in German, identified the possibility of the occurrence of counterintuitive behavior in user-optimized transportation networks. The paper was inspired by a seminar delivered by W. Knoedel in Muenster in 1967 when Braess was 29 years old (see Nagurney and Boyce (2005)). Specifically, he constructed a transportation network example consisting of a single origin/destination pair of nodes and two parallel paths, such that, when expanded with the inclusion of an additional link, which provided another route option for the travelers, the result was an increase in travel cost for all the travelers! The demand and original link cost functions had not changed. This surprising discovery was in contrast to conventional wisdom in that the addition of a link, which yields another route option for travelers between their origin/destination pair, would make each user better-off in terms of travel cost/time. This counterintuitive phenomenon has become known as the Braess Paradox.

What is also remarkable is that, as discussed in Nagurney and Boyce (2005), at the time that Braess described this paradoxical behavior, he was unaware of Wardrop's two principles of travel behavior and their rigorous mathematical formulation given in the book by Beckmann et al. (1956). Nevertheless, and this is also noteworthy, the Braess paper described two different concepts of traffic network utilization that correspond, respectively, to analogues of system-optimization and user-optimization. It is important to mention that the terms "user-optimization" and "system-optimization" were subsequently coined by Dafermos and Sparrow (1969). The Braess (1968) paper was followed by that of Murchland (1970), who elaborated upon the Braess paradox, reflected upon it in the context of Beckmann et

al. (1956) and Beckmann (1967), and brought it to the attention of the English speaking community.

This paradox has come to fascinate researchers and practitioners in transportation and related fields, in which decentralized behavior in congested networks is relevant, such as in computer science, as in the case of the modeling of telecommunication networks and the Internet (see Korilis, Lazar, and Orda (1999), Roughgarden and Tardos (2002), Roughgarden (2005), Nagurney, Parkes, and Daniele (2007)); in electrical engineering, for the study of power systems (Cohen and Horowitz (1991), Blumsack et al. (2007)) and electronic circuits (cf. Nagurney and Nagurney (2016)), as well as in physics in the case of mechanical (Cohen and Horowitz (1991)) and fluid systems (Calvert and Keady (1993)); in biology, as in metabolic networks (see Motter (2010)), ecosystems (Sahasrabudhe and Motter (2011)), and targeted cancer therapy (Kippenberger et al. (2016)), and, surprisingly, in sports analytics as in the study of sports teams, where the Braess paradox analogue corresponds to the removal of a player resulting in better team performance (cf. Skinner (2010) and Gudmunsson and Horton (2017)). Because of the interest from multiple disciplines, a translation of the paper from German to English was published by Braess et al. (2005). The foreword by Nagurney and Boyce (2005) to the translated paper contains additional background of how Braess came up with the counterintuitive phenomenon, along with clarifications of some of the concepts and terms.

The Classic Braess Paradox

The classic Braess paradox example is as follows. Consider first the four node transportation network as illustrated by the left-most network in Figure 1.

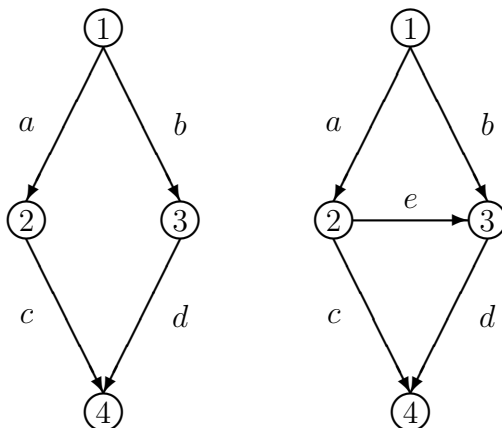


Figure 1: The Braess Paradox Example Topology

There is a single Origin/Destination (O/D) pair of nodes $w = (1, 4)$. A traveler on this network can travel on one of two paths: on path p_1 consisting of the links: a, c or on path p_2 consisting of the links: b, d . We denote the flows on the links by: f_a, f_b , and so on, and their respective user link costs by: c_a, c_b , etc. Specifically, in this network the user link cost functions are:

$$c_a(f_a) = 10f_a, \quad c_b(f_b) = f_b + 50, \quad c_c(f_c) = f_c + 50, \quad c_d(f_d) = 10f_d.$$

Let the travel demand d_w be 6, which represents 6 vehicles of travel per unit time. We denote the flow on a path p by x_p so here we have x_{p_1} and x_{p_2} .

The conservation of flow equations ensure that the demand for each O/D pair is satisfied by the flows on paths that connect the O/D pair and that the link flows capture the flows that utilize the particular link. Specifically, the equations state that the sum of flows on paths connecting each O/D pair must be equal to the demand for that O/D pair and that the flow on a link must be equal to the sum of flows on the paths that contain/utilize that link.

Clearly, under user-optimizing behavior, the resulting equilibrium path flow for the first network is: $x_{p_1}^* = x_{p_2}^* = 3$, with incurred user path costs of: $C_{p_1} = C_{p_2} = 83$. No traveler has any incentive to switch her path since that would result in a higher cost for her. This result is apparent since the costs on the two paths are: $C_{p_1} = c_a + c_c = 10f_a + f_c + 50$ and $C_{p_2} = c_b + c_d = f_b + 50 + 10f_d$ and, therefore, the travelers at a demand of 6 will equally distribute themselves between the two paths, yielding the equilibrium path flows: $x_{p_1}^* = 3$ and $x_{p_2}^* = 3$ and incurred equilibrium link flows: $f_a^* = 3, f_b^* = 3, f_c^* = 3, f_d^* = 3$, with user link costs: $c_a = 30, c_b = 53, c_c = 53, \text{ and } c_d = 30$.

Consider now the addition of a new link, e , joining node 2 with node 3 as in the right-most network in Figure 1. Let the user link cost c_e on link e be

$$c_e(f_e) = f_e + 10.$$

A user on the expanded transportation network now has three path options: the original two paths, p_1 and p_2 , plus the new path $p_3 = (a, e, d)$. The equilibrium flow pattern on the first network will no longer yield an equilibrium for the second network. Indeed, observe that although the costs on paths p_1 and p_2 would be 83, the cost on the new path, C_{p_3} , if it is not used, that is, it has zero flow, would be 70. Clearly, some travelers, under user-optimization, would switch from paths p_1 and p_2 to path p_3 since the cost on path p_3 is less than 83.

Typically, we apply algorithms (cf. Nagurney (1999) and Patriksson (2004)) to determine the user-optimized/equilibrium flows in transportation networks, since they are usually large-scale and a priori it is difficult to identify which paths will and will not be used. In this example, because of its size, we can calculate the solution explicitly. We will assume that all paths are used. Hence, we can set up a system of equations making use of the following:

$$C_{p_1} = C_{p_2} = C_{p_3}$$

and

$$x_{p_1}^* + x_{p_2}^* + x_{p_3}^* = d_w = 6.$$

Furthermore, we know that according to the conservation of flow equations for the link and path flows:

$$f_a^* = x_{p_1}^* + x_{p_3}^*$$

$$f_b^* = x_{p_2}^*$$

$$f_c = x_{p_1}^*$$

$$f_d^* = x_{p_2}^* + x_{p_3}^*$$

$$f_e^* = x_{p_3}^*.$$

Hence, we can rewrite that user costs on paths as functions of path flows as follows

$$C_{p_1} = 10(x_{p_1}^* + x_{p_3}^*) + x_{p_1}^* + 50 = 11x_{p_1}^* + 10x_{p_3}^* + 50,$$

$$C_{p_2} = x_{p_2}^* + 50 + 10(x_{p_2}^* + x_{p_3}^*) = 11x_{p_2}^* + 10x_{p_3}^* + 50,$$

$$C_{p_3} = 10(x_{p_1}^* + x_{p_3}^*) + x_{p_3}^* + 10 + 10(x_{p_2}^* + x_{p_3}^*) = 10x_{p_1}^* + 10x_{p_2}^* + 21x_{p_3}^* + 10.$$

Using then the demand conservation of flow equation above and substituting, we obtain a system of equations, the solution of which yields the equilibrium flow pattern on the expanded network of: $x_{p_1}^* = x_{p_2}^* = x_{p_3}^* = 2$, with the incurred equilibrium path travel costs:

$$C_{p_1} = C_{p_2} = C_{p_3} = 92!$$

Thus, the addition of a new link makes every user worse-off in that each traveler in the expanded network incurs a higher travel cost than before!

It is important to note that the Braess Paradox can only occur under user-optimization and never under system-optimization. Indeed, under system-optimization, in the second

network in Figure 1, only the original paths p_1 and p_2 would be used. It is important to emphasize that tolls can be assigned so that the system-optimizing flow pattern is at the same time user-optimizing; see Dafermos and Sparrow (1971) and Bergendorff, Hearn, and Ramana (1997). In particular, if one assigns link tolls as follows: toll on link a , $r_a = 30$; toll on link b , $r_b = 3$, with link tolls: $r_c = 3$, $r_d = 30$, and $r_e = 0$, then travelers on the second network in Figure 1 will independently distribute themselves according to the system-optimized flow pattern. The formula for tolls, in this case, due to Dafermos and Sparrow (1971), is $r_l = \hat{c}'_l(f_l) - c_l(f_l)$ with the marginal total cost \hat{c}'_l and the user link cost c_l evaluated at the system-optimized flow pattern for all links l in the network and $\hat{c}_l = c_l \times f_l$ corresponding to the total cost on link l .

Generalizations of the Classic Braess Paradox and Related Paradoxes in Transportation

This counterintuitive example has given rise to many questions and the examination of under what conditions and scenarios the Braess Paradox may arise. For example, in the classic example the travel demand $d_w = 6$. Would the Braess Paradox still occur under different values for the travel demand? Pas and Principio (1997) addressed this question and showed that the Braess Paradox occurs only if the demand for travel falls within a certain intermediate range of values, specifically, if $2.58 < d_w < 8.89$. Interestingly, under low levels of demand only the new path p_3 is used and the paradox does not occur, whereas at higher levels of demand only the two original paths are used and the Braess Paradox also does not occur. Subsequently, Nagurney, Parkes, and Daniele (2007), utilizing connections between transportation and telecommunication networks, constructed a dynamic model of the Internet using evolutionary variational inequalities, and cast (cf. Figure 2) the Pas and Principio results into a Braess Paradox with time-varying demands, establishing the same results, but showed also that the curve of path flow equilibria is unique and that the equilibrium trajectories are continuous. Pas and Principio (1997) also showed, when examining linear, separable user link cost functions of the form as in the classic Braess Paradox that, whether or not the paradox occurs, depends on the conditions of the problem; namely, the parameters of the link user cost functions.

Pas and Principio (1997) further suggested that, under higher levels of demand, the Braess Paradox may not occur. Subsequently, Nagurney (2010) considered more general user link cost functions, which could be nonlinear, as well as asymmetric in that $\frac{\partial c_a(f)}{\partial f_b} \neq \frac{\partial c_b(f)}{\partial f_a}$, for all links a, b in the network, where f is the vector of link flows. In her paper, she considered the hypothesis that, in congested networks, the Braess Paradox may “disappear” under higher

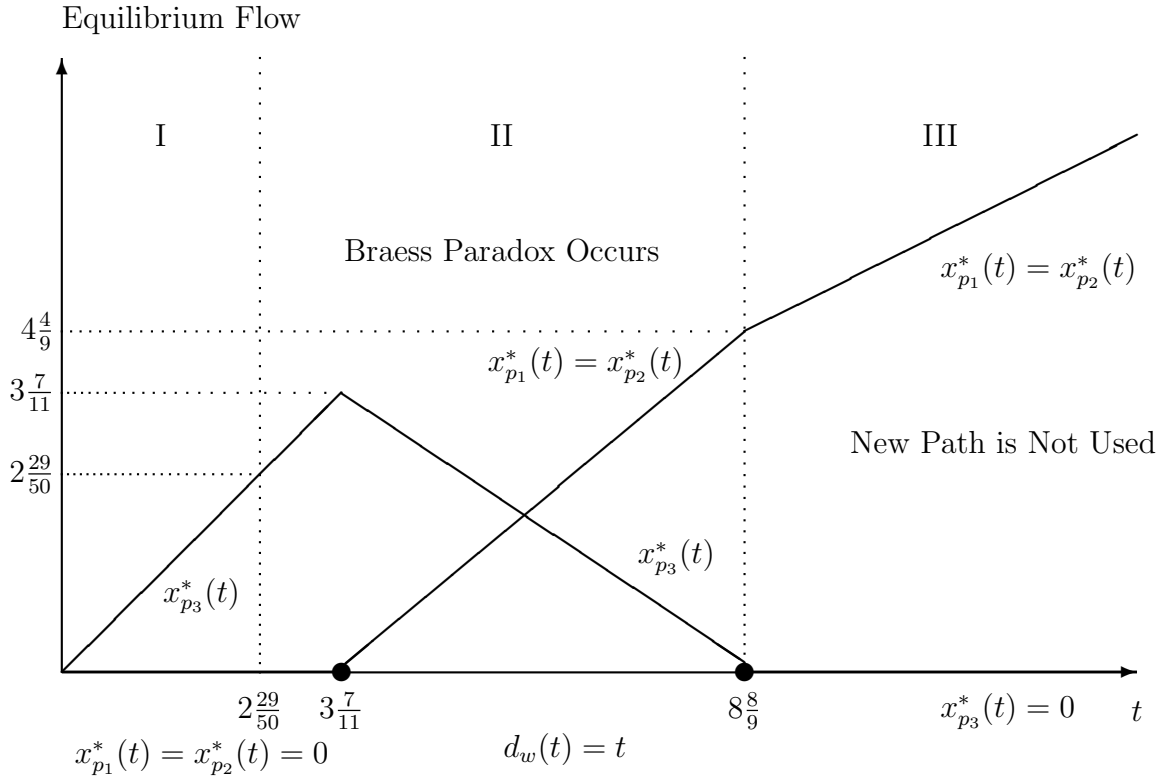


Figure 2: Equilibrium Trajectories of the Braess Network with Time-Dependent Demand

demands, and proved this hypothesis by deriving a formula that provides the increase in demand that will guarantee that the addition of that new route will no longer increase travel cost since the new path will no longer be used. This result was established for any network in which the Braess Paradox originally occurs and suggests that, in the case of congested, noncooperative networks, of which transportation networks are a prime example, a higher demand will negate the Braess Paradox. At the same time, this finding shows that extreme caution should be taken in the design of network infrastructure, including transportation networks, since at higher demands, new routes/pathways may not even be used.

Steinberg and Zangwill (1983) considered linear separable user link cost functions and provided necessary and sufficient conditions, under reasonable assumptions, for the Braess Paradox to occur in a general transportation network. They concluded that the Braess Paradox is about as likely to occur as not occur.

While the classical example of the Braess Paradox uses cost functions that are of the form: a fixed term plus a term proportional to the flow, other possible cost functions have been mathematically investigated in transportation networks, including the Bureau of Public Roads cost function (see Sheffi (1985)), which has a term quartic in the flow. The Braess Paradox has been examined under such functions by LeBlanc (1975), Frank (1981), and Bloy

(2007). Dafermos and Nagurney (1984a) demonstrated how, in terms of traffic networks with general (asymmetric) user link cost functions, the addition of a route connecting an origin-destination pair that shared no links with any other route in the network, could never result in the Braess Paradox. Moreover, the authors provided explicit formulae for the effects of cost function and demand changes on the incurred equilibrium path flows and path travel costs. The generalization of user link cost functions from separable, as well as symmetric ones, to asymmetric ones, made use of the formulation of the governing equilibrium conditions as a variational inequality problem (Smith (1979), Dafermos (1980), Nagurney (1999)). User-optimized networks with symmetric user link cost functions, in contrast, could be reformulated and solved as optimization problems (cf. Beckmann et al. (1956), Dafermos and Sparrow (1969)).

Hallefjord et al. (1994), in turn, presented paradoxes in transportation networks in the case of elastic demand, rather than fixed demands as in the original Braess Paradox example. The authors noted that, in the case of elastic travel demand, it is not apparent what a paradoxical situation is, and in the elastic demand case there is a need for characterizations of different paradoxes. An example is provided in which total flow (demand) decreases while travel time increases due to the addition of a new link to the network, with this being a rather extreme type of paradox. Another highlighted paradox is when the network “improvement” leads to a reduction in the social surplus.

As noted in Boyce et al. (2005), sensitivity analysis is also essential to the effective planning/design of transportation networks and the Braess Paradox motivated much of the subsequent research in sensitivity analysis and networks. Dafermos and Nagurney (1984b), for example, used the variational inequality formulation of traffic network equilibrium with fixed demands to provide directional effects of link cost function changes and to demonstrate that small changes in the data yielded small changes in the resulting equilibrium link flows. For other sensitivity analysis results in the context of traffic network equilibrium problems, see, e.g., Tobin and Friesz (1988), Frank (1992), Yang (1997), Patriksson (2004), and the references therein.

It is worth pointing out additional related paradoxes to that of the Braess Paradox in transportation. Sheffi and Daganzo (1978) presented a counterintuitive result that may occur when stochastic traffic assignment methods are used; see also Yao and Chen (2014). Fisk (1979) constructed examples showing that both origin to destination and global travel costs may decrease for an O/D pair as a result of an increase in travel demands for another O/D pair. Subsequently, Fisk and Pallotino (1981) provided network examples for the city of Winnipeg, demonstrating that the Braess Paradox could occur in real world networks. Yang

and Bell (1998) introduced a new paradox associated with network design problems via a simple network example in which the addition of a new road segment to a road network may reduce the potential capacity of the network. Nagurney (2000) identified emission paradoxes; in particular, three distinct ones that can occur in congested urban transportation networks in terms of the total emissions generated. These emission paradoxes reveal that so-called ‘improvements’ to the transportation network may result in increases in total emissions generated.

Zhang et al. (2016) considered the Downs-Thomson Paradox. The Downs-Thomson Paradox, named after Downs (1962) and Thomson (1977), suggests that highway capacity expansion may produce counterproductive effects in a two-mode (auto and transit) transportation system. Specifically, Zhang et al. (2016) re-examined the paradox when certain assumptions are relaxed while retaining the usual assumption that there is no congestion interaction between the modes. Arnott and Small (1994) reviewed the Downs-Thomson Paradox and also explicated the Pigou-Knight-Downs Paradox, in which expanding road capacity may elicit its own demand with no improvement in congestion.

Observations in Transportation Systems

While it may be challenging to accurately control the demand and to measure the travel time (cost) on real road systems with actual drivers, there are several documented examples of the “inverse” of the Braess Paradox being observed in a transportation network after a link was removed. In other words, travel time improves after the removal of a link. Many of these examples/instances have been written up in the popular press.

For example, Kolata (1990), writing in *The New York Times*, noted that, in 1990, on Earth Day, the New York City’s Transportation Commissioner decided to close 42nd Street, and to “everyone’s surprise” no historic traffic jam was generated and traffic flow actually improved. She stated that this may be a real-world example of the Braess Paradox. That same year, Cohen and Kelly (1990) constructed an example where a Braess-type paradox occurs in a queuing network. The authors cited the paper by Knodel (1969) that noted that the City of Stuttgart had tried to ease downtown traffic by adding a new street. However, congestion only worsened, and hence, in desperation, the authorities closed the street. The result was that the traffic flow improved.

In 1999, according to Vidal (2006), one of the three main traffic tunnels in Seoul, the capital of South Korea, was closed for maintenance. Surprisingly, the result was not chaos and traffic jams, but, rather, the traffic flows improved. Inspired by their experience, Seoul’s

city planners, subsequently, demolished a major motorway leading into the heart of the city and experienced the same strange result, with the added benefit of creating a 5-mile long, 1,000 acre park for the local inhabitants (see also Baker (2009)).

And, in 2009, in an ambitious project in NYC, a part of Broadway in mid-Manhattan was converted to a pedestrian plaza and vehicular travel banned (see Grynbaum (2010)). This redesign of infrastructure was made permanent and has lasted even past the original mayoral administration of Michael Bloomberg. Traffic flows, in parts, improved.

Other Systems that May Exhibit Braess Paradox Behavior

There exists a plethora of realizable physical systems that may exhibit Braess Paradox behavior and, in certain such physical systems, the demand (total flow) may be controllable and the laws of physics ensure that the decentralized network is, in fact, user-optimized. In contrast to congested transportation networks, users are no longer travelers, but correspond to electrons in electric power systems, or to fluid molecules (water, oil, etc.) in the case of pipeline networks, etc. Such analogues of transportation networks are quite natural and we note that Beckmann et al. (1956) hypothesized that electric power generation and distribution networks would behave like congested urban transportation networks. This hypothesis was substantiated by Nagurney et al. (2007); see also the references therein. Moreover, fluid flow models have been developed for traffic; see, e.g., Lighthill and Whitham (1955), Herman et al. (1959), and Herman and Prigogine (1979).

For example, Cohen and Horowitz (1991) suggested that it might be possible to create mechanical, electrical, fluid, and thermal systems that exhibit counterintuitive behavior when a physical component was added. As an illustration of such a mechanical system, they showed that a weight hanging from a coupled pair of springs with safety strings can rise, rather than fall, when the taut coupling string is cut. They also demonstrated that an electrical network with a topology of the Braess (1968) example and consisting of ideal passive components (resistors and zener diodes) can exhibit the counterintuitive behavior of the voltage rising across the network when an additional branch (link) is added.

Details of a spring network that exhibited the Braess Paradox were delineated by PENCHINA and PENCHINA (2003). The authors also noted that the only requirement for the spring network to exhibit the paradox is that the springs must shrink more than the safety strings stretch. PETERS and VONDRACEK (2012) extended the experiments to include variation in the mass and the spring constant.

WITTHAUT and TIMME (2012) studied the addition of single links in a class of oscillator

networks that model modern power grids on coarse scales. They showed that, while, on the average, the additional links stabilized the network, the addition of specific new links could decrease the total grid capacity and, thereby, decrease or even destroy the stability of the grid. In addressing the question of reliability of the power grid, Blumsack et al. (2007) noted that in a power network with the classic Braess topology, adding a line (link) might result in another line becoming capacity limited, which would affect the optimal power dispatch and result in higher costs.

The idealized electrical circuit, as described by Cohen and Horowitz (1991), was converted to a real electrical circuit by Nagurney and Nagurney (2016), who used electric circuit theory to develop matrix equations to describe the voltage drop (equivalent to cost) as a function of the circuit elements and current flow (demand). The authors then used this formulation to build a circuit using standard electrical components in which the voltage and currents could be measured. In addition to constructing an actual physical circuit whose voltage drops were functionally similar to the cost functions suggested by Braess in his classic example, Nagurney and Nagurney (2016) also constructed and measured the parameters of a circuit with more general voltage drops that exhibited behavior consistent with the Braess Paradox.

The concept of the Braess Paradox was also investigated by Pala et al. (2012) in semiconductor networks, whose transport properties are governed by quantum physics. The authors demonstrated theoretically that congestion plays a key role in the occurrence of the Braess Paradox in such networks.

Both macro- and micro-fluidic systems can exhibit behavior analogous to the Braess Paradox. Ayala and Blumsack (2013) considered a macrofluidic system, as in natural gas distribution networks, and studied the existence of paradoxical effects, relevant to network design. In their analysis, the simple addition of a pipe to transport gas may not necessarily increase the ultimate transmission capacity. In terms of a microfluidic system, Case et al. (2019) showed that both the flow rate in a system with the classic Braess topology and the direction of the flow in the linking channel are dependent on the input pressure.

It is clear that, more than 50 years since the publication of the paper by Braess (1968), the paradox and the associated gleaned insights into decentralized noncooperative behavior remain relevant, continue to fascinate, and to inspire research in transportation. Its applicability to practice also continues to this day, in transportation network planning and design. Finally, the Braess Paradox has served as a bridge for broadening perspectives in other scientific disciplines by enabling the advancement of the theory of the behavior of complex network systems with a vast range of important applications.

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