Multicriteria Spatial Price Networks:
Statics and Dynamics

Anna Nagurney
Department of Finance and Operations Management
Isenberg School of Management
University of Massachusetts
Amherst, Massachusetts 01003

June Dong and Ding Zhang
Department of Marketing and Management
School of Business
State University of New York at Oswego
Oswego, New York 13126

revised May, 2002
Appears in *Equilibrium Problems and Variational Models*,
P. Daniele, A. Maugeri, and F. Giannessi, Editors, Kluwer (2003), pp. 299-321.

Abstract:

In this paper, we develop a spatial price network equilibrium model in which consumers at the demand markets consider both the transportation cost and the transportation time associated with obtaining the particular commodity. We provide the governing equilibrium conditions for the multicriteria spatial price problem and derive the variational inequality formulation. We establish existence and uniqueness of the equilibrium commodity shipment and demand price pattern and then propose a dynamic tatonnement process whose set of stationary points coincides with the set of solutions of the variational inequality problem. An iterative scheme is described which provides a time discretization of the continuous time adjustment process and which converges to a stationary point. Numerical examples are given for illustrative purposes.

1. Introduction

Spatial price equilibrium problems have provided a basic formalism for the study of a wide variety of applications arising in agricultural markets, energy, and in interregional and international trade. The rigorous formulation of such problems dates to Samuelson (1952) and Takayama and Judge (1971) who considered problems in which the governing equilibrium conditions could be reformulated as an equivalent mathematical programming (optimization) problem.

In spatial price equilibrium problems, one assumes that the supply and demand markets are spatially separated, that the competition is perfect, and that, in equilibrium, a commodity produced at a supply market will be shipped to a demand market, where it is consumed, provided that the supply price plus the unit transportation cost is equal to the demand price. If the supply price at the supply market plus the unit cost of transportation exceeds the demand price that the consumers are willing to pay for the commodity, then the commodity will not be shipped between the pair of markets.

The basic models were subsequently extended to allow for the treatment of asymmetric price and transportation cost functions and multicommodity situations using a variational inequality framework (cf. Dafermos and Nagurney (1987), Florian and Los (1982), and Nagurney (1987), among others). Refer to Nagurney (1999) and the references therein for recent research on the formulation, theoretical analysis, and computation of solutions to spatial price equilibrium problems.

In this paper, we propose a multicriteria spatial price network equilibrium model. In the model, we assume that consumers in each distinct demand market may be faced with several criteria in selecting the commodity that is produced, specifically, not only the price of the commodity but also the time it takes to receive the commodity. Hence, the consumers are not only price-sensitive but also time-sensitive. We construct explicit demand functions which express these concerns and study the model both from a *static* perspective, from the point of view of the equilibrium pattern, using the theory of variational inequalities, as well as from a *dynamic* perspective through the use of a dynamic tatonnement process which reveals how the producers adjust their commodity shipments to the demand markets and how the generalized prices at the demand markets adjust. The theoretical analysis of the dynamics is conducted using projected dynamical systems theory (see Nagurney and Zhang (1996)).

We note that multicriteria network equilibrium models have been constructed for traffic networks and were introduced by Quandt (1967) and Schneider (1968) and explicitly consider that travelers may be faced with several criteria, notably, travel time and travel cost, in selecting their optimal routes of travel. The ideas were further developed by Dial (1979) who proposed an uncongested model and Dafermos (1981) who introduced congestion effects and derived an infinite-dimensional variational inequality formulation of her multiclass, multicriteria traffic network equilibrium problem, along with some qualitative properties.

Recently, there has been renewed interest in the formulation, analysis, and computation of multicriteria traffic network equilibrium problems. Researchers who have considered an infinite-dimensional variational inequality formulation, motivated by Dafermos' (1981) multiclass model, have included Leurent (1993a) (see also Leurent (1993b)), who presented an elastic demand formulation but did not allow travel cost to be a function of flow. For an overview of multicriteria traffic network equilibrium problems and different formulations, see Leurent (1998) and Marcotte (1998).

In this paper, we build upon the recent work of Nagurney (2000) and Nagurney and Dong (2002) who developed, respectively, a multiclass, multicriteria traffic network equilibrium model with fixed travel demands and with elastic travel demands. However, due to the special structure of the spatial price network problem under consideration here we are able to obtain sharper results in the sense that we are able to establish, under quite reasonable conditions, strict monotonicity of the function that enters the variational inequality problem. Moreover, for the first time, we propose a dynamical system to describe the evolution of the trajectories for a multicriteria network equilibrium problem.

The paper is organized as follows. In Section 2, we present the multicriteria spatial price equilibrium model and derive the variational inequality formulation of the governing equilibrium conditions. In Section 3, we focus on the "statics" and obtain an existence result as well as a uniqueness result. In Section 4, we then describe a dynamic tatonnement process and relate the dynamic and static interpretations of the problem. In Section 5, we propose the Euler method, which is a discrete-time algorithm, and provide convergence results. Section

6 contains numerical examples which illustrate the model and the computational approach. Section 7 summarizes our results and presents the conclusions.

2. The Multicriteria Spatial Price Model

In this Section, we develop the multicriteria spatial price network equilibrium model. The model permits the consumers at each of the demand markets to weight the transportation cost and the transportation time associated with the shipment of the commodity from the supply markets in an individual manner. The equilibrium conditions are then shown to satisfy a finite-dimensional variational inequality problem (see, e.g., Kinderlehrer and Stampacchia (1980) and Nagurney (1999)).

We assume that a certain commodity is produced at m supply markets and is consumed at n demand markets. We denote a typical supply market by i and a typical demand market by j. Let s_i denote the supply of the commodity at supply market i and let Q_{ij} denote the nonnegative commodity shipment from supply market i to demand market j. We group the supplies into a column vector s in R^m and the commodity shipments into a column vector Q in R^{mn}.

We associate with each supply market i a supply price π_i and we group the supply prices into a row vector π in R^m. We assume that, in general, the supply price at a supply market i can depend on the supplies of the commodity at all the supply markets, that is,

$$\pi_i = \pi_i(s), \quad \forall i, \tag{1}$$

where π is a known smooth function.

The supply of the commodity at each supply market i must satisfy the following conservation of flow equation:

$$s_i = \sum_{j=1}^{n} Q_{ij}, \tag{2}$$

that is, the supply of the commodity at a supply market must be equal to the sum of the commodity shipments from the supply market to all the demand markets.

In view of (1) and (2), and, for simplicity of the subsequent derivations and notation, we define the supply price function $\hat{\pi}_i$, for each supply market i, which is a function of the commodity shipment pattern:

$$\hat{\pi}_i = \hat{\pi}_i(Q) \equiv \pi_i(s), \quad \forall i, \tag{3}$$

and we group these functions into the row vector $\hat{\pi} \in R^m$.

We introduce a unit transportation cost c_{ij} associated with shipping the commodity between supply market i and demand market j and the transportation time t_{ij} associated with the shipment. We group the transportation costs and times, respectively, into the row vectors c in R^{mn} and t in R^{mn}. We assume, in turn, that the unit cost of transportation depends on the quantity of the commodity shipped between the pair of markets, that is,

$$c_{ij} = c_{ij}(Q_{ij}), \quad \forall ij, \tag{4}$$

as does the transportation time, i.e.,

$$t_{ij} = t_{ij}(Q_{ij}), \quad \forall ij, \tag{5}$$

where the transportation costs and times are assumed to be known smooth functions.

We assume that each demand market represents a distinct class of consumer who perceives the transportation cost and time in an individual manner. Hence, consumers in one demand market may not be as concerned as to when the commodity is delivered provided that the transportation cost is low, whereas consumers at another demand market may be more time-sensitive and may be willing to pay a higher transportation cost provided that the commodity reaches them in a more timely manner. We let w_j^1 denote the weight associated with the transportation cost for demand market j and we let w_j^2 denote the weight associated with the transportation time to demand market j. We assume that the weights are positive for all demand markets.

We then construct the *generalized* cost associated with link (i,j) and denoted by $\hat{c}_{ij}$ as follows:

$$\hat{c}_{ij} = w_j^1 c_{ij} + w_j^2 t_{ij}. \tag{6}$$

Note that a possible weighting scheme may be one where the weights for each demand market sum to one, that is, $w_j^1 + w_j^2 = 1$, for all j. Dafermos (1981) utilized such a weighting scheme in the context of a traffic network equilibrium model.

We assume that the demand for the commodity at demand market j is determined according to:

$$d_j = d_j(\lambda), \quad \forall j, \tag{7}$$

where λ is the column vector of demand market *generalized* (since it reflects both time and cost) prices with demand market j's generalized price being denoted by λ_j. We group the demand fuctions into the row vector $d(\lambda) \in R^n$.

Multicriteria Spatial Price Network Equilibrium Conditions

The spatial price network equilibrium conditions in the case of known demand functions (see Takayama and Judge (1971), Nagurney and Zhao (1993), and Nagurney, Takayama, and Zhang (1995a, b)), in the generalized context of the multicriteria spatial price network equilibrium problem, take on the form: A pattern $(Q^*, \lambda^*) \in R_+^{mn+n}$ is an equilibrium pattern if for each pair of supply and demand markets (i, j) the following conditions hold:

$$\hat{\pi}_i(Q^*) + w_j^1 c_{ij}(Q_{ij}^*) + w_j^2 t_{ij}(Q_{ij}^*) \begin{cases} = \lambda_j^*, & \text{if} \quad Q_{ij}^* > 0 \\ \geq \lambda_j^*, & \text{if} \quad Q_{ij}^* = 0, \end{cases} \tag{8}$$

and

$$d_j(\lambda^*) \begin{cases} = \sum_{i=1}^{m} Q_{ij}^*, & \text{if} \quad \lambda_j^* > 0 \\ \leq \sum_{i=1}^{m} Q_{ij}^*, & \text{if} \quad \lambda_j^* = 0. \end{cases} \tag{9}$$

In other words, the commodity will be shipped between a pair of supply and demand markets if the supply price plus the generalized cost associated with shipping the commodity is equal to the generalized price at the demand market. If the supply price plus the generalized cost exceeds the generalized price at the demand market, then there will be no trade between the pair of markets. In addition, if the generalized price associated with a demand market is positive, then the market clears for that demand market; that is, the sum of the commodity shipments from the supply markets to that demand market is equal to the demand associated with that demand market; if the generalized price is zero, then the sum of the commodity shipments can exceed the demand for the commodity at the demand market. Henceforth, we refer to the generalized price simply as the price of the commodity at the particular demand market.

We define the feasible set $\mathcal{K}$ underlying the problem as $\mathcal{K} \equiv \{(Q, \lambda) \,|\, (Q, \lambda) \in R_+^{mn+n}\}$.

The equivalence between the multicriteria spatial price network equilibrium conditions and a variational inequality is now established.

Theorem 1: Variational Inequality Formulation

A multicriteria commodity shipment and demand price pattern $(Q^*, \lambda^*) \in \mathcal{K}$ *is a spatial price network equilibrium, that is, satisfies equilibrium conditions (8) and (9) if and only if it satisfies the variational inequality problem:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\hat{\pi}_i(Q^*) + w_j^1 c_{ij}(Q_{ij}^*) + w_j^2 t_{ij}(Q_{ij}^*) - \lambda_j^*) \times (Q_{ij} - Q_{ij}^*)$$

$$+\sum_{j=1}^{n}(\sum_{i=1}^{m} Q_{ij}^* - d_j(\lambda^*)) \times (\lambda_j - \lambda_j^*) \geq 0, \quad \forall (Q, \lambda) \in \mathcal{K}; \tag{10}$$

equivalently, in standard form:

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{11}$$

where $X \equiv (Q, \lambda)$*, and* $F(X) \equiv (F_Q(X), F_\lambda(X))$ *with component* ij *of* $F_Q(X)$*, denoted by* $F_Q(X)_{ij}$*, given by:*

$$F_Q(X)_{ij} = \hat{\pi}_i(Q) + w_j^1 c_{ij}(Q_{ij}) + w_j^2 t_{ij}(Q_{ij}) - \lambda_j, \quad \forall ij,$$

and component j *of* $F_\lambda(X)$*, denoted by* $F_\lambda(X)_j$*, given by:*

$$F_\lambda(X)_j = \sum_{i=1}^{m} Q_{ij} - d_j(\lambda), \quad \forall j.$$

The expression: $\langle \cdot, \cdot \rangle$ *denotes the inner product in* N*-dimensional Euclidean space* R^N *where here* $N = mn + n$*.*

Proof: Assume that (Q^*, λ^*) satisfies equilibrium conditions (8) and (9). Then we have from (8) that, for a fixed pair of supply and demand markets ij:

$$(\hat{\pi}_i(Q^*) + w_j^1 c_{ij}(Q_{ij}^*) + w_j^2 t_{ij}(Q_{ij}^*) - \lambda_j^*) \times (Q_{ij} - Q_{ij}^*) \geq 0, \quad \forall Q_{ij} \geq 0, \tag{12}$$

and from (9), that, for a fixed demand market j:

$$-(d_j(\lambda^*) - \sum_{i=1}^{m} Q_{ij}^*) \times (\lambda_j - \lambda_j^*) \geq 0, \quad \forall \lambda_j \geq 0. \tag{13}$$

Summing inequalities (12) over all pairs of markets ij, and summing (13) over all demand markets j, and adding the two resulting inequalities, yields

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\hat{\pi}_i(Q^*)+w_j^1 c_{ij}(Q_{ij}^*)+w_j^2 t_{ij}(Q_{ij}^*)-\lambda_j^*)\times(Q_{ij}-Q_{ij}^*)$$

$$-\sum_{j=1}^{n}(d_j(\lambda^*)-\sum_{i=1}^{m}Q_{ij}^*))\times(\lambda_j-\lambda_j^*)\geq 0, \quad \forall (Q,\lambda)\in R_+^{mn+n}, \tag{14}$$

which is variational inequality (10).

Assume now that $(Q^*,\lambda^*)\in\mathcal{K}$ is a solution to variational inequality (10). Let $\lambda=\lambda^*$ and let $Q_{kl}=Q_{kl}^*$ for all $kl\neq ij$, and substitute these into variational inequality (10), yielding

$$(\hat{\pi}_i(Q^*)+w_j^1 c_{ij}(Q_{ij}^*)+w_j^2 t_{ij}(Q_{ij}^*)-\lambda_j^*)\times(Q_{ij}-Q_{ij}^*)\geq 0, \quad \forall Q_{ij}\geq 0, \tag{15}$$

which, in turn, implies equilibrium conditions (8). Indeed, since if $Q_{ij}^*>0$, then the term following the multiplication sign in (15) can be either positive, negative, or zero, so for the product to be nonnegative, we must have that the term preceding the multiplication sign in (15) is zero. Hence, the first part of (8) holds true. On the other hand, if $Q_{ij}^*=0$, then the term after the multiplication sign in (15) is nonnegative and for the product of two terms in (15) to be nonnegative, implies that the first term must be nonnegative, which, in turn, is equivalent to the second part of condition (8) being satisfied.

Similarly, let $Q=Q^*$, and let $\lambda_k=\lambda_k^*$, for all $k\neq j$, and substitute into (10), yielding

$$-(d_j(\lambda^*)-\sum_{i=1}^{m}Q_{ij}^*)\times(\lambda_j-\lambda_j^*)\geq 0, \quad \forall \lambda_j\geq 0, \tag{16}$$

which, in turn, arguing as above, implies equilibrium conditions (9). □

3. Qualitative Properties

In this Section, we provide some qualitative properties of the solution to variational inequality (10). In particular, we derive existence and uniqueness results.We also investigate properties of the function F (see (11)) that enters the variational inequality of interest here.

Since the feasible set $\mathcal{K}$ is not compact we cannot derive existence simply from the assumption of continuity of the supply price, transportation cost and time, and demand functions. Nevertheless, we can impose a rather weak condition to guarantee existence of a solution pattern.

Let $r = (r_1, r_2) \in R^2$ and denote by Ω_r the rectangle in R^{mn+n} such that

$$\Omega_r = \{(Q, \lambda)|0 \leq Q \leq r_1, 0 \leq \lambda \leq r_2\}, \tag{17}$$

where $Q \leq r_1, \lambda \leq r_2$ means that $Q_{ij} \leq r_1$ and $\lambda_j \leq r_2$ for all ij. Then $\mathcal{K}_r = \mathcal{K} \cap \Omega_r$, the intersection of original feasible set with the rectangle, is a bounded closed convex subset of R^{mn+n}. Thus, the following variational inequality

$$\langle F(X^r), X - X^r\rangle \geq 0, \quad \forall X^r \in \mathcal{K}_r, \tag{18}$$

admits at least one solution $X^r \in \mathcal{K}_r$, from the standard theory of variational inequalities, since $\mathcal{K}_r$ is compact and F is continuous. Following Kinderlehrer and Stampacchia (1980) (see also Theorem 1.5 in Nagurney (1999)), we then have:

Theorem 2

Variational inequality (10) admits a solution if and only if there exist $r_1 > 0, r_2 > 0$, such that variational inequality (18) admits a solution $X^r = (Q^r, \lambda^r)$ in $\mathcal{K}_r$ with

$$Q^r < r_1, \quad \lambda^r < r_2. \tag{19}$$

Proposition 1

Suppose that there exist positive constants M, N,and R, with $R > 0$, such that:

$$\hat{\pi}_i(Q) + w_j^1 c_{ij}(Q_{ij}) + w_j^2 t_{ij}(Q_{ij}) > R, \quad \forall Q, \textit{ with } Q_{ij} \geq N, \tag{20}$$

$$d_j(\lambda) \leq N, \quad \forall \lambda \text{ with } \lambda_j > M. \tag{21}$$

Then variational inequality (10) admits at least one solution.

Proof: Follows using analogous arguments as the proof of existence for Proposition 1 in Nagurney and Zhao (1993).

Assumptions (20) and (21) are reasonable from an economics perspective, since when the commodity shipment between a pair of markets is large, we can expect the corresponding supply price or the generalized cost to also be large. Moreover, in the case where the generalized price of the commodity at a demand market is high, we can expect that the demand for the commodity will be low at that market.

We now turn to investigating uniqueness of the equilibrium, that is, a solution to variational inequality (10). We first, however, need to establish the following lemmas.

Lemma 1

Assume that the Jacobian matrices of the transportation cost and transportation time functions are both positive definite, for all $Q \in \mathcal{K}$. Then, the generalized cost function $\hat{c}(Q)$ with component ij given by $w_j^1 c_{ij}(Q_{ij}) + w_j^2 t_{ij}(Q_{ij})$ is strictly monotone for such Q, that is,

$$\langle \hat{c}(Q^1) - \hat{c}(Q^2), Q^1 - Q^2 \rangle > 0, \quad \forall Q^1, Q^2 \in \mathcal{K}, \quad Q^1 \neq Q^2. \tag{22}$$

Proof: Recall, from the standard theory of variational inequalities (see Kinderlehrer and Stampacchia (1980) and Theorem 1.7 in Nagurney (1999)), that if the Jacobian matrix of $\hat{c}(Q)$ is positive definite over $\mathcal{K}$, then $\hat{c}(Q)$ is strictly monotone.

The Jacobian of $\hat{c}(Q)$ can be expressed as:

$$\nabla \hat{c}(Q) = [\nabla c] \left[W^1\right] + [\nabla t] \left[W^2\right], \tag{23}$$

where ∇c is the Jacobian of the transportation cost functions, ∇t is the Jacobian of the transportation time functions, and W^i; $i = 1, 2$ are diagonal $mn \times mn$ matrices with the diagonal components $(w_1^i, w_2^i, \ldots, w_n^i)$ repeating m times. Note that since both the transportation cost and time functions are assumed to be separable, their Jacobian matrices are also diagonal.

Clearly, since the weights are assumed to be positive, both W^1 and W^2 are positive definite matrices. Since the Jacobians of c and t are, by assumption, also positive definite matrices, it follows that $[\nabla c]\,[W^1]$ is positive definite since it is the product of two diagonal and positive definite matrices as is $[\nabla t]\,[W^2]$. Finally, $\nabla \hat{c}$ must be positive definite, since it is the sum of two positive definite matrices. □

Lemma 2

Assume that $\hat{\pi}(Q)$ is strictly monotone increasing, that is, that

$$\langle \hat{\pi}(Q^1) - \hat{\pi}(Q^2), Q^1 - Q^2 \rangle > 0, \quad \forall Q^1, Q^2 \in \mathcal{K}, \quad Q^1 \neq Q^2, \tag{24}$$

$d(\lambda)$ is strictly monotone decreasing, that is,

$$-\langle d(\lambda^1) - d(\lambda^2), \lambda^1 - \lambda^2 \rangle > 0, \quad \forall \lambda^1, \lambda^2 \in \mathcal{K}, \quad \lambda^1 \neq \lambda^2, \tag{25}$$

and that the Jacobian matrices ∇c and ∇t are positive definite over the feasible set. Then $F(X)$ is strictly monotone over $\mathcal{K}$.

Proof: Write

$$\langle F(X^{'}) - F(X^{''}), X^{'} - X^{''} \rangle$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}(\hat{\pi}_i(Q') + w_j^1 c_{ij}(Q'_{ij}) + w_j^2 t_{ij}(Q'_{ij}) - \lambda'_j) - (\hat{\pi}_i(Q'') + w_j^1 c_{ij}(Q''_{ij}) + w_j^2 t_{ij}(Q''_{ij}) - \lambda''_j)) \times (Q'_{ij} - Q''_{ij})$$

$$+ \sum_{j=1}^{n}((\sum_{i=1}^{m} Q'_{ij} - d_j(\lambda')) - (\sum_{i=1}^{m} Q''_{ij} - d_j(\lambda''))) \times (\lambda'_j - \lambda''_j)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}((\hat{\pi}_i(Q') + w_j^1 c_{ij}(Q'_{ij}) + w_j^2 t_{ij}(Q'_{ij})) - (\hat{\pi}_i(Q'') + w_j^1 c_{ij}(Q''_{ij}) + w_j^2 t_{ij}(Q''_{ij})) \times (Q'_{ij} - Q''_{ij})$$

$$- \sum_{j=1}^{n}(d_j(\lambda') - d_j(\lambda'')) \times (\lambda'_j - \lambda''_j). \tag{26}$$

We now argue that the right-most term in (26) is strictly greater than zero, and, hence, $F(X)$ is strictly monotone over $\mathcal{K}$. Indeed, by assumption, we have that the demand functions are strictly monotone decreasing, and we have already established in Lemma 1 that the generalized cost functions are also strictly monotone under the assumptions of positive definite Jacobians for the transportation cost and time functions. Finally, it is straightforward to verify that the Jacobian of $\hat{\pi}$ is monotone. Hence, the expression(s) in (26) must be strictly greater than zero and the conclusion follows. □

Theorem 3: Uniqueness

Assume that $\hat{\pi}(Q)$ is strictly monotone increasing and that $d(\lambda)$ is strictly monotone decreasing. Also, assume that ∇c and ∇t are positive definite over $\mathcal{K}$. Then the equilibrium pattern (Q^*, λ^*) satisfying variational inequality (10) is unique.

Proof: Assume that there are two distinct solutions (Q^1, λ^1) and (Q^2, λ^2) to variational inequality (10). Then, we must have that:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\hat{\pi}_i(Q^1) + w_j^1 c_{ij}(Q_{ij}^1) + w_j^2 t_{ij}(Q_{ij}^1) - \lambda_j^1) \times (Q_{ij} - Q_{ij}^1)$$

$$+\sum_{j=1}^{n}(\sum_{i=1}^{m} Q_{ij}^1 - d_j(\lambda^1)) \times (\lambda_j - \lambda_j^1) \geq 0, \quad \forall (Q, \lambda) \in \mathcal{K}, \tag{27}$$

and

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\hat{\pi}_i(Q^2) + w_j^1 c_{ij}(Q_{ij}^2) + w_j^2 t_{ij}(Q_{ij}^2) - \lambda_j^2) \times (Q_{ij} - Q_{ij}^2)$$

$$+\sum_{j=1}^{n}(\sum_{i=1}^{m} Q_{ij}^2 - d_j(\lambda^2)) \times (\lambda_j - \lambda_j^2) \geq 0, \quad \forall (Q, \lambda) \in \mathcal{K}. \tag{28}$$

Let $(Q, \lambda) = (Q^1, \lambda^1)$ and substitute into (28) and let $(Q, \lambda) = (Q^2, \lambda^2)$ and substitute into (27). Adding the two resulting inequalities, after algebraic simplification, yields:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\hat{\pi}_i(Q^1) + w_j^1 c_{ij}(Q_{ij}^1) + w_j^2 t_{ij}(Q_{ij}^1)) - (w_j^1 c_{ij}(Q_{ij}) + w_j^2 t_{ij}(Q_{ij}^2)) \times (Q_{ij}^1 - Q_{ij}^2)$$

$$-\sum_{j=1}^{n}(d_j(\lambda^1) - d_j(\lambda^2)) \times (\lambda_j^1 - \lambda_j^2) \leq 0, \tag{29}$$

but this is in contradiction to the assumptions that the $\hat{\pi}$ functions are strictly monotone increasing, the demand functions d are strictly monotone decreasing, and we know from Lemma 1 that the generalized cost functions are also strictly monotone. Hence, it follows that $(Q^1, \lambda^1) = (Q^2, \lambda^2)$. Thus, uniqueness has been established. $\square$

4. The Dynamics

In this Section, a dynamic counterpart of the multicriteria spatial price network equilibrium model of Section 3 is developed. The set of stationary points coincides with the set of solutions to a variational inequality problem.

The Dynamics of the Commodity Shipments

The dynamic model presented here assumes that the commodity shipments adjust according to the difference between the generalized demand price and the supply price plus the generalized cost associated with a market pair. Mathematically, we have that: For all pairs of markets (i, j) at time τ:

$$\dot{Q}_{ij}(\tau) = \begin{cases} \lambda_j(\tau) - \hat{\pi}_i(Q(\tau)) - w_j^1 c_{ij}(Q_{ij}(\tau)) - w_j^2 t_{ij}(Q_{ij}(\tau)), & \text{when} \quad Q_{ij}(\tau) > 0, \\ \max\{0, \lambda_j(\tau) - \hat{\pi}_i(Q(\tau)) - w_j^1 c_{ij}(Q_{ij}(\tau)) - w_j^2 t_{ij}(Q_{ij}(\tau))\}, & \text{when} \quad Q_{ij}(\tau) = 0. \end{cases} \tag{30}$$

According to (30) the commodity shipment between a pair of supply and demand markets will increase if the generalized demand price at the demand market exceeds the supply price plus the generalized cost associated with shipping the commodity from the supply market. On the other hand, the commodity shipment will decrease if the supply price at the supply market plus the generalized cost exceeds the generalized demand price that the consumers are paying at the demand market. Note also that (30) guarantees that the commodity shipments can never be negative, which would violate feasibility.

The Generalized Demand Price Dynamics

The demand market generalized prices, in turn, evolve at time τ as follows:

$$\dot{\lambda}_j(\tau) \begin{cases} = d_j(\lambda(\tau)) - \sum_{i=1}^m Q_{ij}(\tau), & \text{when} \quad \lambda_j(\tau) > 0, \\ = \max\{0, d_j(\lambda(\tau)) - \sum_{i=1}^m Q_{ij}(\tau)\}, & \text{when} \quad \lambda_j(\tau) = 0. \end{cases} \tag{31}$$

According to (31), the generalized demand price of the commodity at a demand market will increase if the demand exceeds the supply of the commodity at the demand market; it will decrease if the supply exceeds the demand. Moreover, (31) guarantees that the generalized price at the demand market will not be negative.

Hence, this adjustment process is in concert with those proposed by Nagurney, Takayama, and Zhang (1995a, b) for (single-criteria) spatial price equilibrium problems.

The Projected Dynamical System

The dynamic model described by (30) and (31) can now be rewritten as a *projected dynamical system* (cf. Nagurney and Zhang (1996)). For definiteness, we recall some preliminaries and the definition.

The class of ordinary differential equations of interest here takes the form:

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X_0 \in \mathcal{K}, \tag{32}$$

where $\mathcal{K}$ is a closed convex set, corresponding to the constraint set in a particular application, $F(X)$ is a vector field defined on $\mathcal{K}$, and $\Pi_{\mathcal{K}}(X, -F(X))$ is given by:

$$\Pi_{\mathcal{K}}(X, -F(X)) = \lim_{\delta \to 0} \frac{(P_{\mathcal{K}}(X - \delta F(X)) - X)}{\delta} \tag{33}$$

and $P_{\mathcal{K}}$ is the projection map:

$$P_{\mathcal{K}}(X) = \arg\min_{z \in \mathcal{K}} \|X - z\|. \tag{34}$$

We refer to the ordinary differential equation in (32) as ODE$(F, \mathcal{K})$.

Observe that the right-hand side of the ordinary differential equation (32) is associated with a projection operator and is, hence, discontinuous on the boundary of $\mathcal{K}$. Therefore, one needs to explicitly state what one means by a solution to an ODE with a discontinuous right-hand side.

Definition 1: Projected Dynamical System

Define the projected dynamical system (PDS) $X_0(t) : \mathcal{K} \times R \mapsto \mathcal{K}$ *as the family of solutions to the Initial Value Problem* (IVP) *(32) for all* $X_0 \in \mathcal{K}$*.*

It is apparent from the definition that $X_0(0) = X_0$. In the context of the spatial price equilibrium model, $X_0 = (Q_0, \lambda_0)$ is the initial point corresponding to the initial commodity shipment and price pattern. The trajectory of (32) describes the dynamic evolution of and the dynamic interactions between the commodity shipment and the price patterns. A projected dynamical system differs from a classical dynamical system in that the right-hand side in (32) is discontinuous due to the explicit incorporation of the constraint set, where recall that the constraint set here is the nonnegative orthant.

Theorem 4

The set of stationary points of the projected dynamical system (32) coincides with the set of solutions of the variational inequality (11); equivalently, variational inequality (10).

Proof: According to the fundamental theorem of projected dynamical systems (see Nagurney and Zhang (1996)), X^* is a stationary point of the projected dynamical system (32) if and only if it is a solution to variational inequality (11), which is equivalent to (10). □

5. The Discrete-Time Algorithm

In this Section, we propose a discrete-time algorithm, the Euler method, for the computation of the equilibrium pattern. The algorithm provides a discretization of the continuous time adjustment process given in Section 4. Specifically, the Euler method is a special case of the general iterative scheme for projected dynamical systems proposed by Dupuis and Nagurney (1993) (see also Nagurney and Zhang (1996)). The algorithm, hence, computes a solution to variational inequality (10) and also provides a discrete-time approximation to the projected dynamical system (32), the stationary points of which (cf. Theorem 4) coincide with the solutions of variational inequality (10).

Its statement in the general form for the solution of variational inequality (11) and for the time discretization of the corresponding projected dynamical system is given by:

$$X^{k+1} = P_{\mathcal{K}}(X^k - a_k F(X^k)), \tag{35}$$

where k denotes an iteration (or time period) and $\{a_k\}$ is a sequence of positive scalars to be discussed later.

In particular, in the context of the multicriteria spatial price network equilibrium problem formulated as (10), with F as defined following (11) and the feasible set $\mathcal{K}$ being R_+^{mn+n}, the projection operation takes on a very simple form for computational purposes, and, hence, the commodity shipments as well as the demand market prices can be computed at an iteration in closed form as follows:

$$Q_{ij}^{k+1} = \max\{0, a_k(\lambda_j^k - w_j^1 c_{ij}(Q_{ij}^k) - w_j^2 t_{ij}(Q_{ij}^k) - \hat{\pi}_i(Q^k)) + Q_{ij}^k\}, \quad \forall ij, \tag{36}$$

and

$$\lambda_j^{k+1} = \max\{0, a_k(d_j(\lambda^k) - \sum_{i=1}^m Q_{ij}^k) + \lambda_j^k\}, \quad \forall j. \tag{37}$$

We first give the precise conditions for the general convergence theorem, present its statement, and then interpret it for the Euler method applied to the spatial price equilibrium problem. We note that the conditions are given for the general iterative scheme of Dupuis and Nagurney (1993) where an iteration is given by:

$$X^{k+1} = P_{\mathcal{K}}(X^k - a_k F_k(X^k)) \tag{38}$$

with F_k denoting an approximation to F, which, in the case of the Euler method is: $F_k = F$.

Assumption 1

Fix an initial condition $X^0 \in \mathcal{K}$ and define the sequence $\{X^k\}$ by (38). Assume the following conditions.

1. $\sum_{k=1}^{\infty} a_k = \infty, \quad a_k > 0, \quad a_k \to 0$ *as* $k \to \infty$.

2. $d(F_k(X), \bar{F}(X)) \to 0$ *uniformly on compact subsets of $\mathcal{K}$ as $k \to \infty$, where* $d(X, A) = \inf\{\|X - y\|, y \in A\}$ *and where the bar over the F denotes closure.*

3. *Define ϕ_y to be the unique solution to $\dot{X} = \Pi_{\mathcal{K}}(X, -F(X))$ that satisfies $\phi_y(0) = y \in \mathcal{K}$. The $\omega-$limit set*

$$\cup_{y\in\mathcal{K}} \cap_{t\geq 0} \cup_{s\geq t} \{\phi_y(s)\}$$

is contained in the set of stationary points of $\dot{X} = \Pi_{\mathcal{K}}(X, -F(X))$.

4. *The sequence $\{X^k\}$ is bounded.*

5. *The solutions to $\dot{X} = \Pi_{\mathcal{K}}(X, -F(X))$ are stable in the sense that given any compact set $\mathcal{K}_1$ there exists a compact set $\mathcal{K}_2$ such that $\cup_{y\in\mathcal{K}\cap\mathcal{K}_1} \cup_{t\geq 0} \{\phi_y(t)\} \subset \mathcal{K}_2$.*

The assumptions are phrased as they are because they describe more or less what is needed for convergence, and because there are a number of rather different sets of conditions that imply the assumptions, depending on the application (see, e.g., Nagurney and Zhang (1996)).

Theorem 5 (Dupuis and Nagurney (1993))

Let S denote the solutions to the variational inequality (11), and invoke Assumption 1 and Assumption 2, where

Assumption 2

There exists a $B < \infty$ such that the vector field $-F : R^{mn+n} \mapsto R^{mn+n}$ satisfies the linear *growth condition:* $\| -F(X)\| \leq B(1+\|X\|)$ *for* $X \in \mathcal{K}$*, and also*

$$\langle -F(X)+F(y), X-y\rangle \leq B\|X-y\|^2 \tag{39}$$

for all $X, y \in \mathcal{K}$.

Suppose $\{X^k\}$ *is the scheme generated by (38). Then* $d(X^k, S) \to 0$ *as* $k \to \infty$.

Corollary 1 (Dupuis and Nagurney (1993))

Assume the conditions of Theorem 5, and also that S *consists of a finite set of points. Then* $\lim_{k\to\infty} X^k$ *exists and equals a solution to the variational inequality (11).*

We now interpret the meaning of Assumptions 1 and 2 in the context of the spatial price equilibrium problem, in order to establish the convergence of the Euler-type method, which is a special case of the general iterative scheme of Dupuis and Nagurney (1993).

In order to establish convergence of the Euler method we need to adopt the following assumption:

Assumption 3

Assume that there exist sufficiently large constants M_d, M_Q, *and* M_λ, *such that*

$$d_j(\lambda) \leq M_d, \quad \forall \lambda \in R^n_+, \tag{40}$$

$$\lambda_j \leq \hat{\pi}_i(Q) + w^1_j c_{ij}(Q_{ij}) + w^2_j t_{ij}(Q_{ij}), \quad \textit{if } Q_{ij} \geq M_Q, \tag{41}$$

$$d_j(\lambda) \leq \sum_{i=1}^{m} Q_{ij}, \quad \textit{if } \lambda_j \geq M_\lambda, \tag{42}$$

for any j *and* i.

The convergence of the Euler method is stated in the following theorem.

Theorem 6

Suppose that the supply price functions are strictly monotone increasing, the demand functions are strictly monotone decreasing, and the Jacobians of the transportation cost and time *functions are positive definite over the feasible set. Let $\{a_k\}$ be a sequence of positive real numbers that satisfies*

$$\lim_{k\to\infty} a_k = 0 \tag{43}$$

$$\sum_{k=1}^{\infty} a_k = \infty. \tag{44}$$

In addition, assume that Assumption 3 holds true. Then the Euler method given by (35) converges to the unique multicriteria spatial price network equilibrium pattern satisfying conditions (8) and (9).

Proof: In view of Theorem 5 above, we need to verify that Assumptions 1 and 2 above are satisfied here.

First, note that, under strict monotonicity, as established in Lemma 2, the vector field $F(X)$ that governs the projected dynamical system (32) satisfies the linear growth condition, namely,

$$\begin{aligned}\langle -F(X') + F(X''), X' - X'' \rangle &\leq 0 \\ &\leq B\|X' - X''\|^2,\end{aligned} \tag{45}$$

for any positive B.

The first part of Assumption 1 of Dupuis and Nagurney (1993) is automatically satisfied by the selection of the appropriate a_k sequence and the second part of Assumption 1 automatically holds for the Euler method (see also Nagurney and Zhang (1996)).

The third and fifth parts of Assumption 1 are also satisfied (see Propositions 4.1 and 4.2 in Nagurney and Zhang (1996)) since F is strictly monotone.

All that we need to establish now is the fourth part of Assumption 1, that is, we need to show that the sequence generated by the Euler method is bounded. Assumption 3 guarantees that the sequence generated is bounded following the proof of convergence of the Euler method (see Theorem 7.11 in Nagurney and Zhang (1996)) for the traffic network equilibrium problem with given demand functions.

The proof is complete. □

6. Numerical Examples

In this Section, we present numerical examples for illustrative purposes. Specifically, we consider three spatial price equilibrium problems in which there are two supply markets and two demand markets.

In these examples the supply price functions, the transportation cost and time, and demand functions are identical and are given, respectively, by:

$$\pi_1(s) = 5s_1 + s_2 + 2, \quad \pi_2(s) = 2s_2 + 1.5s_1 + 1.5,$$

so that

$$\hat{\pi}_1(Q) = 5\sum_{j=1}^{2} Q_{1j} + \sum_{j=1}^{2} Q_{2j} + 2, \quad \hat{\pi}_2(Q) = 2\sum_{j=1}^{2} Q_{2j} + 1.5\sum_{j=1}^{2} Q_{1j} + 1.5,$$

$$t_{11}(Q_{11}) = Q_{11}, \quad t_{12}(Q_{ij}) = 2Q_{12} + 3.5,$$

$$t_{21}(Q_{21}) = 3Q_{21} + 16.25, \quad t_{22}(Q_{22}) = 2Q_{22} + 11.5,$$

$$c_{11}(Q_{11}) = 2Q_{11} + 5, \quad c_{12}(Q_{12}) = Q_{12} + 2,$$

$$c_{21}(Q_{21}) = 3Q_{21} + 4, \quad c_{22}(Q_{22}) = 5Q_{22} + 1,$$

$$d_1(\lambda) = -2\lambda_1 - 1.5\lambda_2 + 1128.75, \quad d_2(\lambda) = -4\lambda_2 - \lambda_1 + 1241.$$

The weights w_j^1 and w_j^2 for $j = 1, 2$ differ in each example. The generalized costs were constructed according to (6). Hence, these problems illustrate how the equilibrium pattern changes as the weights change.

The Euler method for all the examples was initialized as follows: the commodity shipment pattern Q^1 was set to zero as were the demand prices λ^1. The $\{a_k\}$ sequence that we utilized was: $.1 \times \{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\}$. The convergence criterion used was: $|Q^{k+1} - Q^k| \leq \epsilon$ and $|\lambda^{k+1} - \lambda^k| \leq \epsilon$ with $\epsilon = .0001$. Hence, the Euler method was considered to have converged when the commodity price and shipment pattern had not changed very much between two iterations and had, effectively, reached a stationary; equivalently, an equilibrium point.

We now report the computed results for the examples. The algorithm was coded in FORTRAN and the system used was an IBM SP2 located at the Computer Science Department at the University of Massachusetts at Amherst. The CPU time is reported exclusive of input and output times.

Example 1

In the first example, we assumed that the consumers in each of the two demand markets weighted travel time and travel cost in the same way, and equally. The weights were: $w_1^1 = w_1^2 = w_2^1 = w_2^2 = 0.5$.

The Euler method converged in 729 iterations and required .01 seconds of CPU time. The computed equilibrium commodity shipment pattern was:

$$Q_{11}^* = 44.194, \quad Q_{12}^* = 0.000, \quad Q_{21}^* = 51.475, \quad Q_{22}^* = 7.914,$$

which induced the equilibrium supply pattern:

$$s_1^* = 44.194, \quad s_2^* = 59.389.$$

The computed demand price pattern was:

$$\lambda_1^* = 351.158, \quad \lambda_2^* = 220.484.$$

The equilibrium conditions (8) and (9) were satisfied with good accuracy. Indeed, only the commodity shipment between supply market 1 and demand market 2 was zero. In this case, the supply price at supply market 1 incurred at the computed equilibrium plus the generalized cost between supply market 1 and demand market 2 exceeded the generalized price of the commodity at demand market 2 by 64.24. For the other pairs of supply and demand markets, which were characterized by positive commodity shipments, the difference between the supply price at a supply market plus the generalized cost between the pair of supply and demand markets and the generalized price at the demand market was 0.0.

Also, in terms of equilibrium condition (9), for the first demand market, the sum of the commodity shipments into it was equal to 95.7, which was the demand incurred at the demand market at the computed equilibrium price pattern. In addition, the sum of the computed equilibrium commodity shipments into the second demand market was equal to

7.91, which was equal to the demand at that demand market evaluated at the computed equilibrium price pattern.

Example 2

In the second example, we now modified the weights as follows: $w_1^1 = 0.0$, $w_1^2 = 1.0$, $w_2^1 = 1.0$, $w_2^2 = 0.0$. Hence, in demand market 1 the consumers are transportation time-sensitive and not transportation cost-sensitive, whereas in demand market 2 the consumers are the opposite.

The Euler method converged in 865 iterations and required .01 seconds of CPU time. The computed equilibrium commodity shipment pattern was:

$$Q_{11}^* = 48.629, \quad Q_{12}^* = 0.000, \quad Q_{21}^* = 49.068, \quad Q_{22}^* = 6.797,$$

which induced the equilibrium supply pattern:

$$s_1^* = 48.629, \quad s_2^* = 55.865.$$

The computed demand price pattern was:

$$\lambda_1^* = 359.647, \quad \lambda_2^* = 221.143.$$

Since the consumers in demand market 1 are now more transportation time-sensitive then they were in Example 1, the commodity shipment on the faster (with the lower transportation time function) "link" (between supply market 1 and demand market 1) increased as compared to the corresponding equilibrium commodity shipment in Example 1. As regards demand market 2, in which consumers are now more transportation cost-sensitive than they were in Example 1, the commodity shipment between supply market 2 and demand market 2 decreased.

As in Example 1, there was no trade between supply market 1 and demand market 2. For this pair of markets, the supply price plus the generalized cost now exceeded the generalized price by 81.865. The analogous difference for the three other market pairs, for which there was trade, i.e., a positive commodity shipment, was 0.0, signifying that equilibrium condition

(8) was satisfied to good accuracy, In addition, since the generalized prices were positive at both demand markets, the sum of the commodity shipments into each demand market was (essentially) equal to the computed demand at the respective market at the computed equilibrium price pattern. Indeed, for demand market 1, the sum of commodity shipments into it was 97.7, which was the computed demand. Also, the sum of the commodity shipments into demand market 2 was 6.8 which was equal to the incurred demand at the demand market at the equilibrium price pattern.

Example 3

In the third example, we now modified the weights as follows: $w_1^1 = 1.0$, $w_1^2 = 0.0$, $w_2^1 = 0.0$, $w_2^2 = 1.0$. Hence, in demand market 1 the consumers are now transportation cost-sensitive and not transportation time-sensitive, whereas in demand market 2 the consumers are the opposite.

The Euler method converged in 973 iterations and required .01 seconds of CPU time. The computed equilibrium commodity shipment pattern was:

$$Q_{11}^* = 40.369, \quad Q_{12}^* = 0.000, \quad Q_{21}^* = 53.435, \quad Q_{22}^* = 9.791,$$

which induced the equilibrium supply pattern:

$$s_1^* = 40.360, \quad s_2^* = 63.226.$$

The computed demand price pattern was:

$$\lambda_1^* = 352.798, \quad \lambda_2^* = 219.598.$$

Since the generalized cost now between supply market 2 and demand market 1, which now consists of only the transportation cost has now been reduced either relative to that encountered by the consumers in demand market 1 in Example 1 or Example 2, the commodity shipment Q_{21}^* has increased relative to both equilibrium commodity shipments computed between that pair of markets in the two preceding examples. As regards demand market 2, however, which is now more transportation time-sensitive than in the preceding two examples, the overall commodity shipments into that demand market decrease due to the relatively high transportation times for the commodity.

As in the two preceding examples, there was no trade between supply market 1 and demand market 2, since the supply price at supply market 1 plus the generalized cost between supply market 1 and demand market 2 exceeded the generalized price at demand market 2 by 50.97. The analogous difference for the other market pairs was 0.0, signifying that equilibrium condition (8) held with good accuracy.

In terms of equilibrium condition (9), since the generalized price of the commodity was positive at both demand markets, the computed sum of commodity shipments into each demand market was approximately equal to the incurred demand at the market at the computed equilibrium price pattern. Indeed, for demand market 1, the sum of the commodity shipments was 93.8, which was also the incurred demand for that demand market, whereas for demand market 2, the sum of the commodity shipments into it was 9.8, which was the incurred demand at the computed equilibrium price pattern.

7. Summary and Conclusions

In this paper, we have developed a multicriteria spatial price network equilibrium model and studied it from two perspectives: a static one, with a focus on the equilibrium, and a dynamic one, through a proposed tatonnement process for the evolution of the commodity shipment and price patterns. The model handles consumers who weight the transportation cost and the transportation time associated with the commodity shipment in an individual fashion.

The statics were studied using the finite-dimensional variational inequality formulation of the governing equilibrium conditions whereas the dynamic model was formulated as a projected dynamical system. This is the first time that a multicriteria network equilibrium problem was treated from two such perspectives.

We established that the set of stationary points of the projected dynamical system coincides with the set of solutions of the variational inequality problem. In addition, we proved both existence and uniqueness of the multicriteria spatial price equilibrium pattern under reasonable conditions. These results are sharper than those that have been obtained recently for multicriteria traffic network equilibrium problems.

We provided a discrete-time algorithm, the Euler method, for the approximation of the trajectory, established convergence, and applied the algorithm to several examples for illustrative purposes.

Acknowledgments

The research of the first two authors was supported, in part, by NSF Grant No.: IIS-0002647. The research of the first author was also supported, in part, by NSF Grant No.: CMS-0085720 and by the John F. Smith Memorial Fund at the Isenberg School of Management at the University of Massachusetts at Amherst. The support is gratefully acknowledged.

The authors are grateful to the editor, Patrizia Daniele, for helpful comments and suggestions.

References

S. Dafermos, "A multicriteria route-mode choice traffic equilibrium model," Lefschetz Center for Dynamical Systems, Brown University, Providence, Rhode Island, 1981.

S. Dafermos and A. Nagurney, "Oligopolistic and competitive behavior of spatially separated markets," *Regional Science and Urban Economics* **17** (1987) 245-254.

R. B. Dial, "A model and algorithms for multicriteria route-mode choice," *Transportation Research* **13B** (1979) 311-316.

P. Dupuis and A. Nagurney, "Variational inequalities and dynamical systems," *Annals of Operations Research* **44** (1993) 9-42.

M. Florian and M. Los, "A new look at static spatial price equilibrium models," *Regional Science and Urban Economics* **12** (1982) 579-597.

D. Kinderlehrer and G. Stampacchia, **An Introduction to Variational Inequalities and Their Application**, Academic Press, New York (1980).

F. Leurent, "Modelling elastic, disaggregate demand," in J. C. Moreno Banos, B. Friedrich, M. Papageorgiou, and H. Keller, editors, **Proceedings of the First Meeting of the Euro Working Group on Urban Traffic and Transportation**, Technical University of Munich, Munich, Germany (1993a).

F. Leurent, "Cost versus time equilibrium over a network," *European Journal of Operations Research* **71** (1993b) 205–221.

F. Leurent, "Multicriteria assignment modeling: making explicit the determinants of mode or path choice," in P. Marcotte and S. Nguyen, editors, **Equilibrium and Advanced Transportation Modelling**, Kluwer Academic Publishers, Boston, Massachusetts (1998) 153-174.

P. Marcotte, "Reformulations of a bicriterion equilibrium model," in **Reformulation: Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods**, pp. 269-292, M.

Fukushima and L. Qi, editors, Kluwer Academic Publishers, Dordrecht, The Netherlands.

A. Nagurney, "Computational comparisons of spatial price equilibrium methods, *Journal of Regional Science* **27** (1987) 55-76.

A. Nagurney, **Network Economics: A Variational Inequality Approach**, second and revised edition, Kluwer Academic Publishers, Dordrecht, The Netherlands (1999).

A. Nagurney, "A multiclass, multicriteria traffic network equilibrium model," *Mathematical and Computer Modelling* **32** (2000), 393-411.

A. Nagurney and J. Dong, "A multiclass, multicriteria traffic network equilibrium model with elastic demand," *Transportation Research B* **36** (2002), 445-469.

A. Nagurney, T. Takayama, and D. Zhang, "Massively parallel computation of spatial price equilibrium problems as dynamical systems," *Journal of Economic Dynamics and Control* **18** (1995a) 3-37.

A. Nagurney, T. Takayama, and D. Zhang, "Projected dynamical systems modeling and computation of spatial price network equilibribria," *Networks* **26** (1995b) 69-85.

A. Nagurney and D. Zhang, **Projected Dynamical Systems and Variational Inequalities with Applications**, Kluwer Academic Publishers, Boston, Massachusetts (1996).

A. Nagurney and L. Zhao, "Networks and variational inequalities in the formulation and computation of market disequilibria: the case of direct demand functions," *Transportation Science* **27** (1993) 4-15.

R. E. Quandt, "A probabilistic abstract mode model," in **Studies in Travel Demand VIII**, Mathematica, Inc., Princeton, New Jersey (1967) 127-149.

P. A. Samuelson, "Spatial price equilibrium and linear programming," *American Economic Review* **42** (1952) 283-303.

M. Schneider, "Access and land development," in **Urban Development Models**, Highway Research Board Special Report **97** (1968) 164-177.

T. Takayama and G. G. Judge, **Spatial and Temporal Price and Allocation Models**, North-Holland, Amsterdam The Netherlands (1971).