Influence of Beckmann, McGuire, and Winsten's Studies in the Economics of Transportation on Innovations in Modeling, Methodological Developments, and Applications

Anna Nagurney
Department of Finance and Operations Management
Isenberg School of Management
University of Massachusetts
Amherst, MA 01003, USA

Prepared for the Panel: Studies in the Economics of Transportation: A Retrospective, at the 50th North American Regional Science Association Meeting in Philadelphia, Pennsylvania, November 2003

Abstract

This paper describes the impact and influence of the book, **Studies in the Economics of Transportation**, by M. J. Beckmann, C. B. McGuire, and C. B. Winsten, published in 1956 by Yale University Press. The focus of this paper is on the book's impacts on innovations in modeling, methodological developments, and applications in transportation science and in other disciplines as well.

1. Introduction

The book, **Studies in the Economics of Transportation**, by Beckmann, McGuire, and Winsten was published in 1956 by Yale University Press and was a breakthrough in the rigorous modeling and analysis of transportation problems with a focus on congested highway systems as well as railroad systems. Its impact has been seminal, far-reaching, and continues to this day. In this paper, I focus on the influence of this book on innovations in modeling, methodological developments, and applications. In particular, this paper traces the impacts of the first part of the book, which is on highway transportation.

I first begin with a description of the context in which the writing of this book took place. The book was based on a Rand Corporation report of the same name, RM-1488, and dated May 12, 1955, with an introduction by Tjalling C. Koopmans, who twenty years after was awarded a Nobel Prize in Economics. Koopmans noted that the report consisted of exploratory studies with an intended audience of various professionals, including economists, traffic and railroad engineers, as well as operations researchers/management scientists, and mathematicians. The report resulted from a research project conducted by the Cowles Commission for Research in Economics with funding provided by the Rand Corporation. Koopmans was the project leader. Beckmann, a mathematical economist, was especially interested in linear programming at that time and in economic activity analysis. Winsten, a mathematician and economist, held a particular interest in applying probability concepts to industrial issues, whereas McGuire, an economist, provided a pragmatic and realistic check on the model development. In terms of the study of highway transportation, the main emphasis was on congestion.

The topic of transportation had been addressed earlier in the context of optimal allocation of resources through linear programming by Hitchcock (1941) and Kantorovich (1942) (who later shared the Nobel Prize with Koopmans) as well as by Koopmans (1947) and Dantzig (1951). In such models, however, there was no congestion associated with transportation. The problem of users of a congested transportation network seeking to determine their travel paths from origins to their respective destinations appears as early as Pigou (1920), who considered a two-node, two-link (or path) network, and was further developed in Knight (1924). Both of these references are cited in the Beckmann, McGuire, and Winsten (1956) book. Fascinatingly, Koopmans in his introduction also acknowledged the work of Enke (1951) and Samuelson (1952) in terms of commodity transportation and the determination of interregional price differentials, a topic now known as spatial price equilibrium, and one which we return to later.

In 1952, Wardrop had set forth two principles of transportation network utilization, which have come to be termed, respectively (cf. Dafermos and Sparrow (1969)), user-optimization and system-optimization. The first principle expresses that travelers select their routes of travel from origins to destinations independently and ultimately the journey times of all routes actually used between an origin/destination pair are equal and less than those which would be experienced by a single vehicle on any unused route. The user-optimized solution is also referred to as a *traffic network equilibrium* or a *traffic assignment*. The second principle, in contrast, reflects the situation in which there is a central controller who routes the traffic flows in an optimal manner from origins to the destinations so as to minimize the total cost in the network. That optimum is reached when the marginals of the total costs on used paths connecting an origin/destination pair are equal and minimal. Koopmans noted that completely regulated traffic such as truck convoys of an army could yield higher rates of flow out of a given network than that obtained when individuals make their own choices.

Beckmann, McGuire, and Winsten (1956) were the first to provide a rigorous mathematical formulation of the conditions set forth by Wardrop's first principle that allowed for the ultimate solution of the traffic network equilibrium problem in the context of certain link cost functions which were increasing functions of the flows on the links. In particular, they demonstrated that the optimality conditions in the form of Kuhn-Tucker (1951) conditions of an appropriately constructed mathematical programming/optimization problem coincided with the statement that the travel costs on utilized routes/paths connecting each origin/destination pair of nodes in a transportation network have equal and minimal travel costs. Hence, no traveler, acting unilaterally will have any incentive to alter his path (assuming rational behavior) since his travel cost (travel time) is minimal. Interestingly, Charnes and Cooper (1958, 1961) in their papers had cited the work of Nash (1951), Wardrop (1952), and Prager (1954), with their (1958) paper also noting Duffin (1947), who provided a formulation of the equilibrium in electrical networks, but they did not cite Beckmann, McGuire, and Winsten (1956).

Thus, a problem in which there are numerous decision-makers acting independently and as later also noted by Dafermos and Sparrow (1969) competing in the sense of Nash, could be reformulated (under appropriate assumptions) as a convex optimization problem with a single objective function subject to linear constraints and nonnegativity assumptions of the flow on the network. Prager (1954) had also recognized Wardrop's principles and in his paper emphasized that the traffic cost on a link may depend not only on the flow on that link but on other links in the network as well. Jorgensen (1963) in a report (actually his Master's thesis) did not cite Beckmann, McGuire, and Winsten (1956) but noted the work of Wardrop (1952) and Charnes and Cooper (1961) and developed an optimization reformulation of the traffic network equilibrium conditions in the case of fixed travel demands and link cost functions that were separable. Jorgensen (1963) also influenced the thesis of Dafermos (1968) upon which the paper of Dafermos and Sparrow (1969) is based.

In this paper, I trace the impacts of the book. Such an assignment is challenging and daunting given the almost fifty years that have elapsed since its publication. Nevertheless, it is important to highlight and to emphasize further the impact of this monumental work, even if it is done through the prism of one's own experiences and knowledge of the literature, but accompanied by interactions with many leaders in the transportation science and broader scientific community whose work has been impacted by this volume.

This paper is organized as follows. In Section 2, I identify some of the major innovations in modeling and methodological developments since these two have often been intimately linked. In Section 3, I then focus on innovations in applications which continue to this day. I conclude with Section 4, in which personal reflections and comments are given, which also help to place this paper as well as its citations in the proper context.

2. Innovations in Modeling and Methodological Developments

In this section, we describe some of the major innovations in modeling and methodological developments that were motivated by the work of Beckmann, McGuire, and Winsten (1956). Our emphasis in this section is on transportation and the contributions are presented more or less in chronological order to provide a progression of the intellectual developments and a timeline. In Section 3, we then discuss how the book is referenced in and has contributed to innovations in many additional applications.

Algorithms and Computations for the Standard Models of BMW

The first major methodological innovation in a paper that cites Beckmann, McGuire, and Winsten (BMW) (1956) is the paper by Dafermos and Sparrow (1969) which not only coined the terms "user-opimization" and "system-optimization" to distinguish between Wardrop's first and second principles, respectively, and to help to clarify the underlying behavior of the travelers in these two contexts, but also developed algorithms that explicitly exploited the network structure of these two problems and established convergence results for the schemes. Moreover, that paper, provided not only equilibration algorithms for networks of any topology but also special-purpose ones in the case of special topologies for which the flows could be computed exactly and in closed form. Further, the paper discussed stability of the solution patterns, a topic whose importance was emphasized in BMW. Almond (1967) had constructed an algorithm for the determination of the user-optimized solution but in the case of very simple networks. Tomlin (1966), in turn, considered linear cost (congestion) functions and not nonlinear ones as had Dafermos and Sparrow (1969) and exploited that feature in the development of his algorithm. Almond (1967) cited BMW whereas Tomlin did not although he did refer to Jorgensen (1963). Subsequently, Leventhal, Nemhauser, and Trotter (1973) proposed a column generation procedure that could be embedded in the Dafermos and Sparrow general equilibration procedures to allow for path generation as needed (rather than apriori which could require large computer memory resources).

The first innovations in algorithm development for traffic network equilibrium problems focused on the standard models of BMW where by standard is meant that the link cost functions were separable in that the cost on a link depended upon only the flow on that link. Effective schemes for such problems are important not only for such problems but also in the case of more general network models for which an optimization reformulation of the governing equilibrium conditions is not available and, hence, one must appeal to variational inequality formulations, for example (see below). Variational inequality problems, however, are typically solved as series of optimization problems and, hence, there is a great practical need for efficient optimization-based schemes that can exploit the network structure of the transportation problems. Nagurney (1984) discussed computational experiments conducted on a variety of solution procedures for traffic network equilibrium problems available at that time, many of which are still used today.

Bruynooghe, Gilbert, and Sakarovitch (1969) considered the fixed demand model and also discussed two algorithms and cited the Beckmann, McGuire, and Winsten (1956) book. Netter (1972) further described the properties of system-optimized versus user-optimized solutions and referred to BMW. LeBlanc (1973); see also LeBlanc, Morlok, and Pierskalla (1975), proposed an algorithm based on the Frank-Wolfe (1956) convex programming scheme to solve the traffic assignment problem, and although he cited BMW, he did not cite Dafermos and Sparrow (1969). Nguyen (1974a, b) further pioneered the exploration of appropriate algorithms for the solution of traffic network equilibrium problems, and implemented and tested several schemes.

Ferland (1974) considered the solution of an elastic demand model as did Florian and Nguyen (1974). Gartner (1980) demonstrated how separable elastic demand traffic network models could be transformed and, hence, solved as fixed demand models through reformulations over abstract networks. These authors cited Beckmann, McGuire, and Winsten (1956). The use of concepts formalized in BMW to model traffic network equilibrium problems by defining the appropriate origin/destination pairs, links, paths, and associated link costs as well as travel demands was receiving increasing attention and recognition and has to-date been applied in settings distinct from transportation science, which we elaborate upon in more detail in Section 3. Indeed, it is quite remarkable how the fundamental work of BMW continues to be rediscovered, elaborated upon, and utilized in numerous applications.

We further emphasize the importance of rigorous scientific methodologies for modeling, analysis, and solution of traffic network equilibrium problems, which are not only of theoretical interest, but also of great practical importance due to the growing congestion in developed countries as well as in developing countries. Recently, Bar-Gera (1999) has devised a convergent algorithm based on origin-based assignment which has been applied to solve networks of realistic size. Additional discussions of algorithms for network equilibrium problems can be found in the books by Sheffi (1985), Nagurney (1993), and Patriksson (1994).

Toll Policies

In 1971 Dafermos and Sparrow published a paper on optimal resource allocation and tolls, which would guarantee that once assigned, the user-optimized solution would coincide with the system-optimized solution so that individual travelers would behave in a manner that would also be optimal from a system or societal point of view. BMW had earlier discussed how efficiency toll rates could be determined, whereas Beckmann (1967a) described optimal tolls for highways, tunnels, and bridges. In both works, tolls were viewed as a means of bringing about the best utilization of the transportation network rather than as a means of construction financing per se. Dafermos and Sparrow (1971) proposed two types of toll policies, in link form and in path form, with the latter allowing for more flexibility from the planning perspective but resulting perhaps in subsidies unlike the link policy. That paper, as the paper of Dafermos and Sparrow (1969), was based on the thesis of Dafermos (1968), which, as we have noted earlier, cited BMW.

Pigou in 1920 had proposed tolls which could be imposed by the government in such a way so that the altered user-optimized flow pattern would coincide with the total cost (system-optimized) optimized pattern. Another relevant early reference is that of Vickrey (1952), whose work in the pricing of transportation services later earned him a Nobel Prize, and who was cited in BMW. Walters (1961), subsequently, utilized the network model of BMW for toll determination.

The topic of congestion pricing through tolls has been recently an active area of research and practice with tolls schemes being applied in various parts of the world, including, with some success, in London. For a recent approach and additional references, see Bergendorff, Hearn, and Ramana (1997).

Extended Traffic Network Models Including Models of Urban Location

BMW focused on transportation networks in which the cost (also travel time) on a link, that is, road, depended solely upon the flow on that link. Under such an assumption (i.e., separable functions and necessarily "symmetric") they could then prove their fundamental result. Dafermos in a series of papers in the early 70s, which cited BMW, developed "extended" traffic network models and also formulated tolls in the case of multiclass networks. In particular, Dafermos in her 1971 and 1972 papers, demonstrated that an analogous reformulation of the traffic network equilibrium conditions as a convex optimization problem could be identified in the case of more general user link cost functions in which the cost on a link could depend on the flows on all links in the network, provided that a symmetry condition held either in the single-class user case or the multi-class user case, which allowed for different classes of travelers who perceive the travel cost on a link in an individual manner. She further demonstrated that one could transform that model into an extended, single-class one by constructing appropriate abstract copies of the multiclass network and by redefining the underlying functions and flows. In addition, extensions of the general equilibration algorithms contained in Dafermos and Sparrow (1969), along with convergence results, were obtained in Dafermos (1971, 1972). In 1973, Dafermos further generalized tolls to multiclass traffic networks, which are also now referred to as multimodal networks.

Beckmann, McGuire, and Winsten (1956) clearly delineated that one should distinguish between short-run and long-run decision-making regarding transportation networks. In particular, they noted that if travelers have already made their origin and destination selections, then the decision becomes one where one must determine the optimal path to take between the two. However, in the long-run, travelers may wish to choose not only their routes but also perhaps their origins in the form of residences and/or destinations, say, in the form of places of employment. Motivated by such questions, Dafermos in 1976 demonstrated, through the use of abstract networks that one could capture such decision-making within a network equilibrium context. Again, the fundamental concepts devised and elaborated upon in BMW were now being applied to more complex decision-making which included not only route choice.

In 1980, Boyce proposed a framework for constructing network equilibrium models of urban location which allowed for the incorporation of the trip distribution problem. In 1983, Boyce et al., motivated by the first author's work plus that of BMW and the contributions of Evans (1973, 1976) regarding the efficient and practical solution of network equilibrium problems, presented a unified approach (see also Boyce and Southworth (1979) and Erlander (1980)) to deriving models of urban location, destination, mode, and route choice. Moreover, selected parts of the modeling framework were implemented for the Chicago region. In Boyce et al. (1983), the calibration of the model parameters was described as was the estimation of the coefficients of the generalized link cost functions. See Boyce and Mattsson (1999) for an application of a network equilibrium model for residential location choice in relation to housing location and road tolls, along with additional citations.

Variational Inequality Formulations and Algorithms

Smith (1979) provided an alternative formulation of traffic network equilibrium, which was identified by Dafermos (1980) to be a variational inequality problem. These fundamental papers, which cited BMW, enabled the modeling, analysis, and computation of solutions to traffic network equilibrium problems in which the symmetry assumption no longer held, which, simply stated, means that the cost on a link depends on the flow on another link in the same fashion that the cost on the other link depends on the former link's flow. In this case, important from the application standpoint, one could no longer reformulate the network equilibrium conditions as a solution to an optimization problem. This recognition, along with rigorous computational schemes (cf. Dafermos (1980, 1983)) allowed for the solution of more general traffic network problems that had been possible. Moreover, as noted above, since variational inequality problems were typically solved as series of optimization problems, advances in the development of the solution of symmetric traffic network problems could be applied to more general problem settings. The variational inequality framework would revolutionize the formulation, analysis, and computation of solutions to network equilibrium problems, in general, as well as other equilibrium problems.

BMW specifically emphasized elastic demand traffic network problems and developed a model which allowed for the prediction not only of the traffic volumes on the links or roads of the network but also the travel demand associated with the origin/destination pairs. Hence, there may be times when travelers opt not to travel at all due to the cost associated with congestion. Dafermos in 1982, subsequently, recognizing the generality of the elastic demand traffic network model proposed a multiclass, asymmetric model and formulated and solved it as a variational inequality problem. Aashtiani and Magnanti (1981) had considered a similar model but treated it as a nonlinear complementarity problem. Florian in 1977 had proposed a two mode traffic network mode, whereas Abdulaal and LeBlanc (1979) described heuristic schemes for the computation of the equilibrium flows. Fisk and Nguyen (1979) gave sufficient conditions for the convergence of the scheme devised by Florian (1977). The elastic demand models, as we elaborate upon in Section 3, are rich sources for related models, notably, spatial price equilibrium models. Moreover, they can also be used for combined decision-making on networks in the form of origin/destination/route/mode choice (see also, e.g., Nagurney and Dong (2002a)).

The first book on finite-dimensional variational inequalities, which contains many network-based applications, and fundamentals, is by Nagurney (1993). See also Patriksson (1994).

Multicriteria Decision-Making

The recognition that different criteria in addition to time and cost might be applicable in transportation route choice selection, notably, that of "risk" was explicit in the book of Beckmann, McGuire, and Winsten (1956). This is especially timely given the new world scenario and further underscores the brilliance of this book and the creativity and longevity of the authors' ideas and contributions. Indeed, although Schneider (1968) and Quandt (1967) proposed multicriteria traffic network equilibrium models, it was actually Dial (1979) who further developed such ideas and Dafermos (1981) who introduced congestion effects into such a model and formulated it as a variational inequality problem (in fact, an infinite-dimensional one). Many variations of such models which provide an alternative to multiclass and multimodal traffic network equilibrium models can be found in Nagurney and Dong (2002a, b, c), who cited BMW, and the references therein. In Section 3, we discuss distinct applications of multicriteria, multiclass network equilibrium models.

Stochastic Route Choice Modeling

The first stochastic route choice model was proposed by Dial (1971) who developed a logit model that was flow-independent. Daganzo and Sheffi (1977) constructed a stochastic user equilibrium model in which at the equilibrium state, no traveler can improve upon his *perceived* travel time by unilaterally changing routes. Additional background on such models, can be found in the book by Sheffi (1985). See also the review articles by Boyce, LeBlanc, and Chon (1988) and Florian and Hearn (1995), which also discuss deterministic models. The book by Sheffi (1985) contains both deterministic and stochastic traffic network models and noted the fundamental contributions of Beckmann, McGuire, and Winsten (1956).

Dynamic Transportation Networks

Although Beckmann, McGuire, and Winsten (1956) did not explicitly formulate dynamic traffic network models, the recognition of the importance of such models was explicit in the book. Yagar (1971), Hurdle (1974), and Merchant and Nemhauser (1978a, b) were some of the first contributors to the development of dynamic models with explicit flows, and the work of Merchant and Nemhauser (1978a, b) is often credited with being the first to consider dynamic route choices over general networks. In particular, they studied dynamic system-optimized networks in the case of single destination networks and athough they did not cite BMW, they did reference Dafermos and Sparrow (1969). Carey (1987), in turn, did reference Beckmann, McGuire, and Winsten (1956) and provided a convex programming formulation of a dynamic system-optimized traffic network which could handle multiple destinations and multiple commodities.

Mahmassani and Herman (1984), in turn, citing BMW, build upon the work of Hendrickson and Kocur (1981), and generalize it to the situation where a user can adapt to congestion by not only changing his departure time but also by changing routes.

Today, variational inequality theory has since become the theoretical basis for the analysis and computation of Wardrop equilibria in a within-day static traffic network. Indeed, motivated by realistic concerns, the within-day dynamic traffic assignment problem is receiving increasing attention (cf. Janson (1991), Smith (1993), Friesz et al. (1993), Ran and Boyce (1994), Wu (1994), and Wu et al. (1998), among others). Underlying a dynamic user equilibrium is a "doubly" dynamic system which is comprised of a day-to-day adjustment process and a within-day realization process. The day-to-day adjustment process addresses the users' behavior in acquiring information and in adjusting their departure time and route choices (see, e. g., Smith (1984), Mahmassani (1990), Friesz, et al. (1994), Zhang and Nagurney (1996), Nagurney and Zhang (1997), Zhang and Nagurney (1997)). The within-day realization process addresses the real time dynamic traffic flow as the realization of the users' route choices on the particular day, which, in turn, results in updated information feedback to the day-to-day process. A dynamic loading operation (Wu et al. (1998)) is involved in this realization process that loads the dynamic path departure rates into dynamic link volumes which determine the dynamic link travel times as the feedback to the travelers. For some additional insights, see Zhang, Nagurney, and Wu (2001).

In particular, Dupuis and Nagurney (1993), motivated in great part by the need to introduce dynamics into the formal modeling and analysis of network systems, including transportation networks, that had been studied primarily at an equilibrium state, using, for example, variational inequality theory, developed the basic theory of existence and uniqueness as well as computational procedures for what are now termed "projected dynamical systems" (cf. also Zhang and Nagurney (1995) and Nagurney and Zhang (1996)). Importantly, the set of equilibrium states and, hence, solutions to a variational inequality problem coincides with the set of stationary points of a particular non-classical dynamical system. Such dynamical systems differ from classical ones in that they explicitly incorporate constraints (which in the case of traffic networks would include, for example, nonnegativity assumptions on the flows and the demand constraints), which results in a discontinuous right-hand side. Projected dynamical systems have been used to-date to model and solve fixed demand as well as elastic demand dynamic traffic network problems as well as numerous other applications (see, e.g., Zhang and Nagurney (1996, 1997) and Nagurney and Zhang (1997)). Hence, we see that both the methodologies of (finite-dimensional) variational inequality theory and projected dynamical systems theory can trace the seeds for their ultimate development, evolution, and, finally, application, back to the traffic network equilibrium ideas of Beckmann, McGuire, and Winsten (1956).

Sensitivity Analysis and Stability Analysis

The importance of stability analysis was recognized in Beckmann, McGuire, and Winsten (1956). Dafermos and Sparrow (1969), subsequently, obtained stability analysis results in the context of user-optimized models in the static setting. More recently, Nagurney and Zhang (1996), motivated by the connection between finite-dimensional variational inequality problems and dynamical systems as defined by Dupuis and Nagurney (1993) (see also Zhang and Nagurney (1995)), and as discussed above, obtained local and global stability analysis results for dynamic traffic network problems modeled as projected dynamical systems. Stability analysis using Lyapunov functions was addressed by Smith (1979, 1984) in some of his major works.

We now turn to a discussion of sensitivity analysis which is central to decision-making and, in particular, to the planning of transportation networks. Interestingly, Braess (1968), whose well-known paradox motivated much of the subsequent research in sensitivity analysis and networks, cited neither Wardrop (1952) nor Beckmann, McGuire, and Winsten (1956) and constructed his paradox without familiarity with these publications (cf. Braess (2003)). That paper was followed by the contributions of Murchland (1970), who elaborated upon the Braess paradox and reflected upon it in the context of BMW and Beckmann (1967b). Fisk (1979) also cited BMW and identified additional paradoxical phenomena in traffic networks. Stewart (1980) and Steinberg and Zangwill (1983) further spurred the investigation of sensitivity analysis in network equilibrium problems. The thesis of Nagurney (1983) (see also Dafermos and Nagurney (1984 a, b, c)) addressed such issues as well as computational ones for general network equilibrium problems in a variational inequality framework. Dafermos and Nagurney (1984d) obtained stability and sensitivity analysis results for a general network equilibrium travel choice model with elastic demands using the variational inequality formulation derived therein and noted BMW.

Today, paradoxes on networks, due to alternative behaviors of decision-makers, are garnering increasing attention in other scientific communities, including that of computer science, which we return to in Section 3.

Here, for definiteness, and in order to illustrate some of the above concepts, we recall the Braess paradox; please refer to Figures 1 and 2.

In particular, consider the transportation network depicted in Figure 1 and assume that the user link cost functions are given as follows:

$$c_a(f_a) = 10f_a, \quad c_b(f_b) = f_b + 50, \quad c_c(f_c) = f_c + 50 \quad c_d(f_d) = 10f_d,$$

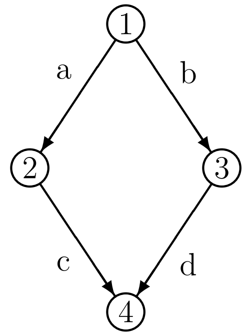


Figure 1: The Braess Network Prior to the Addition of a New Link

with the cost on a link being denoted by c_a for link a, and so on, and the flow on a link a denoted by f_a.

The origin/destination pair w is given by $w = (1, 4)$ and the travel demand is $d_w = 6$. There are two paths available to the travelers and these are paths p_1 and p_2 where we let path $p_1 = (a, c)$ and path $p_2 = (b, d)$. The user-optimized solution in path flows is then:

$$x^*_{p_1} = x^*_{p_2} = 3,$$

which induces the link flow pattern:

$$f^*_a = f^*_b = f^*_c = f^*_d = 3,$$

and the link travel costs:

$$c_a = 30, \quad c_b = 53, \quad c_c = 53, \quad c_d = 30,$$

and the user path travel costs:

$$C_{p_1} = c_a + c_c = 83, \quad C_{p_2} = c_b + c_d = 83.$$

Hence, no user has any incentive to alter his path of travel since all used paths have equal and minimal travel costs and a switch in paths would result in a higher travel cost (Wardrop's (1952) first principle).

Consider the addition of a new road/link e to the network as depicted in Figure 2, with associated user link travel cost:

$$c_e(f_e) = f_e + 10.$$

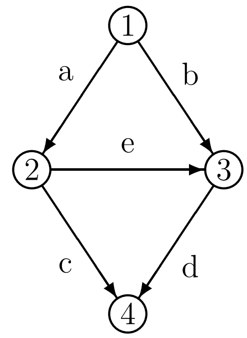


Figure 2: The Braess Network After the Addition of a New Link

A new path $p_3 = (a, e, d)$ is now available to the travelers from origin node 1 to destination node 4. The new user-optimized solution is now, in path flows, given by:

$$x^*_{p_1} = x^*_{p_2} = x^*_{p_3} = 2,$$

which induces the link flow pattern:

$$f^*_a = 4, \quad f^*_b = 2, \quad f^*_c = 2, \quad f^*_d = 4, \quad f^*_e = 2,$$

and associated user path travel costs:

$$C_{p_1} = C_{p_2} = C_{p_3} = 92.$$

Hence, no traveler has any incentive to alter his travel path since all used paths have equal and minimal travel costs. Observe, however, that with the addition of the new link, which provides the travelers with a new path from the origin to their destination, upon reequilibration, the travel cost has increased for all travelers on the network!

In the case of system-optimization, in which the total cost in the network would be minimized, with the total cost on a link a being given by $c_a \times f_a$ for each link a, the system-optimized flow pattern would coincide with the user-optimized one for the network in Figure 1. Moreover, the system-optimized pattern would not change in the case of the network in Figure 2. In other words, if traffic were to be routed in a system-optimal manner, the new path p_3 would not be used. Recall that in the case of system-optimization, all utilized paths connecting an origin/destination pair have the marginals of their total path costs being equal and minimal (which correspond to the Kuhn-Tucker optimality conditions).

In Braess and Koch (1979), the authors establish existence results in the case of multiclass user-optimized networks in which the symmetry assumption does not hold using fixed point arguments and reference Dafermos (1971, 1972). In addition, they highlight the importance of stability in the case of multiple equilibria.

3. Network Equilibrium Applications

In this section, we highlight the many applications whose further development has benefited from the book by Beckmann, McGuire, and Winsten (1956). Beckmann (1967b), in his survey article, noted that there were analogues of the elastic demand network equilibrium model for problems other than road traffic and included examples to the distribution of electric current, steam, water, and natural gas distribution, as well as to the routing of messages in a communications network.

The first application that we discuss in this section, notably, spatial price equilibrium networks, is closely related to traffic network equilibrium problems and this connection was already alluded to, as we mentioned in Section 1, by Koopmans in his introduction in BMW. The subsequent two applications described below, general economic equilibrium and classical market oligopolistic market equilibrium problems, are actually isomorphic to traffic network equilibrium problems on networks with special structure and with fixed demands. Supernetworks, in turn, which we here discuss in the applications of telecommuting decision-making and teleshopping decision-making, are multicriteria network equilibrium problems, in which the concept of path choice is in an abstract setting but in the spirit of user-optimization found in Beckmann, McGuire, and Winsten (and also Dafermos and Sparrow (1969)). We also discuss the relevance of BMW in the setting of supply chain networks. We then turn to knowledge networks which have recently drawn heavily from the work of BMW and reflect upon that application with additional leadership provided by Beckmann (1993, 1994). Finally, the work has been discovered by the computer science community and we highlight some of the relevant activities and applications.

Spatial Price Equilibrium Networks

Koopmans, in his introduction, in discussing the railroad transportation contributions in the BMW book, noted the work of Enke (1951) and Samuelson (1952) in the development of frameworks (the former using analogues to electronic circuits and the latter to a linear programming problem) for the determination of interregional commodity flows and prices in the case of separated markets. Subsequently, Takayama and Judge (1964) in their first major paper on spatial equilibrium demonstrated how, in the case of linear regional supply and demand functions and fixed interregional transportation costs, the governing spatial price equilibrium conditions could be reformulated as the Kuhn-Tucker conditions of a quadratic programming problem. In the paper, the authors thank first Martin Beckmann for helpful comments.

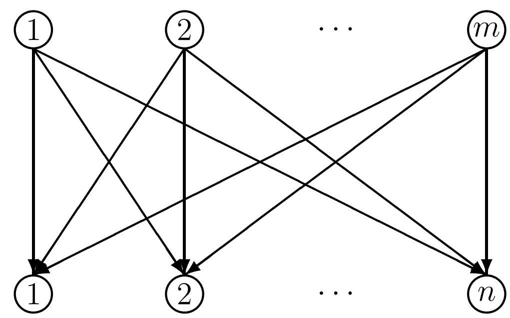


Figure 3: The Structure of Classical Spatial Price Networks

In the case of spatial price equilibrium problems, the governing equilibrium conditions state that a commodity will be produced at a supply market, and shipped to a demand market, where it is consumed, if the price at the supply market plus the unit transportation cost is equal to the price at the demand market; if the price at the supply market plus the transportation cost exceeds the price that the consumers are willing to pay for the commodity at the demand market, then there will be no trade of the commodity between the pair of makets. For the structure of classical spatial price equilibrium networks, see Figure 3. Note that such a network is bipartite.

Florian and Los (1982) provided a synthesis of the Samuelson (1952) model and the BMW network equilibrium model with elastic/variable demand to construct a spatial price equilibrium model on a general network. They also considered multicommodity models and demonstrated that the governing equilibrium conditions satisfy a variational inequality problem akin to those arising in traffic network equilibrium models. Others had also been developing and extending the basic spatial price equilibrium models of Samuelson (1952) and Takayama and Judge (1964, 1971) (for a list of references, see Nagurney (1993)). However, it was researchers in transportation science that truly exploited the connections between the two subjects which had actually been identified as early as the seminal book. Dafermos and Nagurney (1985) established an isomorphism between spatial price and traffic network equilibrium problems which was further elaborated upon by Dafermos (1986) in the context of multicommodity/multiclass networks. In such abstract network constructions there would be a single "super source" node added to the top of the network in Figure 3, along with links emanating from that node to nodes: $1, 2, \ldots, m$. The origin/destination pairs would then be from the super source node to each of the bottom-tiered nodes: $1, 2, \ldots, n$. The "travel" costs on a network with such links would correspond to the supply prices for the top-tiered links and to the unit transportation costs for the next set of links. The "travel disutilities" associated with the origin/destination pairs, in turn, would coincide with the respective demand price functions.

Friesz et al. (1983, 1984), citing BMW, provided additional contributions to the modeling, analysis, and solution of spatial price network equilibrium problems and forged the topic of freight network equilibrium. Nagurney (1987) demonstrated the efficient solution of spatial price equilibrium problems whereas Nagurney, Nicholson, and Bishop (1996) discussed the solution of large-scale such problem in the case of ad valorem tariffs. Finally, utilizing the theory of projected dynamical systems, Nagurney, Takayama, and Zhang (1995) solved dynamic spatial price equilibrium problems using massively parallel computer architecures.

General Economic Equilibrium

Spatial price equilibrium models, in contrast to general economic equilibrium models, are necessarily partial equilibrium models. The network structure of spatial price equilibrium problems considered today often corresponds to the physical transportation network. The general economic equilibrium problem due to Walras (1874) has also been extensively studied (see, e.g., Border (1985)) both from qualitative as well as quantitative perspectives (cf. Dafermos (1990) and the references therein). The Walrasian price equilibrium problem can also be cast into a network equilibrium form as shown in Zhao and Nagurney (1993), who recognized the work of BMW (see also Nagurney (1993)). In this application, cf. Figure 4, there is only a single origin/destination pair of nodes and the links connecting the origin/destination pair correspond to commodities with the flows on the links being now prices. In the context of a traffic network equilibrium problem, hence, this problem is one with a fixed demand and it is the excess demands on used links that are equalized. Again, we get the concept of utilized and nonutilized "paths." Note that this network structure is abstract in that the nodes do not correspond to locations in space and the links to physical routes. Again, we see the generality of network equilibrium due to Beckmann, McGuire, and Winsten (1956) in this application setting. Moreover, algorithms derived for traffic networks have been applied (with the network identification) by Zhao and Nagurney (1993) to solve Walrasian price equilibrium problems (see also Zhao (1988)). Finally, it is fascinating to note that the classical portfolio optimization problem of Markowitz (1959) (see also, e.g., Nagurney and Siokos (1997)) can be transformed into a system-optimized traffic network problem with fixed demand on a network with the structure of the one in Figure 4.

Oligopolistic Market Equilibrium and Game Theory

Game theory, although not explicitly recognized in the sense of Nash (1951) (see also Nash

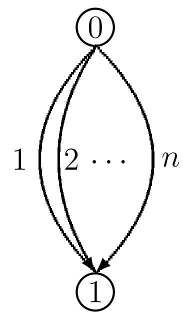


Figure 4: Network Structure of Walrasian Price Equilibrium

(1950)) in the work of BMW, but noted in the Dafermos and Sparrow (1969) paper and cited by Charnes and Cooper (1958, 1961), has had an enormous impact not only on economics but lately also in computer science, which we discuss below. Such problems date to Cournot (1838) and Nash equilibria in the context of oligopoly problems have been shown to satisfy variational inequalities by Gabay and Moulin (1982) and solved thus by Harker (1984, 1986) and by Nagurney (1988) (see also Murphy, Sherali, and Soyster (1982)). Nagurney (1993) demonstrated that the classical aspatial Cournot oligopoly market equilibrium problem could also be cast into a network equilibrium framework on an abstract network (of the same structure as that underlying the Walrasian price equilibrium problem in Figure 4) but with elastic demand and cited Beckmann, McGuire, and Winsten (1956). In the network setting, the links correspond to the firms and the flows on the links are the production outputs. The "costs" on the links correspond to the marginal production cost of the firm minus the marginal product price times the output. Spence (1976) had noted that in the case of linear demand functions and a quadratic production cost function for each firm in the oligopoly, the equilibrium production outputs could be determined as the solution of a convex optimization problem (another example, in which the equilibrium conditions could be reformulated in the case of appropriate assumptions on the underlying model functions as the Kuhn-Tucker conditions of an associated optimization problem). Dafermos and Nagurney (1987) established the connection between spatial oligopolies operating in a Nash-Cournot sense (on networks of the structure in Figure 3) and spatial price equilibrium problems. Devarajan (1981), motivated by the Dafermos and Sparrow (1969) paper, established that a continuous flow, user-optimized network is a pure-strategy Nash equilibrium in a game with a continuum of pure strategies. Haurie and Marcotte (1985) further tightened the connection between Nash-Cournot equilibria and Wardrop equilibria.

Supernetworks: Applications to Telecommuting Decision-Making and Teleshopping Decision-Making

The growing impact of the Information Age, coupled with similarities between traffic networks and communications networks in terms of the relevance of such concepts as system-optimization and user-optimization, along with issues of centralized versus decentralized control, have provided a setting in which the relationships between decision-making on such networks and associated trade-offs could be explored. Towards that end, Nagurney, Dong, and Mokhtarian (2001, 2002), in a series of papers, developed multicriteria network equilibrium models which allowed for distinct classes of decision-makers who weight their criteria associated with utilized transportation versus telecommunications networks in a variety of activities (such as teleshopping and telecommuting) in an individual fashion. Nagurney and Dong (2002b, c) had also proposed multicriteria network equilibrium models in the case of elastic demands as well as for combined location and transportation decision-making, respectively. In such and related models, criteria such as time, cost, risk, as well as opportunity cost (all criteria noted by Beckmann, McGuire, and Winsten (1956)) play a prominent and fresh role. The authors described the governing equilibrium conditions in the case of fixed and elastic demands and provided computational procedures and numerical examples demonstrating that the user-optimizing principle was relevant in the context of these new types of networks termed *supernetworks* in the book by Nagurney and Dong (2002a). That book also traces the origins of the term back to the transportation and computer science literatures.

The decision-makers in the context of the telecommuting versus commuting decision-making application are travelers, who seek to determine their *optimal* routes of travel from their origins, which are residences, to their destinations, which are their places of work. Note that, in the supernetwork framework, a link may correspond to an actual physical link of transportation or an abstract or virtual link corresponding to a telecommuting link. Furthermore, the supernetwork representing the problem under study can be as general as necessary and a path may consist of a set of links corresponding to physical and virtual transportation choices such as would occur if a worker were to commute to a work center from which he could then telecommute. In Figure 5, a conceptualization of this idea is provided.

Of course, the network depicted in Figure 5 is illustrative, and the actual network can be much more complex with numerous paths depicting the physical transportation choices from one's residence to one's work location. Similarly, one can further complexify the telecommunication link/path options. Also, we emphasize, that a *path* within this framework is sufficiently general

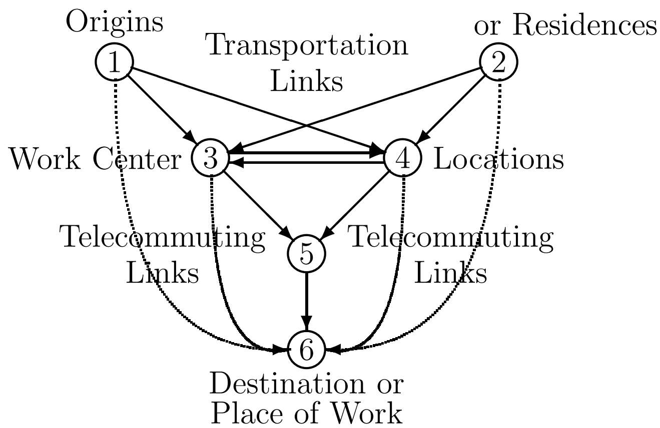


Figure 5: A Supernetwork Conceptualization of Commuting versus Telecommuting

to also capture a choice of mode, which, in the case of transportation, could correspond to busses, trains, or subways (that is, public transit) and, of course, to the use of cars (i.e., private vehicles). Similarly, the concept of path can be used to represent a distinct telecommunications option.

The behavioral assumption is that travelers of a particular class are assumed to choose the paths associated with their origin/destination (O/D) pair so that the generalized cost on that path, which consists of a weighting of the different criteria (which can be different for each class of decision-maker and can also be link-dependent), is minimal. An equilibrium is assumed to be reached when the multicriteria network equilibrium conditions are satisfied whereby only those paths connecting an O/D pair are utilized such that the generalized costs on the paths, as perceived by a class, are equal and minimal.

Now a multicriteria network equilibrium model for teleshopping decision-making is described. For further details, including numerical examples, see Nagurney and Dong (2002a) and the papers by Nagurney, Dong, and Mokhtarian (2001, 2002).

Assume that consumers are engaged in the purchase of a product which they do so in a repetitive fashion, say, on a weekly basis. The product may consist of a single good, such as a book, or a bundle of goods, such as food. Assume also that there are locations, both virtual and physical, where the consumers can obtain information about the product. The virtual locations are accessed through telecommunications via the Internet whereas the physical locations represent more classical shopping venues such as stores and require physical travel to reach.

The consumers may order/purchase the product, once they have selected the appropriate location, be it virtual or physical, with the former requiring shipment to the consumers' locations

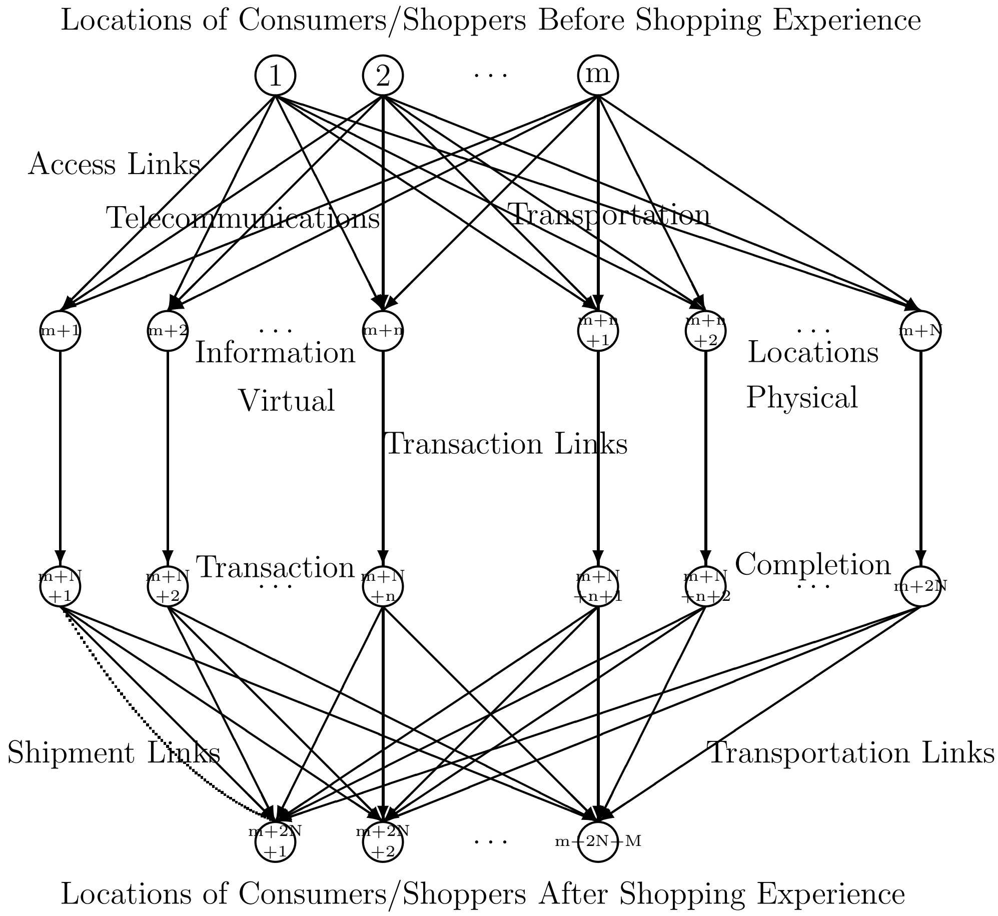


Figure 6: A Supernetwork Framework for Teleshopping versus Shopping

and the latter requiring, after the physical purchase, transportation of the consumer with the product to its final destination (which we expect, typically, to be his residence or, perhaps, place of work).

Refer to the network conceptualization of the problem given in Figure 6. We now identify the above concepts with the corresponding network component. Observe that the network depicted in Figure 6 consists of four levels of nodes with the first (top) level and the last (bottom) level corresponding to the locations (destinations) of the consumers involved in the purchase of the product. An origin/destination pair in this network corresponds to a pair of nodes from the top tier in Figure 6 to the bottom tier. In the shopping network framework, a path consists of a sequence of choices made by a consumer and represents a sequence of possible options for the consumers. The flows, in turn, reflect *how many* consumers of a particular class actually select the particular paths and links, with a zero flow on a path corresponding to the situation that no consumer elects to choose that particular sequence of links.

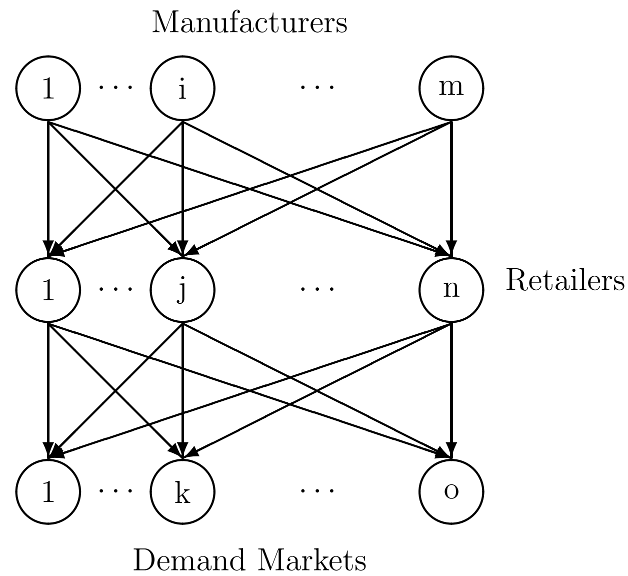


Figure 7: A Supply Chain Network

The criteria that are relevant to decision-making in this application are: time, cost, opportunity cost, and safety or security risk, where, in contrast to the telecommuting application time need not be restricted simply to *travel* time and, depending on the associated link, may include transaction time. In addition, the cost is not exclusively a travel cost but depends on the associated link and can include the transaction cost as well as the product price, or shipment cost. Moreover, the opportunity cost now arises when shoppers on the Internet cannot have the physical experience of trying the good or the actual sociableness of the shopping experience itself. Finally, the safety or security risk cost now can reflect not only the danger of certain physical transportation links but also the potential of credit card fraud, etc.

Supply Chain Networks

Beckmann, McGuire, and Winsten (1956) explicitly recognized the generality of networks as a means of conceptualizing even decision-making of a firm with paths corresponding to production processes and the links corresponding to transformations as the material moved down the path from the origin to the destination. The paths then abstracted the choices or production possibilities available to a firm.

Another application in which the concept of network equilibrium is garnering interest is that of *supply chain networks*. This topic is interdisciplinary by nature since it contains aspects of manufacturing, retailing, transportation, economics, as well as operations research and management science. Zhang, Dong, and Nagurney (2003) have recently generalized Wardrop's second principle to consider not only paths but *chains* in the network to identify the "winning" supply chains. In that application context, paths correspond to production processes and links can be either operation or interface links. Their framework allows for the modeling of competition between supply chains which may entail several firms (producing, transporting, retailing, etc.).

The first work on utilizing network equilibrium concepts in the context of supply chain applications is due to Nagurney, Dong, and Zhang (2001). The depiction of that supply chain network is as given in Figure 7. The decision-makers, now located at the nodes of the network, are faced with their individual objective functions, which can include profit-maximization, and one seeks to determine not only the optimal/equilibrium flows between tiers of nodes but also the prices of the product at the variours tiers. The model therein was subsequently generalized to include electronic commerce by Nagurney et al. (2002).

Knowledge Networks

Indeed, the concept of a network equilibrium first formulated rigorously by Beckmann, McGuire, and Winsten (1956), as the above applications reveal, is much broader than its original application context – that of transportation networks. Its generality was apparent in the book since the authors themselves discussed other application settings, including the application of the concepts to a firm and its production possibilities. Furthermore, above we have already identified other network equilibrium problems and applications whose genesis may be traced, at least, in part, if not entirely, to BMW.

Interestingly, there has been much research conducted in the modeling of knowledge networks from an economic perspective and, notably, by researchers in transportation (cf. Karlqvist and Lundqvist (1972), Batten, Kobayashi, and Andersson (1989), Kobayashi (1995), Nagurney (1999), and the references therein) and even Beckmann (1993, 1994) and the volume edited by Beckmann et al. (1998). Beckmann (1994) noted BMW but in the sense that the topic of transportation networks had been the study of operations researchers, applied mathematicians, and economic theorists while that of knowledge networks had not. Recently, Nagurney and Dong (2003) proposed a framework for the modeling and analysis of knowledge intensive organizations including news organizations, intelligence agencies, and/or global financial institutions. Their perspective used the supernetwork concept of Nagurney and Dong (2002a) and the network equilibrium concept of Beckmann, McGuire, and Winsten (1956) to identify the knowledge products, the origin/destination pairs, the paths and their meanings, along with the links and flows in a

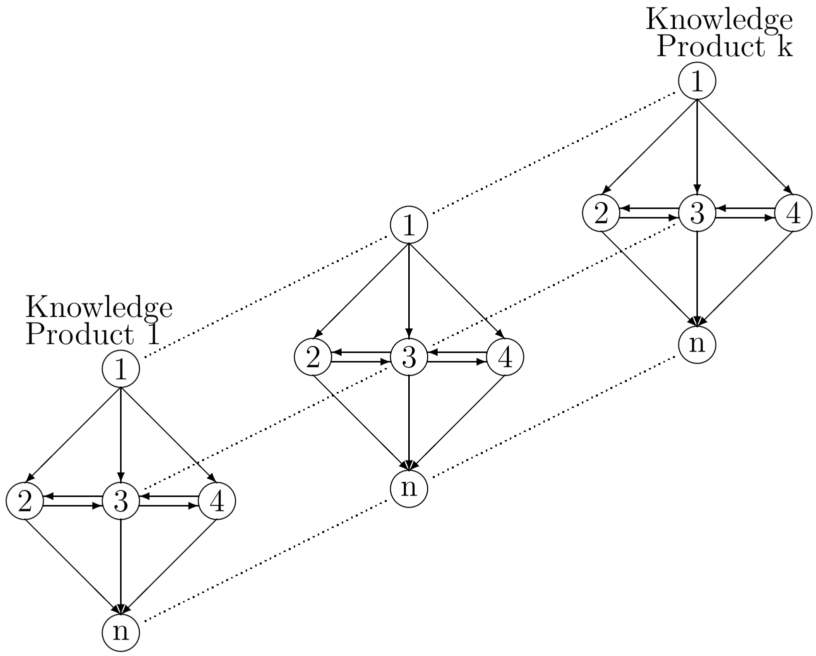


Figure 8: Example of a Knowledge Supernetwork

variety of knowledge organization contexts. Hence, the need for research on such topics as postulated by Beckmann (1994) was now becoming a reality. Moreover, Nagurney and Dong (2003) derived the governing optimality/equilibrium conditions, provided their variational inequality formulations, and, finally, computed several illustrative numerical examples. This application setting further demonstrates the power of the concepts introduced in Beckmann, McGuire, and Winsten (1956). In Figure 8, we show how a knowledge supernetwork may be visualized in the case of multiple knowledge products that need to be produced. Note that knowledge products, unlike many manufactured products, require a substantial human component for their production. In Nagurney and Dong (2003), we discuss the types of factors of production associated with the links that are useful in knowledge production.

Computer Scientists "Discover" Beckmann, McGuire, and Winsten

As mentioned earlier, Beckmann (1967b) noted the relevance of network equilibrium concepts to communication networks. Bertsekas and Gallager (1987) realized the similarities between communication and transportation networks as well and were familiar with the algorithms of Dafermos and Sparrow (1969). Bertsekas and Gafni (1982) had proposed projection-type algorithms for variational inequality formulations of network equilibrium problems earlier.

It was, however, the Braess paradox which, subsequently, provided one of the main linkages between transportation science and computer science. In 1990, Cohen and Kelly described a paradox analogous to that of Braess in the case of a queuing network. Later, Cohen and Horwitz (1991) investigated paradoxical behavior in electrical and mechanical networks. Korilis, Lazar, and Orda (1999), in turn, developed methods to show how resources could be added efficiently to a noncooperative network, including the Internet, so that the Braess paradox would not occur and cited the work of Dafermos and Nagurney (1984a). Roughgarden (2002a), in his thesis, in turn, further elaborated upon the Braess paradox and focused on the quantification of the worst possible loss in network performance arising from noncooperative behavior. He also designed algorithms for the design and management of the networks so that selfish, that is, individual optimizing, behavior, leads to a "socially desirable" outcome. In his thesis, he recognized the importance of the work of Koutsoupias and Papadimitrou (1999), who are computer scientists, and who proposed the idea of bounding of the inefficiency of Nash equilibria, and that of Beckmann, McGuire, and Winsten (1956) and Dafermos and Sparrow (1969). The motivations for his thesis, as well as its foundations, are drawn heavily from the transportation science literature in the form of the traffic network equilibrium problem and its game theoretic aspects. The work is generating much interest among computer scientists and is also often referred to as "selfish routing" by the author and his advisor, Eva Tardos, of Cornell University (cf. Roughgarden and Tardos (2002); see also, e.g., Roughgarden (2001, 2002b)).

Hence, almost 50 years after its publication, Beckmann, McGuire, and Winsten (1956) is finding applications in disciplines that did not even exist when the book was published! I expect that there will be continuing cross-fertilization between many fields in which networks play a prominent role, with BMW serving as one of the fundamental references.

4. Personal Reflections and Comments

I was privileged to have had Martin Beckmann on my doctoral dissertation committee at Brown University with the chair of the committee being Stella Dafermos, who passed away in 1990. Although I could not locate a copy of the Beckmann, McGuire, and Winsten (1956) book for purchase, Stella had given me copies of parts of it for use in my research and it became a reference that has served me well and that I have carried with me on many travels and while living abroad and doing research. Indeed, travel and transportation have been my loves, along with networks ever since I was introduced to the subjects at Brown University where I received 4 degrees.

Amazingly, Brown had been home to such luminaries in transportation as William Prager, who in 1954 published a paper, which discussed the importance of extended type of traffic network models in which the cost on a link could depend not only on its own flow, to Gordon Newell, for a period of time, to Beckmann, as well as to Dafermos.

Gordon Newell (cf. Newell (2002)) had his first exposure to transportation problems in 1954 when he attended a lecture by Prager on the topic of a "fluid theory" of highway traffic. Prager then made available to Newell the paper of Wardrop (1952), whose importance Newell recognized as being a monumental work since prior to Wardrop there was "essentially nothing." Beckmann had become a professor at Brown University in 1959 in the Department of Economics.

Newell recalled (see Newell (2002)) meeting Beckmann in Detroit at an organizational meeting of Highway Research Board (now known as the Transportation Research Board) in the mid 1950s to create a committee on traffic flow theory. Newell subsequently moved to the University of California at Berkeley. Newell noted the book by Beckmann, McGuire, and Winsten as being another major development and also recognized the contributions of Robert Herman and Elliott Montroll. Robert Herman in 1961 edited a volume, **The Theory of Traffic Flow**, which was the proceedings of the first international conference on transportation at which 14 papers were presented by, among others, Wardrop, Charnes and Cooper, Montroll, Newell, and Potts. This conference has continued to take place every three years under the title, "International Symposium on Transportation and Traffic Theory," with the most recent one taking place in Adelaide, Australia in July 2002.

Newell, in taking notes for the above cited paper, which was published posthumously in *Operations Research* upon its 50th anniversary, fondly recalled the visit of Martin Beckmann and his wife to Berkeley in 2000.

I had met Prager who presented a seminar as part of freshman week activities at Brown but did not have a course with Stella Dafermos until becoming a graduate student. Several of my friends, including my college room-mate, did take courses in operations research and transportation from Dafermos so I would hear often about her as an individual (the only female faculty member in either Applied Mathematics or Engineering at that time). According to Stella's husband, Constantine, and confirmed by her thesis advisor, F. Tom Sparrow, Stella had been introduced to operations research at Johns Hopkins University by Sparrow, who, subsequently, moved to Purdue University. Professor Sparrow (see Sparrow (2003)) related to me how the book by Pigou (1920) influenced Dafermos' thesis and her first several papers which were based on her thesis. According to Sparrow, Dafermos set out to derive the user-optimization and system-optimization formulations from first principles. Stella also benefited greatly from the financial and professional support provided by Alan Goldman who, at that time, was with the National Bureau of Standards and from technical assistance from George Nemhauser, a giant in the field of operations research, who is now at Georgia Tech. Upon graduation from Hopkins and prior to following her husband to Brown, Stella spent time at Cornell University, where Nemhauser was also based for a time.

I recall most the memorable party my husband and I hosted after my thesis defense at Brown in which we had baked many Ukrainian tortes but lacked as grad students the proper serving utensil. Martin Beckmann proceeded to apply a plastic spatula to a torte to slice and deliver a piece to his plate. Since that memorable occasion, I have had the pleasure of dining with Beckmann in more gracious settings when our travel itineraries have luckily intersected.

The intellectual journey that these two started me on and influenced numerous others has been fascinating and never dull. It has taken me to many countries, including Canada, Sweden, Russia, Japan, and Australia, and the intellectual inquiries and excitement continue to be fueled by interactions with students, collaborators, and many international colleagues.

Through the Robert Herman Lifetime Achievement Award sponsored by the Transportation and Logistics Section of INFORMS (and named after its first recipient, Robert Herman), the achievements and sustained contributions of innovators in transportation science have been recognized. I have been lucky to have had the opportunity to serve on the committee and to even chair it and to be present at the award ceremonies at which Robert Herman, Martin Beckmann, Michael Florian, Denos Gazis, Amedeo Odoni, and, most recently, David E. Boyce, have received the award. (Newell was selected but declined the award and died in a car accident the year after.) Denos Gazis had been employed by General Motors where he established a great friendship with Robert Herman. He later worked at IBM but continued to make contributions to traffic science. Observe, again, a transfer of knowledge from transportation to communications through an industrial bridging. For some additional historical reflections, see Gazis (2002).

Notes

In this paper, I have attempted to trace some of the major impacts of the Beckmann, McGuire, and Winsten (1956) book in terms of innovations in modeling, methodological developments, and applications. This task has been challenging not only since the book appeared almost half a century ago but due to its depth and breadth of influence. Hence, since there may be contributions that may have been ommitted, I now provide a list of books which can partially fill the gap and in themselves provide a historical evolution of the development of the field of transportation as well as related disciplines. A discussion of transportation networks can be found in Potts and Oliver (1972) and in Newell (1980); see also the volumes of Morlok (1978) and Manheim (1979), which focus on transportation planning. The book by Sheffi (1985) is the first to also include the use of variational inequalities for traffic network equilibrium modeling, analysis, and computation. More recent treatments and additional applications can be found in Nagurney (1993, 1999). The book by Ran and Boyce (see Ran and Boyce (1994, 1996)) is the first book on the modeling of dynamic transportation networks. The book by Nagurney and Dong (2002a), in turn, describes the relationships between transportation and telecommunication networks and decision-making in the Information Age, through the use of supernetworks.

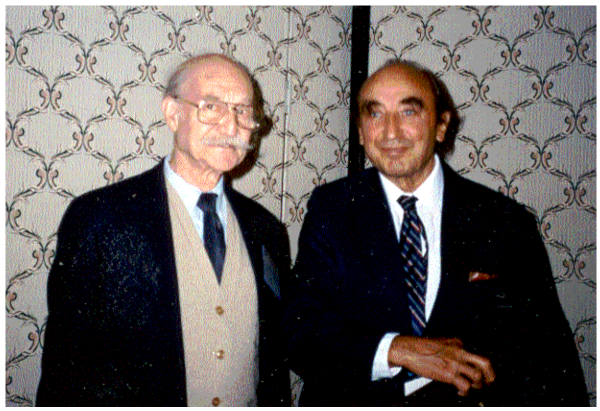
Acknowledgments

The writing of this paper was supported, in part, by NSF grant No.: IIS 0002647 and, in part, by the John F. Smith Memorial Fund at the Isenberg School of Management. This support is gratefully acknowledged.

The author is indebted to Professor David E. Boyce, Professor Emeritus of the University of Illinois at Chicago, Professor Constantine Dafermos of Brown University, and Professor F. Tom Sparrow of Purdue University for helpful and illuminating conversations. The author thanks Professor Boyce for sending her the Jorgensen (1963) report and the Prager (1954) paper and for making the Beckmann, McGuire, and Winsten 1955 Rand Report made available on the Web (see: http://www.rand.org/publications/RM/RM1488.pdf). The author also thanks Professor Dietrich Braess for responding to her questions and for informing her of a reference. Finally, the author thanks Li Zhao for assistance in obtaining references and Tina Wakolbinger for the same and also for carefully proofreading this article.

I now include, as mementos and for historical interest, several photos of Martin Beckmann and several other individuals mentioned in this article.

Photos of Martin Beckmann Receiving
the Robert Herman Lifetime Achievement Award in
Transportation Science - Boston 1994



With Robert Herman at the Award Ceremony

Being Congratulated by Michael Florian

Regional Science Seminar
Mallacoota, Australia, 1992

On the Beach with Anna Nagurney, Mallacoota, Australia - 1992

Stella Dafermos and Anna Nagurney in Athens - 1987

Anna Nagurney and Her PhD Committee,
Professors Majda, Beckmann, and Dafermos
at the Post-Defense Party

References

Aashtiani H. Z. and Magnanti T. L. (1981), "Equilibria on a Congested Transportation Network," *SIAM Journal on Algebraic and Discrete Methods*, **2**, 213-226.

Abdulaal M. and LeBlanc L. J. (1979), "Methods for Combining Modal Split and Equilibrium Assignment Models," *Transportation Science*, **13**, 292-314.

Almond J. (1967), "Traffic Assignment with Flow-Dependent Journey Times," in **Vehicular Traffic Science**, *Proceedings of the 3rd International Symposium on the Theory of Traffic Flow*, L. C. Edie, R. Herman, and R. Rothery, editors, Elsevier, New York, pp. 222-234.

Bar-Gera H. (1999), "Origin-Based Algorithms for Transportation Network," PhD Dissertation, University of Illinois at Chicago.

Batten D. F., Kobayashi K. and Andersson A. E. (1989), "Knowledge, Nodes, and Networks: An Analytical Perspective," in **Knowledge and Industrial Organization**, A. E. Andersson, D. F. Batten, and D. Karlsson, editors, Springer-Verlag, Berlin, Germany, pp. 31-46.

Beckmann M. J. (1967a), "On Optimal Tolls for Highways, Tunnels, and Bridges," in **Vehicular Traffic Science**, American Elsevier, New York, pp. 331-341.

Beckmann, M. J. (1967b), "On the Theory of Traffic Flows in Networks," *Traffic Quarterly*, **21**, 109-116.

Beckmann M. J. (1993), "Knowledge Networks: the Case of Scientific Interaction at a Distance," *The Annals of Regional Science*, **27**, 5-9.

Beckmann M. J. (1994), "On Knowledge Networks in Science: Collaboration among Equals," *The Annals of Regional Science*, **28**, 233-242.

Beckmann M. J., Johannson B., Snickars F. and Thord R. (1998), editors, **Knowledge and Networks in a Dynamic Economy: Festschrift in Honor of Ake E. Andersson**, Springer-Verlag, Berlin, Germany.

Beckmann M. J., McGuire C. B. and Winsten C. B. (1956), **Studies in the Economics of Transportation**, Yale University Press, New Haven, Connecticut; published earlier as Rand-RM 1488, May 12, 1955.

Bergendorff P., Hearn D. W. and Ramana M. V. (1997), "Congestion Toll Pricing of Traffic Networks," in **Network Optimization**, *Lecture Notes in Economics and Mathematical Systems* **450**, P. M. Pardalos, D. W. Hearn, and W. W. Hager, editors, Springer-Verlag, Berlin, Germany, pp. 51-74.

Bertsekas D. P. and Gafni E. M. (1982), "Projection Methods for Variational Inequalities with Application to the Traffic Assignment Problem," *Mathematical Programming*, **17**, 139-159.

Bertsekas D. P. and Gallager R. (1987), **Data Networks**, Prentice-Hall, Englewood Cliffs, New Jersey.

Border K. C. (1985), **Fixed Point Theorems with Application to Economics and Game Theory**, Cambridge University Press, Cambridge, United Kingdom.

Boyce D. E. (1980), "A Framework for Constructing Network Equilibrium Models of Urban Location," *Transportation Science*, **14**, 77-96.

Boyce D. E., Chon K. S., Lee Y. J., Lin K. T. and LeBlanc L. J. (1983), "Implementation and Computational Issues for Combined Models of Location, Destination, Mode, and Route Choice," *Environment and Planning A*, **15**, 1219-1230.

Boyce D. E., LeBlanc L. J. and Chon K. S. (1988), "Network Equilibrium Models of Urban Location and Travel Choices: A Retrospective Survey," *Journal of Regional Science*, **28**, 159-183.

Boyce D. E. and Mattsson L. G. (1999), "Modeling Residential Location Choice in Relation to Housing and Road Tolls on Congested Urban Highway Networks," *Transportation Research B*, **33**, 581-591.

Boyce D. E. and Southworth F. (1979), "Quasi-Dynamic Urban Locations with Endogenously Determined Travel Costs," *Environment and Planning A*, **11**, 575-584.

Braess D. (1968), "Uber ein Paradoxon der Verkehrsplanung," *Unternehmensforschung*, **12**, 258-268.

Braess D. (2003), Personal Communication.

Braess D. and Koch G. (1979), "On the Existence of Equilibria in Asymmetrical Multiclass-User

Transportation Networks," *Transportation Science*, **13**, 56-63.

Bruynooghe M., Gilbert A. and Sakarotvich M. (1969), "Une Methode d'affectation du Traffic," in *Proceedings of the 4th International Symposium on the Theory of Road Traffic*, W. Lemtzback and P. Baron, editors, Karlsruhe, Germany.

Carey M. (1987), "Optimal Time-Varying Flows on Congested Networks," *Operations Research*, **35**, 58-69.

Charnes A. and Cooper W. W. (1958), "Extremal Principles for Simulating Traffic Flow in a Network," *Proceedings of the National Academy of Sciences USA*, Washington DC, pp. 201-204.

Charnes A. and Cooper W. W. (1961), "Multicopy Traffic Networks," in **Theory of Traffic Flow**, R. Herman, editor, Elsevier, Amsterdam, The Netherlands, pp. 85-96.

Cohen J. E. and Kelly F. P. (1990), "A Paradox of Congestion in a Queuing Network," *Journal of Applied Probability*, **27**, 30-34.

Cohen J. E. and Horwitz P. (1991), "Paradoxical Behavior of Mechanical and Electrical Networks," *Nature*, **352**, 699-701.

Cournot A. A. (1838), **Researches into the Mathematical Principles of the Theory of Wealth**, English translation, MacMillan, London, England, 2897.

Dafermos S. C. (1968), "Traffic Assignment and Resource Allocation in Transportation Networks," PhD Thesis, Johns Hopkins University, Baltimore, Maryland.

Dafermos S. (1971), "An Extended Traffic Assignment Model with Applications to Two-Way Traffic," *Transportation Science*, **5**, 366-389.

Dafermos S. (1972), "The Traffic Assignment Problem for Multiclass-User Transportation Networks," *Transportation Science*, **6**, 73-87.

Dafermos S. (1973), "Toll Patterns for Multi-Class User Transportation Networks," *Transportation Science*, **7**, 211-223.

Dafermos S. (1976), "Integrated Equilibrium Flow Models for Transportation Planning," in **Traffic Equilibrium Methods**, *Lecture Notes in Economics and Mathematical Systems*, **118**, M. A. Florian, editor, Springer-Verlag, Berlin, Germany, pp. 106-118.

Dafermos S. (1980), "Traffic Equilibrium and Variational Inequalities," *Transportation Science*, **14**, 42-54.

Dafermos S. (1981), "A Multicriteria Route-Mode Choice Traffic Equilibrium Model," Lefschetz Center for Dynamical Systems, Brown University, Providence, Rhode Island.

Dafermos S. (1982), "The General Multimodal Network Equilibrium Problem with Elastic Demand," *Networks*, **12**, 57-72.

Dafermos S. (1983), "An Iterative Scheme for Variational Inequalities," *Mathematical Programming*, **26**, 40-47.

Dafermos S. (1986), "Isomorphic Multiclass Spatial Price and Multimodal Traffic Network Equilibrium Models," *Regional Science and Urban Economics*, **16**, 197-209.

Dafermos S. (1990), "Exchange Price Equilibria and Variational Inequalities," *Mathematical Programming*, **46**, 391-402.

Dafermos S. and Nagurney A. (1984a), "On Some Traffic Equilibrium Theory Paradoxes," *Transportation Research B*, **18**, 101-110.

Dafermos S. and Nagurney A. (1984b), "Sensitivity Analysis for the Asymmetric Network Equilibrium Problem," *Mathematical Programming*, **28**, 174-185.

Dafermos S. and Nagurney A. (1984c), "Sensitivity Analysis for the General Spatial Economic Equilibrium Problem," *Operations Research*, **32**, 1069-1086.

Dafermos S. and Nagurney A. (1984d), "Stability and Sensitivity Analysis for the General Network Equilibrium-Travel Choice Model, " in *Proceedings of the Ninth International Symposium on Transportation and Traffic Theory*, VNU Science Press, Utrecht, The Netherlands, pp. 217-231.

Dafermos S. and Nagurney A. (1985), "Isomorphism between Spatial Price and Traffic Equilibrium Models," Lefschetz Center for Dynamical Systems Report No. #85-17, Division of Applied Mathematics, Brown University, Providence, Rhode Island.

Dafermos S. and Nagurney A. (1987), "Oligopolistic and Competitive Behavior of Spatially Separated Markets," *Regional Science and Urban Economics*, **17**, 245-254.

Dafermos, S. C. and Sparrow F. T. (1969), "The Traffic Assignment Problem for a General Network," *Journal of Research of the National Bureau of Standards*, **73B**, 91-118.

Dafermos S. C. and Sparrow F. T. (1971), "Optimal Resource Allocation and Toll Patterns in User-Optimized Transportation Networks," *Journal of Transport Economic Policy*, **5**, 184-200.

Daganzo D. F. and Sheffi Y. (1977), "On Stochastic Models of Traffic Assignment," *Transportation Science*, **11**, 253-274.

Dantzig G. B. (1951), "Application of the Simplex Method to the Transportation Problem," in *Activity Analysis of Production and Allocation*, T. C. Koopmans, editor, John Wiley & Sons, New York, pp. 359-373.

Devarajan S. (1981), "A Note of Network Equilibrium and Noncooperative Games," *Transportation Research B*, **15**, 421-426.

Dial R. B. (1971), "Probabilistic Multipath Traffic Assignment Model which Obviates Path Enumeration," *Transportation Research*, **5**, 83-111.

Dial R. B. (1979), "A Model and Algorithms for Multicriteria Route-Mode Choice," *Transportation Research B*, **13**, 311-316.

Duffin R. J. (1947), "Nonlinear Networks II," *Bulletin American Mathematical Society*, **53**, 963-971.

Dupuis P. and Nagurney A. (1993), "Dynamical Systems and Variational Inequalities," *Annals of Operations Research*, **44**, 9-42.

Enke S. (1951), "Equilibrium among Spatially Separated Markets: Solution by Electronic Analogue," *Econometrica*, **10**, 40-47.

Erlander S. (1980), **Optimal Spatial Interaction and the Gravity Model**, Springer, Berlin, Germany.

Evans S. P. (1973), "Some Applications of Mathematical Optimization Theory in Transport Planning," PhD Dissertation, Research Group in Traffic Studies, University College, London, England.

Evans S. P. (1976), "Derivation and Analysis of Some Models for Combining Trip Distribution and Assignment," *Transportation Research*, **10**, 37-57.

Ferland J. A. (1974), "Minimum Cost Multicommodity Circulation Problem with Convex Arc-Costs," *Transportation Science*, **8**, 355-360.

Fisk C. (1979), "More Paradoxes in the Equilibrium Assignment Problem," *Transportation Research B*, **13**, 305-309.

Fisk C. and Nguyen S. (1979), "Properties of the Two Mode Equilibrium Assignment Model with Flow Dependent Transit Costs," Publication #121, Centre de Recherche sur les Transports, Universite de Montreal, Quebec, Canada.

Florian M. (1977), "A Traffic Equilibrium Model of Travel by Car and Public Transit Modes," *Transportation Science*, **8**, 166-178.

Florian M. and Hearn D. (1995), "Network Equilibrium Models and Algorithms," in **Network Routing, Handbooks in Operations Research and Management Science**, **8**, M. O. Ball, T. L. Magnanti, C. L. Monma, and G. L. Nemhauser, editors, Elsevier Science, Amsterdam, The Netherlands, pp. 485-550.

Florian M. and Los M. (1982), "A New Look at Static Spatial Price Equilibrium Models," *Regional Science and Urban Economics*, **12**, 579-597.

Florian M. and Nguyen S. (1974), "A Method for Computing Network Equilibrium with Elastic Demands," *Transportation Science*, **8**, 321-332.

Frank M. and Wolfe P. (1956), "An Algorithm for Quadratic Programming," *Naval Research Logistics Quarterly*, **3**, 95-110.

Friesz T. L., Bernstein D., Smith T. E., Tobin R. L. and Wie B. W. (1993), "A Variational Inequality Formulation of the Dynamic Network User Equilibrium Problem," *Operations Research*, **41**, 179-191.

Friesz T. L., Bernstein D., Mehta J., Tobin R. L. and Ganjalizadeh S. (1994), "Day to Day Dynamic Network Disequilibrium and Idealized Traveler Information Systems," *Operations Research*, **42**, 1120-1136.

Friesz T. L., Harker P. T. and Tobin R. L. (1984), "Alternative Algorithms for the General

Network Spatial Price Equilibrium Problem," *Journal of Regional Science*, **24**, 475-507.

Friesz T. L., Tobin R. L., Smith T. E. and Harker P. T. (1983), "A Nonlinear Complementarity Formulation and Solution Procedure for the General Derived Demand Network Equilibrium Problem," *Journal of Regional Science*, **23**, 337-359.

Gabay D. and Moulin H. (1982), "On the Uniqueness and Stability of Nash Equilibria in Noncooperative Games," in **Applied Stochastic Control in Econometrics and Management Science**, A. Bensoussan, P. Kleindorfer, and C. S. Tapiero, editors, North-Holland, Amsterdam, The Netherlands, pp. 271-294.

Gartner N. H. (1980), "Optimal Traffic Assignment with Elastic Demands: A Review; Part II: Algorithmic Approaches," *Transportation Science*, **14**, 192-208.

Gazis D. C. (2002), "The Origins of Traffic Theory," *Operations Research*, **50**, 69-77.

Harker P. T. (1984), "A Variational Inequality Approach for the Determination of Oligopolistic Market Equilibrium," *Mathematical Programming*, **30**, 105-111.

Harker P. Y. (1986), "Alternative Models of Spatial Competition," *Operations Research*, **34**, 410-425.

Haurie A. and Marcotte P. (1985), "On the Relationship between Nash-Cournot and Wardrop Equilibria," *Networks*, **15**, 295-308.

Hendrickson C. and Kocur G. (1981), "Schedule Delay and Departure Time Decisions in a Deterministic Model," *Transportation Science*, **15**, 62-77.

Hitchcock F. L. (1941), "The Distribution of a Product from Several Sources to Numerous Facilities," *Journal of Mathematical Physics*, **20**, 224-230.

Horowitz J. E. (1984), "The Stability of Stochastic Equilibrium in a Two-Link Transportation Network," *Transportation Research B*, **18**, 13-28.

Hurdle V. F. (1974), "The Effect of Queuing on Traffic Assignment in a Simple Road Network," *Proceedings of the 6th International Symposium on Transportation and Traffic Theory*, Elsevier Science, Amsterdam, The Netherlands, pp. 519-540.

Janson B. N. (1991), "Dynamic Traffic Assignment for an Urban Road Network," *Transportation*

Research, **25B**, 143-161.

Jorgensen, N. O. (1963), "Some Aspects of the Urban Traffic Assignment Problem," Graduate Report, Institute of Transportation and Traffic Engineering, University of California, Berkeley, July.

Kantorovich L. V. (1942), "On the Translocation of Masses," *Comptes Rendus (Doklady) de l'Academie des Sciences de l'URSS*, **37**, reprinted in (1958) in *Management Science*, **5**, 1-4.

Karlqvist A. and Lundqvist L. (1972), "A Contract Model for Spatial Allocation," *Regional Studies*, **4**, 401-419.

Knight F. H. (1924), "Some Fallacies in the Interpretation of Social Costs," *Quarterly Journal of Economics*, **38**, 582-606.

Kobayashi K. (1995), "Knowledge Network and Market Structure: An Analytical Perspective," in **Networks in Action**, D. Batten, J. Casti, and R. Thord, editors, Springer-Verlag, Berlin, Germany, pp. 127-158.

Koopmans T. C. (1947), "Optimum Utilization of the Transportation System," *Proceedings of the International Statistical Conference*, Washington DC; Also in *Econometrica*, **17**, 136-145.

Korilis Y. A., Lazar A. A., and Orda A. (1999), "Avoiding the Braess Paradox in Non-Cooperative Networks," *Journal of Applied Probability*, **36**, 211-222.

Koutsoupias E. and Papadimitrou C. H. (1999), "Worst-Case Equilibria," in *Proceeedings of the 16th Annual Sympoium on the Theoretical Aspects of Computer Science*, pp. 404-413.

Kuhn H. W. and Tucker A. W. (1951), "Nonlinear Programming," in *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, J. Neyman, editor, University of California Press, Berkeley, California, pp. 481-492.

LeBlanc L. J. (1973), "Mathematical Programming Algorithms for Large Scale Network Equilibrium and Network Design Problems," PhD Dissertation, Department of Industrial Engineering and Manageemnt Sciences, Northwestern University, Evanston, Illinois.

LeBlanc L. J., Morlok E. K. and Pierskalla W. P. (1975), "An Efficient Approach to Solving the Road Network Equilibrium Traffic Assignment Problem," *Transportation Research*, **9**, 309-318.

Leventhal T., Nemhauser G., and Trotter, Jr. L. (1973), "A Column Generation Algorithm for Optimal Traffic Assignment," *Transportation Science*, **7**, 168-176.

Mahmassani H. (1990), "Dynamic Models of Commuter Behavior: Experimental Investigation and Application to the Analysis of Planned Traffic Disruptions," *Transportation Research A*, **24**, 465-484.

Mahmassani H. and Herman R. (1984), "Dynamic User Equilibrium Departure Time and Route Choice on Idealized Traffic Arterials," *Transporttaion Science*, **18**, 362-384.

Manheim M. L. (1979), **Fundamentals of Transportation Systems Analysis**, MIT Press, Cambridge, Massachusetts.

Markowitz H. M. (1959), **Portfolio Selection: Efficient Diversification of Investments**, John Wiley & Sons, New York.

Merchant D. K. and Nemhauser G. L. (1978a), "A Model and an Algorithm for the Dynamic Traffic Assignment Problems," *Transportation Science*, **12**, 183-199.

Merchant D. K. and Nemhauser G. L. (1978b), "Optimality Conditions for a Dynamic Traffic Assignment Model," *Transportation Science*, **12**, 200-207.

Morlok E. K. (1978), **Introduction to Transportation Engineering and Planning**, McGraw-Hill, New York.

Murchland J. D. (1970), "Braess' Paradox of Traffic Flow," *Transportation Research*, **4**, 391-394.

Murphy F. H., Sherali H. D. and Soyster A. L. (1982), "A Mathematical Programming Approach for Determining Oligopolistic Market Equilibrium," *Mathematical Programming*, **24**, 92-106.

Nagurney A. (1983), "Stability, Sensitivity Analysis, and Computation of Competitive Network Equilibria," PhD Dissertation, Division of Applied Mathematics, Brown University, Providence, Rhode Island.

Nagurney A. (1984), "Comparative Tests of Multimodal Traffic Equilibrium Methods," *Transportation Research B*, **18**, 469-485.

Nagurney A. (1987), "Computational Comparisons of Spatial Price Equilibrium Methods," *Journal of Regional Science*, **27**, 55-76.

Nagurney A. (1993), **Network Economics: A Variational Inequality Approach**, Kluwer Academic Publishers, Boston, Massachusetts.

Nagurney A. (1988), "Algorithms for Oligopolistic Market Equilibrium Problems," *Regional Science and Urban Economics* **18**, 425-445.

Nagurney A. (1999), **Network Economics: A Variational Inequality Approach**, second and revised edition, Kluwer Academic Publishers, Dordrecht, The Netherlands.

Nagurney A. and Dong J. (2002a), **Supernetworks: Decision-Making for the Information Age**, Edward Elgar Publishing, Cheltenham, England.

Nagurney A. and Dong J. (2002b), "A Multiclass, Multicriteria Traffic Network Equilibrium Model with Elastic Demand," *Transportation Research B*, **36**, 445-469.

Nagurney A. and Dong J. (2002c), "Urban Location and Transportation in the Information Age: A Multiclass, Multicriteria Network Equilibrium Perspective," *Environment & Planning B*, **29**, 53-74.

Nagurney A. and Dong J. (2003), "Management of Knowledge Intensive Systems as Supernetworks: Modeling, Analysis, Computations, and Applications," Isenberg School of Management, University of Massachusetts, Amherst; see: http://supernet.som.umass.edu

Nagurney A., Dong J. and Mokhtarian P. L. (2001), "Teleshopping versus Shopping: A Multicriteria Network Equilibrium Frameowrk," *Mathematical and Computer Modelling*, **34**, 783-798.

Nagurney A., Dong J. and Mokhtarian P. L. (2002), "Multicriteria Network Equilibrium Modeling with Variable Weights for Decision-Making in the Information Age with Applications to Telecommuting and Teleshopping," *Journal of Economic Dynamics and Control*, **26**, 1629-1650.

Nagurney A., Dong J. and Zhang D. (2001), "A Supply Chain Network Equilibrium Model," *Transportation Research E*, **38**, 281-303.

Nagurney A., Loo J., Dong J. and Zhang D. (2002), "Supply Chain Networks and Electronic Commerce: A Theoretical Perspective," *Netnomics*, **4**, 187-220.

Nagurney A., Nicholson C. F. and Bishop P. M. (1996), "Massively Parallel Computation of Large-Scale Spatial Price Equilibrium Models with Discriminatory Ad Valorem Tariffs," *Annals* *of Operations Research*, **68**, 281-300.

Nagurney A. and Siokos S. (1997), **Financial Networks: Statics and Dynamics**, Springer-Verlag, Heidelberg, Germany.

Nagurney A., Takayama T. and Zhang D. (1995), "Projected Dynamical Systems Modeling and Computation of Spatial Network Equilibria," *Networks*, **26**, 69-85.

Nagurney A. and Zhang D. (1996), **Projected Dynamical Systems and Variational Inequalities with Applications**, Kluwer Academic Publishers, Boston, Massachusetts.

Nagurney A. and Zhang D. (1997), "Projected Dynamical Systems in the Formulation, Stability Analysis, and Computation of Fixed-Demand Traffic Network Equilibria," *Transportation Science*, **31**, 147-158.

Nash J. F. (1950), "Equilibrium Points in N-Person Games," *Proceedings of the National Academy of Sciences, USA*, **36**, 48-49.

Nash J. F. (1951), "Noncooperative Games," *Annals of Mathematics*, **54**, 286-298.

Netter M. (1972), "Equilibrium and Marginal Cost Pricing on a Road Network with Several Flow Types," in **Traffic Flow and Transportation**, *Proceedings of the Fifth International Symposium on the Theory of Traffic Flow and Transportation*, Elsevier, New York, pp. 155-163.

Newell G. F. (1980), **Traffic Flow on Transportation Networks**, MIT Press, Cambridge, Massachusetts.

Newell G. F. (2002), "Memoirs on Traffic Flow Theory in the 1950s," *Operations Research*, **50**, 173-178.

Nguyen S. (1974a), "A Unified Approach to Equilibrium Methods for Traffic Assignment," in **Traffic Equilibrium Methods**, *Lecture Notes inEconomics and Mathematical Systems*, Springer-Verlag, New York, pp. 148-182.

Nguyen S. (1974b), "An Algorithm for the Traafic Assignment Problem," *Transportation Science*, **8**, 203-216.

Patriksson M. (1994), **The Traffic Assignment Problem – Models and Methods**, VSP, Utrecht, The Netherlands.

Pigou A. C. (1920), **The Economics of Welfare**, MacMillan, London, England.

Potts R. B. and Oliver R. M. (1972), **Flows in Transportation Networks**, Academic Press, New York.

Prager W. (1954), "Problems in Traffic and Transportation," *Proceedings of a Symposium in Operations Research in Business and Industry*, Midwest Research Institute, Kansas City, Missouri, April.

Quandt R. E. (1967), "A Probabilistic Abstract Mode Model," in **Studies in Travel Demand VIII**, Mathematica Inc., Princeton, New Jersey, pp. 127-149.

Ran B. and Boyce D. E. (1994), **Dynamic Urban Transportation Network Models**, *Lecture Notes in Economics and Mathematical Systems*, **417**, Springer-Verlag, Berlin, Germany.

Ran B. and Boyce D. E. (1996), **Modeling Dynamic Transportation Networks**, second and revised edition, Springer-Verlag, Berlin, Germany.

Roughgarden T. (2001), "Designing Networks for Selfish Users is Hard," in *Proceedings of the 42nd Annual Symposium on the Foundations of Computer Science*, pp. 472-481.

Roughgarden T. (2002a), "Selfish Routing," Doctoral Dissertation, Computer Science, Cornell University, Ithaca, New York.

Roughgarden T. (2002b), "How Unfair is Optimal Routing?" in *Proceedings of the 13th Annual Symposium on Discrete Algorithms*, pp. 203-204.

Roughgarden T. and Tardos E. (2002), "How Bad is Selfish Routing?" *Journal of the ACM*, **49**, 236-259.

Samuelson P. A. (1952), "Spatial Price Equilibrium and Linear Programming," *American Economic Review*, **42**, 283-303.

Schneider M. (1968), "Access and Land Development," in *Urban Development Models*, Highway Research Board Special Report, **97**, pp. 164-177.

Sheffi Y. (1985), **Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods**, Prentice-Hall, Englewood Cliffs, New Jersey.

Smith M. J. (1979), "Existence, Uniqueness and Stability of Traffic Equilibria," *Transportation Research B*, **13**, 259-304.

Smith M. J. (1984), "The Stability of a Dynamic Model of Traffic Assignment – An Application of a Method of Lyapunov," *Transportation Science*, **18**, 245-252.

Smith M. J. (1993), "A New Dynamic Traffic Model and the Existence and Calculation of Dynamic User Equilibria on Congested Capacity Constrained Road Network," *Transportation Research B*, **27**, 49-63.

Spence M. (1976), "The Implicit Maximization of a Function in Monopsolistically Competitive Markets," Harvard Institute of Economic Research, Harvard University Discussion Paper 461, Cambridge, Massachusetts.

Steinberg R. and Zangwill W. (1983), "The Prevalence of Braess' Paradox," *Transportation Science*, **17**, 301-318.

Stewart N. F. (1980), "Equilibrium vs. System-Optimal Flow: Some Examples," *Transportation Research A*, **14**, 81-84.

Takayama T. and Judge G. G. (1964), "An Intertemporal Price Equilibrium Model," *Journal of Farm Economics*, **46**, 477-484.

Takayama T. and Judge G. G. (1971), **Spatial and Temproal Price and Allocation Models**, Morth-Holland, Amsterdam, The Netherlands.

Tomlin J. A. (1966), "Minimum-Cost Multicommodity Network Flows," *Operations Research*, **14**, 45-51.

Vickrey W. B. (1952), "The Revision of the Rapid Transit Fare Structure of the City of New York," Finance Project, Mayor's Committee on Management Survey of the City of New York, Technical Monograph Number Three, New York, February.

Walras L. (1874), **Elements d'Economique Politique Pure**, Corbaz, Lausanne, Switzerland.

Walters A. A. (1961), "Some Theoretical Aspects of Road Traffic Research," *Econometrica*, **29**, 676-699.

Wardrop J. G. (1952) "Some Theoretical Aspects of Road Traffic Research," in *Proceedings of* *the Institute of Civil Engineers*, Part II, pp, 325-378.

Wu J. H. (1994), "Dynamic Network Equilibrium Problem Formulated as an Infinite Dimensional Variational Inequality Problem," *Proceedings of Second Meeting of the EURO Working Group on Urban Traffic and Transportation*, Paris, France, pp. 247-268.

Wu J. H., Chen Y. and Florian M. (1998), " The Continuous Dynamic Network Loading Problem: A Mathematical Formulation and Solution Method," *Transportation Research B*, **32**, 173-187.

Yagar S. (1971), "Dynamic Traffic Assignment by Individual Path Minimisation and Queuing," *Transportation Research*, **5**, 179-196.

Zhang D., Dong J. and Nagurney A. (2003), "A Supply Chain Network Economy: Modeling and Qualitative Analysis," in **Innovations in Financial and Economic Networks**, A. Nagurney, editor, Edward Elgar Publishing, Cheltenham, England, pp. 198-213.

Zhang D. and Nagurney A. (1995), "On the Stability of Projected Dynamical Systems," *Journal of Optimization Theory and Applications*, **85**, 97-124.

Zhang D. and Nagurney A. (1996), "On the Local and Global Stability of a Travel Route Choice Adjustment Process," *Transportation Research B*, **30**, 245-262.

Zhang D. and Nagurney A. (1997), "Formulation, Stability, and Computation of Traffic Network Equilibria as Projected Dynamical Systems," *Journal of Optimization Theory and Applications*, **93**, 417-444.

Zhang D., Nagurney A. and Wu J. H. (2001), "On the Equivalence between Stationary Link Flow Patterns and Traffic Network Equilibria," *Transportation Research B*, **35**, 731-748.

Zhao L. (1988), "Variational Inequalities in General Equilibrium: Analysis and Computation," PhD Dissertation, Division of Applied Mathematics, Brown University, Providence, Rhode Island.

Zhao L. and Nagurney A. (1993), "A Network Formalism for Pure Exchange Economic Equilibria," in **Network Optimization Problems: Algorithms, Complexity, and Applications**, D. Z. Du and P. M. Pardalos, editors, World Scientific Press, Singapore, pp. 363-386.