# Projected Dynamical Systems, Evolutionary Variational Inequalities, Applications, and a Computational Procedure

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**Abstract**

In this paper, we establish the equivalence between the solutions to an evolutionary variational inequality and the critical points of a projected dynamical system in infinite–dimensional spaces. We then present an algorithm, with convergence results, for the computation of solutions to evolutionary variational inequalities based on a discretization method and with the aid of projected dynamical systems theory. A numerical traffic network example is given for illustrative purposes.

# 1 Introduction

Numerous problems in engineering, in operations research and the management sciences, as well as in economics and finance involve interactions among decision-makers and the competition for resources. In such problems, the concept of equilibrium plays a central role and provides a valuable benchmark against which an existing state of such complex systems can be compared. Examples, par excellence, of such equilibrium problems include: congested urban transportation networks, the Internet, multi-sector, multi-instrument financial equilibrium problems as well as a variety of decentralized supply chain networks (see, e.g., [33], [34], and [25]).

Various methodologies have been developed to formulate and solve such problems, which are often large-scale. For example, Dafermos [11] showed that the traffic network equilibrium conditions as formulated by Smith [39] were a finite-dimensional variational inequality and then utilized the theory to establish both existence and uniqueness results of the equilibrium traffic flow pattern as well as to propose an algorithm with convergence results (see also [12]). Finite-dimensional variational inequality theory has been applied to-date to the wide range of equilibrium problems noted above, as well as to game theoretic problems, such as oligopolistic market equilibrium problems (see, e.g., [24], [13], and [34], and the references therein).

As important as the study of the equilibrium state is that of the study of the underlying dynamics or disequilibrium behavior of such systems. Note that since such problems typically involve more than a single decision-maker who is faced with constraints (such as, for example, budgetary, conservation of flow, nonnegativity assumptions on the variables, among others) classical dynamical systems theory is no longer sufficient for the formulation and solution of such problems. Towards that end, Dupuis and Nagurney [23] introduced a new class of dynamical system with a discontinuous right-hand side and provided the foundational theory for such *projected dynamical systems*. Moreover, they established, under suitable conditions, that the set of stationary points of a projected dynamical system coincided with the set of solutions of the associated finite-dimensional variational inequality. This connection allowed for the investigation of the disequilibrium behavior preceding the attainment of the equilibrium. Zhang and Nagurney [42] (see also [36]), subsequently, developed the stability theory for finite-dimensional projected dynamical systems. Such results are relevant since without such a theory the concept of equilibrium may not be valid.

Isac and Cojocaru ([28], [29]) initiated the systematic study of projected dynamical systems on infinite-dimensional Hilbert spaces in 2002 with the fundamental issue of existence of solutions to such problems answered by Cojocaru [6] in her thesis (see also Cojocaru and Jonker [7]).

Evolutionary variational inequalities, which are also infinite-dimensional, were originally introduced by Lions and Stampacchia [31] and by Brezis [3] in order to study problems arising principally from mechanics. They provided a theory for the existence and uniqueness of the solution of such problems. Steinbach [40], on the other hand, studied an obstacle problem with a memory term as a variational inequality problem and established existence and uniqueness results under suitable assumptions on the time-dependent conductivity. Daniele, Maugeri, and Oettli (cf. [19] and [20]), motivated by dynamic traffic network problems, introduced evolutionary (time-dependent) variational inequalities to this application domain and to several others as we shall highlight later.

As noted by Cojocaru, Daniele, and Nagurney [8], the theory and application of evolutionary variational inequalities was developing in parallel to that of projected dynamical systems. That reference reviews the theoretical foundations of both of these methodologies and surveys the historical developments. Moreover, it makes explicit for the first time the connection between projected dynamical systems on Hilbert spaces and evolutionary variational inequalities. Finally, the authors provide an illustrative dynamic traffic network example. In [9], the same authors established further results on the unified theory of projected dynamical systems and evolutionary variational inequalities in the context of double-layered dynamics. Moreover, stability analysis results were provided for the curve of equilibria.

This paper expands upon the theme of that first and second joint paper of ours – that of the synthesis and expansion of the theories of projected dynamical systems and evolutionary variational inequalities to enable the richer modeling and rigorous analysis of a plethora of complex dynamic problems subject to constraints. In particular, here we provide a new proof of the equivalence between solutions to an evolutionary variational inequality and the critical points of a projected dynamical system in infinite dimensions. In addition, we propose a new algorithm for the computation of solutions to evolutionary variational inequalities that exploits the equivalence. Convergence results are also provided.

We now recall some fundamentals and results of our prior work, which, along with the preliminary results in Section 2, will allow us to establish the main contributions of this paper.

Let $\mathbb{K}$ be a convex polyhedral set in $\mathbb{R}^n$, $F : \mathbb{K} \to \mathbb{R}^n$ and let us introduce the operator

$$\Pi_{\mathbb{K}} : \mathbb{R} \times \mathbb{K} \to \mathbb{R}^n$$

defined by means of the directional derivative in the sense of Gâteaux

$$\Pi_{\mathbb{K}}(x, -F(x)) = \lim_{t \to 0^+} \frac{P_{\mathbb{K}}(x - tF(x)) - x}{t}$$

of the projection operator $P_{\mathbb{K}} : \mathbb{R}^n \to \mathbb{K}$ given by

$$\|P_{\mathbb{K}}(z) - z\| = \inf_{y \in \mathbb{K}} \|y - z\|.$$

In [23] Dupuis and Nagurney considered the differential equation with a discontinuous right–hand side

$$\frac{d\,x(t)}{d\,t} = \Pi_{\mathbb{K}}(x(t), -F(x(t)))$$

and the associated Cauchy problem

$$\begin{cases} \dfrac{d\,x(t)}{d\,t} = \Pi_{\mathbb{K}}(x(t), -F(x(t))) \\ \\ x(0) = x_0 \in \mathbb{K}, \end{cases} \tag{1}$$

whose solutions (see also [42]) they called *projected dynamical systems* (PDS). A similar idea, in different contexts, can be found in the papers [10], [27], [1] and in the book [2], as we shall see in Remark 3.1. In [22] and [23] existence theorems of an absolutely continuous solution are shown, provided that $F$ is assumed to be Lipschitz continuous and with linear growth.

The key trait of a projected dynamical system was first found by Dupuis and Nagurney in [23]. In particular, the authors proved the following theorem.

**Theorem 1.1** *The critical points of equation*

$$\frac{d\,x(t)}{d\,t} = \Pi_{\mathbb{K}}(x(t), -F(x(t))), \tag{2}$$

*namely, the solutions such that* $\dfrac{d\,x(t)}{d\,t} \equiv 0$, *are the same as the solutions to the variational inequality*

$$\textit{Find } x \in \mathbb{K}: \;\; \langle F(x), y - x\rangle \geq 0, \quad \forall y \in \mathbb{K}.$$

As noted above, variational inequalities in the finite–dimensional case have been used to formulate a spectrum of problems arising in engineering, operations research and the management sciences, transportation science, economics, and finance, as, for example, in the case of the traffic network equilibrium, spatial price equilibrium, oligopolistic market equilibrium, and financial equilibrium problems. All these applications have also benefited from the theory of projected dynamical systems in terms of analysis and computation (see [33], [8], and the references therein).

As also noted above, projected dynamical systems have been considered in the framework of Hilbert spaces (see [6], [7], [8], [26] and [37]). We now provide a definition of a projected dynamical system.

**Definition 1.1** *A projected dynamical system is given by a mapping* $\Psi : \mathbb{R}_+ \times \mathbb{K} \to \mathbb{K}$ *which solves the initial value problem:*

$$\dot{\Psi}(t, x) = \Pi_{\mathbb{K}}(\Psi(t, x), -F(\Psi(t, x))), \quad \Psi(0, x) = x \in \mathbb{K}.$$

In [6] and [7] the following theorem has been proved.

**Theorem 1.2** *Let* $H$ *be a Hilbert space and let* $\mathbb{K} \subset H$ *be a nonempty, closed and convex subset. Let* $F : \mathbb{K} \to H$ *be a Lipschitz continuous vector field with Lipschitz constant* $b$*. Let* $x_0 \in \mathbb{K}$ *and* $L > 0$ *such that* $\|x_0\| \leq L$*. Then the initial value problem (1) admits a unique solution in the class of the absolutely continuous functions on the interval* $[0, l]$ *where* $l = \dfrac{L}{\|F(x_0)\| + b\,L}$.

In fact, in [6], the author shows that solutions to problem (1) on Hilbert spaces can be extended to $\mathbb{R}_+$, so Definition 1.1 also holds in the context of Hilbert spaces. The important consequence of such a theory in the Hilbert space is that we can establish a connection between the solutions to an evolutionary variational inequality and the stationary solutions to projected dynamical equations in Hilbert spaces (see [6] and [7]).

For completeness and definiteness, we now provide some additional citations to evolutionary variational inequalities and applications. In [19] and [20] Daniele, Maugeri, and Oettli formulated time–dependent traffic equilibria as evolutionary variational inequalities. In [17] Daniele and Maugeri developed a time–dependent spatial equilibrium model (price formulation) in which bounds over the time on the supply and demand market prices and on the commodity shipments between supply and demand market pairs were imposed. Moreover, the authors addressed the time–dependent spatial price equilibrium problem in which the variables were commodity shipments. In [16] Daniele introduced a time–dependent financial network model consisting of multiple sectors, each of which seeks to determine its optimal portfolio given time–depending supplies of the financial holding.

Cojocaru, Daniele, and Nagurney in [8] showed that the all the above considered problems can be formulated into a unified definition as we recall below. We consider the nonempty, convex, closed, bounded subset of the Hilbert space $L^2([0,T],\mathbb{R}^q)$ given by

$$\mathbb{K} = \bigcup_{t\in[0,T]} \Big\{ u \in L^2([0,T],\mathbb{R}^q) : \lambda(t) \leq u(t) \leq \mu(t) \text{ a.e. in } [0,T]; \\ \sum_{i=1}^{q} \xi_{ji}\, u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \\ \xi_{ji} \in \{0,1\}\,,\quad i \in \{1,\ldots,q\}\;\; j \in \{1,\ldots,l\} \Big\}. \tag{3}$$

Let $\lambda, \mu \in L^2([0,T],\mathbb{R}^q)$, $\rho \in L^2([0,T],\mathbb{R}^l)$ be convex functions. For chosen values of the scalars $\xi_{ji}$, of the dimensions $q$ and $l$, and of the constraints $\lambda$, $\mu$, we obtain each of the previous above–cited model constraint set formulations (see [8]), as follows:

- for the traffic network problem (see [19], [20]), we let $\xi_{ji} \in \{0,1\}$, $i \in \{1,\ldots,q\}$, $j \in \{1,\ldots,l\}$, and $\lambda(t) \geq 0$ for all $t \in [0,T]$;

- for the quantity formulation of spatial price equilibrium (see [14]), we let $q = n + m + nm$, $\xi_{ji} \in \{0, 1\}$, $i \in \{1, \ldots, q\}$, $j \in \{1, \ldots, l\}$; $\mu(t)$ large and $\lambda(t) = 0$, for any $t \in [0, T]$;

- for the price formulation of spatial price equilibrium (see [15] and [17]), we let $q = n + m + mn$, $l = 1$, $\xi_{ji} = 0$, $i \in \{1, \ldots, q\}$, $j \in \{1, \ldots, l\}$, and $\lambda(t) \geq 0$ for all $t \in [0, T]$;

- for the financial equilibrium problem (see [16]), we let $q = 2mn+n$, $l = 2m$, $\xi_{ji} = \{0, 1\}$ for $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, l\}$; $\mu(t)$ large and $\lambda(t) = 0$, for any $t \in [0, T]$.

Then, setting

$$\ll \Phi, u \gg = \int_0^T \langle \Phi(t), u(t) \rangle \, dt$$

where $\Phi \in L^2([0, T], \mathbb{R}^q)^*$ and $u \in L^2([0, T], \mathbb{R}^q)$, if $F$ is given such that $F : \mathbb{K} \to L^2([0, T], \mathbb{R}^q)$, we have the following standard form of the evolutionary variational inequality:

$$\text{find } u \in \mathbb{K} : \ll F(u), v - u \gg \geq 0, \quad \forall v \in \mathbb{K}. \tag{4}$$

In [20] sufficient conditions that ensure the existence of a solution to (4) are given.

Now the following general result holds in Hilbert spaces (see [6], [7], [26] and [37]), as we shall prove in Section 4.

**Theorem 1.3** *Assume that the hypotheses of Theorem 1.2 hold. Then the solutions to the variational inequality (4) are the same as the critical points of the projected differential equation (PrDE) (2), that is, the points $x \in \mathbb{K}$ such that*

$$\Pi_{\mathbb{K}}(x(t), -F(x(t))) = 0,$$

*and viceversa.*

As a consequence, and by choosing the Hilbert space $H$ to be $L^2([0, T], \mathbb{R}^p)$, we find that the solutions to the evolutionary variational inequality:

$$\text{find } u \in \mathbb{K} : \int_0^T \langle F(u(t)), v(t) - u(t) \rangle \, dt \geq 0, \quad \forall v \in \mathbb{K} \tag{5}$$

are the same as the critical points of the equation:

$$\frac{d\,u(t,\tau)}{d\tau} = \Pi_{\mathbb{K}}(u(t,\tau), -F(u(t,\tau))), \tag{6}$$

that is, the points such that

$$\Pi_{\mathbb{K}}(u(t,\tau), -F(u(t,\tau))) \equiv 0 \text{ a.e. in } [0,T],$$

which are obviously stationary with respect to $\tau$.

As noted in [8], the meaning of the two "times" in (6) needs to be well understood. Intuitively, at each instant $t \in [0,T]$, the solution of the evolutionary variational inequality (5) represents a static state of the underlying system. As $t$ varies over $[0,T]$, the static states describe one (or more) curves of the equilibria. In contrast, $\tau$ here is the time that describes the dynamics of the system until it reaches one of the equilibria of the curve.

Section 2 is dedicated to the presentation of additional definitions and preliminary results that we need in the subsequent sections. In Section 3 we present a self–contained proof of Theorem 1.3 and we reference similar existing results. In Section 4 we show how a solution to the evolutionary variational inequality (5) can be computed with the aid of the projected dynamical systems theory. In Section 5 we present a proof of the convergence of the algorithm. In Section 6 we present a numerical dynamic traffic network example that is distinct from the one in [8].

## 2 Definitions and Preliminary Results

Following the paper by J. Gwinner [26], let us recall some well-known objects of convex analysis which we need in what follows.
Let $H$ be a real Hilbert space, whose inner product we denote by $\langle \cdot, \cdot \rangle$.

**Definition 2.1** *For a subset $M \subset H$ the polar $M^0$ is defined by*

$$M^0 = \{\xi \in H : \langle \xi, x \rangle \leq 1, \quad \forall x \in M\}.$$

For a cone $C$ Definition 2.1 simplifies into

$$C^0 = C^- = \{\xi \in H : \langle \xi, x \rangle \leq 0, \ \forall x \in C\}.$$

**Definition 2.2** *Let $\mathbb{K}$ be a nonempty, closed, convex subset of $H$. For all $z \in \mathbb{K}$ we define the support cone (or tangent cone, or contingent cone) to $\mathbb{K}$ at $x$ as the set*

$$T_{\mathbb{K}}(x) = \overline{\bigcup_{\lambda>0} \lambda\,(\mathbb{K} - x)}.$$

**Definition 2.3** *We define the normal cone to $\mathbb{K}$ at $x$ as the set*

$$N_{\mathbb{K}}(x) = \left\{\xi \in H : \ \langle \xi, z - x\rangle \leq 0, \ \forall z \in \mathbb{K}\right\}.$$

**Proposition 2.1** *We then have the following result:*

$$\left(T_{\mathbb{K}}(x)\right)^0 = N_{\mathbb{K}}(x) = \left(T_{\mathbb{K}}(x)\right)^-.$$

*Proof.* It is clear (see [2], Proposition 2, page 220) that

$$\left(T_{\mathbb{K}}(x)\right)^0 \subseteq N_{\mathbb{K}}(x) = \left\{\xi \in H : \ \langle \xi, z - x\rangle \leq 0, \ \forall z \in \mathbb{K}\right\},$$

because $z - x \in T_{\mathbb{K}}(x)$, $\forall z \in \mathbb{K}$. Viceversa, $N_{\mathbb{K}}(x) \subseteq \left(T_{\mathbb{K}}(x)\right)^0$, because if $y = \lim_n \lambda_n(z_n - x)$, $z_n \in \mathbb{K}$, $\lambda_n \geq 0$ $\forall n \in \mathbb{N}$, for each $\xi \in N_{\mathbb{K}}(x)$ :

$$\langle \xi, \lambda_n(z_n - x)\rangle \leq 0, \ \forall n \in \mathbb{N}$$

and, hence,

$$\langle \xi, y\rangle \leq 0, \ \forall y \in T_{\mathbb{K}}(x),$$

and the assertion is proved. □

The set $T_{\mathbb{K}}(x)$ is clearly a closed convex cone with vertex 0 and it is the smallest cone $C$ whose translate $x + C$ has vertex $x$ and contains $\mathbb{K}$. The utility of the support cone derives from the following result:

**Theorem 2.1** *If we denote by $P_{\mathbb{K}} = \text{Proj}\,(\mathbb{K}, \cdot)$ the projection onto $\mathbb{K}$ of an element of $H$, then:*

$$P_{\mathbb{K}}(x + \lambda h) = x + \lambda P_{T_{\mathbb{K}}(x)} h + o(\lambda)$$

*for any $x$, $h$, and $\lambda > 0$.*

*Proof.* See [41] Lemma 4.6 page 300. □

**Corollary 2.1** *If we define the projection of* $h$ *at* $x$ *with respect to* $\mathbb{K}$ *as the directional derivative in the sense of Gâteaux*

$$\Pi_{\mathbb{K}}(x,h) = \lim_{\lambda \to 0^+} \frac{P_{\mathbb{K}}(x+\lambda h) - x}{\lambda},$$

*then*

$$\Pi_{\mathbb{K}}(x,h) = P_{T_{\mathbb{K}}(x)} h,$$

*namely,* $\Pi_{\mathbb{K}}(x,h)$ *is the projection of* $h$ *on the support cone* $T_{\mathbb{K}}(x)$.

**Definition 2.4** *The set of unit inward normals to* $\mathbb{K}$ *at* $x$ *is defined by*

$$n_{\mathbb{K}}(x) = \{v : \|v\| = 1 \text{ and } \langle v, x-y \rangle \leq 0, \ \forall y \in \mathbb{K}\}.$$

Then, using Proposition 2.1, we have that

**Proposition 2.2** *The set of unit normals to* $\mathbb{K}$ *at* $x$ *satisfies:*

$$n_{\mathbb{K}}(x) = \partial B(0,1) \cap - (T_{\mathbb{K}}(x))^0,$$

*where* $\partial B(0,1) = \{z : \|z\| = 1\}$.

Now, since in infinite dimensions the interior as well as the relative algebraic interior of a convex set can be empty, we introduce the concepts of quasi interior of $\mathbb{K}$, which may be nonempty.

**Definition 2.5** *We call the quasi interior of* $\mathbb{K}$ *(denoted by* $\text{qi}\,\mathbb{K}$*) the set of those* $x \in \mathbb{K}$ *for which* $T_{\mathbb{K}}(x) = H$.

**Definition 2.6** *We define the quasi boundary of a closed convex set* $\mathbb{K}$ *(denoted by* $\text{qbdry}\,\mathbb{K}$*) as the set* $\mathbb{K} \setminus \text{qi}\,\mathbb{K}$.

Then the following proposition holds.

**Proposition 2.3** $x \in \text{qbdry}\,\mathbb{K}$ *if and only if* $n_{\mathbb{K}}(x) \neq \emptyset$.

*Proof.* Let $x \in \text{qbdry}\,\mathbb{K}$. Then, by virtue of Proposition 2.1 in [4], there exists a $\xi \neq 0$ such that $\langle \xi, x \rangle \leq \langle \xi, y \rangle$, $\forall y \in \mathbb{K}$, and, hence:

$$\langle \frac{\xi}{\|\xi\|}, x-y \rangle \leq 0 \quad \forall y \in \mathbb{K}.$$

Viceversa, if $n_{\mathbb{K}}(x)$ is nonempty, then there exists a $\xi$ with $\|\xi\| = 1$ such that $\langle \xi, x - y\rangle \le 0$, $\forall y \in \mathbb{K}$. Then $x \notin$ qi $\mathbb{K}$, because:
if $x \in$ qi $\mathbb{K}$, then $\langle \xi, x - y\rangle \le 0$, $\forall y \in \mathbb{K}$ implies $\langle \xi, \lambda(x - y)\rangle \le 0$, $\forall \lambda > 0$ and $\forall y \in \mathbb{K}$. If $y \in T_{\mathbb{K}}(x)$, then we can write $y = \lim_n \lambda_n(z_n - x)$ and so $\langle \xi, \lambda_n(z_n - x)\rangle \le 0$, $\forall n \in \mathbb{N}$. When $n \to \infty$, then we get $\langle \xi, y\rangle \le 0$, $\forall y \in T_{\mathbb{K}}(x)$. Therefore, if $x \in$ qi $\mathbb{K}$, then $T_{\mathbb{K}}(x) = H$ and, hence, $\langle \xi, y\rangle \le 0$ $\forall y \in H$. Choosing $-y \in H$, we get $\langle \xi, -y\rangle \le 0$, that is, $\langle \xi, y\rangle = 0$ $\forall y \in H$. Choosing $y = \xi$, we obtain $\|\xi\| = 0$, and then $\xi = 0$, which is an absurdity since $\|\xi\| = 1$. □

Following an idea of Dupuis [21] on Euclidean space, later used in [23] for the theory of finite-dimensional PDS, we present next a generalization of the geometric interpretation of the operator $\Pi_{\mathbb{K}}$ on infinite–dimensional H–spaces. A similar result, also in infinite–dimensional spaces, can be found in Isac and Cojocaru [29] (see also [26] and [38]).

**Theorem 2.2**

1. *If* $x \in$ qi $\mathbb{K}$, *then for any* $h \in H$ *it follows that:* $\Pi_{\mathbb{K}}(x, h) = h$;
2. *If* $x \in$ qbdry $\mathbb{K}$, *then for any* $v \in H \setminus T_{\mathbb{K}}(x)$ *there exists* $n^*(x) \in n_{\mathbb{K}}(x)$ *such that*

$$\beta(x) = -\langle v, n^*(x)\rangle > 0,$$

$$\Pi_{\mathbb{K}}(x, v) = v + \beta(x)\, n^*(x).$$

*Proof.* If $x \in$ qi $\mathbb{K}$, then $T_{\mathbb{K}}(x) = H$, by definition of qi $\mathbb{K}$, and it follows that

$$\Pi_{\mathbb{K}}(x, h) = P_{T_{\mathbb{K}}(x)} h = P_H h = h.$$

If $x \in$ qbdry $\mathbb{K}$, then setting $\hat{v} = \Pi_{\mathbb{K}}(x, v)$, we get:

$$\hat{v} = \Pi_{\mathbb{K}}(x, v) = P_{T_{\mathbb{K}}(x)} v,$$

namely:

$$\langle v - \hat{v}, w - \hat{v}\rangle \le 0, \quad \forall w \in T_{\mathbb{K}}(x).$$

Since $T_{\mathbb{K}}(x)$ is a cone with vertex 0, choosing, in turn, $w = 0$ and $w = 2\hat{v}$, we get:

$$\langle v - \hat{v}, \hat{v}\rangle = 0. \tag{7}$$

Moreover, if we set $w = y + \hat{v}$ with $y \in T_{\mathbb{K}}(x)$, we obtain

$$\langle v - \hat{v}, y + \hat{v} - \hat{v} \rangle = \langle v - \hat{v}, y \rangle \leq 0, \quad \forall y \in T_{\mathbb{K}}(x)$$

and, hence,

$$v - \hat{v} \in (T_{\mathbb{K}}(x))^0 . \tag{8}$$

Since $v \neq \hat{v}$, because $v \in H \setminus T_{\mathbb{K}}(x)$ and $\hat{v} \in T_{\mathbb{K}}(x)$ by assumption, then the relation (8) implies the existence of some $n^* \in n(x)$ and $\beta > 0$ such that

$$\hat{v} - v = \beta \, n^*.$$

Moreover, the orthogonality $\langle n^*, \hat{v} \rangle = 0$ implies

$$\beta = -\langle v, n^* \rangle,$$

and the assertion is proved. □

We also obtain the following characterization (see also [26]).

**Corollary 2.2** *Let $x \in \mathbb{K}$. Then for any $v \in H$:*

$$\Pi_{\mathbb{K}}(x, v) = P_{v - N_{\mathbb{K}}(x)}(0) = (v - N_{\mathbb{K}}(x))^{\#} .$$

*Proof.* If $x \in \text{qi}\, \mathbb{K}$, from Theorem 2.2 we derive

$$\Pi_{\mathbb{K}}(x, v) = v.$$

On the other hand, if $x \in \text{qi}\, \mathbb{K}$, by definition, $T_{\mathbb{K}}(x) = H$ and $N_{\mathbb{K}}(x) = (T_{\mathbb{K}}(x))^- = H^- = \{0\}$. Let us suppose now that $x \in \text{qbdry}\, \mathbb{K}$. From Theorem 2.2 we know that

$$v - \hat{v} \in (T_{\mathbb{K}}(x))^0 = N_{\mathbb{K}}(x),$$

where $\hat{v} = \Pi_{\mathbb{K}}(x, v)$. Then we get

$$\hat{v} \in v - N_{\mathbb{K}}(x).$$

Since $\hat{v} = \Pi_{\mathbb{K}}(x, v) = P_{T_{\mathbb{K}}(x)} v$, then we have $\hat{v} \in T_{\mathbb{K}}(x)$ and, hence, $\langle z, \hat{v} \rangle \leq 0$, $\forall z \in (T_{\mathbb{K}}(x))^0 = N_{\mathbb{K}}(x)$. Taking into account (7), we get

$$\langle \hat{v}, v - \hat{v} - z \rangle \geq 0, \ \forall z \in N_{\mathbb{K}}(x)$$

and, thus, $\hat{v} = P_{v - N_{\mathbb{K}}(x)}(0)$. □

# 3 Proof of Theorem 1.3

We shall now present a new proof of Theorem 1.3, in light of our results in the previous sections. Theorem 1.3 is crucial in the study of projected dynamics and perturbed equilibria. It also has an interesting history: the first proof of this theorem appears in [23] in Euclidean space. In more general spaces, such as Hilbert spaces (finite- or infinite-dimensional), there already exist several proofs of this result, as one can see in [7], Theorem 2.2 [29], Proposition 6. However, we give here a novel proof, independent of the previous ones (see [26]).

Let $x^*$ be a solution to the variational inequality

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \mathbb{K}. \tag{9}$$

Using the characterization of the solution by means of the projection, we get

$$x^* = P_{\mathbb{K}}\left(x^* - \lambda F(x^*)\right), \quad \forall \lambda > 0.$$

Hence,

$$\Pi_{\mathbb{K}}(x^*, -F(x^*)) = \lim_{\lambda \to 0^+} \frac{P_{\mathbb{K}}(x^* - \lambda F(x^*))}{\lambda} = \lim_{\lambda \to 0^+} \frac{x^* - x^*}{\lambda} = 0.$$

Viceversa, let $x^*$ be a stationary point of the projected dynamical system, namely, $x^*$ is such that

$$0 = \Pi_{\mathbb{K}}(x^*, -F(x^*)) = P_{T_{\mathbb{K}(x)}}(-F(x^*)).$$

First, let us consider the case when $x^* \in \text{qbdry}\, \mathbb{K}$ and $-F(x^*) \notin T_{\mathbb{K}}(x)$. By virtue of Theorem 2.2, there exist $\beta^* > 0$ and $n^* \in n_{\mathbb{K}}(x^*)$ such that:

$$F(x^*) = \beta^* \, n^*.$$

Since $n^* \in n_{\mathbb{K}}(x^*)$, we have

$$\langle \beta^* \, n^*, x^* - y \rangle \leq 0, \quad \forall y \in \mathbb{K}$$

and, therefore,

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in \mathbb{K}.$$

Let us consider now the case when $x^* \in \text{qbdry}\,\mathbb{K}$ and $-F(x^*) \in T_{\mathbb{K}}(x^*)$. In this case we get

$$0 = \Pi_{\mathbb{K}}(x, -F(x^*)) = P_{T_{\mathbb{K}}(x^*)}(-F(x^*)) = -F(x^*)$$

and, hence, the variational inequality (9) is satisfied.
Finally, if $x^* \in \text{qi}\,\mathbb{K}$, then $T_{\mathbb{K}}(x^*)$ coincides with $H$ and we get

$$0 = P_H(-F(x^*)) = -F(x^*)$$

as above. □

**Remark 3.1** By virtue of Corollary 2.2, we derive that

$$\frac{d\,\dot{x}(t)}{d\,t} = \Pi_{\mathbb{K}}(x, -F(x)) = P_{-F(x)-N_{\mathbb{K}}(x)}(0) =$$

$$= \left\{ \hat{v} \in -\left(F(x) + N_{\mathbb{K}}(x)\right) : \|\hat{v}\| = \min_{y \in -(F(x)+N_{\mathbb{K}}(x))} \|y\| \right\}.$$

Then, the initial value problem

$$\begin{cases} \dfrac{d\,\dot{x}(t)}{d\,t} = \Pi_{\mathbb{K}}(x(t), -F(x(t))) \\ \\ x(0) = x_0 \in \mathbb{K} \end{cases} \tag{10}$$

consists in finding the "slow" solution (the solution of minimal norm) to the differential variational inequality

$$\dot{x}(t) \in -\left(N_{\mathbb{K}}(x(t)) + F(x(t))\right)$$

under the initial condition

$$x(0) = x_0.$$

Since

$$\Pi_{\mathbb{K}}(x(t), -F(x(t))) = P_{T_{\mathbb{K}}(x(t))}(-F(x(t))),$$

problem (10) is equivalent to finding the "slow" solution to the problem

$$\begin{cases} \dot{x}(t) \in P_{T_{\mathbb{K}}(x)}(-F(x(t))) \\ \\ x(0) = x_0 \end{cases} \tag{11}$$

where the operator $F$ is single–valued.
Then, as already observed in the Introduction, the results of [2] Chapter 6, Section 6, and of [1] Theorem 2, can be applied to our projected dynamical system.

**Remark 3.2** It is worth noting that the variational inequality (4) is equivalent to the problem:

$$\text{find } u \in \mathbb{K} : \ \langle F(u(t)), v(t) - u(t) \rangle \geq 0, \quad \forall v \in \mathbb{K}, \text{ a.e. in } [0, T]. \tag{12}$$

Moreover, this remark is interesting because it means that we may have the possibility of applying to (12), among others, the direct method (that is, finding the explicit closed form solution) in order to find solutions to the variational inequality (4). We illustrate this in the case of a numerical example in Section 6 (see also [18], [32], and [16]).

# 4 Computational Procedure

We now consider the time–dependent variational inequality (5) where $\mathbb{K}$ is given by (3). From Remark 3.2, it is equivalent to (12). Let the operator $F$ be strictly monotone (see, e.g., [30] and [33]), so that the solution $u$ is unique and assume that, using a regularization procedure (for example, one can follow the technique used by Gwinner in [26] page 239 to achieve such a regularization) under regularity assumptions on the data, the variational inequality (12) has the solution $u(t) \in C^0([0, T], \mathbb{R}^q)$. Hence, it follows that:

$$\langle F(u(t)), v(t) - u(t) \rangle \geq 0, \quad \forall t \in [0, T].$$

Consider now a sequence of partitions $\pi_n$ of $[0, T]$, such that:

$$\pi_n = \left(t_n^0, \ldots, t_n^{N_n}\right), \quad 0 = t_n^0 < t_n^1 < \ldots < t_n^{N_n} = T$$

and

$$k_n = \max\left\{t_n^j - t_n^{j-1} : \ j = 1, \ldots, N_n\right\}$$

with $k_n \to 0$ when $n \to \infty$. Then, for each value $t_n^{j-1}$, we consider the variational inequality

$$\langle F\left(u\left(t_n^{j-1}\right)\right), v - u\left(t_n^{j-1}\right) \rangle \geq 0, \quad \forall v \in \mathbb{K}\left(t_n^{j-1}\right) \tag{13}$$

where

$$\mathbb{K}\left(t_n^{j-1}\right) = \left\{ v \in \mathbb{R}^q : \lambda\left(t_n^{j-1}\right) \le v \le \mu\left(t_n^{j-1}\right),\ \sum_{i=1}^{q} \xi_{ji}\, v_i = \rho_j\left(t_n^{j-1}\right) \right\}.$$

We can compute now the unique solution to the finite-dimensional variational inequality (13) by means of the critical point of the projected dynamical system

$$\Pi_{\mathbb{K}}\left(u\left(t_n^{j-1},\tau\right), -F\left(u\left(t_n^{j-1},\tau\right)\right)\right) = 0$$

and we can construct an interpolation function $u_n(t)$ such that

$$\lim \|u_n(t) - u(t)\|_{L^\infty([0,T],\mathbb{R}^q)} = 0.$$

**Remark 4.1** We can overcome the regularization assumption on the solution $u$, by considering a discretization procedure and by computing the solution to the finite–dimensional variational inequality obtained after the discretization (see [38]), using the corresponding projected dynamical system. We will demonstrate how to accomplish this in Section 5.

# 5 Proof of the Convergence

The discretization procedure for the calculus to the solution of the evolutionary variational inequality (5) runs as follows.

We consider a sequence $\{\pi_n\}$ of partitions of $[0,T]$, such that:

$$\pi_n = (t_n^0, \dots, t_n^{N_n}),\ 0 = t_n^0 < t_n^1 < \dots < t_n^{N_n} = T$$

and

$$k_n := \max\left\{ t_n^j - t_n^{j-1} :\ j = 1, \dots, N_n \right\}$$

with $k_n \to 0$ when $n \to \infty$.

We consider the space of $\mathbb{R}^m$–value piecewise constant functions induced by $\pi_n$ :

$$P_n\left([0,T],\mathbb{R}^m\right) := \Big\{ v \in L^\infty\left([0,T],\mathbb{R}^m\right) : \\ v_{(t_n^{j-1},t_n^j]} = v_j \in \mathbb{R}^m,\ j = 1, \dots, N_n \Big\} \quad (14)$$

where $v_j$ denotes the constant value of $v$ on $(t_n^{j-1}, t_n^j]$.

The mean value operators $\mu_n : L^1([0,T], \mathbb{R}^m) \to P_n([0,T], \mathbb{R}^m)$ are then introduced by:

$$\mu_n \, v_{(t_n^{j-1}, t_n^j]} := \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} v(s)\, ds. \tag{15}$$

The following Lemma (see, for instance, [5]) will be useful:

**Lemma 5.1** *Let $1 \leq r < \infty$. Then, the linear operators*

$$\mu_n : L^r([0,T], \mathbb{R}^m) \to L^r([0,T], \mathbb{R}^m)$$

*are uniformly bounded with norm 1 and:*

$$\mu_n v \to v \text{ in } L^r([0,T], \mathbb{R}^m)$$

*as $n \to \infty$, $\forall v \in L^r([0,T], \mathbb{R}^m)$.*

Consider now the following closed and convex set:

$$\begin{aligned} \mathbb{K} := \big\{ F(t) \in L^2([0,T], \mathbb{R}^m) : \ \lambda \leq F(t) \leq \nu, \text{ a.e. in } [0,T], \\ \Phi\, F(t) = \rho(t), \ \lambda, \ \nu \geq 0, \big\} \end{aligned} \tag{16}$$

where, for the time being, the upper and lower bounds and the $\rho(t)$ are constant (i.e. not time-dependent) functions, and a linear mapping $C : [0,T] \times \mathbb{K} \to L^2([0,T], \mathbb{R}^m)$ :

$$C[t, F(t)] = A(t)\, F(t) + B(t), \quad A(t) \in L^\infty, \ B(t) \in L^2.$$

Thus, we are led to solve the problem of finding $H(t) \in \mathbb{K}$ :

$$\int_0^T \langle A(t)\, H(t) + B(t), F(t) - H(t) \rangle \, dt \geq 0, \ \forall F(t) \in \mathbb{K}. \tag{17}$$

In correspondence to each partition we can write:

$$\begin{aligned} &\int_0^T \langle A(t)\, H(t) + B(t), F(t) - H(t) \rangle \, dt = \\ &\sum_{j=1}^{N_n} \int_{t_n^{j-1}}^{t_n^j} \langle A(t)\, H(t) + B(t), F(t) - H(t) \rangle \, dt. \end{aligned} \tag{18}$$

Thus, in each interval $[t_n^{j-1}, t_n^j]$ we can consider the problem of finding $u_j^n(t) \in \mathbb{K}$ :

$$\int_{t_n^{j-1}}^{t_n^j} \langle A(t)\, H_j^n(t) + B(t), F_j^n(t) - H_j^n(t)\rangle\, dt \geq 0, \quad \forall F_j^n(t) \in \mathbb{K}. \tag{19}$$

Instead of (19), consider now the finite–dimensional problem of finding $H_j^n \in \mathbb{K}_m \subset \mathbb{R}^m$ :

$$\langle A_j^n\, H_j^n + B_j^n, F_j^n - H_j^n\rangle \geq 0, \quad \forall F_j^n \in \mathbb{K}_m \tag{20}$$

where

$$A_j^n = \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} A(t)\, dt; \quad B_j^n = \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} B(t)\, dt \tag{21}$$

and consider $H_j^n$ as constant approximations of the solutions $H_j^n(t)$ of (19). Here $\mathbb{K}_m$ is the convex subset of $\mathbb{R}^m$ with same lower and upper bounds and the same demand of $\mathbb{K}$.

Our aim is to prove that the functions:

$$H_n(t) = \sum_{j=1}^{N_n} \chi(t_n^{j-1},\, t_n^j)\, H_j^n \tag{22}$$

are, in a suitable sense, piecewise constant approximations to solutions to the original problem (17). We can then prove the following theorem (see [38]):

**Theorem 5.1** *Let $\mathbb{K}$ be as in (16) and, moreover, let $A(t)$ be positive definite a.e. in $[0,T]$. Then, the set $U = \{H_n\}_{n\in\mathbb{N}}$ is (weakly) compact and its cluster points are feasible. Moreover, if $\bar{H}$ is a weak cluster point for $U$, then $\bar{H}$ solves (17).*

In Theorem 5.1 we have considered the constant convex set (16). Now we turn back to the case of a time-dependent convex set:

$$\begin{aligned}\mathbb{K} := \big\{F(t) \in L^2\left([0,T], \mathbb{R}^m\right) :\ & \lambda(t) \leq F(t) \leq \nu(t),\ \text{a.e. in } [0,T],\\ & \lambda(t),\ \nu(t) \geq 0,\ \Phi F(t) = \rho(t) \text{ a.e. in } [0,T]\big\}\end{aligned} \tag{23}$$

and consider piecewise constant approximations for it. For the sake of clarity and completeness, let us recall some basic definitions of set convergence.

**Definition 5.1** *Let $S$ be a metric space and $\{\mathbb{K}_n\}$ a sequence of sets of $S$. We say that $\mathbb{K}_n$ is Kuratowsky–convergent to $\mathbb{K}$ if and only if:*

$$\liminf_n \mathbb{K}_n = \limsup_n \mathbb{K}_n = \mathbb{K},$$

*where*

$$\limsup_n \mathbb{K}_n := \left\{ y \in S : \exists n_1 < n_2 < \dots, \ \textit{with } y_{n_i} \in \mathbb{K}_n, \ y = \lim_i y_{n_i} \right\}$$

$$\liminf_n \mathbb{K}_n := \left\{ y \in S : \exists n_0 \in \mathbb{N} : \forall n > n_0 \ \exists y_n \in \mathbb{K}_n, \ \textit{and } \lim_n y_n = y \right\}.$$

**Definition 5.2** *Let $S$ be a normed space and $\{\mathbb{K}_n\}$ a sequence of closed and convex subsets therein. We say that $\mathbb{K}_n$ is Mosco convergent to $\mathbb{K}$ if and only if:*

$$w - \limsup_n \mathbb{K}_n \subset \mathbb{K} \subset s - \liminf \mathbb{K}_n \tag{24}$$

*where $w$ and $s$ mean weak and strong topology, respectively.*

We now turn to our set (23) and, in correspondence to each partition $\pi_n$ of $[0, T]$ consider the sets:

$$\begin{aligned} \mathbb{K}_j^n := \big\{ F(t) \in L^2\left([0,T], \mathbb{R}^m\right), \ &\text{piecewise constant:} \\ \bar{\lambda}_{j,n} \le F_j(t) \le \bar{\nu}_{j,n}, \ &\text{a.e. in } (t_{j-1}, t_j), \\ \Phi\, F(t) = \bar{\rho}_{j,n}, \ &\text{a.e. in } (t_{j-1}, t_j) \big\}, \end{aligned} \tag{25}$$

where $\bar{\lambda}_{j,n} = \mu_{j,n}\, \lambda(t)$, $\bar{\nu}_{j,n} = \mu_{j,n}\, \nu(t)$ and $\bar{\rho}_{j,n} = \mu_{j,n}\, \rho(t)$ are the mean values of $\lambda(t)$, $\nu(t)$ and $\rho(t)$ on $(t_{j-1}, t_j)$. Thus, we can consider the set $\mathbb{K}^n = \cap \mathbb{K}_j^n$ which, $\forall n \in \mathbb{N}$, has piecewise constant lower and upper bounds and demand which we denote by $\bar{\lambda}_n$, $\bar{\nu}_n$ and $\bar{\rho}_{j,n}$, respectively. Then, the following result holds (see [26]).

**Lemma 5.2** *The set sequence $\mathbb{K}^n$ converges to $\mathbb{K}$ (in Mosco sense).*

We come back now to our problem of finding $H(t) \in \mathbb{K}(t)$ :

$$\int_0^T \langle C[t, H(t)], F(t) - H(t) \rangle \, dt \ge 0, \quad \forall F(t) \in \mathbb{K}(t) \tag{26}$$

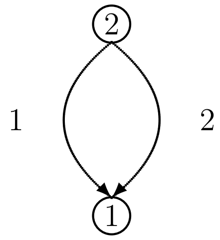


Figure 1: Network Structure of the Numerical Example

and, $\forall F(t) \in \mathbb{K}(t)$, consider $F^n(t) \in \mathbb{K}^n$ such that $F^n(t) \to F(t)$ (strongly). Such $F^n(t)$ does exist thanks to the first part of the proof of Lemma 5.2. Let, $\forall n \in \mathbb{N}$, consider a solution $H^n(t) = \sum_{j=1}^{N_n} \chi(t_n^{j-1}, t_n^j) H_j^n$, where $H_j^n$ is the solution to the finite–dimensional variational inequality:

$$\langle A_j^n H_j^n + B_j^n, F_j^n - H_j^n \rangle \geq 0, \quad \forall F_j^n \in \mathbb{K}_j^n.$$

We are now able to present the final result (see [26]).

**Theorem 5.2** *Let $A(t)$ be positive definite a.e. in $[0, T]$. Then the sequence $H^n(t)$ defined in (22) admits weak cluster points. Each cluster point is feasible and solves the original variational inequality.*

# 6 A Numerical Dynamic Traffic Network Example

In this section, we present a numerical example that is taken from transportation science. For additional background, we refer the reader to [8], [19], [20], and the references therein. We consider a transportation network consisting of a single origin/destination pair of nodes and two paths connecting these nodes of a single link each, as depicted in Figure 1.

The feasible set $\mathbb{K}$ is as in (3), where we take $p := 2$. We also have that $q := 2$, $j := 1$, $T := 2$, $\rho(t) := t$, and $\xi_{ji} := 1$ for $i \in \{1, 2\}$:

$$\mathbb{K} = \bigcup_{t \in [0,2]} \Big\{ u \in L^2([0, 2], \mathbb{R}^2) |$$

$$(0,0) \leq (u_1(t), u_2(t)) \leq \left(t, \frac{3}{2}t\right) \text{ a.e. in } [0,2];$$

$$\left.\sum_{i=1}^{2} u_i(t) = t \text{ a.e. in } [0,2]\right\}.$$

In this application $u(t)$ denotes the vector of path flows at $t$. The cost functions on the paths are defined as: $u_1(t)+1$ for the first path and $u_2(t)+2$ for the second path. We consider a vector field $F$ defined by

$$F : L^2([0,2], \mathbb{R}^2) \to L^2([0,2], \mathbb{R}^2);$$

$$(F_1(u(t), F_2(u(t))) = (u_1(t)+1, u_2(t)+2).$$

The theory of EVI (as described above) states that the system has a unique equilibrium, since $F$ is strictly monotone, for any arbitrarily fixed point $t \in [0,2]$. Indeed, one can easily see that $\langle F(u_1,u_2) - F(v_1,v_2), (u_1 - v_1, u_2 - v_2)\rangle = (u_1-v_1)^2 + (u_2-v_2)^2 > 0$, for any $u \neq v \in L^2([0,2], \mathbb{R}^2)$. With the help of PDS theory, we can compute an approximate curve of equilibria, by selecting $t_0 \in \left\{\frac{k}{4} | k \in \{0, \ldots, 8\}\right\}$. Hence, we obtain a sequence of PDS defined by the vector field $-F(u_1(t_0), u_2(t_0)) = (-u_1(t_0)+1, -u_2(t_0)+2)$ on nonempty, closed, convex, 1–dimensional subsets:

$$\mathbb{K}_{t_0} := \left\{\left\{[0, t_0] \times \left[0, \frac{3}{2}t_0\right]\right\} \cap \{x+y=t_0\}\right\}.$$

For each, we can compute the unique equilibrium of the system at the point $t_0$, that is, the point:

$$(u_1(t_0), u_2(t_0)) \in R^2 \text{ such that } -F(u_1(t_0), u_2(t_0)) \in N_{\mathbb{K}_{t_0}}(u_1(t_0), u_2(t_0)).$$

Proceeding in this manner, we obtain the equilibria consisting of the points:

$$\left\{(0,0), \left(\frac{1}{4}, 0\right), \left(\frac{1}{2}, 0\right), \left(\frac{3}{4}, 0\right), (1,0), \left(\frac{9}{8}, \frac{1}{8}\right),\right.$$

$$\left.\left(\frac{5}{4}, \frac{1}{4}\right), \left(\frac{11}{8}, \frac{3}{8}\right), \left(\frac{3}{2}, \frac{1}{2}\right)\right\}.$$

The interpolation of these points yields the curve of equilibria.

We note that due to the simplicity of the network topology in Figure 1 and the linearity (and separability of the cost functions in this example) we can also obtain explicit formulae for the path flows over time as given below:

$$\begin{cases} u_1(t) = t, \\ & \text{if } \ 0 \leq t \leq 1 \\ u_2(t) = 0 \end{cases}$$

and

$$\begin{cases} u_1(t) = \dfrac{t+1}{2}, \\ & \text{if } \ 1 \leq t \leq 2 \\ u_2(t) = \dfrac{t-1}{2}. \end{cases}$$

The above results demonstrate how the two theories of projected dynamical systems and evolutionary variational inequalities that have been developed in parallel can be connected to enhance the modeling, analysis, and computation of solutions to a plethora of time–dependent equilibrium problems that arise in such disciplines as engineering, operations research/management science, economics, and finance.

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