A Variational Equilibrium Network Framework for Humanitarian Organizations in Disaster Relief: Effective Product Delivery Under Competition for Financial Funds

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Presented at the 3rd International Conference on Dynamics of Disasters Kalamata, Greece, July 5-9, 2017

Abstract: In this paper, we present a new Generalized Nash Equilibrium (GNE) model for post-disaster humanitarian relief by introducing novel financial funding functions and altruism functions, and by also capturing competition on the logistics side among humanitarian organizations. The common, that is, the shared, constraints associated with the relief item deliveries at points of need are imposed by an upper level humanitarian organization or regulatory body and consist of lower and upper bounds to ensure the effective delivery of the estimated volumes of supplies to the victims of the disaster. We identify the network structure of the problem, with logistical and financial flows, and propose a variational equilibrium framework, which allows us to then formulate, analyze, and solve the model using the theory of variational inequalities (rather than quasivariational inequality theory). We then utilize Lagrange analysis and investigate qualitatively the humanitarian organizations' marginal utilities if and when the equilibrium relief item flows are (or are not) at the imposed demand point bounds. We illustrate the game theory model through a case study focused on tornadoes hitting western Massachusetts, a highly unusual event that occurred in 2011. This work significantly extends the original model of Nagurney, Alvarez Flores, and Soylu (2016), which, under the imposed assumptions therein, allowed for an optimization formulation, and adds to the literature of game theory and disaster relief, which is nascent.

Keywords: disaster relief, humanitarian logistics, financial funds, supply chains, competition for funds, NGOs, networks, Generalized Nash Equilibrium, variational equilibrium, variational inequalities

1. Introduction

Disaster relief is fraught with many challenges: the infrastructure, from transportation to communications to energy delivery, may be damaged or destroyed, and services, from healthcare to governmental ones, impacted, all while victims are in desperate need of relief items such as water, food, medicines, and shelter. A timely response to a disaster, hence, can save lives, reduce suffering, and assist in recovery. Moreover, it can also enhance the reputations of humanitarian organizations and their very sustainability in terms of financial donations.

The number of disasters is growing as well as the number of people affected by them (Nagurney and Qiang (2009)) with additional pressures coming from climate change, increasing growth of populations in urban environments, and the spread of diseases brought about by global air travel. The associated costs of the damage and losses due to disasters is estimated at an average $100 billion a year since the turn of the century (Watson et al. (2015)).

Disasters come in many forms, from natural disasters, such as tornadoes, earthquakes, and typhoons, which are often sudden-onset, to famines, which are slow-onset, and can occur not only from changes in weather patterns, resulting in droughts, for example, but also from political situations, including war (cf. Van Wassenhove (2006)). Hence, certain disasters are man-made, as in the case of the Syrian refugee crisis (cf. Sumpf, Isaila, and Najjar (2016)), and terrorist attacks, such as 9/11 (Cox (2008)).

Notable sudden-onset natural disasters have included Hurricane Katrina in 2005, which was the costliest natural disaster in the US, the Haiti earthquake in 2010, the triple disaster in Fukushima, Japan in 2011, consisting of an earthquake, followed by a tsunami and a nuclear meltdown technological disaster, Superstorm Sandy in 2012, tropical cyclone Haiyan in 2013, which was the strongest cyclone ever recorded, the earthquake in Nepal in 2015, and Hurricane Matthew in 2016.

The challenges to disaster relief (humanitarian) organizations, including nongovernmental organizations (NGOs), are immense. The majority operate under a single, common, humanitarian principle of protecting the vulnerable, reducing suffering, and supporting the quality of life, while, at the same time, competing for financial funds from donors to ensure their own sustainability. As noted in Nagurney, Alvarez Flores, and Soylu (2016), competition is intense, with the number of registered US nonprofit organizations increasing from 12,000 in 1940 to more than 1.5 million in 2012. Approximately $300 billion are donated to charities in the United States each year (Zhuang, Saxton, and Wu (2014)). At the same time, many stakeholders believe that humanitarian aid has not been as successful in delivering on the humanitarian principle as might be feasible due to a lack of coordination, which results in duplication of services (see Kopinak (2013)).

We believe that some of the challenges that humanitarian organizations engaged in disaster relief are faced with can be addressed through the use of game theory. Game theory is a methodological framework that captures complex interactions among competing decision-makers (noncooperative games) or cooperating ones (cooperative games). The contributions of John Nash (1950, 1951), in particular, are highly relevant and established some of the foundations of game theory. Specifically, we note that, in the case of noncooperative games, in which the utilities of the competing players, that is, the decision-makers, in the game, depend on the other players' strategies, the governing concept is that of Nash Equilibrium. If, however, the feasible sets, that is, the constraints, are not specific to each player, but, rather, depend also on the strategies of the other players, then we are dealing with a Generalized Nash Equilibrium, introduced by Debreu (1952) (see, also, von Heusinger (2009), Fischer, Herrich, and Schonefeld (2014), and the references therein).

In particular, in this paper, we construct a new Generalized Nash Equilibrium (GNE) network model for disaster relief, which models competition among NGOs for financial funds post-disaster, as well as for the delivery of relief items. The utility function that each NGO seeks to maximize depends on its financial gain from donations plus the weighted benefit accrued from doing good through the delivery of relief items minus the total cost associated with the logistics of delivering the relief items. The model extends the earlier model of Nagurney, Alvarez Flores, and Soylu (2016) in the following significant ways, which means that the optimization reformulation, as done in that paper, no longer applies:

1. The financial funds functions, which capture the amount of donations to each NGO, given their visibility through media of the supplies of relief items delivered at demand points, and under competition, need not take on a particular structure.

2. The altruism or benefit functions, also included in each NGO's utility function, need not be linear.

3. The competition associated with logistics is captured through total cost functions that depend not only on a particular NGO's relief item shipments but also on those of the other NGOs.

In order to guarantee effective product delivery at the demand points, we retain the lower and upper bounds, as introduced in Nagurney, Alvarez Flores, and Soylu (2016). Such common, or shared constraints, assist in coordination (cf. Balcik et al. (2010)) and would be imposed by a higher level humanitarian organization or regulatory authority in order to ensure that the needed volumes of relief items are delivered but are not oversupplied, which can results in congestion, materiel convergence, and wastage. We assume that the NGOs have prepositioned the supplies of the disaster relief items and that the total amount available across all NGOs is sufficient to meet the needs of the victims.

It is important to emphasize that Generalized Nash Equilibrium problems are more challenging to formulate and solve and are usually tackled via quasivariational inequalities (cf. Bensoussan (1974)), the theory of which, as well as the associated computational procedures, are not in as an advanced state as that of variational inequalities (see Kinderlehrer and Stampacchia (1980) and Nagurney (1999)). Here we utilize, for the first time, in the context of humanitarian operations and disaster relief, a *variational equilibrium*. As noted in Nagurney, Yu, and Besik (2017)), a variational equilibrium is a specific kind of GNE (cf. Facchinei and Kanzow (2010), Kulkarni and Shanbhag (2012)). The variational equilibrium allows for alternative variational inequality formulations of our new Generalized Nash Equilibrium network model. What is notable about a variational equilibrium (see also Luna (2013)) is that the Lagrange multipliers associated with the shared or coupling constraints of the NGOs are the same for all NGOs in the disaster relief game. This also provides us with an elegant economic and equity interpretation.

The only other game theory model for disaster relief that includes elements of logistics plus financial funds is that of Nagurney, Alvarez Flores, and Soylu (2016). Zhuang, Saxton, and Wu (2014) proposed a model that showed that the amount of charitable contributions made by donors is positively dependent on the amount of disclosure by the NGOs. The authors emphasized that there is a dearth of existing game-theoretic research on nonprofit organizations. Toyasaki and Wakolbinger (2015) developed game theory models to analyze whether an NGO should establish a special fund after a disaster (in terms of earmarked donations) or allow only unearmarked donations. Nagurney (2016), in turn, presented a network game theory model in which multiple freight service providers are engaged in competition to acquire the business of carrying disaster relief supplies of a humanitarian organization in the amounts desired to the destinations. Coles and Zhuang (2011), on the other hand, argued for the need for cooperative game theory models for disaster recovery operations by highlighting a stream of post-disaster operations. Muggy and Stamm (2014) give an excellent review of game theory in humanitarian operations and note that there are many untapped research opportunities for modeling in this area. See also the dissertation of Muggy (2015). The research in our paper adds to the still nascent literature on game theory and disaster relief

/ humanitarian operations.

This paper is organized as follows. In Section 2, we construct the novel Generalized Nash Equilibrium model for disaster relief, which captures competition both on the financial funds side as well as on the logistics side and we identify the network structure. We present the variational equilibrium framework and also prove the existence of an equilibrium solution. In addition, we provide, for completeness, the variational inequality formulation of a special case of the model, under the Nash equilibrium solution, in the absence of imposed common demand constraints. In Section 3, we then explore, through Lagrange analysis, the humanitarian organizations' marginal utilities when the equilibrium disaster relief flows are at the upper or the lower bounds of the imposed demands of the regulatory body or lie in between. In order to illustrate the framework developed here, Section 4 presents both an algorithmic scheme and a case study, inspired by tornadoes that hit western Massachusetts in June 2011, with devastating impact (cf. Western Massachusetts Regional Homeland Advisory Council (2012)). We summarize our results and present our conclusions in Section 5.

2. The Variational Equilibrium Network Framework for Humanitarian Organizations in Disaster Relief

We now present the new Generalized Nash Equilibrium model for disaster relief, along with the variational equilibrium framework. As mentioned in the Introduction, the model extends the earlier model of Nagurney, Alvarez Flores, and Soylu (2016), which, under the imposed assumptions therein, allowed for an optimization reformulation. Our notation follows closely the notation in the above paper but here we utilize, in contrast, a more general variational equilibrium framework.

We consider m humanitarian organizations, here referred to as nongovernmental organizations (NGOs), with a typical NGO denoted by i, seeking to deliver relief supplies, post a disaster, to n demand points, with a typical demand point denoted by j. The relief supplies can be water, food, or medicine. We assume that the product delivered can be viewed as being homogeneous. We denote the volume of the relief item shipment (flow) delivered by NGO i to demand point j by q_{ij}. We group the nonnegative relief item flows from each NGO i; $i = 1, \ldots, m$, into the vector $q_i \in R^n_+$ and then we group the relief item flows of all the NGOs to all the demand points into the vector $q \in R^{mn}_+$. The vector q_i is the vector of strategies of NGO i.

The NGOs compete for financial funds from donors, while also engaging in competition on the logistic side in terms of costs, since there may be competition for freight services, etc., as well as congestion at the demand sites. The network structure of the problem is given in Figure 1. Note that the links from the first tier nodes representing the NGOs to the bottom tier nodes, corresponding to the demand points, are the shipment links and have relief item flows associated with them. The links from the demand nodes to the NGO nodes (in the opposite direction) are the links with the financial flows from the donors reacting to the visibility of the NGOs in their delivery of the needed supplies through the media. The network structure of this problem differs from the network underlying the model given in Nagurney, Alvarez Flores, and Soylu (2016) since in that model, the financial flows, once collected, were partitioned to each NGO, using a factor representing the portion of the financial funds each humanitarian organization was (likely) to get of the total amount donated.

We emphasize that, in terms of the sequence of events, the humanitarian organizations first decide on the level of relief items to be provided at each demand point and deliver the amounts. Then they receive the corresponding financial flows. Therefore, the financial flows are received after the supplies arrive. As noted in Nagurney, Alvarez Flores, and Soylu

(2016), empirically, these funds are realized and made available quickly, and these two events are almost concurrent in many cases. The justification of this assumption is also provided by the nature of the incentives of the decision-makers in our model, which is to provide humanitarian relief as quickly as possible whenever a disaster strikes.

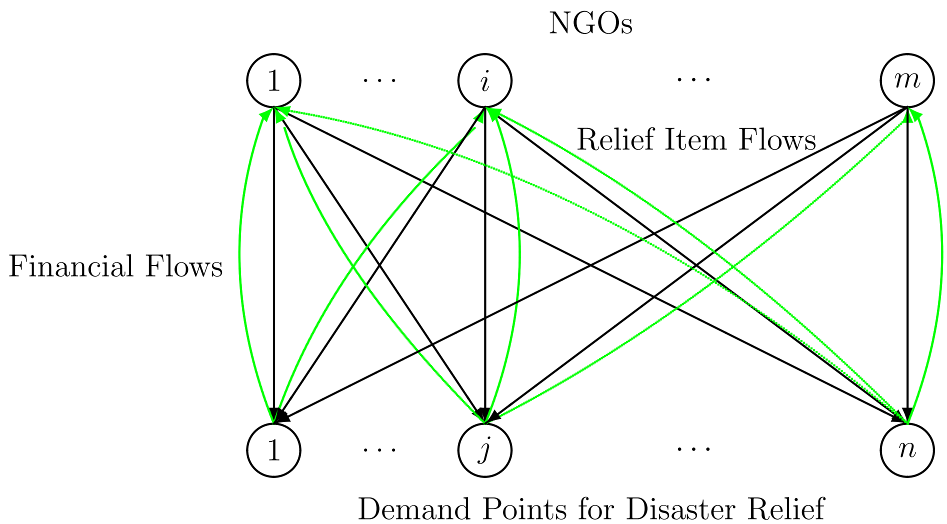

Figure 1: The Network Structure of the Game Theory Model

Each NGO i encumbers a cost, c_{ij}, associated with shipping the relief items to location j, where we assume that

$$c_{ij} = c_{ij}(q), \quad j = 1, \ldots n, \tag{1}$$

with these cost functions being convex and continuously differentiable. These costs also include transaction costs (see also Nagurney (2006)). Note that the cost functions (1) are associated with the logistics aspects and, hence, the cost on a shipment link can depend not only on its flow but also on the flows on the other shipment links associated with the same NGO or with other NGOs.

Each NGO i; $i = 1, \ldots, m$, based on the media attention and the visibility of NGOs at demand point j; $j = 1, \ldots, n$, receives financial funds from donors given by the expression

$$\sum_{j=1}^{n} P_{ij}(q), \tag{2}$$

where $P_{ij}(q)$ denotes the financial funds in donation dollars given to NGO i due to visibility of NGO i at location j. Hence, $P_{ij}(q)$ corresponds to the financial flow on the link joining demand node j with node NGO i in Figure 1. Observe that, according to (2), there is competition among all the NGOs for financial donations since the financial amount of donations that an NGO receives depends not only on its relief item deliveries but also on those delivered by other NGOs. Indeed, according to (2), an NGO may benefit from donations even through visibility of other NGOs providing the product because of, for example, loyalty and support for a specific NGO. We assume that the P_{ij} functions are increasing, concave, and continuously differentiable. Hence, we have positive but decreasing marginal utility of providing aid (in terms of the NGO's effect on attracting donations). It is important to mention that the $P_{ij}(q)$ function contains, as a special case, the financial funds donor function of Nagurney, Alvarez Flores, and Soylu (2016), with $P_{ij}(q) = \beta_i P_j(q)$; $i = 1, \ldots, m$; $j = 1, \ldots, n$. Furthermore, Natsios (1995) noted that the cheapest way for relief organizations to fundraise is to provide early relief in highly visible areas. In our case study in Section 4 we construct explicit $P_{ij}(q)$ functions for all NGOs i and demand points j.

Also, since the NGOs are humanitarian organizations involved in disaster relief, each NGO i also derives some utility from delivering the needed relief supplies. We, hence, introduce an altruism/benefit function B_i; $i = 1, \ldots, m$, such that

$$B_i = B_i(q), \tag{3}$$

and each benefit function is assumed to be concave and continuously differentiable. Previously utilized benefit functions in this application domain were of the form: $B_i = \sum_{j=1}^{n} \gamma_{ij} q_{ij}$; $j = 1, \ldots, n$. Furthermore, when we construct each NGO's full utility function we will also assign a weight ω_i before each $B_i(q)$; $i = 1, \ldots, m$, to represent a monetized weight associated with altruism of i. Such weight concepts are used in multicriteria decision-making; see, e.g., Fishburn (1970), Chankong and Haimes (1983), Yu (1985), Keeney and Raiffa (1993), and Nagurney, Alvarez Flores, and Soylu (2016).

Each NGO i; $i = 1, \ldots, m$, has an amount s_i of the relief item that it can allocate post-disaster, which must satisfy:

$$\sum_{j=1}^{n} q_{ij} \leq s_i. \tag{4}$$

We assume that the relief supplies have been prepositioned so that they are in stock and available, since time is of the essence. According to Roopanarine (2013), prepositioning of supplies can make emergency relief more effective and this is a strategy followed not only by the UNHRD (United Nations Humanitarian Response Depot) but also by the Red Cross and even some smaller relief organizations such as AmeriCares. Gatignon, Van Wassenhove, and Charles (2010) also note the benefits of proper prepositioning of supplies in the case of the International Federation of the Red Cross (IFRC) in terms of cost reduction and a more timely response.

In addition, the relief item flows for each i; $i = 1, \ldots, m$, must be nonnegative, that is:

$$q_{ij} \geq 0, \quad j = 1, \ldots, n. \tag{5}$$

Each NGO i; $i = 1, \ldots, m$, seeks to maximize its utility, U_i, with the utility consisting of the financial gains due to its visibility through media of the relief item flows, $\sum_{j=1}^{n} P_{ij}(q)$, plus the utility associated with the logistical (supply chain) aspects of delivery of the supplies, which consists of the weighted altruism/benefit function minus the logistical costs. For additional background on utility functions for nonprofit and charitable organizations, see Rose-Ackerman (1982) and Malani, Philipson, and David (2003).

Without the imposition of demand bound constraints (which will follow), the optimization problem faced by NGO i; $i = 1, \ldots, m$, is, thus,

$$\text{Maximize} \quad U_i(q) = \sum_{j=1}^{n} P_{ij}(q) + \omega_i B_i(q) - \sum_{j=1}^{n} c_{ij}(q) \tag{6}$$

subject to constraints (4) and (5).

Before imposing the common constraints, we remark that the above model, in the absence of any common constraints, is a Nash Equilibrium problem, which we know can be formulated and solved as a variational inequality problem (cf. Gabay and Moulin (1980) and Nagurney (1999)). Indeed, although the utility functions of the NGOs depend on their strategies and those of the other NGOs, the respective NGO feasible sets do not. However, the NGOs may be faced with several common constraints, which make the game theory problem more complex and challenging. The common constraints, which are imposed by an authority, such as a governmental one or a higher level humanitarian coordination agency, ensure that the needs of the disaster victims are met, while recognizing the negative effects of waste and material convergence. The imposition of such constraints in terms of effectiveness and even gains for NGOs was demonstrated in Nagurney, Alvarez Flores, and Soylu (2016). Later in this section, we present the variational inequality framework. Hence, we will not need to make use of quasivariational inequalities (cf. von Heusinger (2009)) for our new model.

Specifically, the two sets of common imposed constraints, at each demand point j; $j = 1, \ldots, n$, are as follows:

$$\sum_{i=1}^{m} q_{ij} \geq \underline{d}_j, \tag{7}$$

and

$$\sum_{i=1}^{m} q_{ij} \leq \bar{d}_j, \tag{8}$$

where $\underline{d}_j$ is the lower bound on the amount of the relief item needed at demand point j and $\bar{d}_j$ is the upper bound on the amount of the relief item needed at demand point j. The constraints (7) and (8) give flexibility for a regulatory or coordinating body, since it is not likely that the demand will be precisely known in a disaster situation. It is, however, reasonable to assume that, as represented in these equations, estimates for needs assessment for the relief items will be available at the local level.

We assume that

$$\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} \underline{d}_j. \tag{9}$$

Hence, the total supply of the relief item of the NGOs is sufficient to meet the needs at all the demand points.

We define the feasible set K_i for each NGO i as:

$$K_i \equiv \{q_i |\, (4)\,\text{and}\,(5)\,\text{hold}\} \tag{10}$$

and we let $K \equiv \prod_{i=1}^{m} K_i$.

In addition, we define the feasible set $\mathcal{S}$ consisting of the shared constraints as:

$$\mathcal{S} \equiv \{q |\, (7)\,\text{and}\,(8)\,\text{hold}\}. \tag{11}$$

Observe that now not only does the utility of each NGO depend on the strategies, that is, the relief item flows, of the other NGOs, but so does the feasible set because of the common constraints (7) and (8). Hence, the above game theory model, in which the NGOs compete noncooperatively is a Generalized Nash Equilibrium problem. Therefore, we have the following definition.

Definition 1: Disaster Relief Generalized Nash Equilibrium

A relief item flow pattern $q^* \in K = \prod_{i=1}^{m} K_i$, $q^* \in \mathcal{S}$, *constitutes a disaster relief Generalized Nash Equilibrium if for each NGO* i; $i = 1, \ldots, m$:

$$U_i(q_i^*, \hat{q}_i^*) \geq U_i(q_i, \hat{q}_i^*), \quad \forall q_i \in K_i, \forall q \in \mathcal{S}, \tag{12}$$

where $\hat{q}_i^* \equiv (q_1^*, \ldots, q_{i-1}^*, q_{i+1}^*, \ldots, q_m^*)$.

Hence, an equilibrium is established if no NGO can unilaterally improve upon its utility by changing its relief item flows in the disaster relief network, given the relief item flow decisions of the other NGOs, and subject to the supply constraints, the nonnegativity constraints, and the shared/coupling constraints. We remark that both K and $\mathcal{S}$ are convex sets.

If there are no coupling, that is, shared, constraints in the above model, then q and q^* in Definition 1 need only lie in the set K, and, under the assumption of concavity of the utility functions and that they are continuously differentiable, we know that (cf. Gabay and Moulin (1980) and Nagurney (1999)) the solution to what would then be a Nash equilibrium problem (see Nash (1950, 1951)) would coincide with the solution of the following variational inequality problem: determine $q^* \in K$, such that

$$-\sum_{i=1}^{m} \langle \nabla_{q_i} U_i(q^*), q_i - q_i^* \rangle \geq 0, \quad \forall q \in K, \tag{13}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in the corresponding Euclidean space and $\nabla_{q_i} U_i(q)$ denotes the gradient of $U_i(q)$ with respect to q_i.

As emphasized in Nagurney, Yu, and Besik (2017), a refinement of the Generalized Nash Equilibrium is what is known as a variational equilibrium and it is a specific type of GNE (see Kulkarni and Shabhang (2012)). Specifically, in a GNE defined by a variational equilibrium, the Lagrange multipliers associated with the common/shared/coupling constraints are all the same. This feature provides a fairness interpretation and is reasonable from an economic standpoint. More precisely, we have the following definition:

Definition 2: Variational Equilibrium

A strategy vector q^* is said to be a variational equilibrium of the above Generalized Nash Equilibrium game if $q^* \in K, q^* \in \mathcal{S}$ is a solution of the variational inequality:

$$-\sum_{i=1}^{m} \langle \nabla_{q_i} U_i(q^*), q_i - q_i^* \rangle \geq 0, \quad \forall q \in K, \forall q \in \mathcal{S}. \tag{14}$$

By utilizing a variational equilibrium, we can take advantage of the well-developed theory of variational inequalities, including algorithms (cf. Nagurney (1999) and the references therein), which are in a more advanced state of development and application than algorithms for quasivariational inequality problems.

We now expand the terms in variational inequality (14).

Specifically, we have that (14) is equivalent to the variational inequality: determine $q^* \in$

$K, q^* \in \mathcal{S}$, such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\sum_{k=1}^{n}\frac{\partial c_{ik}(q^*)}{\partial q_{ij}} - \sum_{k=1}^{n}\frac{\partial P_{ik}(q^*)}{\partial q_{ij}} - \omega_i\frac{\partial B_i(q^*)}{\partial q_{ij}}\right] \times \left[q_{ij} - q^*_{ij}\right] \geq 0, \quad \forall q \in K, \forall q \in \mathcal{S}. \tag{15}$$

We now put variational inequality (15) into standard variational inequality form (see Nagurney (1999)), that is: determine $X^* \in \mathcal{K} \subset R^N$, such that

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{16}$$

where F is a given continuous function from $\mathcal{K}$ to R^N, $\mathcal{K}$ is a closed and convex set, with both the vectors $F(X)$ and X being column vectors, and $N = mn$.

We define $X \equiv q$ and $F(X)$ where component (i, j); $i = 1, \ldots, m$; $j = 1, \ldots, n$, of $F(X)$, $F_{ij}(X)$, is given by

$$F_{ij}(X) \equiv \left[\sum_{k=1}^{n}\frac{\partial c_{ik}(q^*)}{\partial q_{ij}} - \sum_{k=1}^{n}\frac{\partial P_{ik}(q^*)}{\partial q_{ij}} - \omega_i\frac{\partial B_i(q^*)}{\partial q_{ij}}\right] \tag{17}$$

and $\mathcal{K} \equiv K \cap \mathcal{S}$. Then, clearly, (14) takes on the standard form (16).

Remark: Existence and Uniqueness of an Equilibrium Solution

A solution q^* of disaster relief item flows to the variational inequality problem (15) is guaranteed to exist since the function $F(X)$ in (16) is continuous under the imposed assumptions and the feasible set $\mathcal{K}$ comprised of the constraints is compact. Furthermore, it follows from the classical theory of variational inequalities (cf. Kinderlehrer and Stampacchia (1980) and Nagurney (1999)) that if $F(X)$ is strictly monotone, that is:

$$\langle F(X^1) - F(X^2), X^1 - X^2 \rangle > 0, \quad \forall X^1, X^2 \in \mathcal{K}, X^1 \neq X^2,$$

then the solution to the variational inequality (16) is unique, and we have a unique equilibrium product shipment pattern q^* from the NGOs to the demand points.

3. Lagrange Theory and Analysis of the Marginal Utilities

In this section we explore the Lagrange theory associated with variational inequality (15) and we provide an analysis of the marginal utilities at the equilibrium solution. For an application of Lagrange theory to other models, see: Daniele (2001) (spatial economic models), Barbagallo, Daniele, and Maugeri (2012) (financial networks), Toyasaki, Daniele, and Wakolbinger (2014) (end-of-life products networks), Daniele and Giuffrè (2015) (random traffic networks), Caruso and Daniele (2016) (transplant networks), Nagurney and Dutta (2016) (competition for blood donations).

By setting:

$$C(q)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\sum_{k=1}^{n}\frac{\partial c_{ik}(q^*)}{\partial q_{ij}}-\sum_{k=1}^{n}\frac{\partial P_{ik}(q^*)}{\partial q_{ij}}-\omega_i\frac{\partial B_i(q^*)}{\partial q_{ij}}\right](q_{ij}-q_{ij}^*), \tag{18}$$

variational inequality (15) can be rewritten as a minimization problem as follows:

$$\min_{\mathcal{K}} C(q)=C(q^*)=0. \tag{19}$$

Under the previously imposed assumptions, we know that all the involved functions in (19) are continuously differentiable and convex.

We set:

$$\begin{aligned}
a_{ij} &= -q_{ij}\le 0, && \forall i,\ \forall j,\\
b_i &= \sum_{j=1}^{n} q_{ij}-s_i\le 0, && \forall i,\\
c_j &= \underline{d}_j-\sum_{i=1}^{m} q_{ij}\le 0, && \forall j,\\
e_j &= \sum_{i=1}^{m} q_{ij}-\overline{d}_j\le 0, && \forall j,
\end{aligned} \tag{20}$$

and

$$\Gamma(q)=(a_{ij},b_i,c_j,e_j)_{i=1,\dots,m;\,j=1,\dots,n}\,. \tag{21}$$

As a consequence, we remark that $\mathcal{K}$ can be rewritten as

$$\mathcal{K}=\{q\in R^{mn}:\Gamma(q)\le 0\}. \tag{22}$$

We now consider the following Lagrange function:

$$\begin{aligned}
\mathcal{L}(q,\alpha,\delta,\sigma,\varepsilon) &= \sum_{j=1}^{n} c_{ij}(q)-\sum_{j=1}^{n} P_{ij}(q)-\omega_i B_i(q)\\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{ij}a_{ij}+\sum_{i=1}^{m}\delta_i b_i+\sum_{j=1}^{n}\sigma_j c_j+\sum_{j=1}^{n}\varepsilon_j e_j,
\end{aligned} \tag{23}$$

$$\forall q\in R_+^{mn},\ \forall\alpha\in R_+^{mn},\ \forall\delta\in R_+^{m},\ \forall\sigma\in R_+^{n},\ \forall\varepsilon\in R_+^{n},$$

where α is the vector with components: $\{\alpha_{11},\dots,\alpha_{mn}\}$; δ is the vector with components $\{\delta_1,\dots,\delta_m\}$; σ is the vector with elements: $\{\sigma_1,\dots,\sigma_n\}$, and ϵ is the vector with elements: $\{\epsilon_1,\dots,\epsilon_n\}$.

It is easy to prove that the feasible set $\mathcal{K}$ is convex and that the Slater condition is satisfied. Then, if q^* is a minimal solution to problem (19), there exist $\alpha^* \in R^{mn}_+$, $\delta^* \in R^m_+$, $\sigma^* \in R^n_+$, $\varepsilon^* \in R^n_+$ such that the vector $(q^*, \alpha^*, \delta^*, \sigma^*, \varepsilon^*)$ is a saddle point of the Lagrange function (23); namely:

$$\mathcal{L}(q^*, \alpha, \delta, \sigma, \varepsilon) \leq \mathcal{L}(q^*, \alpha^*, \delta^*, \sigma^*, \varepsilon^*) \leq \mathcal{L}(q, \alpha^*, \delta^*, \sigma^*, \varepsilon^*), \tag{24}$$

$$\forall q \in R^{mn}_+,\ \forall \alpha \in R^{mn}_+,\ \forall \delta \in R^m_+,\ \forall \sigma \in R^n_+,\ \forall \varepsilon \in R^n_+,$$

and

$$\alpha^*_{ij} a^*_{ij} = 0,\ \forall i,\ \forall j,$$

$$\delta^*_i b^*_i = 0,\ \forall i,$$

$$\sigma^*_j c^*_j = 0, \quad \varepsilon^*_j e^*_j = 0,\ \forall j. \tag{25}$$

From the right-hand side of (24), it follows that $q^* \in R^{mn}_+$ is a minimal point of $\mathcal{L}(q, \alpha^*, \delta^*, \sigma^*, \varepsilon^*)$ in the whole space R^{mn}, and hence, for all $i = 1, \ldots, m$, and for all $j = 1, \ldots, n$, we have that:

$$\frac{\partial \mathcal{L}(q^*, \alpha^*, \delta^*, \sigma^*, \varepsilon^*)}{\partial q_{ij}}$$

$$= \sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} - \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} - \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} - \alpha^*_{ij} + \delta^*_i - \sigma^*_j + \varepsilon^*_j = 0, \tag{26}$$

together with conditions (25).

Conditions (25) and (26) represent an equivalent formulation of variational inequality (15). Indeed, if we multiply (26) by $(q_{ij} - q^*_{ij})$ and sum up with respect to i and j, we get:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left[\sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} - \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} - \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} \right] (q_{ij} - q^*_{ij})$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha^*_{ij} q_{ij} - \underbrace{\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha^*_{ij} q^*_{ij}}_{=0} - \sum_{i=1}^{m} \left(\delta^*_i \sum_{j=1}^{n} q_{ij} - \underbrace{\delta^*_i \sum_{j=1}^{n} q^*_{ij}}_{=\delta^*_i s_i} \right)$$

$$+ \sum_{j=1}^{n} \left(\sigma^*_j \sum_{i=1}^{m} q_{ij} - \underbrace{\sigma^*_j \sum_{i=1}^{m} q^*_{ij}}_{=\sigma^*_j \underline{d}_j} \right) - \sum_{j=1}^{n} \left(\varepsilon^*_j \sum_{i=1}^{m} q_{ij} - \underbrace{\varepsilon^*_j \sum_{i=1}^{m} q^*_{ij}}_{=\varepsilon^*_j \bar{d}_j} \right)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} \underbrace{\alpha_{ij}^* q_{ij}}_{\geq 0} - \sum_{i=1}^{m} \delta_i^* \left(\underbrace{\sum_{j=1}^{n} q_{ij} - s_i}_{\leq 0} \right) + \sum_{j=1}^{n} \sigma_j^* \left(\underbrace{\sum_{i=1}^{m} q_{ij} - \underline{d}_j}_{\geq 0} \right)$$

$$- \sum_{j=1}^{n} \varepsilon_j^* \left(\underbrace{\sum_{i=1}^{m} q_{ij} - \overline{d}_j}_{\leq 0} \right) \geq 0. \tag{27}$$

We now discuss the meaning of some of the Lagrange multipliers. We focus on the case where $q_{ij}^* > 0$; namely, the relief item flow from NGO i to demand point j is positive; otherwise, if $q_{ij}^* = 0$, the problem is not interesting. Then, from the first line in (25), we have that $\alpha_{ij}^* = 0$.

Let us consider the situation when the constraints are not active, that is, $b_i^* < 0$ and $\underline{d}_j < \sum_{i=1}^{m} q_{ij}^* < \overline{d}_j$.

Specifically, $b_i^* < 0$ means that $\sum_{j=1}^{n} q_{ij}^* < s_i$; that is, the sum of relief items sent by the i-th NGO to all demand points is strictly less than the total amount s_i at its disposal. Then, from the second line in (25), we get: $\delta_i^* = 0$.

At the same time, from the last line in (25), $\underline{d}_j < \sum_{i=1}^{m} q_{ij}^* < \overline{d}_j$, leads to: $\sigma_j^* = \varepsilon_j^* = 0$.

Hence, (26) yields:

$$\sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} - \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} - \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} = \alpha_{ij}^* - \delta_i^* + \sigma_j^* - \varepsilon_j^* = 0$$

$$\Longleftrightarrow \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} = \sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}}. \tag{28}$$

In this case, the marginal utility associated with the financial donations plus altruism is equal to the marginal costs.

If, on the other hand, $\sum_{i=1}^{m} q_{ij}^* = \underline{d}_j$, then $\sigma_j^* > 0$. Hence, we get:

$$\sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} + \sigma_j^* = \sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}}, \text{ with } \sigma_j^* > 0, \tag{29}$$

and, therefore,

$$\sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} > \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}}, \tag{30}$$

which means that the marginal costs are greater than the marginal utility associated with the financial donations plus altruism and this is a very bad situation.

Finally, if $\sum_{i=1}^{m} q_{ij}^* = \overline{d}_j$, then $\varepsilon_j^* > 0$, we have that:

$$\sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} = \sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} + \varepsilon_j^*, \text{ with } \varepsilon_j^* > 0. \tag{31}$$

Therefore,

$$\sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} < \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}}. \tag{32}$$

In this situation, the relevant marginal utility exceeds the marginal cost and this is a desirable situation.

Analogously, if we assume that the conservation of flow equation is active; that is, if $\sum_{j=1}^{n} q_{ij}^* = s_i$, then $\delta_i^* > 0$. As a consequence, we obtain:

$$\sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} = \sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} + \delta_i^*, \text{ with } \delta_i^* > 0, \tag{33}$$

which means that, once again, the marginal utility associated with the financial donations plus altruism exceeds the marginal cost and this is the desirable situation.

From the above analysis of the Lagrange multipliers and marginal utilities at the equilibrium solution, we can conclude that the most convenient situation, in terms of the marginal utilities, is the one when $\sum_{i=i}^{m} q_{ij}^* = \overline{d}_j$ and $\sum_{j=1}^{n} q_{ij}^* = s_i$.

Taking into account the Lagrange multipliers, an equivalent variational formulation of problem (6) under constraints (4), (5), (7), and (8) is the following one:

$$\begin{gathered}
\text{Find } (q^*, \delta^*, \sigma^*, \varepsilon^*) \in R_+^{mn+m+2n}: \\
\sum_{i=1}^{m} \sum_{j=1}^{n} \left[\sum_{k=1}^{n} \frac{\partial c_{ik}(q^*)}{\partial q_{ij}} - \sum_{k=1}^{n} \frac{\partial P_{ik}(q^*)}{\partial q_{ij}} - \omega_i \frac{\partial B_i(q^*)}{\partial q_{ij}} + \delta_i^* - \sigma_j^* + \varepsilon_j^* \right] (q_{ij} - q_{ij}^*) \\
+ \sum_{i=1}^{m} \left(s_i - \sum_{j=1}^{n} q_{ij}^* \right) (\delta_i - \delta_i^*) \\
+ \sum_{j=1}^{n} \left(\sum_{i=1}^{m} q_{ij}^* - \underline{d}_j \right) (\sigma_j - \sigma_j^*) + \sum_{j=1}^{n} \left(\overline{d}_j - \sum_{i=1}^{m} q_{ij}^* \right) (\varepsilon_j - \varepsilon_j^*) \geq 0,
\end{gathered} \tag{34}$$

$$\forall q \in R_+^{mn}, \ \forall \delta \in R_+^m, \ \forall \sigma \in R_+^n, \ \forall \varepsilon \in R_+^n.$$

4. The Algorithm and Case Study

Before we present the case study, we outline the algorithm that we utilize for the computations, notably, the Euler method of Dupuis and Nagurney (1993), since it nicely exploits the feasible set underlying variational inequality (34), which is simply the nonnegative orthant.

Recall that, as established in Dupuis and Nagurney (1993), for convergence of the general iterative scheme, which induces the Euler method, the sequence $\{a_\tau\}$ must satisfy: $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, as $\tau \to \infty$. Conditions for convergence for a variety of network-based problems can be found in Nagurney and Zhang (1996) and Nagurney (2006).

Specifically, at iteration τ, the Euler method yields the following closed form expressions for the relief item flows and the Lagrange multipliers.

Explicit Formulae for the Euler Method Applied to the Game Theory Model

In particular, we have the following closed form expression for the relief item flows $i = 1, \ldots, m; j = 1, \ldots, n$, at each iteration:

$$q_{ij}^{\tau+1} = \max\{0, q_{ij}^\tau + a_\tau(\sum_{k=1}^{n} \frac{\partial P_{ik}(q^\tau)}{\partial q_{ij}} + \omega_i \frac{\partial B_i(q^\tau)}{\partial q_{ij}} - \sum_{k=1}^{n} \frac{\partial c_{ik}(q^\tau)}{\partial q_{ij}} - \delta_i^\tau + \sigma_j^\tau - \epsilon_j^\tau)\}; \quad (35)$$

the following closed form expressions for the Lagrange multipliers associated with the supply constraints (4), respectively, for $i = 1, \ldots, m$:

$$\delta_i^{\tau+1} = \max\{0, \delta_i^\tau + a_\tau(-s_i + \sum_{j=1}^{n} q_{ij}^\tau)\}; \quad (36)$$

the following closed form expressions for the Lagrange multipliers associated with the lower bound demand constraints (7), respectively, for $j = 1, \ldots, n$:

$$\sigma_j^{\tau+1} = \max\{0, \sigma_j^\tau + a_\tau(-\sum_{i=1}^{m} q_{ij}^\tau + \underline{d}_j)\}, \quad (37)$$

and the following closed form expressions for the Lagrange multipliers associated with the upper bound demand constraints (8), respectively, for $j = 1, \ldots, n$:

$$\epsilon_j^{\tau+1} = \max\{0, \epsilon_j^\tau + a_\tau(-\bar{d}_j + \sum_{i=1}^{m} q_{ij}^\tau)\}. \quad (38)$$

Our case study is inspired by a disaster consisting of a series of tornados that hit western Massachusetts on June 1, 2011 in the late afternoon. The largest tornado was measured at EF3. It was the worst tornado outbreak in the area in a century (see Flynn (2011)). A wide swath from western to central Massachusetts was impacted. According to the Western Massachusetts Regional Homeland Security Advisory Council report (2012): "The tornado caused extensive damage, killed 4 persons, injured more than 200 persons, damaged or destroyed 1,500 homes, left over 350 people homeless in Springfield's MassMutual Center arena, left 50,000 customers without power, and brought down thousands of trees." The same report notes that: FEMA estimated that 1,435 residences were impacted with the following breakdowns: 319 destroyed, 593 sustaining major damage, 273 sustaining minor damage, and 250 otherwise affected. FEMA estimated that the primary impact was damage to buildings and equipment with a cost estimate of \$24,782,299. Total damage estimates from the storm exceeded \$140 million, the majority from the destruction of homes and businesses.

Especially impacted were the city of Springfield and the towns of Monson and Brimfield. It has been estimated that in the aftermath, the Red Cross served about 11,800 meals and the Salvation Army about 20,000 meals (cf. Western Massachusetts Regional Homeland Security Advisory Council (2012)).

The network topology for our case study, Example 1, is depicted in Figure 2. The NGO nodes consist of the American Red Cross and the Salvation Army, respectively. The demand points correspond to Springfield, Monson, and Brimfield, respectively.

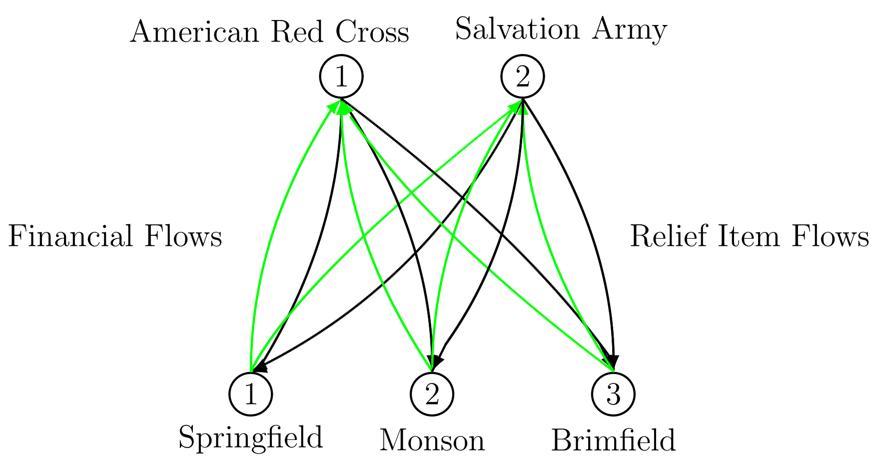

Figure 2: The Network Topology for the Case Study, Example 1

Example 1

The data for our case study, Example 1, are given below. The supplies of meals available for delivery to the victims are:

$$s_1 = 25,000, \quad s_2 = 25,000,$$

with the weights associated with the altruism benefit functions of the NGOs given by:

$$\omega_1 = 1, \quad \omega_2 = 1.$$

The financial funds functions are:

$$P_{11}(q) = 1000\sqrt{(3q_{11} + q_{21})}, \quad P_{12}(q) = 600\sqrt{(2q_{12} + q_{22})}, \quad P_{13}(q) = 400\sqrt{(2q_{13} + q_{23})},$$

$$P_{21}(q) = 800\sqrt{(4q_{21} + q_{11})}, \quad P_{22}(q) = 400\sqrt{(2q_{22} + q_{12})}, \quad P_{23}(q) = 200\sqrt{(2q_{23} + q_{13})}.$$

The altruism functions are:

$$B_1(q) = 300q_{11} + 200q_{12} + 100q_{13}, \quad B_2(q) = 400q_{21} + 300q_{22} + 200q_{23}.$$

The cost functions, which capture distance from the main storage depots in Springfield, are:

$$c_{11}(q) = .15q_{11}^2 + 2q_{11}, \quad c_{12}(q) = .15q_{12}^2 + 5q_{12}, \quad c_{13}(q) = .15q_{13}^2 + 7q_{13},$$

$$c_{21}(q) = .1q_{21}^2 + 2q_{21}, \quad c_{22}(q) = .1q_{22}^2 + 5q_{22}, \quad c_{23}(q) = .1q_{23}^2 + 7q_{23}.$$

The demand lower and upper bounds at the three demand points are:

$$\underline{d}_1 = 10000, \quad \bar{d}_1 = 20000,$$

$$\underline{d}_2 = 1000, \quad \bar{d}_2 = 10000,$$

$$\underline{d}_3 = 1000, \quad \bar{d}_3 = 10000.$$

The Euler method was implemented in FORTRAN and a Linux system at the University of Massachusetts Amherst was used for the computations. The algorithm was initialized as follows: all Lagrange multipliers were set to 0.00 and the initial relief item flows to a given demand point were set to the lower bound divided by the number of NGOs, which here is two.

The Euler method yielded the following Generalized Nash Equilibrium solution:

The equilibrium relief item flows are:

$$q_{11}^* = 3800.24, \quad q_{12}^* = 668.64, \quad q_{13}^* = 326.66,$$

$$q_{21}^* = 6199.59, \quad q_{22}^* = 1490.52, \quad q_{23}^* = 974.97.$$

Since none of the supplies are exhausted, the computed Lagrange multipliers associated with the supply constraints are:

$$\delta_1^* = 0.00, \quad \delta_2^* = 0.00.$$

Since the demand at the first demand point, which is the city of Springfield, is essentially at its lower bound, we have that:

$$\sigma_1^* = 835.22,$$

with

$$\sigma_2^* = 0.00, \quad \sigma_3^* = 0.00.$$

All the Lagrange multipliers associated with the demand upper bound constraints are equal to zero, that is:

$$\epsilon_1^* = \epsilon_2^* = \epsilon_3^* = 0.00.$$

In terms of the gain in financial donations, the NGOs receive the following amounts:

$$\sum_{j=1}^{3} P_{1j}(q^*) = 180,713.23, \quad \sum_{j=1}^{3} P_{2j}(q^*) = 168,996.78.$$

This is reasonable since the American Red Cross tends to have greater visibility post disasters than the Salvation Army through the media and that was the case post the Springfield tornadoes.

We then proceeded to solve the Nash equilibrium counterpart of the above Generalized Nash Equilibrium problem formulated as a variational equilibrium. The variational inequality for the Nash equilibrium is given in (13) and does not include the upper and lower bound demand constraints. We solved it using the Euler method but over the feasible set K as in (13).

The computed equilibrium relief item flows for the Nash equilibrium are:

$$q_{11}^* = 1040.22, \quad q_{12}^* = 668.64, \quad q_{13}^* = 326.66,$$

$$q_{21}^* = 2054.51, \quad q_{22}^* = 1490.52, \quad q_{23}^* = 974.97.$$

The Lagrange multipliers associated with the supply constraints are:

$$\delta_1^* = 0.00, \quad \delta_2^* = 0.00.$$

Observe that, without the imposition of the bounds on the demands, Springfield, which is demand point 1 and is a big city, receives only about one third of the volume of supplies (in this case, meals) as needed, and as determined by the Generalized Nash equilibrium solution.

The American Red Cross now garners financial donations of: 119,985.66, whereas the Salvation Army stands to receive financial donations equal to: 110,683.60. These values are significantly lower than the analogous ones for the Generalized Nash equilibrium model above. Hence, NGOs, by coordinating their deliveries of needed supplies, such as meals, can gain in terms of financial donations and attend to the victims' needs better by delivering in the amounts that have been estimated to be needed in terms of lower and upper bounds. This more general model, for which an optimization reformulation does not exist, in contrast to the model of Nagurney, Alvarez Flores, and Soylu (2016), nevertheless, supports the numerical result findings in the case study for Katrina therein.

Example 2

We now investigate the possible impact of the addition of a new disaster relief organization, such as a church-based one, or the Springfield Partners for Community Action, which also assisted in disaster relief, providing meals post the tornadoes. Hence, the network topology for case study, Example 2, is as in Figure 3. We refer to the added NGO as "Other." It is based in Springfield.

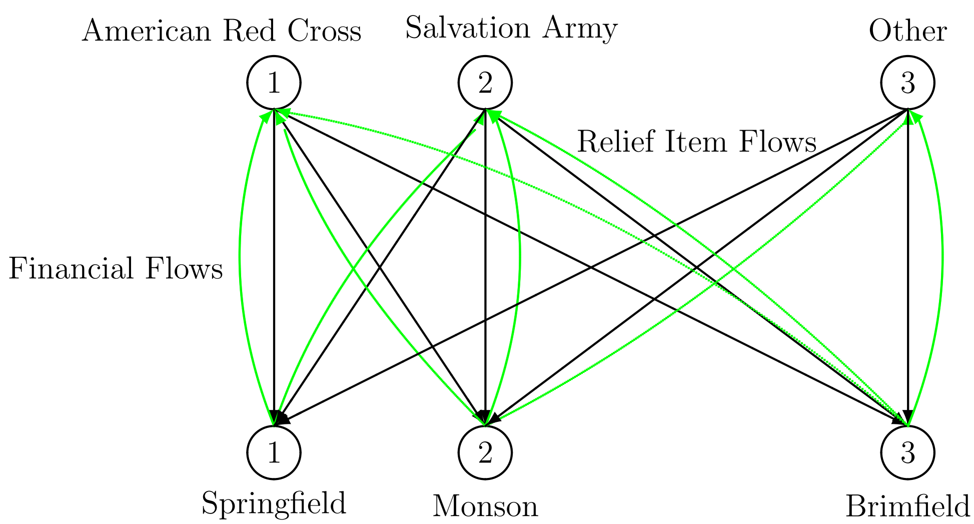

Figure 3: The Network Topology for the Case Study, Example 2

The data are as in Example 1 but with the original $P_{ij}(q)$ functions for the America Red Cross and the Salvation Army expanded as per below and the added data for the "Other" NGO also as given below.

The financial funds functions for are now:

$$P_{11}(q) = 1000\sqrt{(3q_{11} + q_{21} + q_{31})}, \quad P_{12}(q) = 600\sqrt{(2q_{12} + q_{22} + q_{32})}, \quad P_{13}(q) = 400\sqrt{(2q_{13} + q_{23} + q_{33})},$$

$$P_{21}(q) = 800\sqrt{(4q_{21} + q_{11} + q_{31})}, \quad P_{22}(q) = 400\sqrt{(2q_{22} + q_{12} + q_{32})}, \quad P_{23}(q) = 200\sqrt{(2q_{23} + q_{13} + q_{33})},$$

with those for the new NGO:

$$P_{31}(q) = 400\sqrt{(2q_{31} + q_{11} + q_{21})}, \quad P_{32}(q) = 200\sqrt{(2q_{32} + q_{12} + q_{22})}, \quad P_{33}(q) = 100\sqrt{(2q_{33} + q_{13} + q_{23})}.$$

The weight $\omega_3 = 1$ and the altruism/benefit function for the new NGO is:

$$B_3(q) = 200q_{31} + 100q_{32} + 100q_{33}.$$

The cost functions associated with the added NGO are:

$$c_{31}(q) = .1q_{31}^2 + q_{31}, \quad c_{32}(q) = .2q_{32}^2 + 5q_{32}, \quad c_{33}(q) = .2q_{33}^2 + 7q_{33}.$$

The Euler method converged to the following Generalized Nash Equilibrium solution:

The equilibrium relief item flows are:

$$q_{11}^* = 2506.97, \quad q_{12}^* = 667.85, \quad q_{13}^* = 325.59,$$

$$q_{21}^* = 4259.59, \quad q_{22}^* = 1489.98, \quad q_{23}^* = 974.45,$$

$$q_{31}^* = 3233.35, \quad q_{32}^* = 242.42, \quad q_{33}^* = 235.52.$$

Since none of the supplies are exhausted, the computed Lagrange multipliers associated with the supply constraints are:

$$\delta_1^* = 0.00, \quad \delta_2^* = 0.00, \quad \delta_3^* = 0.00.$$

The demand at the first demand point, which is the city of Springfield, is at the lower bound of 10000.00. Hence, we have that:

$$\sigma_1^* = 446.70,$$

with

$$\sigma_2^* = 0.00, \quad \sigma_3^* = 0.00.$$

All the Lagrange multipliers associated with the demand upper bound constraints are equal to zero, that is:

$$\epsilon_1^* = \epsilon_2^* = \epsilon_3^* = 0.00.$$

In terms of the gain in financial donations, the NGOs receive the following amounts:

$$\sum_{j=1}^{3} P_{1j}(q^*) = 173,021.70, \quad \sum_{j=1}^{3} P_{2j}(q^*) = 155,709.50, \quad \sum_{j=1}^{3} P_{3j}(q^*) = 60,504.14.$$

The volumes of relief items from the American Red Cross and the Salvation Army to Springfield are greatly reduced, as compared to the respective volumes in Example 1 and both original NGOs in Example 1 now experience a reduction in financial donations because of the increased competition for financial donations.

For completeness, we also solved the Nash equilibrium counterpart for Example 2.

The Nash equilibrium relief item flows are:

$$q_{11}^* = 1036.27, \quad q_{12}^* = 667.85, \quad q_{13}^* = 325.59,$$

$$q_{21}^* = 2051.17, \quad q_{22}^* = 1489.98, \quad q_{23}^* = 974.45,$$

$$q_{31}^* = 1009.61, \quad q_{32}^* = 242.42, \quad q_{33}^* = 235.52.$$

The financial donations of the NGOs are now the following:

$$\sum_{j=1}^{3} P_{1j}(q^*) = 129,037.42, \quad \sum_{j=1}^{3} P_{2j}(q^*) = 115,964.80, \quad \sum_{j=1}^{3} P_{3j}(q^*) = 43,07.16.$$

In Example 2 of our case study, we, again, see that the NGOs garner greater financial funds through the Generalized Nash Equilibrium solution, rather than the Nash equilibrium one. Moreover, the needs of the victims are met under the Generalized Nash Equilibrium solution.

5. Summary and Conclusions

In this paper, we constructed a new Generalized Nash Equilibrium (GNE) model for disaster relief, which contains both logistical as well as financial funds aspects. The NGOs compete for financial funds through their visibility in the response to a disaster and provide needed supplies to the victims. A coordinating body imposes upper bounds and lower bounds for the supplies at the various demand points to guarantee that the victims receive the amounts at the points of demand that are needed, and without excesses that can add to the congestion and materiel convergence. The model is more general than the one proposed earlier by Nagurney, Alvarez Flores, and Soylu (2016) and no longer is it possible to reformulate the governing equilibrium conditions as an optimization problem.

Here we use a variational equilibrium formulation of the Generalized Nash Equilibrium, which is then amenable to solution via variational inequality algorithms. We provide qualitative properties of the equilibrium pattern and also utilize Lagrange theory for the analysis of the NGOs' marginal utilities.

The proposed computational scheme yields closed form expressions, at each iteration, for the product flows and the Lagrange multipliers. The algorithm is then applied to a case study, inspired by rare tornadoes that caused devastation in parts of western and central Massachusetts in 2011. For completeness, we also compute the solution to the Nash equilibrium counterparts of the two examples making up the case study, in which the common demand bound constraints are removed. The case study reveals that victims may not receive the required amounts of supplies, without the imposition of the demand bounds. These results provide further support for the need for greater coordination in disaster relief. Moreover, by delivering the required amounts of supplies the NGOs can also garner greater financial donations.

Acknowledgments

This paper is dedicated to the memory of Professor Martin J. Beckmann, Professor Emeritus at Brown University, who passed away on April 11, 2017 at the age of 92. He was a renowned scholar in transportation science, regional science, and operations research, and his work on network equilibria have had a profound impact on both theory and practice.

The first author acknowledges support from the Radcliffe Institute for Advanced Study at Harvard University where she was a 2017 Summer Fellow. The second author acknowledges support from the project PON SCN_00451 CLARA - CLoud plAtform and smart underground imaging for natural Risk Assessment, Smart Cities and Communities and Social Innovation.

The authors acknowledge the co-organizers of the 3rd Dynamics of Disasters conference as well as the participants for a stimulating conference and discussions.

References

Balcik, B., Beamon, B., Krejci, C., Muramatsu, K.M., Ramirez, M., 2010. Coordination in humanitarian relief chains: Practices, challenges and opportunities. International Journal of Production Economics, 120(1), 22-34.

Barbagallo, A., Daniele, P., Maugeri, A., 2012. Variational formulation for a general dynamic nancial equilibrium problem: Balance law and liability formula. Nonlinear Analysis, Theory, Methods and Applications, 75(3), 1104-1123.

Bensoussan, A., 1974. Points de Nash dans le cas de fonctionelles quadratiques et jeux differentiels lineaires a N personnes. SIAM Journal on Control, 12, 460-499.

Caruso, V., Daniele, P., 2016. A network model for minimizing the total organ transplant costs. Department of Mathematics and Computer Science, University of Catania, Italy.

Chankong, V., Haimes, Y.Y., 1983. Multiobjective Decision Making: Theory and Methodology, North-Holland, New York.

Coles, J.B., and Zhuang, J., 211. Decisions in disaster recovery operations: A game theoretic perspective on organization cooperation. Journal of Homeland Security and Emergency Management, 8(1), Article 35.

Cox, C.W., 2008. Manmade disasters: A historical review of terrorism and implications for the future. Online Journal of Nursing, 13(1), January 8. Available at:
http://www.nursingworld.org/MainMenuCategories/ANAMarketplace/ANAPeriodicals/OJIN/TableofContents/vol132008/No1Jan08/ArticlePreviousTopic/ManmadeDisasters.html

Daniele, P., 2001. Variational inequalities for static equilibrium market. In: Lagrangean Function and Duality, in Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models. F. Giannessi, A. Maugeri, and P.M. Pardalos Editors, Kluwer Academic Publishers, Norwell, Massachusetts, pp. 43-58.

Daniele, P., Giuffrè, S., 2015. Random variational inequalities and the random traffic equilibrium problem. Journal of Optimization Theory and Applications, 167(1), 363-381.

Debreu, G., 1952. A social equilibrium existence theorem. Proceedings of the National Academy of Sciences of the United States of America, 38, 886-893.

Dupuis, P., Nagurney, A., 1993. Dynamical systems and variational inequalities. Annals of Operations Research, 44, 9-42.

Facchinei, F., Kanzow, C., 2010. Generalized Nash equilibrium problems. Annals of Operations Research, 175(1), 177-211.

Fischer, A., Herrich, M., Schonefeld, K., 2014. Generalized Nash equilibrium problems - Recent advances and challenges. Pesquisa Operacional, 34(3), 521-558.

Fishburn, P.C., 1970. Utility Theory for Decision Making. John Wiley & Son, New York.

Flynn, J., 2011. Tornado tears through Pioneer Valley, killing two, damaging homes in 9

communities and causing widespread power outages. masslive.com, June 2, 2011; available at:
http://www.masslive.com/news/index.ssf/2011/06/tornado_tears_through_pioneer.html

Gabay, D., Moulin, H., 1980. On the uniqueness and stability of Nash equilibria in noncooperative games. In: Applied Stochastic Control of Econometrics and Management Science, A. Bensoussan, P. Kleindorfer, C.S. Tapiero, Editors, North-Holland, Amsterdam, The Netherlands, pp. 271-294.

Gatignon, A., Van Wassenhove, L., Charles, A., 2010. The Yogyakarta earthquake: Humanitarian relief through IFRC's decentralized supply chain. International Journal of Production Economics, 126(1), 102-110.

Keeney, R.L., Raiffa, H., 1993. Decisions with Multiple Objectives: Preferences and Value Tradeoffs. Cambridge University Press, Cambridge, England.

Kinderlehrer, D., Stampacchia, G., 1980. Variational Inequalities and Their Applications. Academic Press, New York.

Kopinak, J.K., 2013. Humanitarian assistance: Are effective and sustainability impossible dreams? Journal of Humanitarian Assistance, March 10.

Kulkarni, A.A., Shanbhag, U.V., 2012. On the variational equilibrium as a refinement of the generalized Nash equilibrium. Automatica, 48, 45-55.

Luna, J.P., 2013. Decomposition and Approximation Methods for Variational Inequalities, with Applications to Deterministic and Stochastic Energy Markets. PhD thesis, Instituto Nacional de Matematica Pura e Aplicada, Rio de Janeiro, Brazil.

Malani, A., Philipson, T., David, G., 2003. Theories of firm behavior in the nonprofit sector. A synthesis and empirical evaluation. In: The Governance of Not-for-Profit Organizations. E.L. Glaeser, Editor, University of Chicago Press, Chicago, Illinois, pp. 181-215.

Muggy, L., 2015. Quantifying and Mitigating Decentralized Decision Making in Humanitarian Logistics Systems. PhD Dissertation, Kansas State University, Manhattan, Kansas.

Muggy L., Heier Stamm, J.L., 2014. Game theory applications in humanitarian operations: A review, Journal of Humanitarian Logistics and Supply Chain Management, 4(1), 4-23.

Nagurney, A., 1999. Network Economics: A Variational Inequality Approach, second and revised edition. Kluwer Academic Publishers, Dordrecht, The Netherlands.

Nagurney, A., 2006. Supply Chain Network Economics: Dynamics of Prices, Flows, and Profits. Edward Elgar Publishing. Cheltenham, England.

Nagurney, A., 2016. Freight service provision for disaster relief: A competitive network model with computations. In: Dynamics of Disasters: Key Concepts, Models, Algorithms, and Insights, I.S. Kotsireas, A. Nagurney, and P.M. Pardalos, Editors, Springer International Publishing Switzerland, pp. 207-229.

Nagurney, A., Alvarez Flores, E., Soylu, C., 2016. A Generalized Nash Equilibrium model for post-disaster humanitarian relief. Transportation Research E, 95, 1-18.

Nagurney, A., Dutta, P., 2016. Competition for blood donations: A Nash equilibrium network framework. Isenberg School of Management, University of Massachusetts Amherst.

Nagurney, A., Qiang, Q., 2009. Fragile Networks: Identifying Vulnerabilities and Synergies in an Uncertain World. John Wiley & Sons, Hoboken, New Jersey.

Nagurney, A., Yu, M., Besik, D., 2017. Supply chain network capacity competition with outsourcing: A variational equilibrium framework. In press in the Journal of Global Optimization.

Nagurney, A., Zhang, D., 1996. Projected Dynamical Systems and Variational Inequalities with Applications. Kluwer Academic Publishers, Boston, Massachusetts.

Nash, J.F., 1950. Equilibrium points in n-person games. Proceedings of the National Academy of Sciences, USA, 36, 48-49.

Nash, J.F., 1951. Noncooperative games. Annals of Mathematics, 54, 286-298.

Natsios, A.S., 1995. NGOs and the UN system in complex humanitarian emergencies: conflict or cooperation?. Third World Quarterly, 16(3), 405419.

Roopanarine, L., 2013. How pre-positioning can make emergency relief more effective. The Guardian, January 17.

Rose-Ackerman, S., 1982. Charitable giving and “excessive” fundraising. The Quarterly Journal of Economics, 97(2), 193-212

Sumpf, D., Isaila, V., Najjar, K., 2016. The impact of the Syria crisis on Lebanon. In: Dynamics of Disasters: Key Concepts, Models, Algorithms, and Insights, I.S. Kotsireas, A. Nagurney, and P.M. Pardalos, Editors, Springer International Publishing Switzerland, pp. 269-308.

Toyasaki, F., Wakolbinger, T., 2014. Impacts of earmarked private donations for disaster fundraising. Annals of Operations Research, 221, 427-447.

Toyasaki, F., Daniele, P., Wakolbinger, T., 2014. A variational inequality formulation of equilibrium models for end-of-life products with nonlinear constraints. European Journal of Operational Research, 236(1), 340-350.

Van Wassenhove, L.N., 2006. Blackett memorial lecture. Humanitarian aid logistics: Supply chain management in high gear. Journal of the Operational Research Society, 57(5), 475-489.

von Heusinger, A., 2009. Numerical Methods for the Solution of the Generalized Nash Equilibrium Problem. PhD Dissertation, University of Wurtburg, Germany.

Watson, C., Caravani, A., Mitchell, T., Kellett, J., Peters, K., 2015. 10 things to know about finance for reducing disaster risk, Overseas Development Institute, London, England.

Western Massachusetts Regional Homeland Security Advisory Council, 2012. June 1, 2011 tornado response: After action report and improvement plan. Available at: http://www.wrhsac.org/frcog/Mass%20Tornado%20AAR%20IP%20Jan%202012.pdf

Yu, P.L., 1985. Multiple Criteria Decision Making Concepts, Techniques, and Extensions. Plenum Press, New York.

Zhuang, J., Saxton, G., Wu, H., 2014. Publicity vs. impact in nonprofit disclosures and donor preferences: A sequential game with one nonprofit organization and N donors. Annals of Operations Research, 221(1), 469-491.