Supply Chains and Transportation Networks

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Abstract: We overview some of the major advances in supply chains and transportation networks, with a focus on their common theoretical frameworks and underlying behavioral principles. We emphasize that the foundations of supply chains as network systems can be found in the regional science and spatial economics literature. In addition, transportation network concepts, models, and accompanying methodologies have enabled the advancement of supply chain network models from a system-wide and holistic perspective.

We discuss how the concepts of system-optimization and user-optimization have underpinned transportation network models and how they have evolved to enable the formulation of supply chain network problems operating (and managed) under centralized or decentralized, that is, competitive, decision-making behavior.

We highlighted some of the principal methodologies, including variational inequality theory, that have enabled the development of advanced transportation network equilibrium models as well as supply chain network equilibrium models.

1. Introduction

Supply chains are networks of suppliers, manufacturers, transportation service providers, storage facility managers, retailers, and consumers at the demand markets. Supply chains are the backbones of our globalized Network Economy and provide the infrastructure for the production, storage, and distribution of goods and associated services as varied as food products, pharmaceuticals, vehicles, computers and other high tech equipment, building materials, furniture, clothing, toys, and even electricity.

Supply chains may operate (and be managed) in a centralized or decentralized manner and be underpinned not only by multimodal transportation and logistical networks but also by telecommunication as well as financial networks. In a centralized supply chain, there is a central entity or decision-maker, such as a firm, that controls the various supply chain network activities whereas in a decentralized supply chain, there are multiple economic decision-makers and the governing paradigm is that of competitive behavior among the relevant stakeholders, with different degrees of cooperation. For example, in a vertically integrated supply chain the same firm may be responsible for production, storage, and distribution of its products. On the other hand, certain industry supply chain network structures may consist of competitive manufacturers, competitive distributors, as well as competing retailers. Nevertheless, the stakeholders involved in supply chains must cooperate to the extent that the products be received and processed as they move downstream in the supply chain (Nagurney (2006)).

The complexity and interconnectivity of some of today's product supply chains have been vividly illustrated through the effects of recent natural disasters, including earthquakes, tsunamis, and even hurricanes, which have severed critical nodes and/or links and have disrupted the production and transportation of products, with major economic implications. Indeed, when supply chain disruptions occur, whether due to natural disasters, human error, attacks, or even market failure, the ramifications can propagate and impact the health and well-being of the citizenry thousands of miles away from the initially affected location (cf. Nagurney and Qiang (2009)).

Since supply chains are network systems, any formalism that seeks to model supply chains and to provide quantifiable insights and measures must be a system-wide one and network-based. Such crucial issues as the stability and resiliency of supply chains, as well as their adaptability and responsiveness to events in a global environment of increasing risk and uncertainty, can only be rigorously examined from the view of supply chains as network systems (Nagurney (2006)).

Supply chains share many of the same characteristics as other network systems; including a large-scale nature and complexity of network topology; congestion, which leads to nonlinearities; alternative behavior of users of the networks, which may lead to paradoxical phenomena (recall the well-known Braess paradox in which the addition of a new road may increase the travel time for all); possibly conflicting criteria associated with optimization (the minimization of time for delivery, for example, may result in higher emissions); interactions among the underlying networks themselves, such as the Internet with electric power networks, financial networks, and transportation and logistical networks, and the growing recognition of their fragility and vulnerability. Moreover, policies surrounding supply chain networks today may have major implications not only economically, but also socially, politically, and security-wise.

Although, historically, supply chain activities of manufacturing, transportation / distribution, as well as inventorying / storage have each, independently, received a lot of attention from both researchers and practitioners, the framework of supply chains views the various activities of production, transportation, and consumption in an integrated, holistic manner. Indeed, without the critical transportation links what is manufactured cannot be delivered to points of demand. Moreover, needed inputs into the production processes / manufacturing links cannot be secured.

While, beginning in the 1980s (cf. Handfield and Nichols, Jr. (1999)), supply chains have captured wide interest among practitioners as well as researchers, it may be argued that the foundations of supply chain networks can be found in regional science and spatial economics, dating to the classical spatial price equilibrium models of Samuelson (1952) and Takayama and Judge (1971) with additional insights as to production processes, transportation and distribution provided by Beckmann, McGuire, and Winsten (1956). For example, in spatial price equilibrium models not only is production of the commodity in question considered at multiple locations or supply markets, with appropriate underlying functions, but also the consumption of the commodity at the demand markets, subject to appropriate functions (either demand or demand price) as well as the cost associated with transporting the commodity between pairs of the spatially separated supply and demand markets. Spatial price equilibrium models have evolved to include multiple commodities, multiple modes of transportation, and may even include general underlying transportation networks. Moreover, with advances in theoretical frameworks, including, for example, the theory of variational inequalities (Nagurney (1999)), one can now formulate and solve complex spatial price equilibrium problems with asymmetric supply price, demand price, and unit transportation/transaction cost functions (for which an optimization reformulation of the governing spatial price equilibrium conditions does not hold).

In addition, versions of spatial equilibrium models that capture oligopolistic behavior under imperfect, as opposed to perfect, competition serve as some of the basic supply chain network models in which competition is included but, at the same time, the important demand/consumption side is also captured (see Nagurney (1999) and the references therein).

Interestingly, spatial price equilibrium problems can be reformulated and solved as transportation network equilibrium problems with elastic demands over appropriately constructed abstract networks or supernetworks (see Nagurney and Dong (2002)). Hence, the plethora of algorithms that have been developed for transportation networks (cf. Sheffi (1985), Patriksson (1994), Nagurney (1999), and Ran and Boyce (1996)) can also be applied to compute solutions to spatial price equilibrium problems. It is worth noting that Beckmann, McGuire, and Winsten in their classical 1956 book, *Studies in the Economics of Transportation*, formulated transportation network equilibrium problems with elastic demands. They proved that, under the assumed user link cost functional forms and the travel disutility functional forms associated with the origin/destination pairs of nodes that the governing equilibrium conditions (now known as user-optimized conditions) in which no traveler has any incentive to alter his route of travel, given that the behavior of others is fixed, could be reformulated and solved as an associated optimization problem. In their book, they also hypothesized that electric power generation and distribution networks, or in today's terminology, electric power supply chains, could be transformed into transportation network equilibrium problems. This has now been established (cf. Nagurney (2006) and the references therein).

Today, the behavior of travelers on transportation networks is assumed to follow one of Wardrop's (1952) two principles of travel behavior, now renamed, according to Dafermos and Sparrow (1969), as user-optimized (selfish or decentralized) or system-optimized (unselfish or centralized). The former concept captures individuals' route-taking decision-making behavior, whereas the latter assumes a central controller that routes the flow on the network so as to minimize the total cost.

Moreover, a plethora of supply chain network equilibrium models, originated by Nagurney, Dong, and Zhang (2002), have been developed in order to address competition among decision-makers in a tier of a supply chain whether among the manufacturers, the distributors, the retailers, and/or even the consumers at the demand markets. Such models capture the behavior of the individual economic decision-makers, as in the case, for example, of profit maximization and acknowledge that consumers also take transaction/transportation costs into consideration in making their purchasing decisions. Prices for the product associated with each decision-maker at each tier are obtained once the entire supply chain network equilibrium problem is solved, yielding also the equilibrium flows of the product on the links of the supply chain network. Such supply chain network equilibrium models also possess (as spatial price equilibrium problems highlighted above) a transportation network equilibrium reformulation.

Supply chain network models have been generalized to include electronic commerce options, multiple products, as well as risk and uncertainty, on the demand-side as well as on the supply-side (cf. Nagurney (2006) and the referenced therein). In addition, and, this is product-specific, supply chain network models have also been constructed to handle time-sensitive products (fast fashion, holiday-based, and even critical needs as in disasters) as well as perishable products (such as food, cut flowers, certain vaccines and medicines, etc.) using multicriteria decision-making formalisms for the former and generalized networks for the latter (see Masoumi, Yu, and Nagurney (2012)). Both static as well as dynamic supply chain network models, including multiperiod ones with inventorying have been formulated, solved, and applied.

It is important to note that not all supply chains are commercial and, in fact, given that the number of disasters is growing, as is the number of people affected by them, humanitarian supply chains have emerged as essential elements in disaster recovery. Unlike commercial or corporate supply chains, humanitarian supply chains are not managed using profit maximization as a decision-making criterion (since donors, for example, would not approve) but rather cost minimization subject to demand satisfaction under uncertainty is relevant (see Nagurney and Qiang (2009)). In addition, such supply chains may need to be constructed quickly and with the cognizant decision-makers working under conditions of damaged, if not destroyed, infrastructure, and limited information.

Supply chain decision-making occurs at different levels – at the strategic, tactical, and operational levels. Strategic decisions may involve where to locate manufacturing facilities and distribution centers, whereas tactical decisions may include with which suppliers to partner and which transportation service providers (carriers) to use. Decisions associated with operational supply chain decision-making would involve how much of the product to produce at which manufacturing plants, which storage facilities to use and how much to store where, as well as how much of the product should be supplied to the different retailers or points of demand. In addition, because of globalization, supply chain decision-making may now involve outsourcing decisions as well as the accompanying risk management.

Today, it has been argued that, increasingly, in the Network Economy it is not only competition within a product supply chain that is taking place but, rather, supply chain versus supply chain competition. Zhang, Dong, and Nagurney (2003) generalized Wardrop's first principle of travel behavior to formulate competition among supply chains.

Location-based decisions are fundamental to supply chain decision-making, design, and management. Furthermore, such decisions affect spatial competition as well as trade, with Ohlin (1933) and Isard (1954) noting the need to integrate industrial location and international trade in a common framework.

Nagurney (2010) constructed a system-optimization model that can be applied to the design or redesign of a supply chain network and has as endogenous variables both the capacities associated with the links (corresponding to manufacturing, transportation, and storage), as well as the operational flows of the product in order to meet the demands. The model has been extended in various directions to handle oligopolistic competition as well as product perishability in specific applications (cf. Masoumi, Yu, and Nagurney (2012) and the references therein).

At the same time that supply chains have become increasingly globalized, environmental concerns due to global warming and associated risks have drawn the attention of numerous constituencies. Firms are increasingly being held accountable not only for their own performance in terms of their environmental performance, but also for that of their suppliers, subcontractors, joint venture partners, distribution outlets and, ultimately, even for the disposal of their products. Consequently, poor environmental performance at any stage of the supply chain may damage the most important asset that a company has, which is its reputation. Hence, the topic of sustainable supply chain network modeling and analysis has emerged as an essential area for research, practice, as well as for policy analysis (see Boone, Jayaraman, and Ganeshan (2012)).

2. Fundamental Decision-Making Concepts and Models

In this section of this essay, we interweave fundamental concepts in transportation that have been used successfully and with wide application in supply chain network modeling, analysis, operations management, and design. Our goal is to provide the necessary background from which additional explorations and advances can be made using a readable and accessible format.

As noted in the Introduction, over half a century ago, Wardrop (1952) considered alternative possible behaviors of users of transportation networks, notably, urban transportation networks, and stated two principles, which are named after him:

First Principle: The journey times of all routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

Second Principle: The average journey time is minimal.

The first principle corresponds to the behavioral principle in which travelers seek to (unilaterally) determine their minimal costs of travel; the second principle corresponds to the behavioral principle in which the total cost in the network is minimal.

Beckmann, McGuire, and Winsten (1956) were the first to rigorously formulate these conditions mathematically and proved the equivalence between the *transportation network equilibrium* conditions, which state that all used paths connecting an origin/destination (O/D) pair will have equal and minimal travel times (or costs) (corresponding to Wardrop's first principle), and the Kuhn-Tucker conditions of an appropriately constructed optimization problem, under a symmetry assumption on the underlying functions. Hence, in this case, the equilibrium link and path flows could be obtained as the solution of a mathematical programming problem. Their fundamental result made the formulation, analysis, and subsequent computation of solutions to transportation network problems based on actual transportation networks realizable.

Dafermos and Sparrow (1969) coined the terms *user-optimized* (U-O) and *system-optimized* (S-O) transportation networks to distinguish between two distinct situations in which, respectively, travelers act unilaterally, in their own self-interest, in selecting their routes, and

Table 1: Distinct Behavior on Transportation Networks

User-Optimization	**System-Optimization**
$\Downarrow$	$\Downarrow$
User Equilibrium Principle:	**System Optimality Principle:**
User travel costs on used paths for each O/D pair are equalized and minimal.	Marginals of the total travel cost on used paths for each O/D pair are equalized and minimal.

in which travelers choose routes/paths according to what is optimal from a societal point of view, in that the total cost in the network system is minimized. In the latter problem, marginal total costs rather than average costs are equilibrated. As noted in the Introduction, the former problem coincides with Wardrop's first principle, and the latter with Wardrop's second principle. Table 1 highlights the two distinct behavioral principles underlying transportation networks.

The concept of "system-optimization" is also relevant to other types of "routing models" in transportation, including those concerned with the routing of freight. Dafermos and Sparrow (1969) also provided explicit computational procedures, that is, *algorithms*, to compute the solutions to such network problems in the case where the user travel cost on a link was an increasing (in order to handle congestion) function of the flow on the particular link, and linear. Today, the concepts of user-optimization versus system-optimization also capture, respectively, decentralized versus centralized decision-making on supply chain networks after the proper identifications are made (Boyce, Mahmassani, and Nagurney (2005) and Nagurney (2006)).

2.1 User-Optimization versus System-Optimization

In this section, the basic transportation network models are first recalled, under distinct assumptions as to their operation and the underlying behavior of the users of the network. The models are classical and are due to Beckmann, McGuire, and Winsten (1956) and Dafermos and Sparrow (1969). In subsequent sections, we present more general models in which the user link cost functions are no longer separable but, rather, are asymmetric.

For such models we also provide the variational inequality formulations of the governing equilibrium conditions, since, in such cases, the governing equilibrium conditions can no longer be reformulated as the Kuhn-Tucker conditions of a convex optimization problem. The presentation follows that in Nagurney (2007) with addition of material on supply chains with synthesis.

For easy accessibility, we recall the classical user-optimized network model in Section 2.1.1 and then the classical system-optimized network model in Section 2.1.2. The Braess (1968) paradox is, subsequently, highlighted in Section 2.1.3.

2.1.1 The User-Optimized Problem

The user-optimized network problem is also commonly referred to in the transportation literature as the *traffic assignment* problem or the *traffic network equilibrium* problem.

Consider a general network $\mathcal{G} = [\mathcal{N}, \mathcal{L}]$, where $\mathcal{N}$ denotes the set of nodes, and $\mathcal{L}$ the set of directed links. Links connect pairs of nodes in the network and are denoted by a, b, etc. Let p denote a path consisting of a sequence of links connecting an origin/destination (O/D) pair of nodes. Paths are assumed to be acyclic and are denoted by p, q, etc. In transportation networks, nodes correspond to origins and destinations, as well as to intersections. Links, on the other hand, correspond to roads/streets in the case of urban transportation networks and to railroad segments in the case of train networks. A path in its most basic setting, thus, is a sequence of "roads" which comprise a route from an origin to a destination. In the supply chain network context, links correspond to supply chain activities (with appropriate associated cost functions) and represent manufacturing, transportation/shipment, storage, etc. In addition, links can correspond to outsourcing links (see Nagurney (2006)).

Here we consider *paths*, rather than *routes*, since the former subsumes the latter. The network concepts presented here are sufficiently general to abstract not only transportation decision-making but also combined/integrated location-transportation decision-making as well as a spectrum of supply chain decisions. In addition, in the setting of *supernetworks*, that is, abstract networks, in which nodes need to correspond to locations in space (see Nagurney and Dong (2002)), a path is viewed more broadly and need not be limited to a route-type decision but may, in fact, correspond to not only transportation but also to manufacturing and inventorying/storage decision-making.

Let P_ω denote the set of paths connecting the origin/destination (O/D) pair of nodes ω. Let P denote the set of all paths in the network and assume that there are J origin/destination pairs of nodes in the set Ω. Let x_p represent the nonnegative flow on path p and let f_a denote the flow on link a. All vectors here are assume to be column vectors. The path flows on the network are grouped into the vector $x \in R^{n_P}_+$, where n_P denotes the number of paths in the network. The link flows, in turn, are grouped into the vector $f \in R^{n_L}_+$, where n_L denotes the number of links in the network.

Assume, as given, the demand associated with each O/D pair ω, which is denoted by d_ω, for $\omega \in \Omega$. In the network, the following conservation of flow equations must hold:

$$d_\omega = \sum_{p \in P_\omega} x_p, \quad \forall \omega \in \Omega, \tag{1}$$

where $x_p \geq 0$, $\forall p \in P$; that is, the sum of all the path flows between an origin/destination pair ω must be equal to the given demand d_ω.

In addition, the following conservation of flow equations must also hold:

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in \mathcal{L}, \tag{2}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. Expression (2) states that the flow on a link a is equal to the sum of all the path flows on paths p that contain (traverse) link a.

Equations (1) and (2) guarantee that the flows in the network (be they travelers, products, etc.) are conserved, that is, do not disappear (or are lost) in the network and arrive at the designated destinations from the origins.

Let c_a denote the user link cost associated with traversing link a, and let C_p denote the user cost associated with traversing the path p. Assume that the user link cost function is given by the *separable* function in which the cost on a link depends only on the flow on the link, that is,

$$c_a = c_a(f_a), \quad \forall a \in \mathcal{L}, \tag{3}$$

where c_a is assumed to be continuous and an increasing function of the link flow f_a in order to model the effect of the link flow on the cost and, in particular, congestion.

The cost on a path is equal to the sum of the costs on the links that make up that path, that is,

$$C_p = \sum_{a \in \mathcal{L}} c_a(f_a)\delta_{ap}, \quad \forall p \in P. \tag{4}$$

Transportation Network Equilibrium Conditions

In the case of the user-optimization (U-O) problem one seeks to determine the path flow pattern x^* (and the corresponding link flow pattern f^*) which satisfies the conservation of flow equations (1) and (2), and the nonnegativity assumption on the path flows, and which also satisfies the transportation network equilibrium conditions given by the following statement. For each O/D pair $\omega \in \Omega$ and each path $p \in P_\omega$:

$$C_p \begin{cases} = \lambda_\omega, & \text{if} \quad x_p^* > 0 \\ \geq \lambda_\omega, & \text{if} \quad x_p^* = 0. \end{cases} \tag{5}$$

In the user-optimization problem there is no explicit optimization criterion, since users of the transportation network system act independently, in a non-cooperative manner, until they cannot improve on their situations unilaterally and, thus, an equilibrium is achieved, governed by the above equilibrium conditions. Conditions (5) are simply a restatement of Wardrop's (1952) first principle mathematically and mean that only those paths connecting an O/D pair will be used which have equal and minimal user costs. In (5) the minimal cost for O/D pair ω is denoted by λ_ω and its value is obtained once the equilibrium flow pattern is determined. Otherwise, a user of the network could improve upon his situation by switching to a path with lower cost.

Beckmann, McGuire, and Winsten (1956) established that the solution to the network equilibrium problem, in the case of user link cost functions of the form (3), in which the cost on a link only depends on the flow on that link and is assumed to be continuous and an increasing function of the flow, could be obtained by solving the following optimization problem:

$$\text{Minimize} \quad \sum_{a\in\mathcal{L}} \int_0^{f_a} c_a(y)dy \tag{6}$$

subject to:

$$\sum_{p\in P_\omega} x_p = d_\omega, \quad \forall \omega \in \Omega, \tag{7}$$

$$f_a = \sum_{p\in P} x_p \delta_{ap}, \quad \forall a \in \mathcal{L}, \tag{8}$$

$$x_p \geq 0, \quad \forall p \in P. \tag{9}$$

The objective function given by (6) is simply a device constructed to obtain a solution using general purpose convex programming algorithms. It does not possess the economic meaning of the objective function encountered in the system-optimization problem which will be recalled below. Note that in the case of separable, as well as nonseparable, but symmetric (which we come back to later), user link cost functions, the λ_ω term in (5) corresponds to the Lagrange multiplier associated with the constraint (7) for that O/D pair ω. However, in the case of nonseparable and asymmetric functions there is no optimization reformulation of the transportation network equilibrium conditions (5) and the λ_ω term simply reflects the minimum user cost associated with the O/D pair ω at the equilibrium. As noted as early as Dafermos and Sparrow (1969), the above network equilibrium conditions also correspond to a Nash equilibrium (see Nash (1951)). The equilibrium link flow pattern is unique for problem (6), subject to (7) – (9), if the objective function (6) is strictly convex (for additional background on optimization theory.

It has also been established (cf. Nagurney (2006) and the references therein) that multitiered supply chain network problems in which decision-makers (manufacturers, retailers, and even consumers) compete across a tier of the supply chain network but cooperate between tiers, as depicted in Figure 1, could be transformed into a transportation network equilibrium problem using a supernetwork transformation, as in Figure 2. In Figure 2, the activities of manufacturing and retailer handling/storage are associated with the top-most and the third sets of links, respectively. The second and fourth sets of links from the top in Figure 2 are the transportation links (as is the case with the links in Figure 1). This connection provides us with a path flow efficiency interpretation of supply chain network equilibria. She utilized variational inequality theory (see below) to establish the equivalence.

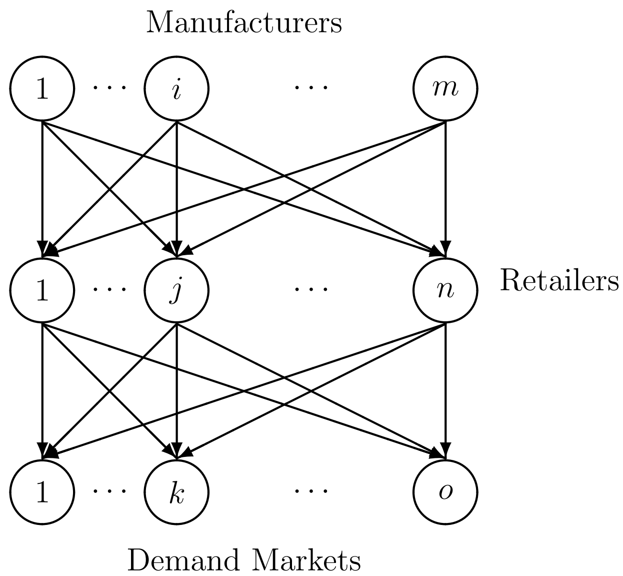

Figure 1: The Multitiered Network Structure of the Supply Chain

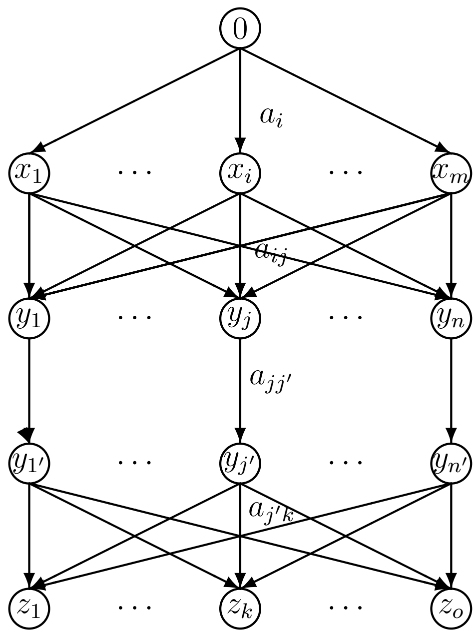

Figure 2: The Supernetwork Representation of Supply Chain Network Equilibrium

2.1.2 The System-Optimized Problem

We now recall the system-optimized problem. As in the user-optimized problem of Section 2.1.1, the network $\mathcal{G} = [\mathcal{N}, \mathcal{L}]$, the demands associated with the origin/destination pairs, and the user link cost functions are assumed as given. In the system-optimized problem, there is a central controller who routes the flows in an optimal manner so as to minimize the total cost in the network. This problem has direct relevance to the management of operations of a supply chain.

The total cost on link a, denoted by $\hat{c}_a(f_a)$, is given by:

$$\hat{c}_a(f_a) = c_a(f_a) \times f_a, \quad \forall a \in \mathcal{L}, \tag{10}$$

that is, the total cost on a link is equal to the user link cost on the link times the flow on the link. As noted earlier, in the system-optimized problem, there exists a central controller who seeks to minimize the total cost in the network system, which can correspond to a supply chain, where the total cost is expressed as

$$\sum_{a \in \mathcal{L}} \hat{c}_a(f_a), \tag{11}$$

and the total cost on a link is given by expression (10).

The system-optimization (S-O) problem is, thus, given by:

$$\text{Minimize} \quad \sum_{a \in \mathcal{L}} \hat{c}_a(f_a) \tag{12}$$

subject to the same conservation of flow equations as for the user-optimized problem, as well as the nonnegativity assumption of the path flows; that is, constraints (7), (8), and (9) must also be satisfied for the system-optimized problem.

The total cost on a path, denoted by $\hat{C}_p$, is the user cost on a path times the flow on a path, that is,

$$\hat{C}_p = C_p x_p, \quad \forall p \in P, \tag{13}$$

where the user cost on a path, C_p, is given by the sum of the user costs on the links that comprise the path (as in (4)), that is,

$$C_p = \sum_{a \in \mathcal{L}} c_a(f_a)\delta_{ap}, \quad \forall a \in \mathcal{L}. \tag{14}$$

In view of (2), (3), and (4), one may express the cost on a path p as a function of the path flow variables and, hence, an alternative version of the above system-optimization problem with objective function (12) can be stated in path flow variables only, where one has now the problem:

$$\text{Minimize} \quad \sum_{p \in P} C_p(x) x_p \tag{15}$$

subject to constraints (7) and (9).

System-Optimality Conditions

Under the assumption of increasing user link cost functions, the objective function (12) in the S-O problem is convex, and the feasible set consisting of the linear constraints (7) – (9) is also convex. Therefore, the optimality conditions, that is, the Kuhn-Tucker conditions are: for each O/D pair $\omega \in \Omega$, and each path $p \in P_\omega$, the flow pattern x (and corresponding link flow pattern f), satisfying (7)–(9) must satisfy:

$$\hat{C}'_p \begin{cases} = \mu_\omega, & \text{if} \quad x_p > 0 \\ \geq \mu_\omega, & \text{if} \quad x_p = 0, \end{cases} \tag{16}$$

where $\hat{C}'_p$ denotes the marginal of the total cost on path p, given by:

$$\hat{C}'_p = \sum_{a \in \mathcal{L}} \frac{\partial \hat{c}_a(f_a)}{\partial f_a} \delta_{ap}, \tag{17}$$

evaluated in (16) at the solution and μ_ω is the Lagrange multiplier associated with constraint (7) for that O/D pair ω.

The system-optimization approach has been applied to supply chain networks in order to assess synergy associated with a possible merger or acquisition before such a decision, which may be very costly, is made. Nagurney and Qiang (2009) overview such an approach, which assesses the total cost prior to the merger and post.

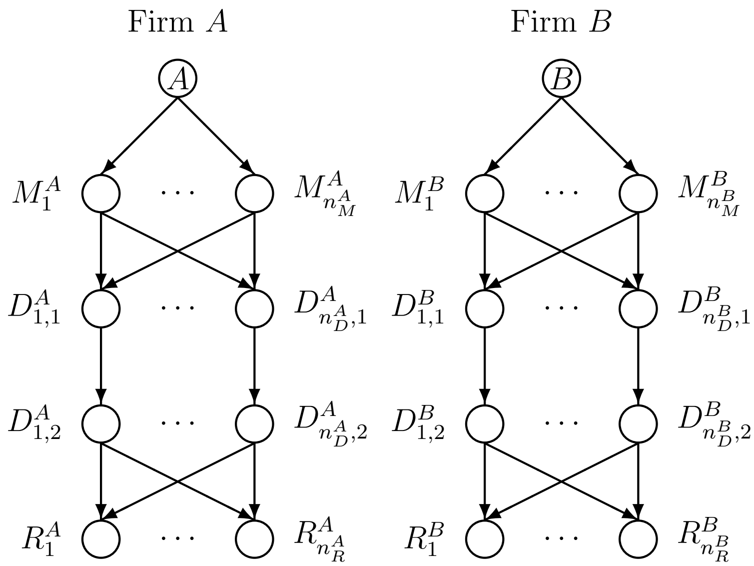

Figure 3: Case 0: Firms A and B Pre-Merger

The pre-merger supply chains corresponding to the individual firms, prior to the merger, are depicted in Figure 3, whereas the post-merger supply chain network is given in Figure 4. The top-most links correspond to the manufacturing links in Figure 3, followed by the transportation links ending in the storage/distribution facility links, followed by additional shipment links to the demand markets. In Figure 4, on the other hand, the topmost links represent the merger/acquisition with appropriate total cost functions assigned to those links.

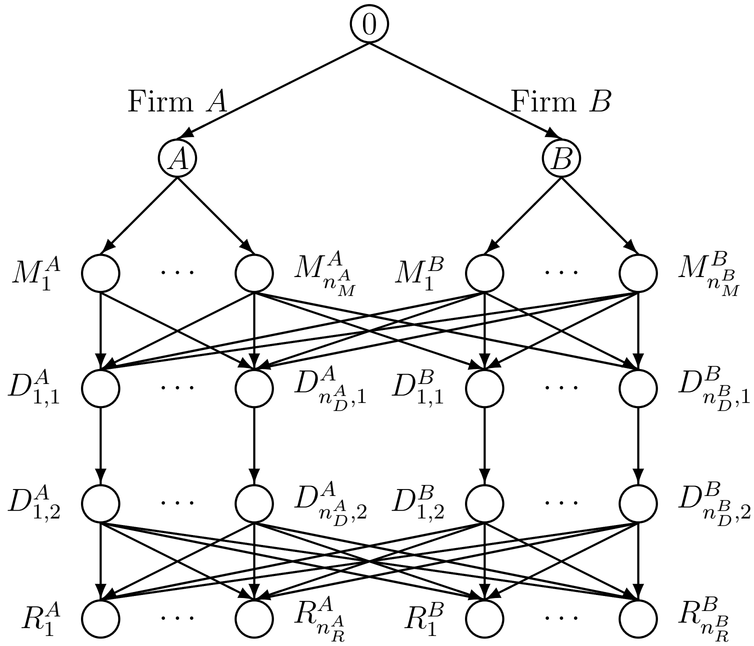

Figure 4: Post-Merger Network

2.1.3 The Braess Paradox

In order to illustrate the difference between user-optimization and system-optimization in a concrete example, and to reinforce the above concepts, we now recall the well-known Braess (1968) paradox; see also Braess, Nagurney, and Wakolbinger (2005). Assume a network as the first network depicted in Figure 5 in which there are four nodes: $1, 2, 3, 4$; four links: a, b, c, d; and a single O/D pair $\omega_1 = (1, 4)$. There are, hence, two paths available to travelers between this O/D pair: $p_1 = (a, c)$ and $p_2 = (b, d)$.

The user link travel cost functions are:

$$c_a(f_a) = 10f_a, \quad c_b(f_b) = f_b + 50, \quad c_c(f_c) = f_c + 50, \quad c_d(f_d) = 10f_d.$$

Assume a fixed travel demand $d_{\omega_1} = 6$.

It is easy to verify that the equilibrium path flows are: $x^*_{p_1} = 3$, $x^*_{p_2} = 3$, the equilibrium link flows are: $f^*_a = 3$, $f^*_b = 3$, $f^*_c = 3$, $f^*_d = 3$, with associated equilibrium path travel costs: $C_{p_1} = c_a + c_c = 83$, $C_{p_2} = c_b + c_d = 83$.

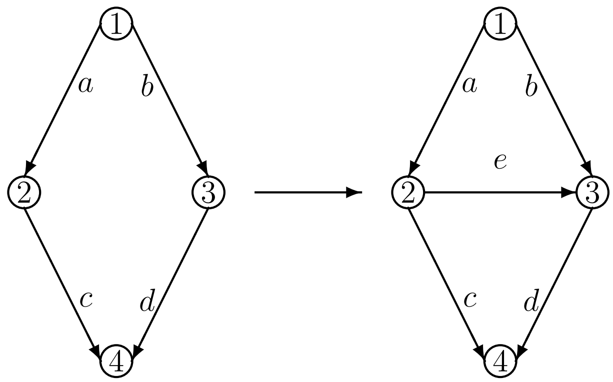

Figure 5: The Braess Network Example

Assume now that, as depicted in Figure 5, a new link "e", joining node 2 to node 3 is added to the original network, with user link cost function $c_e(f_e) = f_e + 10$. The addition of this link creates a new path $p_3 = (a, e, d)$ that is available to the travelers. The travel demand d_{ω_1} remains at 6 units of flow. The original flow pattern $x_{p_1} = 3$ and $x_{p_2} = 3$ is no longer an equilibrium pattern, since, at this level of flow, the user cost on path p_3, $C_{p_3} = c_a + c_e + c_d = 70$. Hence, users on paths p_1 and p_2 would switch to path p_3.

The equilibrium flow pattern on the new network is: $x^*_{p_1} = 2$, $x^*_{p_2} = 2$, $x^*_{p_3} = 2$, with equilibrium link flows: $f^*_a = 4$, $f^*_b = 2$, $f^*_c = 2$, $f^*_e = 2$, $f^*_d = 4$, and with associated equilibrium user path travel costs: $C_{p_1} = 92$, $C_{p_2} = 92$. Indeed, one can verify that any reallocation of the path flows would yield a higher travel cost on a path.

Note that the travel cost increased for every user of the network from 83 to 92 without a change in the travel demand!

The system-optimizing solution, on the other hand, for the first network in Figure 5 is: $x_{p_1} = x_{p_2} = 3$, with marginal total path costs given by: $\hat{C}'_{p_1} = \hat{C}'_{p_2} = 116$. This would remain the system-optimizing solution, even after the addition of link e, since the marginal cost of path p_3, $\hat{C}'_{p_3}$, at this feasible flow pattern is equal to 130.

The addition of a new link to a network cannot increase the total cost of the network system, but can, of course, increase a user's cost since travelers act individually.

3. Models with Asymmetric Link Costs

In this section, we consider network models in which the user cost on a link is no longer dependent solely on the flow on that link. We present a fixed demand transportation network equilibrium model in Section 3.1 and an elastic demand one in Section 3.2.

We note that fixed demand supply chain network problems are relevant to applications in which there are good estimates of the demand as would be the case in certain healthcare applications. Elastic demand supply chain network problems can capture price sensitivity associated with the product and are used in profit-maximizing settings (cf. Nagurney (2006)). Asymmetric link costs are relevant also in the case of competitive supply chain network equilibrium problems.

Assume that user link cost functions are now of a general form, that is, the cost on a link may depend not only on the flow on the link but on other link flows on the network, that is,

$$c_a = c_a(f), \quad \forall a \in \mathcal{L}. \tag{18}$$

In the case where the symmetry assumption exists, that is, $\frac{\partial c_a(f)}{\partial f_b} = \frac{\partial c_b(f)}{\partial f_a}$, for all links $a, b \in \mathcal{L}$, one can still reformulate the solution to the network equilibrium problem satisfying equilibrium conditions (5) as the solution to an optimization problem, albeit, again, with an objective function that is artificial and simply a mathematical device. However, when the symmetry assumption is no longer satisfied, such an optimization reformulation no longer exists and one must appeal to *variational inequality theory* (cf. Nagurney (1999) and the references therein). Models of supply chains and transportation networks with asymmetric cost functions are important since they allow for the formulation, qualitative analysis, and, ultimately, solution to problems in which the cost on a link may depend on the flow on another link in a different way than the cost on the other link depends on that link's flow.

It was in the domain of such network equilibrium problems that the theory of finite-dimensional variational inequalities realized its earliest success, beginning with the contributions of Smith (1979) and Dafermos (1980). For an introduction to the subject, as well as applications ranging from transportation network and spatial price equilibrium problems to financial equilibrium problems, see the book by Nagurney (1999). Below we present variational inequality formulations of both fixed demand and elastic demand network equilibrium problems.

The system-optimization problem, in turn, in the case of nonseparable (cf. (18)) user link cost functions becomes (see also (12)):

$$\text{Minimize} \quad \sum_{a \in \mathcal{L}} \hat{c}_a(f), \tag{19}$$

subject to (7)–(9), where $\hat{c}_a(f) = c_a(f) \times f_a$, $\forall a \in \mathcal{L}$.

The system-optimality conditions remain as in (16), but now the marginal of the total cost on a path becomes, in this more general case:

$$\hat{C}'_p = \sum_{a,b \in \mathcal{L}} \frac{\partial \hat{c}_b(f)}{\partial f_a} \delta_{ap}, \quad \forall p \in P. \tag{20}$$

3.1 Variational Inequality Formulations of Fixed Demand Problems

As mentioned earlier, in the case where the user link cost functions are no longer symmetric, one cannot compute the solution to the U-O, that is, to the network equilibrium, problem using standard optimization algorithms. We emphasize, again, that such general cost functions are very important from an application standpoint since they allow for asymmetric interactions on the network. For example, allowing for asymmetric cost functions permits one to handle the situation when the flow on a particular link affects the cost on another link in a different way than the cost on the particular link is affected by the flow on the other link.

First, the definition of a variational inequality problem is recalled. For further background, theoretical formulations, derivations, and the proofs of the results below, see the books by Nagurney (1999) and by Nagurney and Dong (2002) and the references therein. We provide the variational inequality of the network equilibrium conditions in path flows as well as in link flows since different formulations suggest different computational methods for solution.

Specifically, the variational inequality problem (finite-dimensional) is defined as follows:

Definition 1: Variational Inequality Problem

The finite-dimensional variational inequality problem, $\mathrm{VI}(F, \mathcal{K})$*, is to determine a vector* $X^* \in \mathcal{K}$ *such that*

$$\langle F(X^*)^T, X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{21}$$

where F *is a given continuous function from* $\mathcal{K}$ *to* R^N*,* $\mathcal{K}$ *is a given closed convex set, and* $\langle \cdot, \cdot \rangle$ *denotes the inner product in* R^N*.*

Variational inequality (21) is referred to as being in *standard form*. Hence, for a given problem, typically an *equilibrium* problem, one must determine the function F that enters the variational inequality problem, the vector of variables X, as well as the feasible set $\mathcal{K}$.

The variational inequality problem contains, as special cases, such well-known problems as systems of equations, optimization problems, and complementarity problems. Thus, it is a powerful unifying methodology for equilibrium analysis and computation and continues to be utilized for the formulation, analysis, and solution of a spectrum of supply chain network problems (cf. Nagurney (2006)).

A geometric interpretation of the variational inequality problem $\mathrm{VI}(F, \mathcal{K})$ is given in Figure 6. Specifically, $F(X^*)$ is "orthogonal" to the feasible set $\mathcal{K}$ at the point X^*.

Theorem 1: Variational Inequality Formulation of Network Equilibrium with Fixed Demands – Path Flow Version

A vector $x^* \in K^1$ *is a network equilibrium path flow pattern, that is, it satisfies equilibrium conditions (5) if and only if it satisfies the variational inequality problem:*

$$\sum_{\omega \in \Omega} \sum_{p \in P_\omega} C_p(x^*) \times (x - x^*) \geq 0, \quad \forall x \in K^1, \tag{22}$$

or, in vector form:

$$\langle C(x^*)^T, x - x^* \rangle \geq 0, \quad \forall x \in K^1, \tag{23}$$

where C *is the* n_P*-dimensional vector of path user costs and* K^1 *is defined as:* $K^1 \equiv \{x \geq 0,$ *such that* (7) *holds*$\}$.

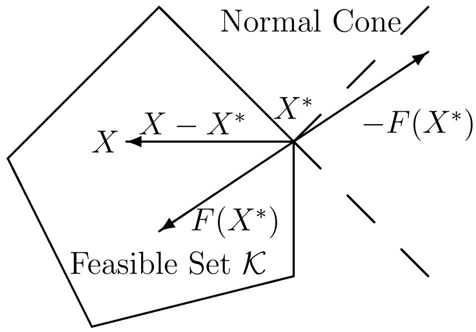

Figure 6: Geometric Interpretation of VI$(F, \mathcal{K})$

Theorem 2: Variational Inequality Formulation of Network Equilibrium with Fixed Demands – Link Flow Version

A vector $f^* \in K^2$ is a network equilibrium link flow pattern if and only if it satisfies the variational inequality problem:

$$\sum_{a \in \mathcal{L}} c_a(f^*) \times (f_a - f_a^*) \geq 0, \quad \forall f \in K^2, \tag{24}$$

or, in vector form:

$$\langle c(f^*)^T, f - f^* \rangle \geq 0, \quad \forall f \in K^2, \tag{25}$$

where c is the n_L-dimensional vector of link user costs and K^2 is defined as:
$K^2 \equiv \{f \mid \textit{there exists an}\, x \geq 0 \textit{ and satisfying}\, (7) \textit{ and}\, (8)\}$.

One may put variational inequality (23) into standard form (21) by letting $F \equiv C$, $X \equiv x$, and $\mathcal{K} \equiv K^1$. One may also put variational inequality (25) into standard form where now $F \equiv c$, $X \equiv f$, and $\mathcal{K} \equiv K^2$. Hence, fixed demand transportation network equilibrium problems in the case of asymmetric user link cost functions can be solved as variational inequality problems, as given above.

The theory of variational inequalities (see Kinderlehrer and Stampacchia (1980) and Nagurney (1999)) allows one to qualitatively analyze the equilibrium patterns in terms of existence, uniqueness, as well as sensitivity and stability of solutions, and to apply rigorous algorithms for the numerical computation of the equilibrium patterns. Variational inequality algorithms usually resolve the variational inequality problem into series of simpler subproblems, which, in turn, are often optimization problems, which can then be effectively solved using a variety of algorithms.

We emphasize that the above network equilibrium framework is sufficiently general to also formalize the entire transportation planning process (consisting of origin selection, or destination selection, or both, in addition to route selection, in an optimal fashion) as path choices over an appropriately constructed *abstract* network or supernetwork. Further discussion can be found in the books by Nagurney (1999, 2000) and Nagurney and Dong (2002) who also developed more general models in which the costs (as described above) need not be separable nor asymmetric.

3.2 Variational Inequality Formulations of Elastic Demand Problems

We now describe a general network equilibrium model with elastic demands due to Dafermos (1982) but we present the single-modal version, for simplicity. It is assumed that one has associated with each O/D pair ω in the network a travel disutility function λ_ω, where here the general case is considered in which the disutility may depend upon the entire vector of demands, which are no longer fixed, but are now variables, that is,

$$\lambda_\omega = \lambda_\omega(d), \quad \forall \omega \in \Omega, \tag{26}$$

where d is the J-dimensional vector of the demands.

The notation is as described earlier, except that here we also consider user link cost functions which are general, that is, of the form (18). The conservation of flow equations (see also (1) and (2)), in turn, are given by

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in \mathcal{L}, \tag{27}$$

$$d_\omega = \sum_{p \in P_\omega} x_p, \quad \forall \omega \in \Omega, \tag{28}$$

$$x_p \geq 0, \quad \forall p \in P. \tag{29}$$

In the elastic demand case, the demands in expression (28) are variables and no longer given, in contrast to the fixed demand expression in (1).

Network Equilibrium Conditions in the Case of Elastic Demands

The network equilibrium conditions (see also (5)) take on in the elastic demand case the following form. For every O/D pair $\omega \in \Omega$, and each path $p \in P_\omega$, a vector of path flows and demands (x^*, d^*) satisfying (28) and (29) (which induces a link flow pattern f^* through (27)) is a network equilibrium pattern if it satisfies:

$$C_p(x^*) \begin{cases} = \lambda_\omega(d^*), & \text{if} \quad x_p^* > 0 \\ \geq \lambda_\omega(d^*), & \text{if} \quad x_p^* = 0. \end{cases} \tag{30}$$

Equilibrium conditions (30) state that the costs on used paths for each O/D pair are equal and minimal and equal to the disutility associated with that O/D pair. Costs on unutilized paths can exceed the disutility. Observe that in the elastic demand model users of the network can forego travel altogether for a given O/D pair if the user costs on the connecting paths exceed the travel disutility associated with that O/D pair. This model, hence, allows one to ascertain the attractiveness of different O/D pairs based on the ultimate equilibrium demand associated with the O/D pairs. In addition, this model can handle such situations as the equilibrium determination of employment location and route selection, or residential location and route selection, or residential and employment selection as well as route selection through the appropriate transformations via the addition of links and nodes, and given, respectively, functions associated with the residential locations, the employment locations, and the network overall (cf. Nagurney (1999) and Nagurney and Dong (2002)).

In the next two theorems, both the path flow version and the link flow version of the variational inequality formulations of the network equilibrium conditions (30) are presented. These are analogues of the formulations (22) and (23), and (24) and (25), respectively, for the fixed demand model, and are due to Dafermos (1982).

Theorem 3: Variational Inequality Formulation of Network Equilibrium with Elastic Demands – Path Flow Version

A vector $(x^*, d^*) \in K^3$ *is a network equilibrium path flow pattern, that is, it satisfies equilibrium conditions (30) if and only if it satisfies the variational inequality problem:*

$$\sum_{\omega \in \Omega} \sum_{p \in P_\omega} C_p(x^*) \times (x - x^*) - \sum_{\omega \in \Omega} \lambda_\omega(d^*) \times (d_\omega - d^*_\omega) \geq 0, \quad \forall (x, d) \in K^3, \tag{31}$$

or, in vector form:

$$\langle C(x^*)^T, x - x^* \rangle - \langle \lambda(d^*)^T, d - d^* \rangle \geq 0, \quad \forall (x, d) \in K^3, \tag{32}$$

where λ *is the* J*-dimensional vector of disutilities and* K^3 *is defined as:* $K^3 \equiv \{x \geq 0, \textit{ such that } (28) \textit{ holds}\}$.

Theorem 4: Variational Inequality Formulation of Network Equilibrium with Elastic Demands – Link Flow Version

A vector $(f^*, d^*) \in K^4$ *is a network equilibrium link flow pattern if and only if it satisfies the variational inequality problem:*

$$\sum_{a \in \mathcal{L}} c_a(f^*) \times (f_a - f^*_a) - \sum_{\omega \in \Omega} \lambda_\omega(d^*) \times (d_\omega - d^*_\omega) \geq 0, \quad \forall (f, d) \in K^4, \tag{33}$$

or, in vector form:

$$\langle c(f^*)^T, f - f^* \rangle - \langle \lambda(d^*)^T, d - d^* \rangle \geq 0, \quad \forall (f, d) \in K^4, \tag{34}$$

where $K^4 \equiv \{(f, d), \textit{ such that there exists an } x \geq 0 \textit{ satisfying } (27), (28)\}$.

Under the symmetry assumption on the disutility functions, that is, if $\frac{\partial \lambda_w}{\partial d_\omega} = \frac{\partial \lambda_\omega}{\partial d_w}$, for all w, ω, in addition to such an assumption on the user link cost functions (see following (18)), one can obtain (see Beckmann, McGuire, and Winsten (1956)) an optimization reformulation of the network equilibrium conditions (30), which in the case of separable user link cost functions and disutility functions is given by:

$$\text{Minimize} \quad \sum_{a \in \mathcal{L}} \int_0^{f_a} c_a(y) dy - \sum_{\omega \in \Omega} \int_0^{d_\omega} \lambda_\omega(z) dz \tag{35}$$

subject to: (27)–(29).

Variational inequality theory has become a fundamental methodological framework for the formulation and solution of competitive supply chain problems in which the governing concept is that of Nash equilibrium (see, e.g., Masoumi, Yu, and Nagurney (2012)).

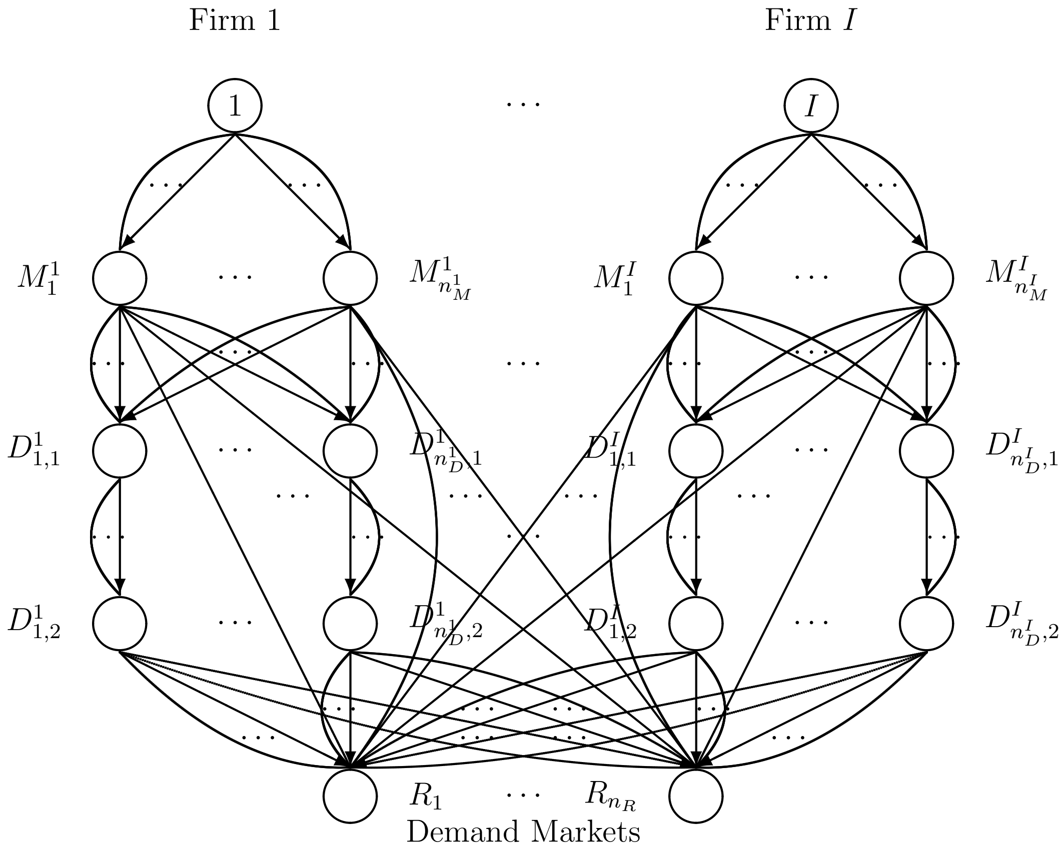

Figure 7: The Competitive Supply Chain Network Topology

In Figure 7, a competitive supply chain network is depicted in which the firms have vertically integrated supply chains but compete in common demand markets. The top-most links represent manufacturing activities at different plants with different such links denoting alternative manufacturing technologies. The second set of links from the top reflects transportation and alternative links depict the possibility of alternative modes of transportation. The next set of links corresponds to storage at the distribution centers and the final set of links the transportation to the demand markets. Here we also use multiple links to denote alternative technologies and transportation modes, respectively. The costs on the links can be separable or not and asymmetric, depending on the specific product application. Product differentiation and branding has also been incorporated into such supply chain networks using variational inequality theory. Observe that in the supply chain network depicted in Figure 7, direct shipments from the manufacturing plants to the demand points / retailers are allowed and depicted by the corresponding links.

Finally, it is important to emphasize that the dynamics of the underlying interactions can be formulated and has been done so using projected dynamical systems (Nagurney and Zhang (1996)).

4. Conclusions

In this essay, we have highlighted some of the major advances in supply chains and transportation networks, with a focus on the common elements as to the theoretical frameworks and underlying behavioral principles. We have also argued that the foundations of supply chains as network systems can be found in the regional science and spatial economics literature.

Specifically, we have discussed how the concepts of system-optimization and user-optimization have underpinned transportation network models and, more recently, have evolved to enable the formulation of supply chain network problems operating (and managed) under centralized or decentralized, that is, competitive, decision-making behavior.

We have also highlighted some of the principal methodologies, including variational inequality theory that have enabled the development not only of advanced transportation network equilibrium models but also supply chain network equilibrium models.

We have aimed to include both primary references as well as tertiary references; the interested reader can delve further, at his/her convenience and according to interest.

In conclusion, transportation network concepts, models, and accompanying methodologies have enabled the advancement of supply chain network models from a system-wide and holistic perspective.

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