A Transportation Network Efficiency Measure that Captures Flows, Behavior, and Costs with Applications to Network Component Importance Identification and Vulnerability

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Abstract: In this paper, we propose a transportation network efficiency measure that can be used to assess the performance of a transportation network and which differs from other proposed measures, including complex network measures, in that it captures flows, costs, and travel behavior information, along with the topology. The new transportation network efficiency measure allows one to determine the criticality of various nodes (as well as links) as we demonstrate through a network component importance definition, which is well-defined even if the network becomes disconnected. Several illustrative transportation network examples are provided in which the efficiencies and importance of network components are explicitly computed, and their rankings tabulated.

This framework can be utilized to assess the vulnerability of network components in terms of their criticality to network efficiency/performance and to, ultimately, enhance security.

1. Introduction

Networks and, in particular, *complex networks*, have been the subject of intense research activity in recent years although the topic, which is based on graph theory, is centuries old. Indeed, the subject of networks, with its rich applications has been tackled by operations researchers/management scientists, applied mathematicians, economists, engineers, physicists, biologists, and sociologists; see, for some examples: Beckmann, McGuire, and Winsten (1956), Sheffi (1985), Ahuja, Magnanti, and Orlin (1993), Nagurney (1993), Patriksson (1994), Ran and Boyce (1996), Watts and Strogatz (1998), Barabási and Albert (1999), Latora and Marchiori (2001), Newman (2003), Roughgarden (2005), and the references therein. Three types of networks, in particular, have received recent intense attention, especially in regards to the development of network measures, and we note, specifically, the random network model (Erdös-Rényi, 1960), the small-world model (Watts and Strogatz, 1998), and scale-free networks (Barabási and Albert, 1999).

The importance of studying and identifying the vulnerable components of a network, in turn, has been linked to events such as 9/11 and to Hurricane Katrina, as well as to the biggest blackout in North America that occurred on August 14, 2003 (cf. Sheffi, 2005; Nagurney, 2006). In order to hedge against terrorism and natural disasters, a majority of the associated complex network (sometimes also referred to as *network science*) literature (cf. the survey by Newman, 2003) focuses on the graph characteristics (e.g. connectivity between nodes) of the associated application in order to evaluate the network reliability and vulnerability; see also, for example, Chassin and Posse (2005) and Holme et al. (2002).

However, in order to be able to evaluate the vulnerability and the reliability of a network, a measure that can quantifiably capture the efficiency/performance of a network must be developed. For example, in a series of papers, beginning in 2001, Latora and Marchiori discussed the network performance issue by measuring the "global efficiency" in a weighted network as compared to that of the simple non-weighted small-world network. In a weighted network, the network is not only characterized by the edges that connect different nodes, but also by the weights associated with different edges in order to capture the relationships between different nodes. The network efficiency E of a network G is defined in the paper of Latora and Marchiori (2001) as $E = \frac{1}{n(n-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}}$, where n is the number of nodes in G and d_{ij} is the shortest path length (the geodesic distance) between nodes i and j. This measure has been applied by the above authors to a variety of networks, including the (MBTA) Boston subway transportation network and the Internet (cf. Latora and Marchiori 2002, 2004).

Although the topological structure of a network obviously has an impact on network performance and the vulnerability of the network, we believe that the flow on a network is also an important indicator, as are the induced costs, and the behavior of users of the network(s). Indeed, flows represent the usage of a network and which paths and links have positive flows and the magnitude of these flows are relevant in the case of network disruptions. Interestingly, although recently a few papers have appeared in the complex network literature that emphasize flows on a transportation network, with a focus on airline networks (cf. Barrat, Barthélemy, and Vespignani, 2005, and Dall'Asta et al., 2006), the aforementioned papers only consider the importance of nodes and not that of links and ignore the behavior of users. It is well-known in the transportation literature that the users' perception of the travel costs will affect the traffic pattern on the network (see, e.g., Beckmann, McGuire, and Winsten, 1956; Dafermos and Sparrow, 1969; Boyce et al., 1983; Ran and Boyce, 1996, and Nagurney, 1993). Therefore, a network efficiency measure that captures flows, the costs associated with "travel," and user behavior, along with the network topology, is more appropriate in evaluating networks such as transportation networks, which are the classical critical infrastructure. Indeed, in the case of disruptions, which can affect either nodes, or links, or both, we can expect travelers to readjust their behavior and the usage of the network accordingly. Furthermore, as noted by Jenelius, Petersen, and Mattsson (2006), the criticality of a network component, consisting of a node, link, or combination of nodes and links, is related to the vulnerability of the network system in that the more critical (or, as we consider, the more important) the component, the greater the damage to the network system when this component is removed, be it through natural disasters, terrorist attacks, structural failures, etc.

In this paper, we propose a new transportation network performance measure that can be used to evaluate the efficiency of a transportation network as well as the importance of its network components. We also relate the new measure to the Latora and Marchiori (2001) measure used in the "complex" network literature. In addition, we compare the resulting network component importance definitions derived from our measure to those recently proposed by Jenelius, Petersen, and Mattsson (2006) (see also Taylor and D'Este, 2004) and also provide illustrative examples. Our measure has the additional notable feature that it is applicable, as is our proposed importance definition of network components, even in the case that the network becomes disconnected (after the removal of the component).

The paper is organized as follows. In Section 2, we present some preliminaries. The new transportation network efficiency measure is introduced in Section 3, along with the associated definition of the importance of network components. We also prove that the new measure contains, as a special case, an existing measure, due to Latora and Marchiori (2001), that has been much studied and applied in the complex network literature. Section 4 then presents three network examples for which the efficiency measures are computed and the node and link importance rankings determined using the new transportation network efficiency measure. Comparisons with the Latora and Marchiori (2001) measure are also provided, for completeness, and with the Jenelius, Petersen, and Mattsson (2006) importance indicators in the case of the link components. Section 5 then applies the new measure to a larger scale network to further illustrate the applicability of the proposed measure. Section 6 summarizes the results in this paper and provides suggestions for future research.

2. Some Preliminaries

In this Section, we recall the transportation network equilibrium model with fixed demands, due to Dafermos (1980) (see also Smith, 1979). We consider a network G with the set of links L with n_L elements and the set of nodes N with n elements. The set of origin/destination (O/D) pairs of nodes is denoted by W and consists of n_W elements. We denote the set of paths joining an O/D pair of nodes w by P_w and the set of all paths by P. Links are denoted by a, b, etc; paths by p, q, etc., and O/D pairs by w_1, w_2, etc. Links are assumed to be directed and paths to be acyclic.

We denote the nonnegative flow on path p by x_p and the flow on link a by f_a. The link flows are related to the path flows through the following conservation of flow equations:

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in L, \tag{1}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and $\delta_{ap} = 0$, otherwise. Hence, the flow on a link is equal to the sum of the flows on paths that contain that link. We group the link flows into the vector $f \in R_+^{n_L}$ and the path flows into the vector $x \in R_+^{n_P}$.

The demand for O/D pair w is denoted by d_w and is assumed to be positive. We assume that the following conservation of flow equations hold:

$$\sum_{p \in P_w} x_p = d_w, \quad \forall w \in W, \tag{2}$$

that is, the sum of flows on paths connecting each O/D pair w must be equal to the demand for w.

The cost on a path p is denoted by C_p and the cost on a link a by c_a.

The costs on paths are related to costs on links through the following equations:

$$C_p = \sum_{a \in L} c_a \delta_{ap}, \quad \forall p \in P, \tag{3}$$

that is, the cost on a path is equal to the sum of costs on links that make up the path.

Furthermore, we allow the cost on a link to depend, in general, upon the flows on the network links, so that

$$c_a = c_a(f), \quad \forall a \in L, \tag{4}$$

and we assume that the link cost functions are continuous and strictly monotonically increasing (cf. Nagurney, 1993) so that the equilibrium link flows, defined below, will be unique.

Definition 1: Transportation Network Equilibrium

A path flow pattern $x^* \in \mathcal{K}^1$, where $\mathcal{K}^1 \equiv \{x | x \in R_+^{n_P}$ and (2) holds with d_w known and fixed for each $w \in W\}$, is said to be a network equilibrium, if the following condition holds for each O/D pair $w \in W$ and each path $p \in P_w$:

$$C_p(x^*) \begin{cases} = \lambda_w, & \text{if} \quad x_p^* > 0, \\ \geq \lambda_w, & \text{if} \quad x_p^* = 0, \end{cases} \tag{5}$$

where λ_w denotes the equilibrium cost associated with O/D pair w.

The interpretation of conditions (5) is that all used paths connecting an O/D pair w have equal and minimal costs, which corresponds to Wardrop's well-known first principle (see Wardrop, 1952; Beckmann, McGuire, and Winsten, 1956; Dafermos and Sparrow, 1969). As proved in Smith (1979) and Dafermos (1980), the network equilibrium conditions (5) (in the case of user link cost functions of the form (4)) correspond to the solution of the following variational inequality problem(s) in path flows and link flows, respectively.

Theorem 1

A path flow pattern $x^* \in \mathcal{K}^1$ is a transportation network equilibrium according to Definition 1 if and only if it satisfies the variational inequality problem in path flows: determine $x^* \in \mathcal{K}^1$ such that

$$\sum_{w \in W} \sum_{p \in P_w} C_p(x^*) \times (x_p - x_p^*) \geq 0, \quad \forall x \in \mathcal{K}^1, \tag{6a}$$

or, equivalently, a link flow pattern $f^* \in \mathcal{K}^2$ is a network equilibrium if and only if it satisfies the variational inequality problem in link flows: determine $f^* \in \mathcal{K}^2$ such that

$$\sum_{a \in L} c_a(f^*) \times (f_a - f_a^*) \geq 0, \quad \forall f \in \mathcal{K}^2, \tag{6b}$$

where $\mathcal{K}^2 \equiv \{f | \text{there exists an } x \in R_+^{n_P} \text{ satisfying (1) and (2)}\}$.

Existence of solutions to variational inequalities (6a) and (6b) is guaranteed from the standard theory of variational inequalities (cf. Nagurney, 1993) since the link cost (and, hence, also the path cost) functions are assumed to be continuous and the feasible sets $\mathcal{K}^1$ and $\mathcal{K}^2$ are compact.

Finally, in the case where the link cost functions (4) are separable, that is, $c_a = c_a(f_a)$, for all links $a \in L$, then the equilibrium solution can be obtained by solving (cf. Beckmann, McGuire, and Winsten (1956) and Dafermos and Sparrow (1969)) the following nonlinear optimization problem:

$$\text{Minimize}_{f\in\mathcal{K}^2} \sum_{q\in L} \int_0^{f_a} c_a(y)dy. \tag{6c}$$

The importance of the reformulation of the transportation network equilibrium problem as an optimization problem, under the appropriate assumptions, and due to Beckmann, McGuire, and Winsten (1956) and the impacts of this classical work on theory and practice, are overviewed in Boyce, Mahmassani, and Nagurney (2005). Problem (6c) is now solved daily in practice using a variety of algorithms. For recent approaches for the effective solution of such, often-times, very large-case problems, see Bar-Gera (2002).

Algorithms for the solutions of variational inequalities (6a) and (6b) can be found in Nagurney (1993), Nagurney and Zhang (1996), and the references therein. It is important to emphasize that many variational inequality algorithms are based on the sequential solution of optimization problems and, hence, efficient algorithms for (6c) can be embedded in algorithms to solve (6a) and (6b).

3. A Transportation Network Efficiency Measure and the Importance of a Network Component

We now present the new transportation network efficiency measure with additional results.

First, recall that the network efficiency measure of Latora and Marchiori (the L-M measure) (2001), which was proposed to measure the efficiency of networks in which the links may have associated weights or costs, is defined as follows:

Definition 2: The L-M Measure

The L-M network performance/efficiency measure, $E(G)$, according to Latora and Marchiori (2001) for a given network topology G, is defined as:

$$E(G) = \frac{1}{n(n-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}}, \tag{7a}$$

where n is the number of nodes in the network and d_{ij} is the shortest path length between node i and node j.

Our transportation network efficiency measure is given in the following definition.

Definition 3: A Transportation Network Efficiency Measure

The network transportation efficiency measure, $\mathcal{E}(G,d)$, for a given network topology G and vector of O/D demands, d, is defined as follows:

$$\mathcal{E} = \mathcal{E}(G,d) = \frac{\sum_{w \in W} \frac{d_w}{\lambda_w}}{n_W}, \tag{7b}$$

where λ_w denotes the cost on the minimum cost (shortest) used path(s), that is, ones with positive flow, connecting O/D pair w.

Remark

The transportation network efficiency measure given in (7b) has a meaningful economic interpretation which is that the efficiency of a transportation network is equal to the average, in terms of O/D pairs, traffic to price ratio with the traffic per O/D pair being given by d_w and the equilibrium price of travel between O/D pair w by λ_w. The higher the traffic that can be handled at a given price (which also reflects the cost and, from an engineering perspective, the travel time), the higher the efficiency or performance of the transportation network.

Interestingly, we will show in the following theorem that, under appropriate assumptions, our measure and the L-M measure are equivalent. However, the latter measure considers neither flows nor demands and does not incorporate any underlying users' behavior.

Theorem 2

If positive demands exist for all pairs of nodes in the network G, and each of these demands is equal to 1 and if d_{ij} is set equal to λ_w, where $w = (i, j)$, for all $w \in W$ then the proposed network efficiency measure (7b) and the L-M measure (7a) are one and the same.

Proof: Let n be the number of nodes in G. Hence, the total number of O/D pairs, n_W, is equal to $n(n-1)$ given the assumption that there exist positive demands for all pairs of nodes in G. Furthermore, by assumption, we have that $d_w = 1$, $\forall w \in W$, $w = (i, j)$, and $d_{ij} = \lambda_w$, where $i \neq j$, $\forall i, j \in G$. Then our network efficiency measure (7b) becomes as follows:

$$E(G) = \frac{1}{n(n-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}} = \frac{\sum_{i \neq j \in G} \frac{1}{d_{ij}}}{n_W} = \frac{\sum_{w \in W} \frac{d_w}{\lambda_w}}{n_W} = \mathcal{E}(G, d), \tag{8}$$

and the conclusion follows. □

Note that, from the definition, λ_w is the value of the cost of the minimum or "shortest" used paths for O/D pair w and d_{ij}, according to Latora and Marchiori (2001), is the shortest path length (the geodesic distance) between nodes i and j. Therefore, the assumption of d_{ij} being equal to λ_w is not unreasonable. Our measure, however, is a more general measure which also captures flows and behavior on the network, according to Definition 1.

With our transportation network efficiency measure, we can investigate the importance of network components by studying their impact on the transportation network efficiency through their removal. We define the importance of a node or a link (or a subset of nodes and links) as follows:

Definition 4: Importance of a Network Component

The importance, $I(g)$ of a network component $g \in G$, is measured by the relative network efficiency drop after g is removed from the network:

$$I(g) = \frac{\triangle \mathcal{E}}{\mathcal{E}} = \frac{\mathcal{E}(G, d) - \mathcal{E}(G - g, d)}{\mathcal{E}(G, d)}, \tag{9}$$

where $G - g$ is the resulting network after component g is removed from network G.

The upper bound of the importance of a network component is 1.

The elimination of a link is treated in our measure by removing that link while the removal of a node is managed by removing the links entering or exiting that node. In the case that the removal results in no path connecting an O/D pair, we simply assign the demand for that O/D pair to an abstract path with a cost of infinity. Hence, our measure is well-defined even in the case of disconnected networks. Notably, Latora and Marchiori (2001) also mention this important characteristic which gives their measure an attractive property over the measure used for the small-world network model (cf. Watts and Strogatz, 1998).

Furthermore, we note that Jenelius, Petersen, and Mattsson (2006) proposed several link importance indicators and applied them to the road network in northern Sweden. In particular, they proposed distinct link importance indicators depending upon whether the removal of the link would cause the network to become disconnected or not. The definitions of their link importance indicators are briefly recalled in the following (with the notation adapted to ours, for clarity):

Definition 5: Link Importance Indicators According to Jenelius, Petersen, and Mattsson (2006)

In a network G, the global importance, I^1, the demand-weighted importance, I^2, and the relative unsatisfied demand, I^3, of link $k \in G$ are defined, respectively, as follows:

$$I^1(k) = \frac{1}{n_W} \sum_{w \in W} (\lambda_w(G-k) - \lambda_w(G)), \tag{10}$$

$$I^2(k) = \frac{\sum_{w \in W} d_w(\lambda_w(G-k) - \lambda_w(G))}{\sum_{w \in W} d_w}, \tag{11}$$

$$I^3(k) = \frac{\sum_{w \in W} u_w(G-k)}{\sum_{w \in W} d_w}, \tag{12}$$

where $\lambda_w(G)$ is the original equilibrium cost of O/D pair w while $\lambda_w(G-k)$ is the equilibrium cost of O/D pair w after link k is removed; $u_w(G-k)$ is the unsatisfied demand for O/D pair w after link k is removed.

According to the above definitions, I^1 and I^2 are defined for a link whose removal will not cause any O/D pairs to be disconnected while I^3 is defined for the opposite situation. It is worth pointing out that the above measures also take cost and flow into consideration when evaluating a link's importance. However, our transportation network efficiency measure has three major advantages in gauging the importance of a network component:

(1). it is simple and can be applied to any network component, be it a node, or a link, or a set of nodes and links;

(2). the induced network component importance definition (cf. (9)) does not depend on whether or not the removal of the component will yield a disconnected network, which provided a uniform and unified indication of the importance of the network components, and

(3). our network component importance definition is based on a well-defined network efficiency measure, given by (7b).

The advantages of our network efficiency measure will be further illustrated in the next section through concrete examples.

4. Transportation Network Examples

In this Section, three network examples are presented. The importance values of individual nodes and links are calculated and also ranked, respectively, according to our measure and the measure proposed by Latora and Marchiori (2001). Furthermore, the link importance indicators proposed by Jenelius, Petersen, and Mattsson (2006) are also reported. In all the examples below, in computing the Latora and Marchiori measure; henceforth, referred to as the L-M measure, and the associated importance rankings, we assume that for each O/D pair w, where $w = (i, j)$, that $d_{ij} = \lambda_w$ (cf. Latora and Marchiori, 2001, 2004).

Example 1

Consider the network in Figure 1 in which there are two O/D pairs: $w_1 = (1, 2)$ and $w_2 = (1, 3)$ with demands given, respectively, by $d_{w_1} = 100$ and $d_{w_2} = 20$. We have that path $p_1 = a$ and path $p_2 = b$. Assume that the link cost functions are given by: $c_a(f_a) = .01f_a + 19$ and $c_b(f_b) = .05f_b + 19$. Clearly, we must have that $x^*_{p_1} = 100$ and $x^*_{p_2} = 20$ so that $\lambda_{w_1} = \lambda_{w_2} = 20$. The network efficiency measure $\mathcal{E} = 3.0000$ whereas the L-M measure $E = .0167$.

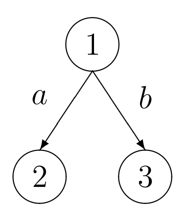


Figure 1: Example 1

In regards to the importance indices proposed by Jenelius, Petersen, and Mattsson (2006), since the removal of link a or b will cause the O/D pairs to become disconnected, only their I^3 measure can be applied to this example. Recall also that their importance measure is for links only and not for nodes. The importance values and the rankings of the links and the nodes for Example 1 are given, respectively, in Tables 1 and 2, using the relevant importance measures.

Table 1: Importance Values and Ranking of Links in Example 1

Link	Importance Value from Our Measure	Importance Ranking from Our Measure	Importance Value from the L-M Measure	Importance Ranking from the L-M Measure	Importance Value from I^3	Importance Ranking from I^3
a	0.8333	1	0.5000	1	0.8333	1
b	0.1667	2	0.5000	1	0.1667	2

Table 2: Importance Values and Ranking of Nodes in Example 1

Node	Importance Value from Our Measure	Importance Ranking from Our Measure	Importance Value from the L-M Measure	Importance Ranking from the L-M Measure
1	1.0000	1	1.0000	1
2	0.8333	2	0.5000	2
3	0.1667	3	0.5000	2

Clearly, our measure, which captures flow information is the most general, reasonable, and precise since, in the case of a disruption, the destruction of link a, with which was associated a flow 5 times the flow of link b, would result in a greater loss of efficiency! The same qualitative analysis holds for the destruction of node 2 versus node 3.

Example 2

Consider now the network in Figure 2 in which there are six O/D pairs: $w_1 = (1, 2)$, $w_2 = (2, 3)$, $w_3 = (3, 1)$, $w_4 = (2, 1)$, $w_5 = (3, 2)$, and $w_6 = (1, 3)$. The demands are given, respectively, by $d_{w_1} = 1$, $d_{w_2} = 5$, $d_{w_3} = 10$, $d_{w_4} = 1$, $d_{w_5} = 1$, and $d_{w_6} = 1$. The paths for the O/D pairs are: for w_1: $p_1 = a$; for w_2: $p_2 = b$; for w_3: $p_3 = c$; for w_4: $p_4 = (b, c)$; for w_5: $p_5 = (c, a)$, and for w_6: $p_6 = (a, b)$.

Assume that the link cost functions are given by: $c_a(f_a) = f_a$, $c_b(f_b) = 2f_b$, and $c_c(f_c) = 0.5f_c$. Obviously, we must have that $x^*_{p_1} = 1$, $x^*_{p_2} = 5$, $x^*_{p_3} = 10$, $x^*_{p_4} = 1$, $x^*_{p_5} = 1$, and $x^*_{p_6} = 1$. We also must have that $\lambda_{w_1} = 3$, $\lambda_{w_2} = 14$, $\lambda_{w_3} = 6$, $\lambda_{w_4} = 20$, $\lambda_{w_5} = 9$, and $\lambda_{w_6} = 17$. Our network efficiency measure is then $\mathcal{E} = 0.4295$ whereas the L-M measure $E = 0.1319$.

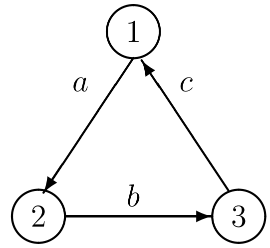


Figure 2: Example 2

Again, since the removal of any link in the network in Example 2 will cause an O/D pair to be disconnected, only I^3 from Jenelius, Petersen, and Mattsson (2006) is applicable. The importance values and the rankings of the links and the nodes in Example 2 are given, respectively, in Tables 3 and 4, using the relevant and applicable importance measures.

Table 3: Importance Values and Ranking of Links in Example 2

Link	Importance Value from Our Measure	Importance Ranking from Our Measure	Importance Value from the L-M Measure	Importance Ranking from the L-M Measure	Importance Value from I^3	Importance Ranking from I^3
a	0.1106	2	0.5927	1	0.1579	3
b	0.0487	3	-0.0301	3	0.3684	2
c	0.6166	1	0.1726	2	0.6316	1

Table 4: Importance Values and Ranking of Nodes in Example 2

Node	Importance Value from Our Measure	Importance Ranking from Our Measure	Importance Value from the L-M Measure	Importance Ranking from the L-M Measure
1	0.8060	1	0.8736	1
2	0.2239	3	0.7473	2
3	0.6120	2	-0.2636	3

Our measure gives the highest ranking to link c which has the highest volume of flow and a relatively low link cost. On the other hand, node 2 is the least important node according to our measure while it is quite important according to the L-M measure (in second place). This is due to the fact that the removal of node 2 is equivalent to the elimination of links a and b and links a and b are not important in our measure (ranked, respectively, second and third) but very important according to the L-M measure.

Example 3

We now consider the network in Figure 3. There are five O/D pairs: $w_1 = (1, 2)$, $w_2 = (1, 3)$, $w_3 = (1, 4)$, $w_4 = (3, 2)$, and $w_5 = (3, 4)$. The demands are given, respectively, by $d_{w_1} = 5$, $d_{w_2} = 1$, $d_{w_3} = 8$, $d_{w_4} = 1$, and $d_{w_5} = 1$. The paths for each O/D pair are as follows: for w_1: $p_1 = a$, $p_2 = (e, d)$; for w_2: $p_3 = e$; for w_3: $p_4 = b$, $p_5 = (e, c)$; for w_4: $p_6 = d$, and for w_5: $p_7 = c$.

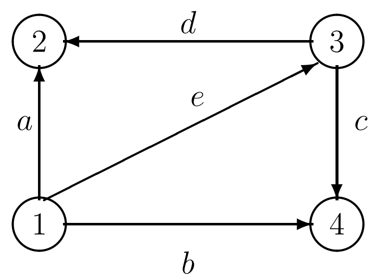


Figure 3: Example 3

Assume that the link cost functions are given by: $c_a(f_a) = f_a$, $c_b(f_b) = f_b$, $c_c(f_c) = 0.1f_c + 2$, $c_d(f_d) = 0.1f_d + 2$, and $c_e(f_e) = 0.1f_e$. According to the discussion in Section 3, we must have $x^*_{p_1} = 3.0559$, $x^*_{p_2} = 1.9441$, $x^*_{p_3} = 1$, $x^*_{p_4} = 3.3287$, $x^*_{p_5} = 4.6713$, $x^*_{p_6} = 1$ and $x^*_{p_7} = 1$. We also must have that $\lambda_{w_1} = 3.0559$, $\lambda_{w_2} = 0.7615$, $\lambda_{w_3} = 3.3287$, $\lambda_{w_4} = 2.2944$, and $\lambda_{w_5} = 2.5671$. Our network efficiency measure is then $\mathcal{E} = 1.2356$ whereas the L-M measure $E = 0.2305$.

In the Example 3 network, only the removal of a or b will not cause any O/D pair to be disconnected. Therefore, links a and b can be measured by I^1 and I^2, whereas the rest of the links can only be measured by I^3 (cf. (10) – (12)).

Table 5: Importance Values and Ranking of Links from I^1 and I^2 in Example 3

Link	Importance Value from I^1	Importance Ranking from I^1	Importance Value from I^2	Importance Ranking from I^2
a	0.2802	2	1.1052	2
b	0.3052	1	1.4203	1

The importance values and the ranking by I^1 and I^2 of the links in Example 3 are given in Table 5 while in Table 6, the importance and ranking of the links by I^3 are reported in Table 6. The importance values and the ranking of the links and the nodes in Example 3 according to our measure and the L-M measure are given in Tables 7 and 8.

Table 6: Importance and Ranking of Links from I^3 in Example 3

Link	Importance Value from I^3	Importance Ranking from I^3
c	0.0625	1
d	0.0625	1
e	0.0625	1

Table 7: Importance Values and Ranking of Links in Example 3

Link	Importance Value from Our Measure	Importance Ranking from Our Measure	Importance Value from the L-M Measure	Importance Ranking from the L-M Measure
a	0.1351	3	0.0706	3
b	0.1518	2	0.1774	2
c	-0.0235	5	-1.2487	5
d	0.0891	4	-0.0603	4
e	0.5221	1	0.5382	1

Note that node 1 is the most important node according to our measure. This is due to the fact that node 1 is the origin node with the two largest demands, namely, d_{w_1} and d_{w_3}. Therefore, the removal of this node should bring the largest negative impact to the network efficiency. However, the L-M measure only places node 1 in second place, which is not reasonable.

Moreover, from the above examples, we can see that our measure generates similar ranking results to those obtained, where applicable, via I^1, I^2, and I^3. However, in the cases where the O/D pairs become disconnected (which is very much a possibility with a spectrum of transportation network disruptions), our measure gives more reasonable and unified results. More importantly, due to the fact that the removal of a node will usually cause some O/D pairs to be disconnected, the ability to gauge the importance of a node is essential.

Table 8: Importance Values and Ranking of Nodes in Example 3

Node	Importance Value from Our Measure	Importance Rank from Our Measure	Importance Value from the L-M Measure	Importance Rank from the L-M Measure
1	0.8458	1	0.6557	2
2	0.8371	3	0.1267	3
3	0.6763	2	0.8825	1
4	0.2552	4	-0.1375	4

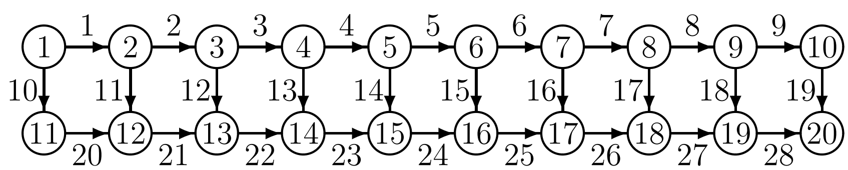


Figure 4: Network for Example 4

5. A Larger Transportation Network Example

The fourth example consisted of 20 nodes, 28 links, and 8 O/D pairs, and is depicted in Figure 4.

A similar transportation network had been used previously in Nagurney (1984) where it is referred to as Network 20; see also Dhanda, Nagurney, and Ramanujam (1999). For simplicity, and easy reproducibility, we considered separable user link cost functions, which were adapted from Network 20 in Nagurney (1984) with the cross-terms removed.

The O/D pairs were: $w_1 = (1, 20)$ and $w_2 = (1, 19)$ and the travel demands: $d_{w_1} = 100$, and $d_{w_2} = 100$. The link cost functions are given in Table 9.

We utilized the projection method (cf. Dafermos, 1980 and Nagurney, 1993) with the embedded Dafermos and Sparrow (1969) equilibration algorithm (see also, e.g., Nagurney, 1984) to compute the equilibrium solutions and to determine the network efficiency according to (7b) as well as the Importance Values and the Importance Rankings of the links according to (9).

The computed efficiency measure for this network was: $\mathcal{E} = .002518$. The computed importance values of the links and their ranking for this transportation network are reported in Table 9.

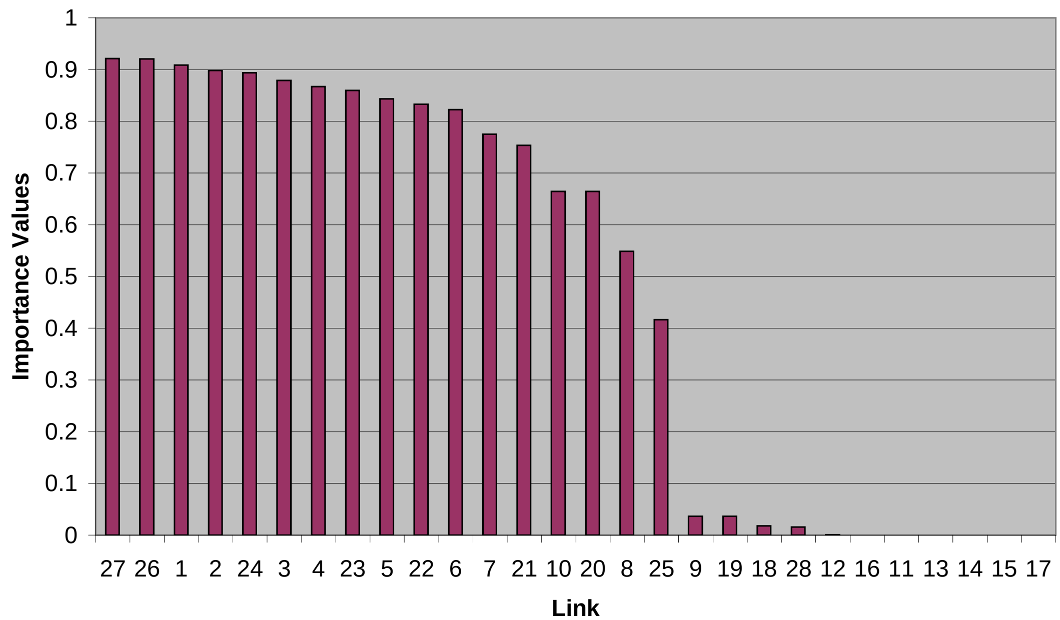
Table 9: Example 4 - Links, Link Cost Functions, Importance Values, and Importance Rankings

Link a	Link Cost Function $c_a(f_a)$	Importance Value	Importance Ranking
1	$.00005f_1^4 + 5f_1 + 500$	0.9086	3
2	$.00003f_2^4 + 4f_2 + 200$	0.8984	4
3	$.00005f_3^4 + 3f_3 + 350$	0.8791	6
4	$.00003f_4^4 + 6f_4 + 400$	0.8672	7
5	$.00006f_5^4 + 6f_5 + 600$	0.8430	9
6	$7f_6 + 500$	0.8226	11
7	$.00008f_7^4 + 8f_7 + 400$	0.7750	12
8	$.00004f_8^4 + 5f_8 + 650$	0.5483	15
9	$.00001f_9^4 + 6f_9 + 700$	0.0362	17
10	$4f_{10} + 800$	0.6641	14
11	$.00007f_{11}^4 + 7f_{11} + 650$	0.0000	22
12	$8f_{12} + 700$	0.0006	20
13	$.00001f_{13}^4 + 7f_{13} + 600$	0.0000	22
14	$8f_{14} + 500$	0.0000	22
15	$.00003f_{15}^4 + 9f_{15} + 200$	0.0000	22
16	$8f_{16} + 300$	0.0001	21
17	$.00003f_{17}^4 + 7f_{17} + 450$	0.0000	22
18	$5f_{18} + 300$	0.0175	18
19	$8f_{19} + 600$	0.0362	17
20	$.00003f_{20}^4 + 6f_{20} + 300$	0.6641	14
21	$.00004f_{21}^4 + 4f_{21} + 400$	0.7537	13
22	$.00002f_{22}^4 + 6f_{22} + 500$	0.8333	10
23	$.00003f_{23}^4 + 9f_{23} + 350$	0.8598	8
24	$.00002f_{24}^4 + 8f_{24} + 400$	0.8939	5
25	$.00003f_{25}^4 + 9f_{25} + 450$	0.4162	16
26	$.00006f_{26}^4 + 7f_{26} + 300$	0.9203	2
27	$.00003f_{27}^4 + 8f_{27} + 500$	0.9213	1
28	$.00003f_{28}^4 + 7f_{28} + 650$	0.0155	19

In Figure 5, we display graphically the Importance Values and Importance Rankings of the links for Example 4.

From the above results, it is clear that transportation planners and security officials should pay most attention to links: 1, 2, and 26, 27, since these are the top four links in terms of Importance Rankings. On the other hand, the elimination of links: 11, 13, 14, 15, and 17 should have no impact on the network performance/efficiency.

Figure 5: Example 4 Link Importance Rankings



6. Summary and Conclusions

In this paper, we have proposed a transportation network efficiency measure, the Nagurney-Qiang measure, that has several notable features:

(1). it captures flows, costs, and behavior of travelers, in addition to network topology;

(2). the resulting importance definition of network components is applicable and well-defined even in the case of disconnected networks;

(3). it contains an existing efficiency measure that has been widely used and applied in the complex network literature, as a special case;

(4). it can be used to identify the importance (and ranking) of either nodes, or links, or both, and

(5). it is readily computable as are the network component importance values and their rankings, given the state-of-the-art of transportation network modeling and computing technology.

We note that the proposed transportation network measure is also relevant to assess the efficiency/performance of other critical infrastructure networks, since it has been established (see Nagurney, 2006a, and the references therein) that supply chain networks (Nagurney, 2006b), electric power generation and distribution networks (Wu et al., 2006), as well as financial networks with intermediation (see Liu and Nagurney, 2006) can all be reformulated and solved as transportation network equilibrium problems. Further application of the new transportation network measure to these applications domains will be the subject of future research.

In addition, our measure is applicable not only to fixed (or inelastic) demand transportation networks but also to elastic demand networks. This will be the topic of a future publication.

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