Network Economics

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Handbook of Computational Econometrics

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1. Introduction

Networks throughout history have provided the foundations by which humans conduct their economic activities. Transportation networks and logistical networks make possible the movement of individuals, goods, and services, whereas communication networks enable the exchange of messages and information. Energy networks provide the fuel to support economic activities.

As noted in Nagurney (2003), the emergence and evolution of physical networks over space and time, such as transportation and communication networks, and the effects of human decision-making on such networks, have given rise, in turn, to the development of elegant theories and scientific methodologies that are network-based. Networks, as a science, have impacted disciplines ranging from economics, engineering, computer science, applied mathematics, physics, sociology, and even biology. The novelty of networks is that they are pervasive, providing the medium for connectivity of our societies and economies, while, methodologically, network theory has developed into a powerful and dynamic mechanism for abstracting complex problems, which, at first glance, may not even appear to be networks, with associated nodes, links, and flows.

The topic of networks, as a scientific subject of inquiry, dates to the classical paper by Euler (1736), the earliest paper on *graph* theory. By a graph in this context is meant, mathematically, a means of abstractly representing a system by its representation in terms of vertices (or nodes) and edges (or arcs, equivalently, links) connecting various pairs of vertices. Euler was interested in determining whether it was possible to walk around Königsberg (later called Kaliningrad) by crossing the seven bridges over the River Pregel precisely once. The problem was represented as a graph (cf. Figure 1) in which the vertices corresponded to land masses and the edges to bridges.

Quesnay (1758), in his *Tableau Economique*, conceptualized the circular flow of financial funds in an economy as a network and this work can be identified as the first paper on the topic of financial networks. Quesnay's basic idea has been utilized in the construction of financial flow of funds accounts, which are a statistical description of the flows of money and credit in an economy (see Cohen (1987) and Hughes and Nagurney (1992)).

The concept of a network in economics was also implicit as early as the classical work of Cournot (1838), who seems to have first explicitly stated that a competitive price is determined by the intersection of supply and demand curves, and had done so in the context of two spatially separated markets in which the cost associated with transporting the goods was also included. Pigou (1920) studied a network system in the form of a transportation network consisting of two routes and noted that the decision-making behavior of the the users of such a system would lead to different flow patterns. The network of concern therein consists of the graph, which is now directed, with the edges or links represented by arrows, as well as the resulting flows on the links. Transportation networks will be the topic in Sections 3 and 8 of this chapter.

As noted in Nagurney (2003), Monge, who had worked under Napoleon in providing infrastructure support for his army, published in 1781 what is probably the first paper on the transportation model (see, e.g., Buckard, Klinz, and Rudolf (1996)). He was interested in minimizing the cost associated with backfilling n places from m other places with surplus brash with cost c_{ij} being proportional to the distance between origin i and destination j.

After the first book on graph theory by König (1936) was published, the economists Kantorovich (1939), Hitchcock (1941), and Koopmans (1947) considered the network flow problem associated

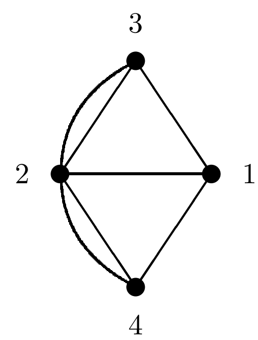

Figure 1: The Euler Graph Representation of the Seven Bridge Königsberg Problem

with this classical minimum cost transportation problem. They provided insights into the special network structure of such problems, which yielded network-based algorithmic approaches. Interestingly, the study of network flows precedes that of optimization techniques, in general, with seminal work done by Dantzig in 1948 in linear programming with the simplex method and, subsequently, adapted for the classical transportation problem in 1951.

Copeland (1952) conceptualized the interrelationships among financial funds as a network and raised the question, "Does money flow like water or electricity?" He provided a "wiring diagram for the main money circuit." Enke (1951) proposed electronic circuits as a means of solving spatial price equilibrium problems, in which goods are produced, consumed, and traded, in the presence of transportation costs. Such analog computational machines, however, were soon to be superseded by digital computers along with appropriate algorithms, based on mathematical programming, which included not only the aforementioned linear programming techniques but other optimization techniques, as well.

Samuelson (1952) revisited the spatial price equilibrium problem and derived a rigorous mathematical formulation of the spatial price equilibrium problem and explicitly recognized and utilized the network structure, which was bipartite (the same structure as in the classical transportation problems), that is, consisting of two sets of nodes (cf. Figure 2). In spatial price equilibrium problems, unlike classical transportation problems, the supplies and the demands are variables, rather than fixed quantities. The work was, later, extended by Takayama and Judge (1964, 1971) and others (cf. Florian and Los (1982), Harker (1985), Dafermos and Nagurney (1987), Nagurney and Kim (1989), Nagurney (1999), and the references therein) to include, respectively, multiple commodities, and asymmetric supply price and demand functions, as well as other extensions, made possible by such advances as quadratic programming techniques, complementarity theory, as well as variational inequality theory (which allowed for the formulation and solution of equilibrium problems for which no optimization reformulation of the governing equilibrium conditions was available). We discuss spatial price equilibrium models in greater detail in Section 5 in this chapter. Section 6 then considers game theoretic foundations and Nash equilibria, along with oligopolistic market equilibrium problems, both aspatial and spatial.

Beckmann, McGuire, and Winsten (1956) provided a rigorous treatment of congested transportation networks, and formulated their solution as mathematical programming problems. Their contributions added significantly to the topic of *network equilibrium problems*, which was later

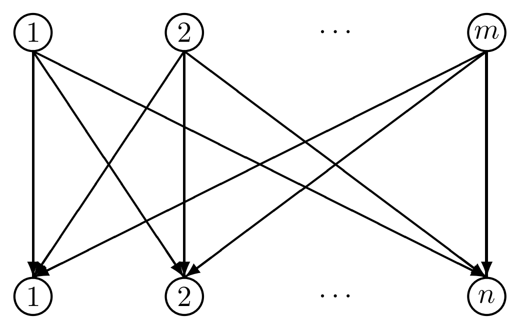

Figure 2: A Bipartite Network with Directed Links

advanced by the contributions of Dafermos and Sparrow (1969), who coined the terms *user-optimization* versus *system-optimization* to correspond, respectively, to the first and second principles of Wardrop (1952). Dafermos and Sparrow (1969) also provided computational methods for the determination of the resulting flows on such networks. Subsequent notable contributions were made by Smith (1979) and Dafermos (1980) who allowed for asymmetric interactions associated with the link travel costs (resulting in no equivalent optimization reformulation of the equilibrium conditions) and established the methodology of variational inequalities as a primary tool for both the qualitative analysis and the solution of such and other related problems. For additional background, see the book by Nagurney (1999). Today, these concepts are as relevant to the Internet, which is a telecommunications network par excellence, and characterized by decentralized decision-making (cf. Roughgarden (2005) and Boyce, Mahmassani, and Nagurney (2005)), as they are to transportation networks, as well as to electric power generation and distribution networks (cf. Wu et al. (2006)), supply chain networks (see Nagurney (2006a)), and financial networks with intermediation (see Liu and Nagurney (2006) and the references therein). We return to supply chain networks in Section 10 of this chapter.

Indeed, many complex systems in which decision-makers/agents compete for scarce resources on a network, be it a physical one, as in the case of congested urban transportation systems, or an abstract one, as in the case of certain economic and financial problems, can be formulated and studied as network equilibrium problems. Applications of network equilibrium problems are common in many disciplines, in particular, in economics and engineering and in operations research and management science (cf. Nagurney (1999, 2006b) and Florian and Hearn (1995)). For a reference to network flows with a focus on linear, rather than nonlinear, problems and numerous applications, see the book by Ahuja, Magnanti, and Orlin (1993).

In this chapter we provide the foundations for the study of network economics, emphasizing both the scientific methodologies, such as variational inequalities and optimization theory in Section 2, and projected dynamical systems in Section 7, as well as numerous applications, accompanied by examples.

2. Variational Inequalities

Equilibrium in a fundamental concept in the study of competitive problems and also central to network economics. Methodologies that have been applied to the study of equilibrium problems include: systems of equations, optimization theory, complementarity theory, as well as fixed point theory. Variational inequality theory, in particular, has become a powerful technique for equilibrium analysis and computation, and has garnered wide application in network-based problems.

Variational inequalities were introduced by Hartman and Stampacchia (1966), principally, for the study of partial differential equation problems drawn from mechanics. That research focused on infinite-dimensional variational inequalities. An exposition of infinite-dimensional variational inequalities and references can be found in Kinderlehrer and Stampacchia (1980).

Smith (1979) provided a formulation of the transportation network equilibrium problem which was then shown by Dafermos (1980) to satisfy a finite-dimensional variational inequality problem. This connection allowed for the construction of more realistic models as well as rigorous computational techniques for equilibrium problems including: transportation network equilibrium problems, spatial price equilibrium problems, oligopolistic market equilibrium problems, as well as economic and financial equilibrium problems (see, e.g., Nagurney (1999, 2003, 2006b) and the references therein). We will overview some of these network economics-based applications in this chapter.

Many mathematical problems can be formulated as variational inequality problems and, hence, this formulation is particularly convenient since it allows for a unified treatment of equilibrium, including network equilibrium, problems and optimization problems.

In this chapter, we assume that the vectors are column vectors, except where noted. We begin with a fundamental definition.

Definition (Variational Inequality Problem) *The finite - dimensional variational inequality problem,* VI$(F, \mathcal{K})$*, is to determine a vector* $X^* \in \mathcal{K} \subset R^n$*, such that*

$$\langle F(X^*)^T, X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K},$$

where F *is a given continuous function from* $\mathcal{K}$ *to* R^n*,* $\mathcal{K}$ *is a given closed convex set, and* $\langle \cdot, \cdot \rangle$ *denotes the inner product in* R^n*, where* R^n *in the* n*-dimensional Euclidean space.*

We now discuss some basic problem types and their relationships to the variational inequality problem. We also provide examples. Proofs of the theoretical results may be found in books by Kinderlehrer and Stampacchia (1980) and Nagurney (1999). For algorithms for the computation of variational inequalities, see also the books by Bertsekas and Tsitsiklis (1989), Nagurney (1999), Patriksson (1994), Nagurney and Zhang (1996).

We begin with systems of equations, which have been used to formulate certain equilibrium problems. We then discuss optimization problems, both unconstrained and constrained, as well as complementarity problems. We conclude with a fixed point problem and its relationship with the variational inequality problem.

Problem Classes

We here briefly review certain problem classes, which appear frequently in equilibrium modeling, and identify their relationships to the variational inequality problem.

Systems of Equations

Systems of equations are common in equilibrium analysis, expressing, for example, that the demand is equal to the supply of various commodities at the equilibrium price levels. Let $\mathcal{K} = R^n$ and let $F : R^n \mapsto R^n$ be a given function. A vector $X^* \in R^n$ is said to solve a *system of equations* if

$$F(X^*) = 0.$$

The relationship to a variational inequality problem is stated in the following proposition.

Proposition *Let $\mathcal{K} = R^n$ and let $F : R^n \mapsto R^n$ be a given vector function. Then $X^* \in R^n$ solves the variational inequality problem* $\mathrm{VI}(F, \mathcal{K})$ *if and only if X^* solves the system of equations*

$$F(X^*) = 0.$$

An Example (Market Equilibrium with Equalities Only)

As an illustration, we now present an example of a system of equations. Consider m consumers, with a typical consumer denoted by j, and n commodities, with a typical commodity denoted by i. Let p denote the n-dimensional vector of the commodity prices with components: $\{p_1, \ldots, p_n\}$.

Assume that the demand for a commodity i, d_i, may, in general, depend upon the prices of all the commodities, that is,

$$d_i(p) = \sum_{j=1}^{m} d_i^j(p),$$

where $d_i^j(p)$ denotes the demand for commodity i by consumer j at the price vector p.

Similarly, the supply of a commodity i, s_i, may, in general, depend upon the prices of all the commodities, that is,

$$s_i(p) = \sum_{j=1}^{m} s_i^j(p),$$

where $s_i^j(p)$ denotes the supply of commodity i of consumer j at the price vector p.

We group the aggregate demands for the commodities into the n-dimensional column vector d with components: $\{d_1, \ldots, d_n\}$ and the aggregate supplies of the commodities into the n-dimensional column vector s with components: $\{s_1, \ldots, s_n\}$.

The market equilibrium conditions that require that the supply of each commodity must be equal to the demand for each commodity at the equilibrium price vector p^*, are equivalent to the following system of equations:

$$s(p^*) - d(p^*) = 0.$$

Clearly, this expression can be put into the standard nonlinear equation form, if we define the vectors $X \equiv p$ and $F(X) \equiv s(p) - d(p)$.

Note, however, that the problem class of nonlinear equations is not sufficiently general to guarantee, for example, that $X^* \geq 0$, which may be desirable in this example in which the vector X refers to prices.

Optimization Problems

Optimization problems, on the other hand, consider explicitly an objective function to be minimized (or maximized), subject to constraints that may consist of both equalities and inequalities. Let f be a continuously differentiable function where $f : \mathcal{K} \mapsto R$. Mathematically, the statement of an *optimization problem* is:

$$\text{Minimize} \quad f(X)$$

subject to:

$$X \in \mathcal{K}.$$

The relationship between an optimization problem and a variational inequality problem is now given.

Proposition *Let X^* be a solution to the optimization problem:*

$$\textit{Minimize} \quad f(X)$$

$$\textit{subject to:} \quad X \in \mathcal{K},$$

where f is continuously differentiable and $\mathcal{K}$ is closed and convex. Then X^* is a solution of the variational inequality problem:

$$\langle \nabla f(X^*)^T, X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}.$$

Furthermore, we have the following:

Proposition *If $f(X)$ is a convex function and X^* is a solution to* $\mathrm{VI}(\nabla f, \mathcal{K})$*, then X^* is a solution to the above optimization problem.*

If the feasible set $\mathcal{K} = R^n$, then the unconstrained optimization problem is also a variational inequality problem.

On the other hand, in the case where a certain symmetry condition holds, the variational inequality problem can be reformulated as an optimization problem. In other words, in the case that the variational inequality formulation of the equilibrium conditions underlying a specific problem is characterized by a function with a symmetric Jacobian, then the solution of the equilibrium conditions and the solution of a particular optimization problem are one and the same. We first introduce the following definition and then fix this relationship in a theorem.

Definition *An $n \times n$ matrix $M(X)$, whose elements $m_{ij}(X)$; $i = 1, \ldots, n$; $j = 1, \ldots, n$, are functions defined on the set $S \subset R^n$, is said to be positive semidefinite on S if*

$$v^T M(X) v \geq 0, \quad \forall v \in R^n, X \in S.$$

It is said to be positive definite on S if

$$v^T M(X) v > 0, \quad \forall v \neq 0, v \in R^n, X \in S.$$

It is said to be strongly positive definite on S if

$$v^T M(X) v \geq \alpha \|v\|^2, \quad \textit{for some} \quad \alpha > 0, \quad \forall v \in R^n, X \in S.$$

Note that if $\gamma(X)$ is the smallest eigenvalue, which is necessarily real, of the symmetric part of $M(X)$, that is, $\frac{1}{2}\left[M(X) + M(X)^T\right]$, then it follows that (i). $M(X)$ is positive semidefinite on S if and only if $\gamma(X) \geq 0$, for all $X \in S$; (ii). $M(X)$ is positive definite on S if and only if $\gamma(X) > 0$, for all $X \in S$; and (iii). $M(X)$ is strongly positive definite on S if and only if $\gamma(X) \geq \alpha > 0$, for all $X \in S$.

Theorem *Assume that $F(X)$ is continuously differentiable on $\mathcal{K}$ and that the Jacobian matrix*

$$\nabla F(X) = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial X_1} & \cdots & \frac{\partial F_n}{\partial X_n} \end{bmatrix}$$

is symmetric and positive semidefinite. Then there is a real-valued convex function $f : \mathcal{K} \mapsto R^1$ satisfying

$$\nabla f(X) = F(X)$$

with X^* the solution of VI$(F, \mathcal{K})$ *also being the solution of the mathematical programming problem:*

$$\textit{Minimize} \quad f(X)$$

$$\textit{subject to:} \quad X \in \mathcal{K}.$$

Hence, although the variational inequality problem encompasses the optimization problem, a variational inequality problem can be reformulated as a convex optimization problem, only when the symmetry condition and the positive semidefiniteness condition hold.

The variational inequality is the more general problem in that it can also handle a function $F(X)$ with an asymmetric Jacobian. Historically, many equilibrium problems were reformulated as optimization problems, under precisely such a symmetry assumption. The assumption, however, in terms of applications was restrictive and precluded the more realistic modeling of multiple commodities, multiple modes and/or classes in competition. Moreover, the objective function that resulted was sometimes artificial, without a clear economic interpretation, and simply a mathematical device.

Complementarity Problems

The variational inequality problem also contains the complementarity problem as a special case. Complementarity problems are defined on the nonnegative orthant.

Let R^n_+ denote the nonnegative orthant in R^n, and let $F : R^n \mapsto R^n$. The *nonlinear complementarity problem* over R^n_+ is a system of equations and inequalities stated as:

Find $X^* \geq 0$ such that

$$F(X^*) \geq 0 \quad \text{and} \quad \langle F(X^*)^T, X^* \rangle = 0.$$

Whenever the mapping F is affine, that is, whenever $F(X) = MX + b$, where M is an $n \times n$ matrix and b an $n \times 1$ vector, the problem is then known as the *linear complementarity* problem.

The relationship between the complementarity problem and the variational inequality problem is as follows.

Proposition VI(F, R^n_+) *and the complementarity problem have precisely the same solutions, if any.*

An Example (Market Equilibrium with Equalities and Inequalities)

A nonlinear complementarity formulation of market equilibrium is now presented. Assume that the prices must now be nonnegative in equilibrium. Hence, we consider the following situation, in which the demand functions are given as previously as are the supply functions, but now, instead of the market equilibrium conditions, which are represented of a system of equations, we have the following equilibrium conditions. For each commodity i; $i = 1, \ldots, n$:

$$s_i(p^*) - d_i(p^*) \begin{cases} = 0, & \text{if } p^*_i > 0 \\ \geq 0, & \text{if } p^*_i = 0. \end{cases}$$

These equilibrium conditions state that if the price of a commodity is positive in equilibrium then the supply of that commodity must be equal to the demand for that commodity. On the other hand, if the price of a commodity at equilibrium is zero, then there may be an excess supply of that commodity at equilibrium, that is, $s_i(p^*) - d_i(p^*) > 0$, or the market clears. Furthermore, this system of equalities and inequalities guarantees that the prices of the instruments do not take on negative values, which may occur in the system of equations expressing the market clearing conditions.

The nonlinear complementarity formulation of this problem is as follows.

Determine $p^* \in R^n_+$, satisfying:

$$s(p^*) - d(p^*) \geq 0 \quad \text{and} \quad \langle (s(p^*) - d(p^*))^T, p^* \rangle = 0.$$

Moreover, since a nonlinear complementarity problem is a special case of a variational inequality problem, we may rewrite the nonlinear complementarity formulation of the market equilibrium problem above as the following variational inequality problem:

Determine $p^* \in R^n_+$, such that

$$\langle (s(p^*) - d(p^*))^T, p - p^* \rangle \geq 0, \quad \forall p \in R^n_+.$$

Note, first, that in the *special* case of demand functions and supply functions that are separable, the Jacobians of these functions are symmetric since they are diagonal and given, respectively, by

$$\nabla s(p) = \begin{pmatrix} \frac{\partial s_1}{\partial p_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{\partial s_2}{\partial p_2} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\partial s_n}{\partial p_n} \end{pmatrix},$$

$$\nabla d(p) = \begin{pmatrix} \frac{\partial d_1}{\partial p_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{\partial d_2}{\partial p_2} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\partial d_n}{\partial p_n} \end{pmatrix}.$$

Indeed, in this special case model, the supply of a commodity depends only upon the price of that commodity and, similarly, the demand for a commodity depends only upon the price of that commodity.

Hence, in this special case, the price vector p^* that satisfies the equilibrium conditions can be obtained by solving the following optimization problem:

$$\text{Minimize} \sum_{i=1}^{n} \int_0^{p_i} s_i(x)dx - \sum_{i=1}^{n} \int_0^{p_i} d_i(y)dy$$

subject to:

$$p_i \geq 0, \quad i = 1, \dots, n.$$

One also obtains an optimization reformulation of the equilibrium conditions, provided that the following symmetry condition holds: $\frac{\partial s_i}{\partial p_k} = \frac{\partial s_k}{\partial p_i}$ and $\frac{\partial d_i}{\partial p_k} = \frac{\partial d_k}{\partial p_i}$ for all commodities i, k. In other words, the price of a commodity k affects the supply of a commodity i in the same way that the price of a commodity i affects the price of a commodity k. A similar situation must hold for the demands for the commodities.

However, such conditions are limiting from the application standpoint and, hence, the appeal of variational inequality problem that enables the formulation and, ultimately, the computation of equilibria where such restrictive symmetry assumptions on the underlying functions need no longer hold. Indeed, such symmetry assumptions were not imposed in the variational inequality problem.

An Example (Market Equilibrium with Equalities and Inequalities and Policy Interventions)

We now provide a generalization of the preceding market equilibrium model to allow for price policy interventions in the form of price floors and ceilings. Let p_i^C denote the imposed price ceiling on the price of commodity i, and we let p_i^F denote the imposed price floor on the price of commodity i. Then we have the following equilibrium conditions. For each commodity i; $i = 1, \dots, n$:

$$s_i(p^*) - d_i(p^*) \begin{cases} \leq 0, & \text{if } p_i^* = p_i^C \\ = 0, & \text{if } p_i^F < p_i^* < p_i^C \\ \geq 0, & \text{if } p_i^* = p_i^F. \end{cases}$$

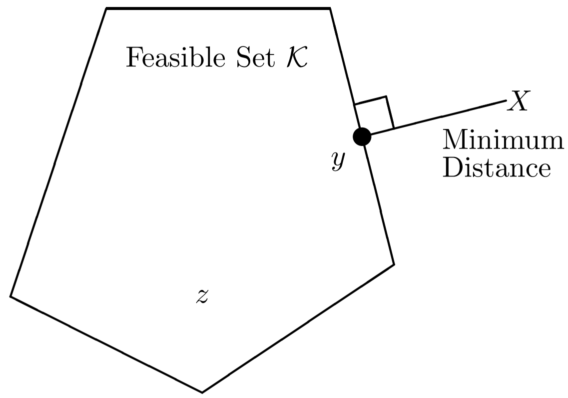

Figure 3: The Norm Projection y of X on the Set $\mathcal{K}$

These equilibrium conditions state that if the price of a commodity in equilibrium lies between the imposed price floor and ceiling, then the supply of that commodity must be equal to the demand for that commodity. On the other hand, if the price of a commodity at equilibrium is at the imposed floor, then there may be an excess supply of that commodity at equilibrium, that is, $s_i(p^*) - d_i(p^*) > 0$, or the market clears. In contrast, if the price of a commodity in equilibrium is at the imposed ceiling, then there may be an excess demand of the commodity in equilibrium.

The variational inequality formulation of the governing equilibrium conditions is:

Determine $p^* \in \mathcal{K}$, such that

$$\langle (s(p^*) - d(p^*))^T, p - p^* \rangle \geq 0, \quad \forall p \in \mathcal{K},$$

where the feasible set $\mathcal{K} \equiv \{p | p^F \leq p \leq p^C\}$, where p^F and p^C denote, respectively, the n-dimensional column vectors of imposed price floors and ceilings.

Fixed Point Problems

We turn to a discussion of fixed point problems in conjunction with variational inequality problems. We also provide the geometric interpretation of the variational inequality problem and its relationship to a fixed point problem.

We first define a norm projection. For a graphical depiction, see Figure 3.

Definition (A Norm Projection) *Let $\mathcal{K}$ be a closed convex set in R^n. Then for each $X \in R^n$, there is a unique point $y \in \mathcal{K}$, such that*

$$\|X - y\| \leq \|X - z\|, \quad \forall z \in \mathcal{K},$$

and y is known as the orthogonal projection of X on the set $\mathcal{K}$ with respect to the Euclidean norm, that is,

$$y = P_{\mathcal{K}}(X) = \arg\min_{z \in \mathcal{K}} \|X - z\|.$$

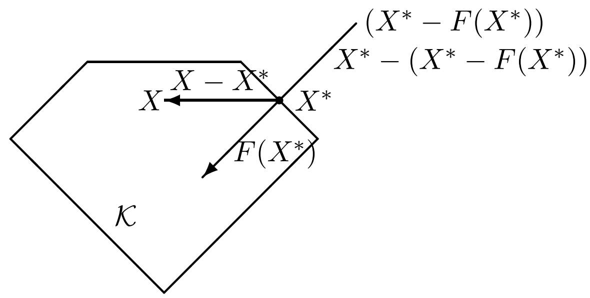

Figure 4: Geometric Depiction of the Variational Inequality Problem and its Fixed Point Equivalence (with $\gamma = 1$)

In other words, the closest point to X lying in the set $\mathcal{K}$ is given by y.

We now present a property of the projection operator that is useful both in the qualitative analysis of equilibria and in their computation. Let $\mathcal{K}$ again be a closed convex set. Then the projection operator $P_{\mathcal{K}}$ is *nonexpansive*, that is,

$$\|P_{\mathcal{K}}X - P_{\mathcal{K}}X'\| \leq \|X - X'\|, \quad \forall X, X' \in R^n.$$

The relationship between a variational inequality and a fixed point problem can now be stated (see Figure 4).

Theorem *Assume that $\mathcal{K}$ is closed and convex. Then $X^* \in \mathcal{K}$ is a solution of the variational inequality problem* VI$(F, \mathcal{K})$ *if and only if X^* is a fixed point of the map: $P_{\mathcal{K}}(I - \gamma F) : \mathcal{K} \mapsto \mathcal{K}$, for $\gamma > 0$, that is,*

$$X^* = P_{\mathcal{K}}(X^* - \gamma F(X^*)).$$

As noted above, variational inequality theory is a powerful tool in the qualitative analysis of equilibria. Here now further discuss the geometric interpretation of the variational inequality problem and conditions for existence and uniqueness of solutions. For proofs of such theoretical results, see the books by Kinderlehrer and Stampacchia (1980) and Nagurney (1999). For stability and sensitivity analysis of variational inequalities, including applications, see Nagurney (1999), and the references therein.

From the definition of a variational inequality problem, one can deduce that the necessary and sufficient condition for X^* to be a solution to VI$(F, \mathcal{K})$ is that

$$-F(X^*) \in C(X^*),$$

where $C(X)$ denotes the *normal cone* of $\mathcal{K}$ at X defined by

$$C(X) \equiv \{y \in R^n : \langle y^T, X' - X \rangle \ \leq 0, \ \forall X' \in \mathcal{K}\}.$$

Existence of a solution to a variational inequality problem follows from continuity of the function F entering the variational inequality, provided that the feasible set $\mathcal{K}$ is compact. Indeed, we have the following:

Theorem *If $\mathcal{K}$ is a compact (closed and bounded) convex set and $F(X)$ is continuous on $\mathcal{K}$, then the variational inequality problem admits at least one solution X^*.*

In the case of an unbounded feasible set $\mathcal{K}$, this Theorem is no longer applicable; the existence of a solution to a variational inequality problem can, nevertheless, be established under the subsequent condition.

Let $B_R(0)$ denote a closed ball with radius R centered at 0 and let $\mathcal{K}_R = \mathcal{K} \cap B_R(0)$. $\mathcal{K}_R$ is then bounded. By VI_R is denoted then the variational inequality problem:

Determine $X_R^* \in \mathcal{K}_R$, such that

$$\langle F(X_R^*)^T, y - X_R^* \rangle \geq 0, \quad \forall y \in \mathcal{K}_R.$$

We now state

Theorem $\mathrm{VI}(F, \mathcal{K})$ *admits a solution if and only if there exists an $R > 0$ and a solution of VI_R, X_R^*, such that $\|X_R^*\| < R$.*

Although $\|X_R^*\| < R$ may be difficult to check, one may be able to identify an appropriate R based on the particular application.

Existence of a solution to a variational inequality problem may also be established under the coercivity condition, as in the subsequent corollary.

Corollary *Suppose that $F(X)$ satisfies the coercivity condition:*

$$\frac{\langle (F(X) - F(X_0))^T, X - X_0 \rangle}{\|X - X_0\|} \rightarrow \infty$$

as $\|X\| \rightarrow \infty$ for $X \in \mathcal{K}$ and for some $X_0 \in \mathcal{K}$. Then $\mathrm{VI}(F, \mathcal{K})$ *always has a solution.*

Corollary *Suppose that X^* is a solution of* $\mathrm{VI}(F, \mathcal{K})$ *and $X^* \in \mathcal{K}^0$, the interior of $\mathcal{K}$. Then $F(X^*) = 0$.*

Qualitative properties of existence and uniqueness become easily obtainable under certain monotonicity conditions. First we outline the definitions and then present the results.

Definition (Monotonicity) *$F(X)$ is monotone on $\mathcal{K}$ if*

$$\langle (F(X^1) - F(X^2))^T, X^1 - X^2 \rangle \geq 0, \quad \forall X^1, X^2 \in \mathcal{K}.$$

Definition (Strict Monotonicity) *$F(X)$ is strictly monotone on $\mathcal{K}$ if*

$$\langle (F(X^1) - F(X^2))^T, X^1 - X^2 \rangle > 0, \qquad \forall X^1, X^2 \in \mathcal{K}, \quad X^1 \neq X^2.$$

Definition (Strong Monotonicity) *$F(X)$ is strongly monotone if for some $\alpha > 0$*

$$\langle (F(X^1) - F(X^2))^T, X^1 - X^2 \rangle \geq \alpha \|X^1 - X^2\|^2, \quad \forall X^1, X^2 \in \mathcal{K}.$$

Definition (Lipschitz Continuous) *$F(X)$ is Lipschitz continous if there exists an $L > 0$, such that*

$$\|F(X^1) - F(X^2)\| \leq L \|X^1 - X^2\|, \quad \forall X^1, X^2 \in \mathcal{K}.$$

Similarly, one may define local monotonicity (strict monotonicity, strong monotonicity) if one restricts the points: X^1, X^2 in the neighborhood of a certain point $\bar{X}$. Let $B(\bar{X})$ denote a ball in R^n centered at $\bar{X}$.

Definition (Local Monotonicity) *$F(X)$ is locally monotone at $\bar{X}$ if*

$$\langle (F(X^1) - F(X^2))^T, X^1 - X^2 \rangle \geq 0, \quad \forall X^1, y^1 \in \mathcal{K} \cap B(\bar{X}).$$

Definition (Local Strict Monotonicity) *$F(X)$ is locally strictly monotone at $\bar{X}$ if*

$$\langle (F(X^1) - F(X^2))^T, X^1 - X^2 \rangle > 0, \quad \forall X^1, X^2 \in \mathcal{K} \cap B(\bar{X}), \quad X^1 \neq X^2.$$

Definition (Local Strong Monotonicity) *$F(X)$ is locally strongly monotone at $\bar{X}$ if for some $\alpha > 0$*

$$\langle (F(X^1) - F(X^2))^T, X^1 - X^2 \rangle \geq \alpha \|X^1 - X^2\|^2, \quad \forall X^1, X^2 \in \mathcal{K} \cap B(\bar{X}).$$

A uniqueness result is presented in the subsequent theorem.

Theorem *Suppose that $F(X)$ is strictly monotone on $\mathcal{K}$. Then the solution is unique, if one exists.*

Similarly, one can show that if F is locally strictly monotone on $\mathcal{K}$, then VI$(F, \mathcal{K})$ has at most one local solution.

Monotonicity is closely related to positive definiteness.

Theorem *Suppose that $F(X)$ is continuously differentiable on $\mathcal{K}$ and the Jacobian matrix*

$$\nabla F(X) = \begin{bmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial X_1} & \cdots & \frac{\partial F_n}{\partial X_n} \end{bmatrix},$$

which need not be symmetric, is positive semidefinite (positive definite). Then $F(X)$ is monotone (strictly monotone).

Proposition *Assume that $F(X)$ is continuously differentiable on $\mathcal{K}$ and that $\nabla F(X)$ is strongly positive definite. Then $F(X)$ is strongly monotone.*

One obtains a stronger result in the special case where $F(X)$ is linear.

Corollary *Suppose that $F(X) = MX + b$, where M is an $n \times n$ matrix and b is a constant vector in R^n. The function F is monotone if and only if M is positive semidefinite. F is strongly monotone if and only if M is positive definite.*

Proposition *Assume that $F : \mathcal{K} \mapsto R^n$ is continuously differentiable at $\bar{X}$. Then $F(X)$ is locally strictly (strongly) monotone at $\bar{X}$ if $\nabla F(\bar{X})$ is positive definite (strongly positive definite), that is,*

$$v^T F(\bar{X}) v > 0, \quad \forall v \in R^n, v \neq 0,$$

$$v^T \nabla F(\bar{X}) v \geq \alpha \|v\|^2, \quad \textit{for some} \quad \alpha > 0, \quad \forall v \in R^n.$$

The following theorem provides a condition under which both existence and uniqueness of the solution to the variational inequality problem are guaranteed. Here no assumption on the compactness of the feasible set $\mathcal{K}$ is made.

Theorem *Assume that $F(X)$ is strongly monotone. Then there exists precisely one solution X^* to* VI$(F, \mathcal{K})$.

Hence, in the case of an unbounded feasible set $\mathcal{K}$, strong monotonicity of the function F guarantees both existence and uniqueness. If $\mathcal{K}$ is compact, then existence is guaranteed if F is continuous, and only the strict monotonicity condition needs to hold for uniqueness to be guaranteed.

Assume now that $F(X)$ is both strongly monotone and Lipschitz continuous. Then the projection $P_{\mathcal{K}}[X - \gamma F(X)]$ is a contraction with respect to X, that is, we have the following:

Theorem *Fix $0 < \gamma \leq \frac{\alpha}{L^2}$ where α and L are the constants appearing, respectively, in the strong monotonicity and the Lipschitz continuity condition definitions. Then*

$$\|P_{\mathcal{K}}(X - \gamma F(X)) - P_{\mathcal{K}}(y - \gamma F(y))\| \leq \beta \|X - y\|$$

for all $X, y \in \mathcal{K}$, where

$$(1 - \gamma\alpha)^{\frac{1}{2}} \leq \beta < 1.$$

An immediate consequence of this theorem and the Banach Fixed Point Theorem is:

Corollary *The operator $P_{\mathcal{K}}(X - \gamma F(X))$ has a unique fixed point X^*.*

3. Transportation Networks: User-Optmization versus System-Optimization

The transportation network equilibrium problem, sometimes also referred to as the *traffic assignment* problem, addresses the problem of users of a congested transportation network seeking to determine their minimal cost travel paths from their origins to their respective destinations. It is a classical *network equilibrium* problem and was studied by Pigou (1920) (see also Kohl (1841)), who considered a two-node, two-link (or path) transportation network, and was further developed by Knight (1924). The congestion on a link is modeled by having the travel cost as perceived by the user be a nonlinear function; in many applications the cost is convex or monotone.

The main objective in the study of transportation network equilibria is the determination of traffic patterns characterized by the property that, once, established, no user or potential user may decrease his travel cost or disutility by changing his travel arrangements. This behavioral principle is actually the first principle of travel behavior proposed by Wardrop (1952). Indeed, Wardrop (1952) stated two principles of travel behavior, which are now named after him:

First Principle: The journey times of all routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

Second Principle: The average journey time is minimal.

As noted in the Introduction, Dafermos and Sparrow (1969) coined the terms *user-optimized* and *system-optimized* transportation networks to distinguish between these two distinct situations. In the user-optimized problem users act unilaterally, in their own self-interest, in selecting their routes, and the equilibrium pattern satisfies Wardrop's first principle, whereas is the system-optimized problem users select routes according to what is optimal from a societal point of view, in that the total costs in the system are minimized. In the latter problem, the marginal total costs rather than the average user costs are equalized. Dafermos and Sparrow (1969) also introduced *equilibration* algorithms based on the path formulation of the problem which exploited the network structure of the problem (see also Leventhal, Nemhauser, and Trotter (1973)). Another algorithm that is widely used in practice for the symmetric transportation network equilibrium problem is the Frank-Wolfe (1956) algorithm. Today, solutions to large-scale transportation network equilibrium problems are commonly computed in practice (cf. Bar-Gera (2002) and Boyce and Bar-Gera (2005)).

In the *standard* transportation network equilibrium problem, the travel cost on a link depends solely upon the flow on that link whereas the travel demand associated with an O/D pair may be either *fixed*, that is given, or *elastic*, that is, it depends upon the travel cost associated with the particular origin/destination (O/D) pair. Beckmann, McGuire, and Winsten (1956) in their seminal work showed that the equilibrium conditions in the case of separable (and increasing) functions coincided with the optimality conditions of an appropriately constructed convex optimization problem. Such a reformulation also holds in the nonseparable case provided that the Jacobian of the functions is symmetric. The reformulation of the equilibrium conditions in the *symmetric* case as a convex optimization problem was also done in the case of the spatial price equilibrium problem by Samuelson (1952).

Such a symmetry assumption was limiting, however, from both modeling and application standpoints. The discovery of Dafermos (1980) that the transportation network equilibrium conditions as formulated by Smith (1979) defined a variational inequality problem allowed for such modeling extensions as: asymmetric link travel costs, link interactions, and multiple modes of transportation and classes of users. It also stimulated the development of rigorous algorithms for the computation of solutions to such problems as well as the qualitative study of equilibrium patterns in terms of the existence and uniqueness of solutions in addition to sensitivity analysis and stability issues.

Algorithms that have been applied to solve general transportation network equilibrium problems include projection and relaxation methods (cf. Nagurney (1984), Patriksson (1994), Florian and Hearn (1995), Nagurney (1999)) and simplicial decomposition (cf. Lawphongpanich and Hearn (1984), and the references therein). Projection and relaxation methods resolve the variational inequality problem into a series of convex optimization problems, with projection methods yielding quadratic programming problems and relaxation methods, typically, nonlinear programming problems. Hence, the overall effectiveness of a variational inequality-based method for the computation of traffic network equilibria will depend upon the algorithm used at each iteration.

Sensitivity analysis for traffic networks was conducted by Hall (1978) and Steinberg and Zangwill (1983) and in a variational inequality framework by Dafermos and Nagurney (1984a,b,c,d). Some of the work was, in part, an attempt to explain traffic network paradoxes such as the Braess (1968) paradox (see also Frank (1981), Pas and Principio (1997), Nagurney (1999), Braess, Nagurney, and Wakolbinger (2005), and Nagurney, Parkes, and Daniele (2006))) in which the addition of a link results in all users of the transportation network being worse off. We will overview the Braess paradox later in this Section.

A variety of transportation network equilibrium models are now presented along with the variational inequality formulations of the governing equilibrium conditions.

Transportation Network Equilibrium with Travel Disutility Functions

The first model described here is due to Dafermos (1982a). In this model, the travel demands are not known and fixed but are variables. This model is also commonly referred to as the elastic demand model with travel disutiliy functions.

Consider a network $\mathcal{G} = [\mathcal{N}, \mathcal{L}]$ consisting of nodes $\mathcal{N}$ and directed links $\mathcal{L}$. Let a denote a link of the network connecting a pair of nodes, and let p denote a path (assumed to be acyclic) consisting of a sequence of links connecting an O/D pair w. P_w denotes the set of paths connecting the O/D pair w with n_{P_w} paths. Let W denote the set of O/D pairs and P the set of paths in the network. There are J O/D pairs, n_A links, and n_p paths in the network.

Let x_p represent the flow on path p and let f_a denote the flow on link a. The following conservation of flow equation must hold for each link a:

$$f_a = \sum_{p \in P} x_p \delta_{ap},$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. Hence, the flow on a link a is equal to the sum of all the path flows on paths that contain the link a.

Moreover, if we let d_w denote the demand associated with an O/D pair w, then we must have that for each O/D pair w:

$$d_w = \sum_{p \in P_w} x_p,$$

where $x_p \geq 0$, for all p, that is, the sum of all the path flows on paths connecting the O/D pair w must be equal to the demand d_w. We refer to this expression as the demand feasibility condition.

The *conservation of flow* equations correspond, hence, to the above two expressions relating the link flows to the path flows and the demands to the path flows, with the path flows being nonnegative.

Let x denote the column vector of path flows with dimension n_P.

Let c_a denote the user cost associated with traversing link a, and let C_p the user cost associated with traversing path p. Then

$$C_p = \sum_{a \in \mathcal{L}} c_a \delta_{ap}.$$

In other words, the cost of a path is equal to the sum of the costs on the links comprising that path. We group the link costs into the vector c with n_A components, and the path costs into the vector C with n_P components. In practice, the user cost on a link often reflects the travel time and such cost functions are commonly estimated in practice. For example, the form of the user link cost functions that is often used is that of the Bureau of Public Roads (cf. Sheffi (1985) and Nagurney (2000)). It is a nonlinear cost function that captures the capacity of a link, and the congestion effects on travel time.

We also assume that we are given a travel disutility function λ_w for each O/D pair w. We group the travel disutilities into the vector λ with J components.

We assume that, in general, the cost associated with a link may depend upon the entire link flow pattern, that is, $c_a = c_a(f)$ and that the travel disutility associated with an O/D pair may depend upon the entire demand pattern, that is, $\lambda_w = \lambda_w(d)$, where f is the n_A-dimensional vector of link flows and d is the J-dimensional vector of travel demands.

Definition (Transportation Network Equilibrium with Elastic Demands) (Beckmann, McGuire, and Winsten (1956), Dafermos (1982a)) *A vector $x^* \in R_+^{n_P}$, which induces a vector d^*, through the demand feasibility condition, is a transportation network equilibrium if for each path $p \in P_w$ and every O/D pair w:*

$$C_p(x^*) \begin{cases} = \lambda_w(d^*), & \text{if} \quad x_p^* > 0 \\ \geq \lambda_w(d^*), & \text{if} \quad x_p^* = 0. \end{cases}$$

In equilibrium, only those paths connecting an O/D pair that have minimal user costs are used, and their costs are equal to the travel disutility associated with traveling between the O/D pair.

The equilibrium conditions have been formulated as a variational inequality problem by Dafermos (1982a). In particular, we have:

Theorem *$(x^*, d^*) \in K^1$ is a transportation network equilibrium pattern, that is, satisfies the equilibrium conditions if and only if it satisfies the variational inequality problem:*

Path Flow Formulation

$$\langle C(x^*)^T, x - x^* \rangle - \langle \lambda(d^*)^T, d - d^* \rangle \geq 0, \quad \forall\, (x, d) \in K^1,$$

where $K^1 \equiv \{(x, d) : x \geq 0;$ and the demand feasibility condition holds$\}$, or, equivalently, $(f^*, d^*) \in K^2$ satisfies the variational inequality problem:

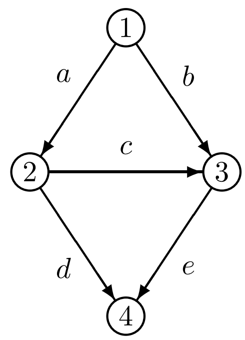

Figure 5: A Transportation Network Equilibrium Example

Link Flow Formulation

$$\langle c(f^*)^T, f - f^* \rangle - \langle \lambda(d^*)^T, d - d^* \rangle \geq 0, \quad \forall\, (f, d) \in K^2,$$

where $K^2 \equiv \{(f, d) : x \geq 0;$ *and the conservation of flow and demand feasibility conditions hold*$\}$ *and* $\langle \cdot, \cdot \rangle$ *denotes the inner product.*

Both these variational inequality problems can be put into standard variational inequality form; existence and uniqueness results can be obtained under strong monotonicity assumptions on the functions $c(f)$ and $-\lambda(d)$ (see Section 2).

An Example – An Elastic Demand Transportation Network Equilibrium Problem

For illustrative purposes, we now present an example, which is illustrated in Figure 5. Assume that there are four nodes and five links in the network as depicted in the figure and a single O/D pair $w = (1, 4)$. We define the paths connecting the O/D pair w: $p_1 = (a, d)$, $p_2 = (b, e)$, and $p_3 = (a, c, e)$.

Assume that the link travel cost functions are given by:

$$c_a(f) = 5f_a + 5f_c + 5, \quad c_b(f) = 10f_b + f_a + 5, \quad c_c(f) = 10f_c + 5f_b + 10,$$

$$c_d(f) = 7f_d + 2f_e + 1, \quad c_e(f) = 10f_e + f_c + 21,$$

and the travel disutility function is given by:

$$\lambda_w(d) = -3d_w + 181.$$

The equilibrium path flow pattern is: $x^*_{p_1} = 10$, $x^*_{p_2} = 5$, $x^*_{p_3} = 0$, with induced link flows: $f^*_a = 10$, $f^*_b = 5$, $f^*_c = 0$, $f^*_d = 10$, $f^*_e = 5$, and the equilibrium travel demand: $d^*_w = 15$.

The incurred travel costs are: $C_{p_1} = C_{p_2} = 136$, $C_{p_3} = 161$, and the incurred travel disutility $\lambda_w = 136$. Hence, as stated by the elastic demand transportation network equilibrium conditions, all used paths have equal and minimal travel costs with the costs equal to the travel disutility associated with the O/D pair that the paths join.

In the special case (cf. Beckmann, McGuire, and Winsten (1956) and Section 2), where the user link cost functions are separable, that is, $c_a = c_a(f_a)$, for all links $a \in \mathcal{L}$, and the travel disutility functions are also separable, that is, $\lambda_w = \lambda_w(d_w)$, for all $w \in W$, the transportation network equilibrium pattern can be obtained as the solution to the optimization problem:

$$\text{Minimize}_{(f,d)\in K^2} \sum_{a\in\mathcal{L}} \int_0^{f_a} c_a(y)dy - \sum_{w\in W} \int_0^{d_w} \lambda_w(z)dz.$$

Elastic Demand Transportation Network Problems with Known Travel Demand Functions

We now consider elastic demand transportation network problems in which the travel demand functions rather than the travel disutility functions are assumed to be given. It is important to have formulations of both such cases, since, in practice, it may be easier to estimate one form of function over the other. The model is due to Dafermos and Nagurney (1984d). We retain the notation of the preceding model except for the following changes. We assume now that the demand d_w, associated with traveling between O/D pair w, is now a function, in general, of the travel disutilities associated with traveling between all the O/D pairs, that is, $d_w = d_w(\lambda)$. The vector of demands d is a row vector and the vector of travel disutilities λ is a column vector.

Note that the expression relating the link loads to the path flows is still valid, as is the nonnegativity assumption on the path flows. In addition, the link cost and path cost functions are as defined previously.

The transportation network equilibrium conditions are now the following:

Definition (Transportation Network Equilibrium with Elastic Demands and Known Demand Functions) *A path flow pattern x^* and a travel disutility pattern λ^* is a transportation network equilibrium pattern in the case of elastic demands but known demand functions, if, for every O/D pair w and each path $p \in P_w$, the following equalities and inequalities hold:*

$$C_p(x^*) \begin{cases} = \lambda_w^*, & \text{if } \ x_p^* > 0 \\ \geq \lambda_w^*, & \text{if } \ x_p^* = 0, \end{cases}$$

and

$$d_w(\lambda^*) \begin{cases} = \sum_{p\in P_w} x_p^*, & \text{if } \ \lambda_w^* > 0 \\ \leq \sum_{p\in P_w} x_p^*, & \text{if } \ \lambda_w^* = 0. \end{cases}$$

The first system of equalities and inequalities above is analogous to the transportation network equilibrium conditions for the preceding elastic demand model, but now the equilibrium travel disutilities λ^* are to be determined, rather than the equilibrium travel demands d^*.

The second set of equalities and inequalities, in turn, has the following interpretation: if the travel disutility (or price) associated with traveling between an O/D pair w is positive, then the "market" clears for that O/D pair, that is, the sum of the path flows on paths connecting that O/D pair are equal to the demand associated with that O/D pair; if the travel disutility (or price) is zero, then the sum of the path flows can exceed the demand.

Here we can immediately write down the governing variational inequality formulation in path flow and travel disutility variables (see, also, e.g., Dafermos and Nagurney (1984d)).

Theorem (Variational Inequality Formulation) $(x^*, \lambda^*) \in R_+^{n_P+J}$ *is a transportation network equilibrium if and only if it satisfies the variational inequality problem:*

$$\sum_{w\in W}\sum_{p\in P_w}[C_p(x^*)-\lambda_w^*]\times\left[x_p-x_p^*\right]-\sum_{w\in W}\left[d_w(\lambda^*)-\sum_{p\in P_w}x_p^*\right]\times[\lambda_w-\lambda_w^*]\geq 0, \quad \forall (x,\lambda)\in R_+^{n_P+J},$$

or, in vector form:

$$\langle (C(x^*)-\tilde{B}^T\lambda^*)^T, x-x^*\rangle - \langle (d(\lambda^*)-\tilde{B}x^*)^T, \lambda-\lambda^*\rangle \geq 0, \quad \forall (x,\lambda)\in R_+^{n_P+J},$$

where $\tilde{B}$ *is the* $J\times n_P$*-dimensional matrix with element* $(w,p)=1$, *if* $p\in P_w$, *and 0, otherwise.*

This variational inequality can also be put into standard form, as in Section 2, if we defne $X\equiv(x,\lambda)$, $F(X)\equiv(C(x)-\tilde{B}^T\lambda),(d(\lambda)-\tilde{B}x)$, and $\mathcal{K}\equiv R_+^{n_P+J}$.

Fixed Demand Transportation Network Problems

We now present the path flow and link flow variational inequality formulations of the transportation network equilibrium conditions in the case of fixed travel demands, introduced in Smith (1979) and Dafermos (1980).

We retain the notation of the preceding two models. However, in contrast, it is assumed now that there is a fixed and known travel demand associated with traveling between each O/D pair in the network. Let d_w denote the traffic demand between O/D pair w, which is assumed to be known and fixed. The demand must satisfy, for each $w\in W$,

$$d_w = \sum_{p\in P_w} x_p,$$

where $x_p\geq 0, \forall p$, that is, the sum of the path flows between an O/D pair w must be equal to the demand d_w; such a path flow pattern is termed feasible.

Following Wardrop (1952) and Beckmann, McGuire, and Winsten (1956), the transportation network equilibrium conditions are given as follows.

Definition (Fixed Demand Transportation Network Equilibrium) *A path flow pattern* x^*, *which satisfies the demand, is a transportation network equilibrium, if, for every O/D pair* w *and each path* $p\in P_w$, *the following equalities and inequalities hold:*

$$C_p(x^*)\begin{cases} =\lambda_w, & \text{if} \quad x_p^*>0\\ \geq\lambda_w, & \text{if} \quad x_p^*=0,\end{cases}$$

where λ_w *is the travel disutility incurred in equilibrium.*

Again, as in the elastic demand models, in equilibrium, only those paths connecting an O/D pair that have minimal user travel costs are used, and those paths that are not used have costs that are higher than or equal to these minimal travel costs. However, here the demands and travel disutilities are no longer functions.

The equilibrium conditions have been formulated as a variational inequality problem by Smith (1979) and Dafermos (1980). In particular, we present two formulations, in path flows and link flows, respectively:

Theorem (Variational Inequality Formulation in Path Flows) $x^* \in K^3$ *is a transportation network equilibrium in path flows if and only if it solves the following variational inequality problem:*

$$\langle C(x^*)^T, x - x^* \rangle \geq 0, \quad \forall x \in K^3,$$

where $K^3 \equiv \{x \in R_+^{n_P}:$ *and the path flow pattern is feasible*$\}$.

This variational inequality can be put into standard form (cf. Section 2) if we define: $X \equiv x$, $F(X) \equiv C(x)$, and $\mathcal{K} \equiv K^3$.

Theorem (Variational Inequality Formulation in Link Flows) $f^* \in K^4$ *is a transportation network equilibrium in link flows if and only if it satisfies the following variational inequality problem:*

$$\langle c(f^*)^T, f - f^* \rangle \geq 0 \quad \forall f \in K^4,$$

where $K^4 \equiv \{f : \exists x \geq 0 :$ *the path flow pattern is feasible and induces a link flow pattern*$\}$.

This variational inequality can also be put into standard form if we let $X \equiv f$, F(X)$\equiv c(f)$, and $\mathcal{K} \equiv K^4$.

Since both the feasible sets K^3 and K^4 are compact (see Section 2), existence of a solution to the respective variational inequality problems follows under the sole assumption that the user link cost functions are continuous (see Section 2). Uniqueness of an equilibrium link flow pattern, in turn, is then guaranteed if the user link cost functions are strictly monotone (see Section 2).

In the case where the Jacobian of the link travel cost functions is symmetric, i. e., $\frac{\partial c_a(f)}{\partial f_b} = \frac{\partial c_b(f)}{\partial f_a}$, for all links $a, b \in \mathcal{L}$, then by Green's Lemma the vector $c(f)$ is the gradient of the line vector $\int_0^f c(y)dy$. Moreover, if the Jacobian is positive semidefinite, then the transportation equilibrium pattern (f^*) coincides with the solution of the convex optimization problem:

$$\text{Minimize}_{f \in K^4} \int_0^f c(y)dy.$$

In particular, when the link travel cost functions c_a are separable, that is, $c_a = c_a(f_a)$ for all links $a \in \mathcal{L}$, then one obtains the objective function:

$$\text{Minimize}_{f \in K^4} \sum_a \int_0^{f_a} c_a(y)dy$$

which is the classical and standard transportation network equilibrium problem with fixed travel demands (cf. Dafermos and Sparrow (1969)).

The System-Optimized Problem

We now describe and discuss the system-optimized problem, which is, typically, stated in the case of fixed travel demands. Again, as in the user-optimized (or transportation network equilibrium) problem with fixed demands, we assume, as given: the network $\mathcal{G} = [\mathcal{N}, \mathcal{L}]$, the demands associated with the origin/destination pairs, and the user link cost functions. In the system-optimized problem, there is a central controller of the traffic who routes the traffic in an optimal manner so as to minimize the total cost in the network.

The total cost on link a, denoted by $\hat{c}_a(f_a)$, is given by:

$$\hat{c}_a(f_a) = c_a(f_a) \times f_a, \quad \forall a \in \mathcal{L},$$

that is, the total cost on a link is equal to the user link cost on the link times the flow on the link. As noted earlier, in the system-optimized problem, there exists a central controller who seeks to minimize the total cost in the network system, where the total cost is expressed as

$$\sum_{a \in \mathcal{L}} \hat{c}_a(f_a).$$

The system-optimization (S-O) problem is, thus, given by:

$$\text{Minimize} \quad \sum_{a \in \mathcal{L}} \hat{c}_a(f_a)$$

subject to the same conservation of flow equations as for the user-optimized problem, as well as the nonnegativity assumption of the path flows.

The total cost on a path, denoted by $\hat{C}_p$, is the user cost on a path times the flow on a path, that is,

$$\hat{C}_p = C_p x_p, \quad \forall p \in P,$$

where the user cost on a path, C_p, is given by the sum of the user costs on the links that comprise the path, that is,

$$C_p = \sum_{a \in \mathcal{L}} c_a(f_a)\delta_{ap}, \quad \forall a \in \mathcal{L}.$$

One may also express the cost on a path p as a function of the path flow variables and, hence, an alternative version of the above system-optimization problem can be stated in path flow variables only, where one has now the problem:

$$\text{Minimize} \quad \sum_{p \in P} C_p(x) x_p$$

subject to nonnegativity constraints on the path flows and the conservation of flow equations relating the demands and the path flows.

System-Optimality Conditions

Under the assumption of increasing user link cost functions, the objective function in the S-O problem is convex, and the feasible set consisting of the linear constraints is also convex. Therefore, the optimality conditions, that is, the Kuhn-Tucker (1951) conditions are: for each O/D pair w, and each path $p \in P_w$, the flow pattern x (and corresponding link flow pattern f), satisfying the underlying constraints, must satisfy:

$$\hat{C}'_p \begin{cases} = \mu_w, & \text{if} \quad x_p > 0 \\ \geq \mu_w, & \text{if} \quad x_p = 0, \end{cases}$$

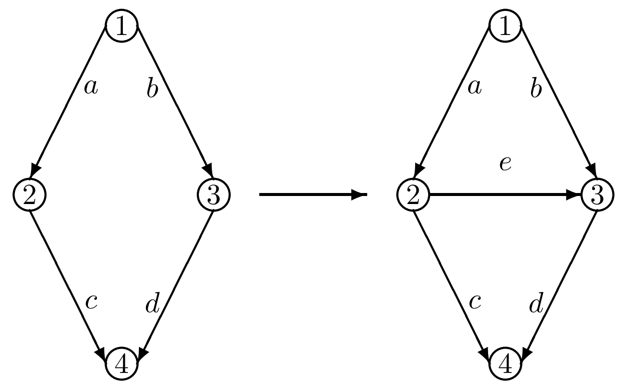

Figure 6: The Braess Network Example

where $\hat{C}'_p$ denotes the marginal of the total cost on path p, given by:

$$\hat{C}'_p = \sum_{a \in \mathcal{L}} \frac{\partial \hat{c}_a(f_a)}{\partial f_a} \delta_{ap},$$

evaluated at the solution and μ_ω is the Largrange multiplier associated with constraint the demand constraint for that O/D pair w.

Observe that the optimality conditions above may be rewritten so that there exists an ordering of the paths for each O/D pair whereby all used paths (that is, those with positive flow) have equal and minimal marginal total costs and the unused paths (that is, those with zero flow) have higher (or equal) marginal total costs than those of the used paths. Hence, in the S-O problem it is the marginal of the total cost on each used path connecting an O/D pair which is equalized and minimal (see also, e.g., Dafermos and Sparrow (1969)).

The Braess Paradox

In order to illustrate the difference between user-optimization and system-optimization in a concrete example, and to reinforce the above concepts, we now recall the well-known Braess (1968) paradox; see also Murchland (1970) and Braess, Nagurney, and Wakolbinger (2005). This paradox is as relevant to transportation networks as it is to telecommunication networks, and, in particular, to the Internet, since such networks are subject to traffic operating in a decentralized decision-making manner (cf. Korilis, Lazar, and Orda (1999), Nagurney, Parkes, and Daniele (2006), and the references therein).

Assume a network as the first network depicted in Figure 6 in which there are four nodes: $1, 2, 3, 4$; four links: a, b, c, d; and a single O/D pair $w = (1, 4)$. There are, hence, two paths available to travelers between this O/D pair: $p_1 = (a, c)$ and $p_2 = (b, d)$.

The user link travel cost functions are:

$$c_a(f_a) = 10 f_a, \quad c_b(f_b) = f_b + 50, \quad c_c(f_c) = f_c + 50, \quad c_d(f_d) = 10 f_d.$$

Assume a fixed travel demand $d_w = 6$.

It is easy to verify that the equilibrium path flows are: $x^*_{p_1} = 3$, $x^*_{p_2} = 3$, the equilibrium link flows are: $f^*_a = 3$, $f^*_b = 3$, $f^*_c = 3$, $f^*_d = 3$, with associated equilibrium path travel costs: $C_{p_1} = c_a + c_c = 83$, $C_{p_2} = c_b + c_d = 83$.

Assume now that, as depicted in Figure 6, a new link "e", joining node 2 to node 3 is added to the original network, with user link cost function $c_e(f_e) = f_e + 10$. The addition of this link creates a new path $p_3 = (a, e, d)$ that is available to the travelers. Assume that the travel demand d_{w_1} remains at 6 units of flow. Note that the original flow distribution pattern $x_{p_1} = 3$ and $x_{p_2} = 3$ is no longer an equilibrium pattern, since at this level of flow the user cost on path p_3, $C_{p_3} = c_a + c_e + c_d = 70$. Hence, users from paths p_1 and p_2 would switch to path p_3.

The equilibrium flow pattern on the new network is: $x^*_{p_1} = 2$, $x^*_{p_2} = 2$, $x^*_{p_3} = 2$, with equilibrium link flows: $f^*_a = 4$, $f^*_b = 2$, $f^*_c = 2$, $f^*_e = 2$, $f^*_d = 4$, and with associated equilibrium user path travel costs: $C_{p_1} = 92$, $C_{p_2} = 92$. Indeed, one can verify that any reallocation of the path flows would yield a higher travel cost on a path.

Note that the travel cost increased for every user of the network from 83 to 92 without a change in the travel demand!

The increase in travel cost on the paths is due, in part, to the fact that in this network two links are shared by distinct paths and these links incur an increase in flow and associated cost. Hence, the Braess paradox is related to the underlying topology of the networks and, of course, to the behavior of the travelers, which here is that of user-optimization. One may show, however, that the addition of a path connecting an O/D pair that shares no links with the original O/D pair will never result in the Braess paradox for that O/D pair (cf. Dafermos and Nagurney (1984c)).

Recall that a system-optimizing solution, which corresponds to Wardrop's (1952) second principle, is one that minimizes the total cost in the network, and all utilized paths connecting each O/D pair have equal and minimal marginal total travel costs.

The system-optimizing solution for the first network in Figure 6 is: $x_{p_1} = x_{p_2} = 3$, with marginal total path costs given by: $\hat{C}'_{p_1} = \hat{C}'_{p_2} = 116$. This would remain the system-optimizing solution, even after the addition of link e, since the marginal cost of path p_3, $\hat{C}'_{p_3}$, at this feasible flow pattern is equal to 130.

The addition of a new link to a network cannot increase the total cost of the network system, but can, of course, increase a user's cost since travelers act individually.

System-Optimizing Problem in the Case of Non-Separable User Link Cost Functions

The system-optimization problem, in turn, in the case of nonseparable user link cost functions becomes:

$$\text{Minimize} \quad \sum_{a \in \mathcal{L}} \hat{c}_a(f),$$

subject to the conservation of flow equations, where $\hat{c}_a(f) = c_a(f) \times f_a$, $\forall a \in \mathcal{L}$.

The system-optimality conditions remain as before, but now the marginal of the total cost on a path becomes, in this more general case:

$$\hat{C}'_p = \sum_{a,b\in\mathcal{L}} \frac{\partial \hat{c}_b(f)}{\partial f_a}\delta_{ap}, \quad \forall p \in P.$$

Tolls can be added to guarantee that the system-optimizing traffic flow pattern is also user-optimizing (cf. Dafermos (1973), Nagurney (2000), and the references therein). Tolls are increasingly applied in practice and have had success in various cities, including London (see Lawphongpanich, Hearn, and Smith (2006)). It is also worth mentioning the *price of anarchy* (cf. Roughgarden (2005) and Perakis (2004) and the references therein), which is the ratio of the system-optimization objective function evaluated at the user-optimized solution divided by that objective function evaluated at the system-optimized solution. This measure is being applied to transportation networks as well as to telecommunication networks, including the Internet. Later, in this chapter we discuss a network efficiency measure, due to Nagurney and Qiang (2007), which allows for also the determination of the importance of nodes and links in a given transportation network, and which is also applicable to other critical infrastructure networks, such as the Internet.

4. Spatial Price Equilibria

In the *spatial price equilibrium problem*, one seeks to compute the commodity supply prices, demand prices, and trade flows satisfying the equilibrium condition that the demand price is equal to the supply price plus the cost of transportation, if there is trade between the pair of supply and demand markets; if the demand price is less than the supply price plus the transportation cost, then there will be no trade. Spatial price equilibrium problems arise in agricultural markets, energy markets, and financial markets and such models provide the basis for interregional and international trade modeling (see, e.g., Samuelson (1952), Takayama and Judge (1964, 1971), Harker (1985, 1986), Nagurney (1999), and the references therein).

The first reference in the literature to such problems was by Cournot in 1838, who considered two spatially separated markets. As noted in the Introduction, Enke (1951), more than a century later, used an analogy between spatial price equilibrium and electronic circuits to give the first computational approach, albeit analogue, to such problems, in the case of linear and separable supply and demand functions.

As also noted in the Introduction, Samuelson (1952) initiated the rigorous treatment of such problems by establishing that the solution to the spatial price equilibrium problem, as posed by Enke, could be obtained by solving an optimization problem in which the objective function, although artificial, had the interpretation of a net social pay-off function. The spatial price equilibrium, in this case, coincided with the Kuhn-Tucker conditions of the appropriately constructed optimization problem. Samuelson also related Enke's specification to a standard problem in linear programming, the Hitchcock-Koopmans transportation problem and noted that the spatial price equilibrium problem was more general in the sense that the supplies and demands were not known a priori. Finally, Samuelson also identified the network structure of such problems, which is bipartite, that is of the form given in Figure 2.

Takayama and Judge (1964, 1971) expanded on the work of Samuelson (1952) and showed that the prices and commodity flows satisfying the spatial price equilibrium conditions could be determined by solving a quadratic programming problem in the case of linear supply and demand price functions for which the Jacobians were symmetric and not necessarily diagonal. This theoretical advance enabled not only the qualitative study of equilibrium patterns, but also opened up the possibility for the development of effective computational procedures, based on convex programming, as well as, the exploitation of the network structure (see Nagurney (1999)).

Takayama and Judge (1971), developed a variety of spatial price equilibrium models, and emphasized that distinct model formulations are needed, in particular, both quantity and price formulations. In a *quantity* formulation it is assumed that the supply price functions and demand price functions are given (and these are a function, respectively, the quantities produced and consumed) whereas in a *price* formulation it is assumed that the supply and demand functions are given and these are a function, respectively, of the supply and demand prices. Moreover, Takayama and Judge (1971) realized that a pure optimization framework was not sufficient to handle, for example, multicommodity spatial price equilibrium problems in which the Jacobians of the supply and demand price functions were no longer symmetric.

Novel formulations were proposed for the spatial price equilibrium problem under more general settings, including fixed point, complementarity, and variational inequality formulations. MacKinnon (1975) gave a fixed point formulation (see also, Kuhn and MacKinnon (1975)). Asmuth, Eaves, and Peterson (1979) considered the linear asymmetric spatial price quilibrium problem formulated as a linear complementarity problem and proposed Lemke's algorithm for the computation of the spatial price equilibrium. Pang and Lee (1981) developed special-purpose algorithms based on the complementarity formulation of the problem. Florian and Los (1982) and Dafermos and Nagurney (1984a) addressed variational inequality formulations of general spatial price equilibrium models with the latter authors providing sensitivity analysis results.

Dafermos and Nagurney (1985) established the equivalence of the spatial price equilibrium problem with the transportation network equilibrium problem. This identification stimulated further research in network equilibria and in algorithm development for such problems. Computational testing of different algorithms for spatial price equilibrium problems problems can be found in Nagurney (1987a), Dafermos and Nagurney (1989), Friesz, Harker, and Tobin (1984), Nagurney and Kim (1989), Guder, Morris, and Yoon (1992). Spatial price equilbrium models have also been used for policy analysis (see, e.g., Nagurney, Nicholson, and Bishop (1996), Judge and Takayama (1973), and the references therein).

For definiteness, we first present the quantity model and then the price model and provide the variational inequality formulations of the governing equilibrium conditions. Refer to Figure 2 for a graphical depiction of the underlying network structure. The presentation here follows that of Nagurney (2001a).

The Quantity Model

Consider the spatial price equilibrium problem in quantity variables with m supply markets and n demand markets involved in the production and consumption of a homogeneous commodity under perfect competition. Denote a typical supply market by i and a typical demand market by j. Let s_i denote the supply and π_i the supply price of the commodity at supply market i. Let d_j denote the demand and ρ_j the demand price at demand market j. Group the supplies and supply prices, respectively, into a vector $s \in R^m$ and a vector $\pi \in R^m$. Similarly, group the demands and demand prices, respectively, into a vector $d \in R^n$ and a vector $\rho \in R^n$. Let Q_{ij} denote the nonnegative commodity shipment between the supply and demand market pair (i, j), and let c_{ij} denote the unit transaction cost associated with trading the commodity between (i, j). The unit transaction costs are assumed to include the unit costs of transportation from supply markets to demand markets, and, depending upon the application, may also include a tax/tariff, duty, or subsidy incorporated into these costs. Group the commodity shipments into a column vector $Q \in R^{mn}$ and the transaction costs into a row vector $c \in R^{mn}$.

Assume that the supply price at any supply market may, in general, depend upon the supply of the commodity at every supply market, that is, $\pi = \pi(s)$, where π is a known smooth function. Similarly, the demand price at any demand market may depend upon, in general, the demand of the commodity at every demand market, that is, $\rho = \rho(d)$, where ρ is a known smooth function. The unit transaction cost between a pair of supply and demand markets may depend upon the shipments of the commodity between every pair of markets, that is, $c = c(Q)$, where c is a known smooth function.

The supplies, demands, and shipments of the commodity, in turn, must satisfy the following feasibility conditions, which are also referred to as the *conservation of flow equations*:

$$s_i = \sum_{j=1}^{n} Q_{ij}, \quad i = 1, ..., m$$

$$d_j = \sum_{i=1}^{m} Q_{ij}, \quad j = 1, ..., n$$

$$Q_{ij} \geq 0, \quad i = 1, ..., m; j = 1, ..., n.$$

In other words, the supply at each supply market is equal to the commodity shipments out of that supply market to all the demand markets. Similarly, the demand at each demand market is equal to the commodity shipments from all the supply markets into that demand market.

Definition (Spatial Price Equilibrium) *Following Samuelson (1952) and Takayama and Judge (1971), the supply, demand, and commodity shipment pattern (s^*, Q^*, d^*) constitutes a spatial price equilibrium, if it is feasible, and for all pairs of supply and demand markets (i, j), it satisfies the conditions:*

$$\pi_i(s^*) + c_{ij}(Q^*) \begin{cases} = \rho_j(d^*), & \text{if} \quad Q^*_{ij} > 0 \\ \geq \rho_j(d^*), & \text{if} \quad Q^*_{ij} = 0. \end{cases}$$

Hence, if the commodity shipment between a pair of supply and demand markets is positive at equilibrium, then the demand price at the demand market must be equal to the supply price at the originating supply market plus the unit transaction cost. If the commodity shipment is zero in equilibrium, then the supply price plus the unit transaction cost can exceed the demand price.

The spatial price equilibrium can be formulated as a variational inequality problem (cf. Dafermos and Nagurney (1985) and Nagurney (1999) for proofs). Precisely, we have

Theorem (Variational Inequality Formulation) *A commodity supply, shipment, and demand pattern $(s^*, Q^*, d^*) \in \mathcal{K}$ is a spatial price equilibrium if and only if it satisfies the following variational inequality problem:*

$$\langle \pi(s^*)^T, s - s^* \rangle + \langle c(Q^*)^T, Q - Q^* \rangle + \langle -\rho(d^*)^T, d - d^* \rangle \geq 0, \quad \forall (s, Q, d) \in \mathcal{K},$$

where $\mathcal{K} \equiv \{(s, Q, d) :$ feasibility conditions hold$\}$ and $\langle \cdot, \cdot \rangle$ denotes the inner product.

An Example

For illustrative purposes, we now present a small example. Consider the spatial price equilibrium problem consisting of two supply markets and two demand markets. Assume that the functions are as follows:

$$\pi_1(s) = 5s_1 + s_2 + 1, \quad \pi_2(s) = 4s_2 + s_1 + 2$$

$$c_{11}(Q) = 2Q_{11} + Q_{12} + 3, \ c_{12}(Q) = Q_{12} + 5, \ c_{21}(Q) = 3Q_{21} + Q_{22} + 5, \ c_{22}(Q) = 3Q_{22} + 2Q_{21} + 9$$

$$\rho_1(d) = -2d_1 - d_2 + 21, \quad \rho_2(d) = -5d_2 - 3d_1 + 29.$$

It is easy to verify that the spatial price equilibrium pattern is given by:

$$s_1^* = 2, \quad s_2^* = 1, \quad Q_{11}^* = 1, \quad Q_{12}^* = 1, \quad Q_{21}^* = 1, \quad Q_{22}^* = 0, \quad d_1^* = 2, \quad d_2^* = 1.$$

In one of the simplest models, in which the Jacobians of the supply price functions, $\left[\frac{\partial \pi}{\partial s}\right]$, the transportation (or transaction) cost functions, $\left[\frac{\partial c}{\partial Q}\right]$, and minus the demand price functions, $-\left[\frac{\partial \rho}{\partial d}\right]$ are diagonal and positive definite, then the spatial price equilibrium pattern coincides with the Kuhn-Tucker conditions of the strictly convex optimization problem:

$$\text{Minimize}_{Q\in R_+^{mn}} \quad \sum_{i=1}^{n}\int_0^{\sum_{j=1}^{n} Q_{ij}} \pi_i(x)dx + \sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^{Q_{ij}} c_{ij}(y)dy - \sum_{j=1}^{n}\int_0^{\sum_{i=1}^{m} Q_{ij}} \rho_j(z)dz.$$

The Price Model

We now describe briefly the price model (see also Nagurney (1999)). The notation is as for the quantity model except now we consider the situation where the supplies at the supply markets, denoted by the row vector s may, in general, depend upon the column vector of supply prices π, that is, $s = s(\pi)$. Similarly, assume that the demands at the demand markets, denoted by the row vector d, may, in general, depend upon the column vector of demand prices ρ, that is, $d = d(\rho)$. The transaction/transportation costs are of the same form as in the quantity model.

The spatial equilibrium conditions now take the following form: For all pairs of supply and demand markets $(i, j) : i = 1, \ldots, m; j = 1, \ldots, n$:

$$\pi_i^* + c_{ij}(Q^*) \begin{cases} = \rho_j^*, & \text{if } Q_{ij}^* > 0 \\ \geq \rho_j^*, & \text{if } Q_{ij}^* = 0, \end{cases}$$

where

$$s_i(\pi^*) \begin{cases} = \sum_{j=1}^{n} Q_{ij}^*, & \text{if } \pi_i^* > 0 \\ \geq \sum_{j=1}^{n} Q_{ij}^*, & \text{if } \pi_i^* = 0, \end{cases}$$

and

$$d_j(\rho^*) \begin{cases} = \sum_{i=1}^{m} Q_{ij}^*, & \text{if } \rho_j^* > 0 \\ \leq \sum_{i=1}^{m} Q_{ij}^*, & \text{if } \rho_j^* = 0. \end{cases}$$

The first equilibrium condition is as in the quantity model with the exception that the prices are now variables. The other two conditions allow for the possibility that if the equilibrium prices are zero, then one may have excess supply and/or excess demand at the respective market(s). If the prices are positive, then the markets will clear.

The variational inequality formulation of the equilibrium conditions governing the price model is now given (for a proof, see Nagurney (1999)).

Theorem (Variational Inequality Formulation) *The vector $X^* \equiv (Q^*, \pi^*, \rho^*) \in R_+^{mn+m+n}$ is an equilibrium shipment and price vector if and only if it satisfies the variational inequality:*

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in R_+^{mn+m+n},$$

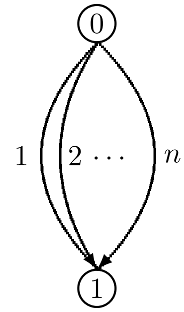

Figure 7: Network Structure of Walrasian Price Equilibrium

where $F : \mathcal{K} \mapsto R^{mn+m+n}$ *is the row vector:* $F(X) \equiv (T(X), S(X), D(X))$*, where* T*:* $R^{mn+m+n}_+ \mapsto R^{mn}$*,* $S : R^{mn+m}_+ \mapsto R^m$*,*$D$*:* $R^{mn+n}_+ \mapsto R^n$*, with the* i, j*-th term of* $T(X)$ *given by* $\pi_i + c_{ij}(Q) - \rho_j(d)$*, the* i*-th term of* $S(X)$ *given by* $s_i - \sum_{j=1}^n Q_{ij}$*, and the* j*-th term of* $D(X)$ *given by:* $-d_j(\rho) + \sum_{i=1}^m Q_{ij}$*.*

5. General Economic Equilibrium

Spatial price equilibrium models, in contrast to general economic equilibrium models, are necessarily partial equilibrium models. The network structure of spatial price equilibrium problems considered today often corresponds to the physical transportation network (cf. Dafermos and Nagurney (1984a)). The general economic equilibrium problem due to Walras (1874) has also been extensively studied (see, e.g., Border (1985)) both from qualitative as well as quantitative perspectives (cf. Dafermos (1990) and the references therein). The Walrasian price equilibrium problem can also be cast into a network equilibrium form as shown in Zhao and Nagurney (1993), who recognized the work of Beckmann, McGuire, and Winsten (1956) (see also Nagurney (1999)). In this application, cf. Figure 7, there is only a single origin/destination pair of nodes and the links connecting the origin/destination pair correspond to commodities with the flows on the links being now prices. In the context of a transportation network equilibrium problem, hence, this problem is one with a fixed demand and it is the excess demands on used links that are equalized. Again, we get the concept of utilized and nonutilized "paths." Note that this network structure is abstract in that the nodes do not correspond to locations in space and the links to physical routes. We see, once more, the generality of network equilibrium due to Beckmann, McGuire, and Winsten (1956) in this application setting. Moreover, algorithms derived for traffic networks have been applied (with the network identification) by Zhao and Nagurney (1993) to solve Walrasian price equilibrium problems. Finally, it is fascinating to note that the classical portfolio optimization problem of Markowitz (1952, 1959) (see also, e.g., Nagurney and Siokos (1997)) can be transformed into a system-optimized transportation network problem with fixed demand on a network with the structure of the one in Figure 7. For additional financial network models, see Nagurney and Siokos (1997). For more recent approaches to financial networks with intermediation including the incorporation of electronic transactions, see the edited volume by Nagurney (2003).

6. Oligopolistic Market Equilibria

Oligopolies are a fundamental market structure. An oligopoly consists of a finite number (usually few) firms involved in the production of a good. Examples of oligopolies range from large firms in automobile, computer, chemical, or mineral extraction industries to small firms with local markets. Oligopolies are examples of *imperfect competition* in that the producers or firms are sufficiently large that they affect the prices of the goods. A monopoly, on the other hand, consists of a single firm which has full control of the market.

Cournot (1838) considered competition between two producers, the so-called *duopoly* problem, and is credited with being the first to study *noncooperative* behavior, in which the agents act in their own self-interest. In his study, the decisions made by the producers or firms are said to be in equilibrium if no producer can increase his income by unilateral action, given that the other producer does not alter his decision.

Nash (1950, 1951) generalized Cournot's concept of an equilibrium for a behavioral model consisting of several agents or players, each acting in his own self-interest, which has come to be called a *noncooperative game*. Specifically, consider m players, each player i having at his disposal a strategy vector $X_i = \{X_{i1}, \ldots, X_{in}\}$ selected from a closed, convex set $\mathcal{K}^i \subset R^n$, with a utility function u_i: $\mathcal{K} \mapsto R^1$, where $\mathcal{K} = \mathcal{K}^1 \times \mathcal{K}^2 \times \ldots \times \mathcal{K}^m \subset R^{mn}$. The rationality postulate is that each player i selects a strategy vector $X_i \in \mathcal{K}^i$ that maximizes his utility level $u_i(X_1, \ldots, X_{i-1}, X_i, X_{i+1}, \ldots, X_m)$ given the decisions $(X_j)_{j \neq i}$ of the other players. In this framework one then has:

Definition (Nash Equilibrium) *A Nash equilibrium is a strategy vector* $X^* = (X_1^*, \ldots, X_m^*) \in \mathcal{K}$, *such that*

$$u_i(X_i^*, \hat{X}_i^*) \geq u_i(X_i, \hat{X}_i^*), \quad \forall X_i \in \mathcal{K}^i, \forall i,$$

where $\hat{X}_i^* = (X_1^*, \ldots, X_{i-1}^*, X_{i+1}^*, \ldots, X_m^*)$.

It has been shown (cf. Hartman and Stampacchia (1966) and Gabay and Moulin (1980)) that Nash equilibria satisfy variational inequalities. In the present context, under the assumption that each u_i is continuously differentiable on $\mathcal{K}$ and concave with respect to X_i, one has

Theorem (Variational Inequality Formulation) *Under the previous assumptions,* X^* *is a Nash equilibrium if and only if* $X^* \in \mathcal{K}$ *is a solution of the variational inequality*

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K},$$

where $F(X) \equiv (-\nabla_{X_1} u_1(X), \ldots, -\nabla_{X_m} u_m(X))$, *and is assumed to be a row vector, where* $\nabla_{X_i} u_i(X)$ *denotes the gradient of* u_i *with respect to* X_i.

If the feasible set $\mathcal{K}$ is compact, then existence is guaranteed under the assumption that each utility function u_i is continuously differentiable. Rosen (1965) proved existence under similar conditions. Karamardian (1969) relaxed the assumption of compactness of $\mathcal{K}$ and provided a proof of existence and uniqueness of Nash equilibria under the strong monotonicity condition. As shown by Gabay and Moulin (1980), the imposition of a coercivity condition on $F(X)$ (see Section 2) will guarantee existence of a Nash equilibrium X^* even if the feasible set is no longer compact. Moreover, if $F(X)$ satisfies the strict monotonicity condition then uniqueness of X^* is guaranteed, provided that the equilibrium exists (see also Section 2).

We begin with the presentation of a classical oligopoly model and then present a spatial oligopoly model which is related to the spatial price equilibrium problem. The presentation of these models follows that in Nagurney (2001b).

The Classical Oligopoly Problem

We now describe the classical oligopoly problem (cf. Gabzewicz and Vial (1972), Manas (1972), Friedman (1977), Harker (1984), Haurie and Marcotte (1985), Flam and Ben-Israel (1990)) in which there are m producers involved in the production of a homogeneous commodity. The quantity produced by firm i is denoted by q_i, with the production quantities grouped into a column vector $q \in R^m$. Let f_i denote the cost of producing the commodity by firm i, and let ρ denote the demand price associated with the good. Assume that

$$f_i = f_i(q_i)$$

and

$$\rho = \rho(\sum_{i=1}^{m} q_i).$$

The profit for firm i, u_i, which is the difference between the revenue and cost, can then be expressed as

$$u_i(q) = \rho(\sum_{i=1}^{m} q_i)q_i - f_i(q_i).$$

Given that the competitive mechanism is one of noncooperative behavior, one can write down immediately:

Theorem (Variational Inequality Formulation) *Assume that the profit function $u_i(q)$ for each firm i is concave with respect to q_i, and that $u_i(q)$ is continuously differentiable. Then $q^* \in R^m_+$ is a Nash equilibrium if and only if it satisfies the variational inequality:*

$$\sum_{i=1}^{m} \left[\frac{\partial f_i(q_i^*)}{\partial q_i} - \frac{\partial \rho(\sum_{i=1}^{m} q_i^*)}{\partial q_i} q_i^* - \rho(\sum_{i=1}^{m} q_i^*) \right] \times [q_i - q_i^*] \geq 0, \quad \forall q \in R^m_+.$$

An example is now presented.

An Example (Nagurney (2001b)

In this oligopoly example there are three firms. The data are as follows: the producer cost functions are given by:

$$f_1(q_1) = q_1^2 + q_1 + 10, \quad f_2(q_2) = \frac{1}{2}q_2^2 + 4q_2 + 12, \quad f_3(q_3) = q_3^2 + \frac{1}{2}q_3 + 15,$$

and the inverse demand or price function is given by: $\rho(\sum_{i=1}^{3} q_i) = -\sum_{i=1}^{3} q_i + 5$.

The equilibrium production outputs are as follows:

$$q_1^* = \frac{23}{30}, \quad q_2^* = 0, \quad q_3^* = \frac{14}{15}; \quad \sum_{i=1}^{3} q_i^* = \frac{17}{10}.$$

We now verify that the variational inequality is satisfied: $-\frac{\partial u_1(q^*)}{\partial q_1}$ is equal to zero, as is $-\frac{\partial u_3(q^*)}{\partial q_3}$, whereas $-\frac{\partial u_2(q^*)}{\partial q_2} = \frac{7}{10}$. Since both q_1^* and q_3^* are greater than zero, and $q_2^* = 0$, one sees that, indeed, the above variational inequality is satisfied.

Computational approaches can be found in Okuguchi (1976), Murphy, Sherali, and Soyster (1982), Harker (1984), Okuguchi and Szidarovsky (1990), Nagurney (1999), and the references therein.

In the special case where the production cost functions are quadratic (and separable) and the inverse demand or price function is linear, one can reformulate the Nash equilibrium conditions of the Cournot oligopoly problem as the solution to an optimization problem (see Spence (1976) and Nagurney (1999)).

A Spatial Oligopoly Model

We now describe a generalized version of the oligopoly model due to Dafermos and Nagurney (1987) (see, also, Nagurney (1999)), which is *spatial* in that the firms are now located in different regions and there are transportation costs associated with shipping the commodity between the producers and the consumers. For the relationship between this model and the perfectly competitive spatial price equilibrium problem, see Dafermos and Nagurney (1987). Algorithms for the computation of solutions to this model can be foun din Nagurney (1987b, 1988).

Assume that there are m firms and n demand markets that are generally spatially separated. Hence, the network structure of this problem is also bipartitie, of the form in Figure 2. Assume that the homogeneous commodity is produced by the m firms and is consumed at the n markets. As before, let q_i denote the nonnegative commodity output produced by firm i and now let d_j denote the demand for the commodity at demand market j. Let Q_{ij} denote the nonnegative commodity shipment from supply market i to demand market j. Group the production outputs into a column vector $q \in R^m_+$, the demands into a column vector $d \in R^n_+$, and the commodity shipments into a column vector $Q \in R^{mn}_+$.

The following *conservation of flow* equations must hold:

$$q_i = \sum_{j=1}^{n} Q_{ij}, \quad \forall i$$

$$d_j = \sum_{i=1}^{m} Q_{ij}, \quad \forall j$$

where $Q_{ij} \geq 0$, $\forall i, j$.

We associate with each firm i a production cost f_i, but allow now for the more general situation where the production cost of a firm i may depend upon the entire production pattern, that is,

$$f_i = f_i(q).$$

Similarly, we allow the demand price for the commodity at a demand market to depend, in general, upon the entire consumption pattern, that is,

$$\rho_j = \rho_j(d).$$

Let c_{ij} denote the transaction cost, which includes the transportation cost, associated with trading (shipping) the commodity between firm i and demand market j. Here the transaction cost may depend, in general, upon the entire shipment pattern, that is,

$$c_{ij} = c_{ij}(Q).$$

The profit u_i of firm i is then given by:

$$u_i = \sum_{j=1}^{n} \rho_j c_{ij} - f_i - \sum_{j=1}^{n} c_{ij} Q_{ij},$$

which, in view of the conservation of flow equations and the functions, one may write as

$$u = u(Q).$$

Now consider the usual oligopolistic market mechanism, in which the m firms supply the commodity in a noncooperative fashion, each one trying to maximize his own profit. We seek to determine a nonnegative commodity distribution pattern Q for which the m firms will be in a state of equilibrium as defined below.

Definition (Spatial Cournot-Nash Equilibrium) *A commodity shipment distribution $Q^* \in R^{mn}_+$ is said to constitute a Cournot-Nash equilibrium if for each firm $i; i = 1, \ldots, m$,*

$$u_i(Q_i^*, \hat{Q}_i^*) \geq u_i(Q_i, \hat{Q}_i^*), \quad \forall Q_i \in R^n_+,$$

where

$$Q_i \equiv \{Q_{i1}, \ldots, Q_{in}\} \quad \text{and} \quad \hat{Q}_i^* \equiv (Q_1^*, \ldots, Q_{i-1}^*, Q_{i+1}^*, \ldots, Q_m^*).$$

The variational inequality formulation of the Cournot-Nash equilibrium is given in the following theorem.

Theorem (Variational Inequality Formulation) (Dafermos and Nagurney (1987)) *Assume that for each firm i the profit function $u_i(Q)$ is concave with respect to the variables $\{Q_{i1}, \ldots, Q_{in}\}$, and continuously differentiable. Then $Q^* \in R^{mn}_+$ is a Cournot-Nash equilibrium if and only if it satisfies the variational inequality*

$$-\sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\partial u_i(Q^*)}{\partial Q_{ij}} \times (Q_{ij} - Q_{ij}^*) \geq 0, \quad \forall Q \in R^{mn}_+.$$

Using the expressions for the utility functions for this model and the conservation of flow equations this variational inequality may be rewritten as:

$$\sum_{i=1}^{m} \frac{\partial f_i(q^*)}{\partial q_i} \times (q_i - q_i^*) + \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}(Q^*) \times (Q_{ij} - Q_{ij}^*) - \sum_{j=1}^{n} \rho_j(d^*) \times (d_j - d_j^*)$$

$$-\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{l=1}^{n} \left[\frac{\partial \rho_l(d^*)}{\partial d_j} - \frac{\partial c_{il}(Q^*)}{\partial Q_{ij}} \right] Q_{il}^* (Q_{ij} - Q_{ij}^*) \geq 0, \quad \forall (q, Q, d) \in K,$$

where $K \equiv \{(q, Q, d) | Q \geq 0,$ *and the conservation of flow equations hold*$\}$.

Note that, in the special case, where there is only a single demand market and the transaction costs are identically equal to zero, this variational inequality collapses to the variational inequality governing the *aspatial* or the classical oligopoly problem.

7. Variational Inequalities and Projected Dynamical Systems

We have demonstrated that a plethora of equilibrium problems in network economics, including network equilibrium problems, can be uniformly formulated and studied as finite-dimensional variational inequality problems.

Usually, using this methodology, one first formulates the governing equilibrium conditions as a variational inequality problem. Qualitative properties of existence and uniqueness of solutions to a variational inequality problem can then be studied using the standard theory or by exploiting problem structure.

Finite-dimensional variational inequality theory by itself, however, provides no framework for the study of the dynamics of competitive systems. Rather, it captures the system at its equilibrium state and, hence, the focus of this tool is static in nature.

Dupuis and Nagurney (1993) established that, given a variational inequality problem, there is a naturally associated dynamical system, the stationary points of which correspond precisely to the solutions of the variational inequality problem. This association was first noted by Dupuis and Ishii (1991). This dynamical system, first referred to as a *projected dynamical system* by Zhang and Nagurney (1995), is non-classical in that its right-hand side, which is a projection operator, is discontinuous. The discontinuities arise because of the constraints underlying the variational inequality problem modeling the application in question. Hence, classical dynamical systems theory (cf. Coddington and Levinson (1955), Lefschetz (1957), Hartman (1964), Hirsch and Smale (1974)) is no longer applicable.

Nevertheless, as demonstrated rigorously in Dupuis and Nagurney (1993), a projected dynamical system may be studied through the use of the Skorokhod Problem (1961), a tool originally introduced for the study of stochastic differential equations with a reflecting boundary condition. Existence and uniqueness of a solution path, which is essential for the dynamical system to provide a reasonable model, were also established therein.

Here we present some results in the development of *projected dynamical systems* theory (cf. Nagurney and Zhang (1996)). One of the notable features of this tool, whose rigorous theoretical foundations were laid in Dupuis and Nagurney (1993), is its relationship to the variational inequality problem. Projected dynamical systems theory, however, goes further than finite-dimensional variational inequality theory in that it extends the static study of equilibrium states by introducing an additional time dimension in order to allow for the analysis of disequilibrium behavior that precedes the equilibrium.

In particular, we associate with a given variational inequality problem, a nonclassical dynamical system, called a projected dynamical system. The projected dynamical system is interesting both as a dynamical model for the system whose equilibrium behavior is described by the variational inequality, and, also, because its set of stationary points coincides with the set of solutions to a variational inequality problem. In this framework, the feasibility constraints in the variational inequality problem correspond to discontinuities in the right-hand side of the differential equation, which is a projection operator. Consequently, the projected dynamical system is not amenable to analysis via the classical theory of dynamical systems.

The stationary points of a projected dynamical system are identified with the solutions to the corresponding variational inequality problem with the same constraint set. We then state in a theorem the fundamental properties of such a projected dynamical system in regards to the existence and uniqueness of solution paths to the governing ordinary differential equation. We subsequently provide an interpretation of the ordinary differential equation that defines the projected dynamical system, along with a description of how the solutions may be expected to behave.

For additional qualitative results, in particular, stability analysis results, see Nagurney and Zhang (1996). For a discussion of the general iterative scheme and proof of convergence, see Dupuis and Nagurney (1993). For applications to dynamic spatial price equilibrium problems, oligopolistic market equilibrium problems, and traffic network equilibrium problems, see Nagurney and Zhang (1996), and the references therein. For extensions of these results to infinite-dimensional projected dynamical systems and evolutionary variational inequalities, see Cojocaru, Daniele, and Nagurney (2005, 2006) and the books by Daniele (2006) and Nagurney (2006b).

The Projected Dynamical System

As noted, in the preceding Sections, finite-dimensional variational inequality theory provides no framework for studying the underlying dynamics of systems, since it considers only equilibrium solutions in its formulation. Hence, in a sense, it provides a static representation of a system at its "steady state." One would, therefore, like a theoretical framework that permits one to study a system not only at its equilibrium point, but also in a dynamical setting.

The definition of a projected dynamical system (PDS) is given with respect to a closed convex set $\mathcal{K}$, which is usually the constraint set underlying a particular application, such as, for example, network equilibrium problems, and a vector field F whose domain contains $\mathcal{K}$. Such projected dynamical systems provide mathematically convenient approximations to more "realistic" dynamical models that might be used to describe non-static behavior. The relationship between a projected dynamical system and its associated variational inequality problem with the same constraint set is then highlighted. For completeness, we also recall the fundamental properties of existence and uniqueness of the solution to the ordinary differential equation (ODE) that defines such a projected dynamical system.

Let $\mathcal{K} \subset R^n$ be closed and convex. Denote the boundary and interior of $\mathcal{K}$, respectively, by $\partial\mathcal{K}$ and $\mathcal{K}^0$. Given $X \in \partial\mathcal{K}$, define the set of inward normals to $\mathcal{K}$ at X by

$$N(X) = \{\gamma : \|\gamma\| = 1, \text{ and } \langle \gamma^T, X - y\rangle \leq 0, \forall y \in \mathcal{K}\}.$$

We define $N(X)$ to be $\{\gamma : \|\gamma\| = 1\}$ for X in the interior of $\mathcal{K}$.

When $\mathcal{K}$ is a convex polyhedron (for example, when $\mathcal{K}$ consists of linear constraints), $\mathcal{K}$ takes the form $\cap_{i=1}^{Z}\mathcal{K}_i$, where each $\mathcal{K}_i$ is a closed half-space with inward normal N_i. Let $P_{\mathcal{K}}$ be the norm projection (see Section 2). Then $P_{\mathcal{K}}$ projects onto $\mathcal{K}$ "along N," in that if $y \in \mathcal{K}$, then $P(y) = y$, and if $y \notin \mathcal{K}$, then $P(y) \in \partial\mathcal{K}$, and $P(y) - y = \alpha\gamma$ for some $\alpha > 0$ and $\gamma \in N(P(y))$.

Definition *Given $X \in \mathcal{K}$ and $v \in R^n$, define the projection of the vector v at X (with respect to*

$\mathcal{K}$) by

$$\Pi_{\mathcal{K}}(X, v) = \lim_{\delta \to 0} \frac{(P_{\mathcal{K}}(X + \delta v) - X)}{\delta}.$$

The class of ordinary differential equations that are of interest here take the following form:

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)),$$

where $\mathcal{K}$ is a closed convex set, corresponding to the constraint set in a particular application, and $F(X)$ is a vector field defined on $\mathcal{K}$.

Note that a classical dynamical system, in contrast, is of the form

$$\dot{X} = -F(X).$$

We have the following results (cf. Dupuis and Nagurney (1993)):

(i). If $X \in \mathcal{K}^0$, then

$$\Pi_{\mathcal{K}}(X, -F(X)) = -F(X).$$

(ii). If $X \in \partial\mathcal{K}$, then

$$\Pi_{\mathcal{K}}(X, -F(X)) = -F(X) + \beta(X)N^*(X),$$

where

$$N^*(X) = \arg \max_{N \in N(X)} \langle (-F(X))^T, -N \rangle,$$

and

$$\beta(X) = \max\{0, \ \langle (-F(X))^T, -N^*(X) \rangle\}.$$

Note that since the right-hand side of the ordinary differential equation is associated with a projection operator, it is discontinuous on the boundary of $\mathcal{K}$. Therefore, one needs to explicitly state what one means by a solution to an ODE with a discontinuous right-hand side.

Definition *We say that the function $X : [0, \infty) \mapsto \mathcal{K}$ is a solution to the equation $\dot{X} = \Pi_{\mathcal{K}}(X, -F(X))$ if $X(\cdot)$ is absolutely continuous and $\dot{X}(t) = \Pi_{\mathcal{K}}(X(t), -F(X(t)))$, save on a set of Lebesgue measure zero.*

In order to distinguish between the pertinent ODEs from the classical ODEs with continuous right-hand sides, we refer to the above as ODE$(F, \mathcal{K})$.

Definition (An Initial Value Problem) *For any $X_0 \in \mathcal{K}$ as an initial value, we associate with* ODE$(F, \mathcal{K})$ *an initial value problem,* IVP$(F, \mathcal{K}, X_0)$*, defined as:*

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X_0.$$

Note that if there is a solution $\phi_{X_0}(t)$ to the initial value problem IVP$(F, \mathcal{K}, X_0)$, with $\phi_{X_0}(0) = X_0 \in \mathcal{K}$, then $\phi_{X_0}(t)$ always stays in the constraint set $\mathcal{K}$ for $t \geq 0$.

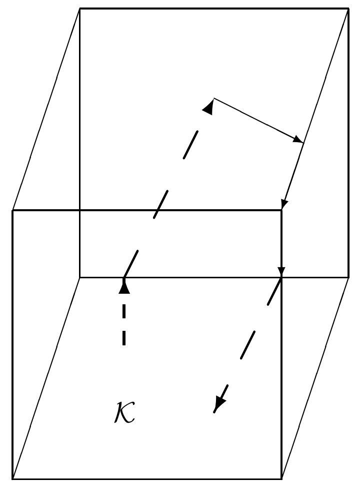

Figure 8: A Trajectory of a PDS that Evolves both on the Interior and on the Boundary of the Constraint Set $\mathcal{K}$

We now present the definition of a projected dynamical system, governed by such an ODE$(F, \mathcal{K})$, which, correspondingly, will be denoted by PDS$(F, \mathcal{K})$.

Definition (The Projected Dynamical System) *Define the projected dynamical system* PDS $(F, \mathcal{K})$ *as the map* $\Phi : \mathcal{K} \times R \mapsto \mathcal{K}$ *where*

$$\Phi(X, t) = \phi_X(t)$$

solves the IVP$(F, \mathcal{K}, X)$*, that is,*

$$\dot{\phi}_X(t) = \Pi_{\mathcal{K}}(\phi_X(t), -F(\phi_X(t))), \quad \phi_X(0) = X.$$

The behavior of the dynamical system is now described. One may refer to Figure 8 for an illustration of this behavior. If $X(t) \in \mathcal{K}^0$, then the evolution of the solution is directly given in terms of $F : \dot{X} = -F(X)$. However, if the vector field $-F$ drives X to $\partial \mathcal{K}$ (that is, for some t one has $X(t) \in \partial \mathcal{K}$ and $-F(X(t))$ points "out" of $\mathcal{K}$) the right-hand side of the ODE becomes the projection of $-F$ onto $\partial \mathcal{K}$. The solution to the ODE then evolves along a "section" of $\partial \mathcal{K}$, e. g., $\partial \mathcal{K}_i$ for some i. At a later time the solution may re-enter $\mathcal{K}^0$, or it may enter a lower dimensional part of $\partial \mathcal{K}$, e.g., $\partial \mathcal{K}_i \cap \partial \mathcal{K}_j$. Depending on the particular vector field F, it may then evolve within the set $\partial \mathcal{K}_i \cap \partial \mathcal{K}_j$, re-enter $\partial \mathcal{K}_i$, enter $\partial \mathcal{K}_j$, etc.

We now define a stationary or an equilibrium point.

Definition (A Stationary Point or an Equilibrium Point) *The vector* $X^* \in \mathcal{K}$ *is a stationary point or an equilibrium point of the projected dynamical system* PDS$(F, \mathcal{K})$ *if*

$$0 = \Pi_{\mathcal{K}}(X^*, -F(X^*)).$$

In other words, we say that X^* is a stationary point or an equilibrium point if, once the projected dynamical system is at X^*, it will remain at X^* for all future times.

From the definition it is apparent that X^* is an equilibrium point of the projected dynamical system PDS$(F, \mathcal{K})$ if the vector field F vanishes at X^*. The contrary, however, is only true when X^* is an interior point of the constraint set $\mathcal{K}$. Indeed, when X^* lies on the boundary of $\mathcal{K}$, we may have $F(X^*) \neq 0$.

Note that for classical dynamical systems, the necessary and sufficient condition for an equilibrium point is that the vector field vanish at that point, that is, that $0 = -F(X^*)$.

The following theorem states a basic connection between the static world of finite-dimensional variational inequality problems and the dynamic world of projected dynamical systems.

Theorem (Dupuis and Nagurney (1993)) *Assume that $\mathcal{K}$ is a convex polyhedron. Then the equilibrium points of the* PDS$(F, \mathcal{K})$ *coincide with the solutions of* VI$(F, \mathcal{K})$. *Hence, for $X^* \in \mathcal{K}$ and satisfying*

$$0 = \Pi_{\mathcal{K}}(X^*, -F(X^*))$$

also satisfies

$$\langle F(X^*)^T, X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}.$$

This Theorem establishes the equivalence between the set of equilibria of a projected dynamical system and the set of solutions of a variational inequality problem. Moreover, it provides a natural underlying dynamics (out of equilibrium) of such systems.

Before stating the fundamental theorem of projected dynamical systems, we introduce the following assumption needed for the theorem.

Assumption (Linear Growth Condition) *There exists a $B < \infty$ such that the vector field $-F : R^n \mapsto R^n$ satisfies the linear growth condition: $\|F(X)\| \leq B(1 + \|X\|)$ for $X \in \mathcal{K}$, and also*

$$\langle (-F(X) + F(y))^T, X - y \rangle \leq B\|X - y\|^2, \quad \forall X, y \in \mathcal{K}.$$

Theorem (Existence, Uniqueness, and Continuous Dependence) *Assume that the linear growth condition holds. Then*
(i). *For any $X_0 \in \mathcal{K}$, there exists a unique solution $X_0(t)$ to the initial value problem;*
(ii). *If $X_k \to X_0$ as $k \to \infty$, then $X_k(t)$ converges to $X_0(t)$ uniformly on every compact set of $[0, \infty)$.*

The second statement of this Theorem is sometimes called the *continuous dependence* of the solution path to ODE$(F, \mathcal{K})$ on the initial value. By virtue of the Theorem, the PDS$(F, \mathcal{K})$ is well-defined and inhabits $\mathcal{K}$ whenever the Assumption holds.

Lipschitz continuity (see also Section 2) is a condition that plays an important role in the study of variational inequality problems. It also is a critical concept in the classical study of dynamical systems.

Lipschitz continuity implies the Assumption and is, therefore, a sufficient condition for the fundamental properties of projected dynamical systems stated in the Theorem.

We now present an example.

An Example (A Tatonnement or Adjustment Process)

Consider the market equilibrium model introduced in Section 2 in which there are n commodities for which we are interested in determining the equilibrium pattern that satisfies the following equilibrium conditions:

Market Equilibrium Conditions

For each commodity i; $i = 1, \dots, n$:

$$s_i(p^*) - d_i(p^*) \begin{cases} = 0, & \text{if } p_i^* > 0 \\ \geq 0, & \text{if } p_i^* = 0. \end{cases}$$

For this problem we propose the following adjustment or tatonnement process. For each commodity i; $i = 1, \dots, n$:

$$\dot{p}_i = \begin{cases} d_i(p) - s_i(p), & \text{if } p_i > 0 \\ \max\{0, d_i(p) - s_i(p)\}, & \text{if } p_i = 0. \end{cases}$$

In other words, a price of a commodity will increase if the demand for that commodity exceeds the supply of that commodity; the price will decrease if the demand for that commodity is less than the supply for that commodity. If the price of an commodity is equal to zero, and the supply of that commodity exceeds the demand, then the price will not change since one cannot have negative prices according to equilibrium conditions.

In vector form, we may express the above as

$$\dot{p} = \Pi_{\mathcal{K}}(p, d(p) - s(p)),$$

where $\mathcal{K} = R^n_+$, $s(p)$ is the n-dimensional column vector of supply functions, and $d(p)$ is the n-dimensional column vector of demand functions. Note that this adjustment process can be put into the standard form of a PDS, if we define the column vectors: $X \equiv p$ and $F(X) \equiv s(p) - d(p)$.

On the other hand, if we do not constrain the commodity prices to be nonnegative, then $\mathcal{K} = R^n$, and the above tatonnement process would take the form:

$$\dot{p} = d(p) - s(p).$$

This would then be an example of a classical dynamical system.

In the context of the Example, we have then that, according to the Theorem, the stationary point of prices, p^*, that is, those prices that satisfy

$$0 = \Pi_{\mathcal{K}}(p^*, d(p^*) - s(p^*))$$

also satisfy the variational inequality problem (see also Section 2):

$$\langle (s(p^*) - d(p^*))^T, p - p^* \rangle \geq 0, \quad \forall p \in \mathcal{K}.$$

Hence, there is a natural underlying dynamics for the prices, and the equilibrium point satisfies the variational inequality problem; equivalently, is a stationary point of the projected dynamical system.

8. Dynamic Transportation Networks

The study of dynamic travel path/route choice models on general transportation networks, where time is explicitly incorporated into the framework, was initiated by Merchant and Nemhauser (1978), who focused on dynamic system-optimal (see also Section 3) networks with the characteristic of many origins and a single destination. In system-optimal networks, in contrast to user-optimal networks, one seeks to determine the path flow and link flow patterns that minimize the total cost in the network, rather than the individual path travel costs.

Smith (1984) proposed a dynamic traffic user-optimized model with fixed demands. Mahmassani (1991) also proposed dynamic traffic models and investigated them experimentally; see also, Mahmassani et al. (1993). The book by Ran and Boyce (1996) provides an overview of the history of dynamic traffic network models and discusses distinct approaches for their analysis and computation. See also Lesort (1996), Marcotte and Nguyen (1998), and Mahmassani (2005).

Here we present a dynamic transportation model with elastic demands proposed by Dupuis and Nagurney (1993).

The adjustment process overviewed here models the travelers' *day-to-day* dynamic behavior of making trip decisions and path/route choices associated with a travel disutility perspective. Subsequently, some of the stability results of this adjustment process obtained by Zhang and Nagurney (1996, 1997) are highlighted, which address whether and how the travelers' dynamic behavior in attempting to avoid congestion leads to a transportation network equilibrium pattern. Finally, we recall some of the discrete time algorithms devised for the computation of transportation network equilibria with elastic demands and with known travel disutility functions. The convergence of these discrete time algorithms was established by Nagurney and Zhang (1996, 1997). The presentation here follows that in Nagurney (2001c).

A Dynamic Transportation Network Model

The model that we present is due to Dupuis and Nagurney (1993). It is a dynamic counterpart to the static transportation network equilibrium model with elastic travel demands outlined in Section 3, and the notation for the dynamic version follows that given for the static counterpart in Section 3.

The Path Choice Adjustment Process

The dynamical system, whose stationary points correspond to solutions of the variational inequality problem governing the elastic demand transportation network equilibrium model with travel disutility functions (see Section 3), is given by:

$$\dot{X} = \Pi_{\mathcal{K}}(X, \bar{\lambda}(X) - C(X)), \quad X(0) = X_0 \in \mathcal{K},$$

where $X \equiv x$, corresponds to the vector of path flows, and $\bar{\lambda}(X)$ is simply the vector of travel disutilities but expressed now as a function of path flows and re-dimensioned accordingly. The vector $C(X)$ is the vector of path travel costs. The feasible set $\mathcal{K} \equiv R^{n_P}_+$.

This dynamical system is a projected dynamical system, since the right-hand side, which is a projection operator, is discontinuous.

The adjustment process interpretation of the dynamical system, as discussed in Dupuis and Nagurney (1993), is as follows. Users of a transportation network choose, at the greatest rate, those paths whose differences between the travel disutilities (demand prices) and path costs are maximal; in other words, those paths whose costs are minimal relative to the travel disutilities. If the travel cost on a path exceeds the travel disutility associated with the O/D pair, then the flow on that path will decrease; if the travel disutility exceeds the cost on a path, then the flow on that path will increase. If the difference between the travel disutility and the path cost drives the path flow to be negative, then the projection operator guarantees that the path flow will be zero. The process continues until there is no change in path flows, that is, until all used paths have path costs equal to the travel disutilities, whereas unused paths will have costs which exceed the disutilities. Specifically, the travelers adjust their path choices until an equilibrium is reached.

The following example, given in a certain discrete time realization, shows how the dynamic mechanism of the above path choice adjustment would reallocate the traffic flow among the paths and would react to changes in the travel disutilities.

An Example (cf. Nagurney (2001c))

Consider a simple transportation network consisting of two nodes, with a single O/D pair w, and two links a and b representing the two disjoint paths connecting the O/D pair. Suppose that the link costs are:

$$c_a(f_a) = f_a + 2, \quad c_b(f_b) = 2f_b,$$

and the travel disutility function is given by:

$$\lambda_w(d_w) = -d_w + 5.$$

Note that here a path consists of a single link and, hence, we can use x and f interchangeably. Suppose that, at time $t = 0$, the flow on link a is 0.7, the flow on link b is 1.5; hence, the demand is 2.2, and the travel disutility is 2.8, that is,

$$x_a(0) = 0.7, \quad x_b(0) = 1.5, \quad d_w(0) = 2.2, \quad \lambda_w(0) = 2.8,$$

which yields travel costs: $c_a(0) = 2.7$ and $c_b(0) = 3.0$.

According to the above path choice adjustment process, the flow changing rates at time $t = 0$ are:

$$\dot{x}_a(0) = \lambda_w(0) - c_a(0) = 0.1, \quad \dot{x}_b(0) = \lambda_w(0) - c_b(0) = -0.2.$$

If a time increment of 0.5 is used, then at the next moment $t = 0.5$, the flows on link a and link b are:

$$x_a(0.5) = x_a(0) + 0.5\dot{x}_a(0) = 0.7 + 0.5 \times 0.1 = 0.75,$$

$$x_b(0.5) = x_b(0) + 0.5\dot{x}_b(0) = 1.5 - 0.5 \times 0.2 = 1.4,$$

which yields travel costs: $c_a(0.5) = 2.75$ and $c_b(0.5) = 2.8$, a travel demand $d_w(0.5) = 2.15$, and a travel disutility $\lambda_w(0.5) = 2.85$. Now, the flow changing rates are given by:

$$\dot{x}_a(0.5) = \lambda_w(0.5) - c_a(0.5) = 2.85 - 2.75 = 0.1,$$

$$\dot{x}_b(0.5) = \lambda_w(0.5) - c_b(0.5) = 2.85 - 2.8 = 0.05.$$

The flows on link a and link b at time $t = 1.0$ would, hence, then be:

$$x_a(1.0) = x_a(0.5) + 0.5\dot{x}_a(0.5) = 0.75 + 0.5 \times 0.1 = 0.80,$$

$$x_b(1.0) = x_b(0.5) + 0.5\dot{x}_b(0.5) = 1.4 + 0.5 \times 0.05 = 1.425,$$

which yields travel costs: $c_a(1.0) = 2.80$ and $c_b(1.0) = 2.85$, a travel demand $d_w(1.0) = 2.225$, and a travel disutility $\lambda_w(1.0) = 2.775$. Now, the flow changing rates are given by:

$$\dot{x}_a(1.0) = \lambda_w(1.0) - c_a(1.0) = 2.775 - 2.800 = 0.025,$$

$$\dot{x}_b(1.0) = \lambda_w(1.0) - c_b(1.0) = 2.775 - 2.850 = -0.075.$$

The flows on link a and link b at time $t = 1.5$ would be:

$$x_a(1.5) = x_a(1.0) + 0.5\dot{x}_a(1.0) = 0.8 - 0.5 \times 0.025 = 0.7875,$$

$$x_b(1.5) = x_b(1.0) + 0.5\dot{x}_b(1.0) = 1.425 - 0.5 \times 0.075 = 1.3875,$$

which yields travel costs: $c_a(1.5) = 2.7875$ and $c_b(1.5) = 2.775$, a travel demand $d_w(1.5) = 2.175$, and a travel disutility $\lambda_w(1.0) = 2.82$.

In this example, as time elapses, the path choice adjustment process adjusts the flow volume on the two links so that the difference between the travel costs of link a and link b is being reduced, from 0.3, to 0.05, and, finally, to 0.0125; and, the difference between the disutility and the travel costs on the used links is also being reduced from 0.2, to 0.1, and to 0.045. In fact, the transportation network equilibrium with: $x_a^* = 0.8$ and $x_b^* = 1.4$, which induces the demand $d_w^* = 2.2$, is almost attained in only 1.5 time units.

Stability Analysis

We now present the stability results of the path choice adjustment process. The results described here are due to Zhang and Nagurney (1996, 1997). As noted therein, the questions that motivate transportation planners and analysts to study the stability of a transportation system include: Will any initial flow pattern be driven to an equilibrium by the adjustment process? In addition, will a flow pattern near an equilibrium always stay close to it? These concerns of *system stability* are important in traffic assignment and form, indeed, a critical base for the very concept of an equilibrium flow pattern.

For the specific application of transportation network problems, the following definitions of stability of the transportation system and the local stability of an equilibrium are adapted from the general stability concepts of projected dynamical systems (cf. Zhang and Nagurney (1995)).

Definition (Stability at an Equilibrium) *An equilibrium path flow pattern X^* is stable if it is a global monotone attractor for the corresponding path choice adjustment process.*

Definition (Asymptotical Stability at an Equilibrium) *An equilibrium path flow pattern X^* is asymptotically stable if it is a strictly global monotone attractor for the corresponding route choice adjustment process.*

Definition (Stability of the System) *A path choice adjustment process is stable if all its equilibrium path flow patterns are stable.*

Definition (Asymptotical Stability of the System) *A path choice adjustment process is asymptotically stable if all its equilibrium flow patterns are asymptotically stable.*

We now present the stability results in Zhang and Nagurney (1996) for the path choice adjustment process.

Theorem (Zhang and Nagurney (1996)) *Suppose that the link cost functions c are monotone increasing in the link flow pattern f and that the travel disutility functions λ are monotone decreasing in the travel demand d. Then the path choice adjustment process is stable.*

Theorem (Zhang and Nagurney (1996)) *Assume that there exists some equilibrium path flow pattern. Suppose that the link cost functions c and negative disutility functions $-\lambda$ are strictly monotone in the link flow f and the travel demand d, respectively. Then, the path choice adjustment process is asymptotically stable.*

The first theorem states that, provided that monotonicity of the link cost functions and the travel disutility functions holds true, then any flow pattern near an equilibrium will stay close to it forever. Under the strict monotonicity assumption, on the other hand, the second theorem can be interpreted as saying that any initial flow pattern will eventually be driven to an equilibrium by the path choice adjustment process.

Discrete Time Algorithms

The Euler method and the Heun method were employed in Nagurney and Zhang (1996) for the computation of solutions to dynamic elastic demand transportation network problems with known travel disutility functions, and their convergence was also established therein. We refer the reader to these references for numerical results, including transportation network examples that are solved on a massively parallel computer architecture. Applications of such algorithms to other network-based problems, are also given in Nagurney and Zhang (2006). Additional novel applications of projected dynamical systems theory, including extensions to infinite-dimensions and relationships to evolutionary variational ienqualities and double-layered dynamics, can be found in the book by Nagurney (2006b), which focuses on supply chain network economics.

In particular, at iteration τ, the Euler method computes

$$X^{\tau+1} = P_{\mathcal{K}}(X^{\tau} - a_{\tau}F(X^{\tau})),$$

whereas, according to the Heun method, at iteration τ one computes

$$X^{\tau+1} = P_{\mathcal{K}}(X^{\tau} - a_{\tau}\frac{1}{2}\left[F(X^{\tau}) + F(P(X^{\tau} - a_{\tau}F(X^{\tau})))\right]).$$

In the case that the sequence $\{a_{\tau}\}$ in the Euler method is fixed, say, $\{a_{\tau}\} = \rho$, for all iterations τ, then the Euler method collapses to a projection method (cf. Dafermos (1980, 1983), Bertsekas and Gafni (1982), Nagurney (1999)).

In the context of the dynamic transportation network problem with known travel disutility functions, the projection operation in the above discrete time algorithms can be evaluated explicitly and in closed form. Indeed, each iteration τ of Euler method takes the form: For each path $p \in P$ in the transportation network, compute the path flow $x_p^{\tau+1}$ according to:

$$x_p^{\tau+1} = \max\{0, x_p^\tau + a_\tau(\lambda_w(d^\tau) - C_p(x^\tau))\}.$$

Each iteration of the Heun method, in turn, consists of two steps. First, at iteration τ one computes the approximate path flows:

$$\bar{x}_p^\tau = \max\{0, x_p^\tau + a_\tau(\lambda_w(d^\tau) - C_p(x^\tau))\}, \; \forall p \in P,$$

and updates the approximate travel demands:

$$\bar{d}_w^\tau = \sum_{p \in P_w} \bar{x}_p^\tau, \; \forall w \in W.$$

Let

$$\bar{x}^\tau = \{\bar{x}_p^\tau, p \in P\}$$

and

$$\bar{d}^\tau = \{\bar{d}_w^\tau, w \in W\}.$$

Then, for each path $p \in P$ in the transportation network one computes the updated path flows $x_p^{\tau+1}$ according to:

$$x_p^{\tau+1} = \max\{0, x_p^\tau + \frac{a_\tau}{2}\left[\lambda_w(d^\tau) - C_p(x^\tau) + \lambda_w(\bar{d}^\tau) - C_p(\bar{x}^\tau)\right]\}, \; \forall p \in P,$$

and updates the travel demands $d_w^{\tau+1}$ according to:

$$d_w^{\tau+1} = \sum_{p \in P_w} x_p^{\tau+1}, \; \forall w \in W.$$

It is worth noting that both the Euler method and the Heun method at each iteration yield subproblems in the path flow variables, each of which can be solved not only in closed form, but also, simultaneously. Hence, these algorithms in the context of this model can be interpreted as massively parallel algorithms and can be implemented on massively parallel architectures. Indeed, this has been done so by Nagurney and Zhang (1996).

In order to establish the convergence of the Euler method and the Heun method, one regularizes the link cost structures.

Definition (A Regular Cost Function) *The link cost function c is called regular if, for every link $a \in L$,*

$$c_a(f) \longrightarrow \infty, \; \text{as } f_a \longrightarrow \infty,$$

holds uniformly true for all link flow patterns.

We note that the above regularity condition on the link cost functions is natural from a practical point of view and it does not impose any substantial restrictions. In reality, any link has an upper bound in the form of a capacity. Therefore, letting $f_a \longrightarrow \infty$ is an artificial device under which one can reasonably deduce that $c_a(f) \longrightarrow \infty$, due to the congestion effect. Consequently, any practical link cost structure can be theoretically extended to a regular link cost structure to allow for an infinite load.

The theorem below shows that both the Euler method and the Heun method converge to the transportation network equilibrium under reasonable assumptions.

Theorem (Nagurney and Zhang (1996)) *Suppose that the link cost function c is regular and strictly monotone increasing, and that the travel disutility function λ is strictly monotone decreasing. Let $\{a_\mathcal{T}\}$ be a sequence of positive real numbers that satisfies*

$$\lim_{\mathcal{T}\to\infty} a_\mathcal{T} = 0$$

and

$$\sum_{\mathcal{T}=0}^{\infty} a_\mathcal{T} = \infty.$$

Then both the Euler method and the Heun method produce sequences $\{X^\mathcal{T}\}$ that converge to some transportation network equilibrium path flow pattern.

A Dynamic Spatial Price Model

We now present the projected dynamical system (cf. Section 7) model of the static spatial price model in price (and quantity) variables presented in Section 4. In view of the variational inequality governing the price model, we may write the projected dynamical system immediately as:

$$\begin{pmatrix} \dot{Q} \\ \dot{\pi} \\ \dot{\rho} \end{pmatrix} = \Pi_{R_+^{mn+m+n}} \left(\begin{pmatrix} Q \\ \pi \\ \rho \end{pmatrix}, \begin{pmatrix} -T(Q,\pi,\rho) \\ -S(Q,\pi) \\ -D(Q,\rho) \end{pmatrix} \right).$$

More explicitly, if the demand price at a demand market exceeds the supply price plus the unit transaction cost associated with shipping the commodity between a pair of supply and demand markets, then the commodity shipment between this pair of markets will increase. On the other hand, if the supply price plus unit transaction cost exceeds the demand price, then the commodity shipment between the pair of supply and demand markets will decrease. If the supply at a supply market exceeds (is exceeded by) the commodity shipments out of the market, then the supply price will decrease (increase). In contrast, if the demand at a demand market exceeds (is exceeded by) the commodity shipments into the market, then the demand price will increase (decrease).

If the commodity shipments, and/or the supply prices, and/or the demand prices are driven to be negative, then the projection ensures that the commodity shipments and the prices will be nonnegative, by setting the values equal to zero. The solution to the projected dynamical system then evolves along a "section" of the boundary of the feasible set. At a later time, the solution may re-enter the interior of the constraint set, or it may enter a lower dimensional part of its boundary, with, ultimately, the spatial price equilibrium conditions being reached at a stationary point, that is, when $\dot{X} = 0$.

It is worth noting that there are also relevant applications of projected dynamical systems to evolutionary games and evolutionary dynamics; see Sandholm (2005).

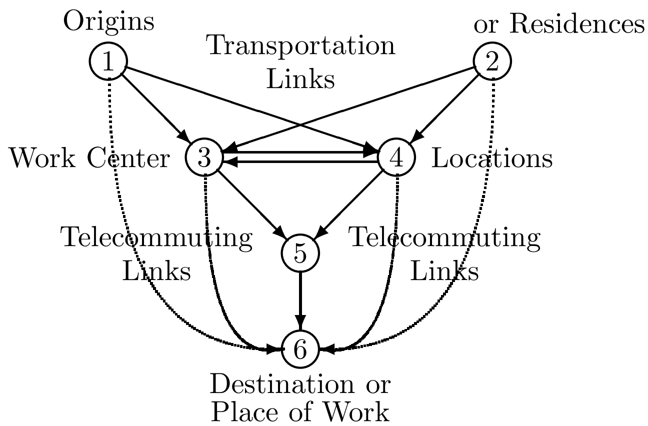

Figure 9: A Supernetwork Conceptualization of Commuting versus Telecommuting

9. Supernetworks: Applications to Telecommuting Decision-Making and Teleshopping Decision-Making

The growing impact of the Information Age, coupled with similarities between transportation networks and communications networks in terms of the relevance of such concepts as system-optimization and user-optimization, along with issues of centralized versus decentralized control, have provided a setting in which the relationships between decision-making on such networks and associated trade-offs could be explored. Towards that end, Nagurney, Dong, and Mokhtarian (2002a, b), developed multicriteria network equilibrium models which allowed for distinct classes of decision-makers who weight their criteria associated with utilized transportation versus telecommunications networks in a variety of activities (such as teleshopping and telecommuting) in an individual fashion. Nagurney and Dong (2002b, c) had also proposed multicriteria network equilibrium models in the case of elastic demands as well as for combined location and transportation decision-making, respectively. In such and related models, criteria such as time, cost, risk, as well as opportunity cost (all criteria noted by Beckmann, McGuire, and Winsten (1956)) play a prominent and fresh role. The authors described the governing equilibrium conditions in the case of fixed and elastic demands and provided computational procedures and numerical examples demonstrating that the user-optimizing principle was relevant in the context of these new types of networks termed *supernetworks* in the book by Nagurney and Dong (2002a). That book also traces the origins of the term back to the transportation and computer science literatures.

The decision-makers in the context of the telecommuting versus commuting decision-making application are travelers, who seek to determine their *optimal* routes of travel from their origins, which are residences, to their destinations, which are their places of work. Note that, in the supernetwork framework, a link may correspond to an actual physical link of transportation or an abstract or virtual link corresponding to a telecommuting link. Furthermore, the supernetwork representing the problem under study can be as general as necessary and a path may consist of a set of links corresponding to physical and virtual transportation choices such as would occur if a worker were to commute to a work center from which he could then telecommute. In Figure 9, a conceptualization of this idea is recalled.

Of course, the network depicted in Figure 9 is illustrative, and the actual network can be much more complex with numerous paths depicting the physical transportation choices from one's residence to one's work location. Similarly, one can further complexify the telecommunication link/path options. Also, we emphasize, that a *path* within this framework is sufficiently general to also capture a choice of mode, which, in the case of transportation, could correspond to busses, trains, or subways (that is, public transit) and, of course, to the use of cars (i.e., private vehicles). The concept of path can be used to represent a distinct telecommunications option.

The behavioral assumption is that travelers of a particular class are assumed to choose the paths associated with their origin/destination (O/D) pair so that the generalized cost on that path, which consists of a weighting of the different criteria (which can be different for each class of decision-maker and can also be link-dependent), is minimal. An equilibrium is assumed to be reached when the multicriteria network equilibrium conditions are satisfied whereby only those paths connecting an O/D pair are utilized such that the generalized costs on the paths, as perceived by a class, are equal and minimal.

Now a multicriteria network equilibrium model for teleshopping decision-making is described. For further details, including numerical examples, see Nagurney and Dong (2002a) and the papers by Nagurney, Dong, and Mokhtarian (2002a, b). Assume that consumers are engaged in the purchase of a product which they do so in a repetitive fashion, say, on a weekly basis. The product may consist of a single good, such as a book, or a bundle of goods, such as food. Assume also that there are locations, both virtual and physical, where the consumers can obtain information about the product. The virtual locations are accessed through telecommunications via the Internet whereas the physical locations represent more classical shopping venues such as stores and require physical travel to reach.

The consumers may order/purchase the product, once they have selected the appropriate location, be it virtual or physical, with the former requiring shipment to the consumers' locations and the latter requiring, after the physical purchase, transportation of the consumer with the product to its final destination (which we expect, typically, to be his residence or, perhaps, place of work).

Refer to the network conceptualization of the problem given in Figure 10. We now identify the above concepts with the corresponding network component. Observe that the network depicted in Figure 10 consists of four levels of nodes with the first (top) level and the last (bottom) level corresponding to the locations (destinations) of the consumers involved in the purchase of the product. An origin/destination pair in this network corresponds to a pair of nodes from the top tier in Figure 10 to the bottom tier. In the shopping network framework, a path consists of a sequence of choices made by a consumer and represents a sequence of possible options for the consumers. The flows, in turn, reflect *how many* consumers of a particular class actually select the particular paths and links, with a zero flow on a path corresponding to the situation that no consumer elects to choose that particular sequence of links.

The criteria that are relevant to decision-making in this context are: time, cost, opportunity cost, and safety or security risk, where, in contrast to the telecommuting application time need not be restricted simply to *travel* time and, depending on the associated link, may include transaction time. In addition, the cost is not exclusively a travel cost but depends on the associated link and can include the transaction cost as well as the product price, or shipment cost. Moreover, the opportunity cost now arises when shoppers on the Internet cannot have the physical experience of

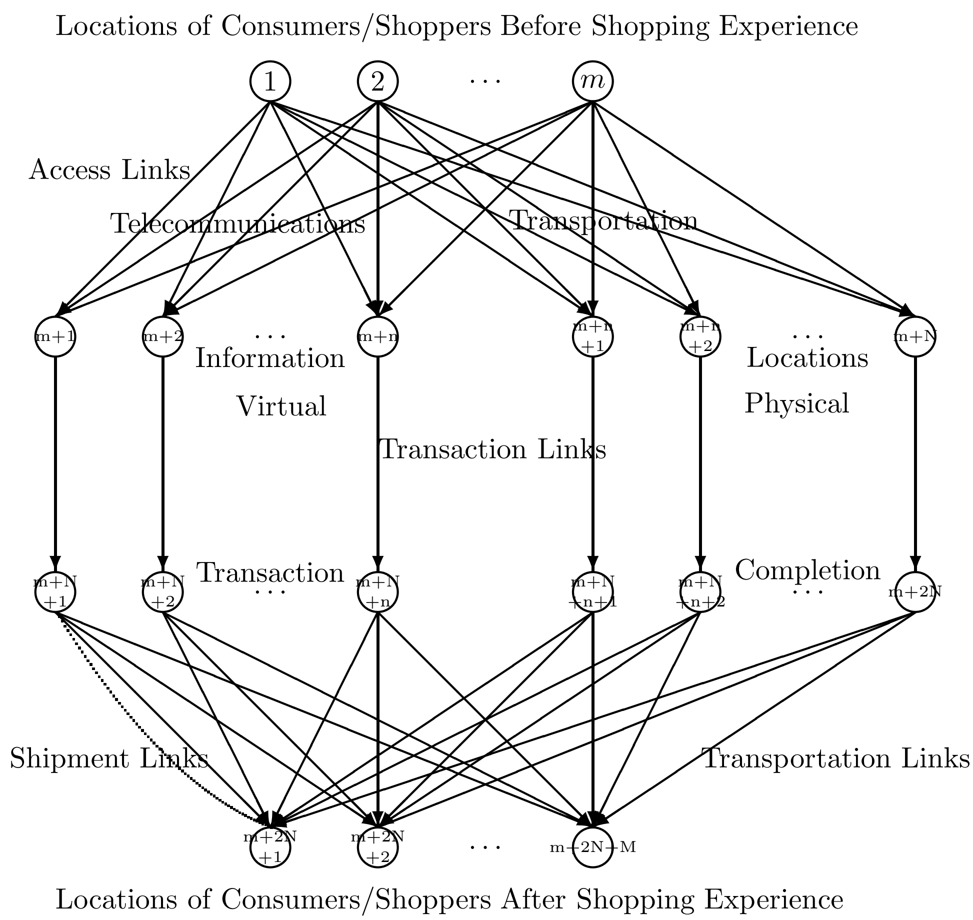

Figure 10: A Supernetwork Framework for Teleshopping versus Shopping

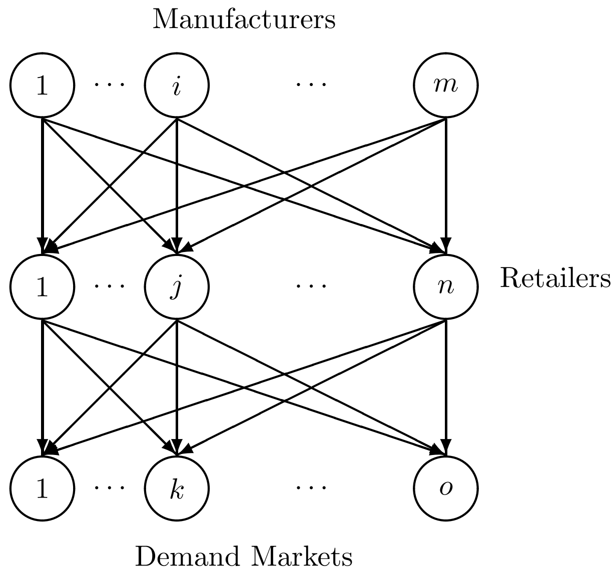

Figure 11: A Supply Chain Network

trying the good or the actual sociableness of the shopping experience itself. Finally, the safety or security risk cost now can reflect not only the danger of certain physical transportation links but also the potential of credit card fraud, etc.

10. Supply Chain Networks and Other Applications

Beckmann, McGuire, and Winsten (1956) also explicitly recognized the generality of networks as a means of conceptualizing even decision-making of a firm with paths corresponding to production processes and the links corresponding to transformations as the material moved down the path from the origin to the destination. The paths then abstracted the choices or production possibilities available to a firm.

As mentioned in the Introduction, another application in which the concept of network equilibrium is garnering interest is that of *supply chain networks*. This topic is interdisciplinary by nature since it contains aspects of manufacturing, retailing, transportation, economics, as well as operations research and management science. Zhang, Dong, and Nagurney (2003) have recently generalized Wardrop's principle(s) to consider not only paths but *chains* in the network to identify the "winning" supply chains. In that application context, paths correspond to production processes and links can be either operation or interface links. Their framework allows for the modeling of competition between supply chains which may entail several firms (producing, transporting, retailing, etc.).

The first work on utilizing network equilibrium concepts in the context of supply chain applications is due to Nagurney, Dong, and Zhang (2002). The depiction of that supply chain network is as given in Figure 11. The decision-makers, now located at the nodes of the network, are faced with their individual objective functions, which can include profit-maximization, and one seeks to determine not only the optimal/equilibrium flows between tiers of nodes but also the prices of the product at the various tiers. The model therein was, subsequently, generalized to include electronic commerce by Nagurney et al. (2002).

Recently, Nagurney (2006a) proved that supply chain network equilibrium problems, in which decision-makers compete across a tier of the network, but cooperate between tiers, could be reformulated as transportation network equilibrium problems over appropriately constructed abstract networks or supernetworks. This result provided a new interpretation of the supply chain equilibrium conditions in terms of paths, path flows, and associated costs (as featured in Section 3). A similar connection was established by Nagurney and Liu (2006); see also Wu et al. (2006), but in the case of electric power generation and distribution networks, thus, resolving an open hypothesis raised over fifty years ago by Beckmann, McGuire and Winsten (1956). Indeed, electric power networks can be transformed and solved as transportation network equilibrium problems. Liu and Nagurney (2006) were able to obtain a similar result for financial networks with intermediation.

Finally, given the importance of network economics, in general, and its associated numerous applications, some of which have been discussed in this chapter, we conclude this chapter with a brief discussion of network efficiency and the identification of the importance of the network components, that is, nodes and links, in a network. Recently, Nagurney and Qiang (2007) applied a network efficiency measure which captures flows, costs, and behavior to several transportation networks in order to not only determine the efficiency of the network but to also identify and rank the nodes and links in terms of their importance. The measure provides an economic evaluation of the network in terms of the demand for the resources versus the associated costs. We expect that the Nagurney and Qiang measure will be very useful for policy makers, planners, and security experts, since it identitifes which nodes and links, if removed, affect the efficiency of the network the most. Hence, the network is most vulnerable when such nodes and links are destroyed due to, for example, natural disasters, structural failures, terrorist attacks. etc. Such a measure, in view of the above discussion, is also relevant to the Internet, supply chains, electric power generation and distribution networks, as well as to financial networks with intermediation. Moreover, it also includes, as a special case, the network measure used in the complex (cf. Newman (2003)) literature, due to Latora and Marchiori (2001), which, however, although based on shortest paths, does not explicitly incorporate behavior of the users of the network and the associated flows.

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