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The negation of the Braess paradox as demand increases: The wisdom of crowds in transportation networks

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Abstract. - In the well-known Braess paradox [D. Braess, Unternehmenforschung 12, 258 (1968)], the addition of a new route in a specific congested transportation network made all the travelers worse off in terms of their individual travel cost (time). In this paper, we consider the hypothesis that, in congested networks, the Braess Paradox may "disappear" under higher demands, and we prove this hypothesis by deriving a formula that provides the increase in demand that will guarantee that the addition of that new route will no longer increase travel cost since the new path will no longer be used. This result is established for any network in which the Braess Paradox originally occurs. This suggests that, in the case of congested, noncooperative networks, of which transportation networks are a prime example, a higher demand will negate the counterintuitive phenomenon known as the Braess Paradox. At the same time, this result demonstrates that extreme caution should be taken in the design of network infrastructure, including transportation networks, since at higher demands, new routes/pathways may not even be used!

Introduction. – Congestion is a fundamental problem in a variety of network systems, ranging from urban transportation networks to electric power generation and distribution networks and the Internet [1]-[16]. Congestion leads to increases in travel time, wear and tear on our infrastructure, higher emissions due to vehicular idling, as well as to losses in productivity.

Congested networks are flow-dependent with induced flows being the result of the behavior of the users of the particular network. Historically, there have been two principles of travel behavior, dating to Wardrop [8]-[10], corresponding to user-optimizing (U-O) behavior, in which travelers select their optimal routes of travel individually and unilaterally, leading to an equilibrium, and system-optimizing (S-O) behavior, in which a central controller routes or assigns the flows to particular paths in the network so that the total cost is minimized.

The Braess Paradox (cf. [1] and [11] for the translation of the article from German to English) in which the cost on used paths increases for all after the addition of a new route occurs only under user-optimizing or selfish behavior. Such behavior, however, is characteristic of commuting behavior, decentralized routing on the Internet, as well as the behavior of a spectrum of network systems, including electric power generation and distribution networks, in which decision-makers act independently and noncooperatively [2]-[5]. The recognition of the existence of the Braess paradox has led, in practice, to major policy decisions such as road closures in Seoul, Korea; in Stuttgart, Germany, as well as in New York City [6].

Interestingly, it was established that, for the specific Braess network [1], the paradox no longer occurred as the demand for travel increased [5]-[7]. This leads us to the hypothesis that, under a higher demand, the Braess Paradox is negated in that the new route, which resulted in increased travel time at a particular demand, will no longer be used. We establish this result through the derivation of a formula that is applicable to any network of general topology in which the Braess paradox originally occurs. For definiteness, we consider transportation networks in which the travel cost on each link is an increasing linear function of the flows (volume of traffic on the links) and we do not limit the analysis to separable functions as was the case in the original Braess Paradox network(s).

We first briefly review the U-O transportation model, referred to, henceforth, as the traffic network equilibrium model, which has been widely studied and applied in practice to congested urban transportation networks globally and the Internet, and which is also closely related to electric power generation and distribution networks [17, 18]. We then derive the formula and establish our result. Subsequently, we revisit the Braess Paradox example.

Traffic Network Equilibrium and the Braess Paradox. – Following [16], let $\mathcal{G}$ denote a traffic network with the set of directed links L with n_L elements and the set of origin/destination (O/D) pairs of nodes W with n_W elements. We denote the set of acyclic paths joining O/D pair w by P_w. The set of (acyclic) paths for all O/D pairs is denoted by P and there are n_P such paths in the network. Links are denoted by a, b, etc; paths by p, q, etc., and O/D pairs by w_1, w_2, etc.

We assume that the demand d_w is known for all $w \in W$ and group the demands into the vector $d \in R_+^{n_W}$. We denote the nonnegative flow on path p by x_p, the flow on link a by f_a, and group the path flows into the vector $x \in R_+^{n_P}$ and the link flows into the vector $f \in R_+^{n_L}$.

The following conservation of flow equations must hold:

$$\sum_{p \in P_w} x_p = d_w, \quad \forall w \in W, \tag{1}$$

which means that the sum of path flows on paths connecting each O/D pair must be equal to the demand for that O/D pair, or, in matrix form,

$$Bx = d, \tag{2}$$

where B is the $n_W \times n_P$ matrix whose (w, p) entry is 1 if path p connects w and 0, otherwise.

The link flows are related to the path flows, in turn, through the following conservation of flow equations:

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in L, \tag{3}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and $\delta_{ap} = 0$, otherwise. Hence, the flow on a link is equal to the sum of the flows on paths that contain that link. In matrix form, we may write

$$f = Ax, \tag{4}$$

where A is the $n_L \times n_P$ link-path incidence matrix whose (a, p) entry is 1 if link a is contained in path p and 0, otherwise.

The user (travel) cost on a path p is denoted by C_p and the user (travel) cost on a link a by c_a. The user costs on paths are related to user costs on links through the following expression:

$$C_p = \sum_{a \in L} c_a \delta_{ap}, \quad \forall p \in P, \tag{5}$$

that is, the user cost on a path is equal to the sum of user costs on links that make up the path. In engineering practice [15], the travel time on a link is used as a proxy for the travel cost.

Since we are concerned with congested networks, we allow the user link cost function on each link to depend on the flow and, in general, upon the vector of link flows, so that

$$c_a = c_a(f), \quad \forall a \in L. \tag{6}$$

We assume that the link cost functions are continuous and monotonically increasing.

In the case where the user link cost functions are affine, we let

$$c_a = \sum_{b \in L} g_{ab} f_b + h_a, \quad \forall a \in L, \tag{7}$$

and denote the Jacobian of the user link cost functions by the $n_L \times n_L$-dimensional matrix G where

$$G = [g_{ab}], \tag{8}$$

which is assumed to be positive definite. This condition is reasonable in congested networks and will also guarantee uniqueness of the equilibrium link flow pattern.

In view of (1), (3), and (5), we may write

$$C_p = C_p(x), \quad \forall p \in P. \tag{9}$$

According to Wardrop's first principle of travel behavior: a U-O or equilibrium flow pattern is defined as follows. A path flow pattern $x^* \in \mathcal{K}$, where $\mathcal{K} \equiv \{x | x \in R_+^{n_P} \text{ and } (2) \text{ holds}\}$, is said to be in equilibrium, if the following conditions hold for each O/D pair $w \in W$ and each path $p \in P_w$:

$$C_p(x^*) \begin{cases} = \lambda_w, & \text{if} \quad x_p^* > 0, \\ \geq \lambda_w, & \text{if} \quad x_p^* = 0. \end{cases} \tag{10}$$

The interpretation of conditions (10) is that all used paths connecting an O/D pair w have equal and minimal costs (with these minimal path costs equal to the equilibrium travel disutility, denoted by λ_w). These conditions are also referred to as the user-optimized conditions ([10, 12]) since no traveler has any incentive to switch his travel path. As established in [14], the equilibrium pattern according to above definition is also the solution of a variational inequality problem.

In the classical traffic network equilibrium problem, in which the cost on each link depends solely on the flow on that link, the traffic network equilibrium conditions (10) can be reformulated as the solution to an appropriately constructed optimization problem [9]. For additional background on this model, along with its impacts, see [12].

Consider now the linear system (2) and (4) expressed as:

$$\begin{pmatrix} A \\ B \end{pmatrix} x = \begin{pmatrix} f \\ d \end{pmatrix}, \quad x \geq 0, \tag{11}$$

and assume that it admits only one solution, that is, rank $\begin{pmatrix} A \\ B \end{pmatrix} = n_P$. A necessary condition for this is that $n_L + n_W \geq n_P$ (cf. [16]).

A change in the travel demand, Δd, will induce uniquely determined changes Δf, Δx, Δc, ΔC, and $\Delta\lambda$, in the U-O link flow and path flow patterns, and the incurred U-O link cost, path cost, and O/D pair travel disutilities.

By virtue of (2), (4), (5), and (7):

$$\Delta d = B\Delta x,\ \Delta f = A\Delta x,\ \Delta c = G\Delta f,\ \Delta C = A^T\Delta c. \quad (12)$$

We assume that the flow on every path is positive before and after the change Δd. It follows then from (10) that:

$$\Delta C_p = \Delta\lambda_w, \quad \forall p \in P_w, \forall w. \quad (13)$$

Hence,

$$\Delta C = B^T\Delta\lambda. \quad (14)$$

Combining now (12) and (14), we get

$$B^T\Delta\lambda = \Delta C = A^T\Delta c = A^TG\Delta f = A^TGA\Delta x. \quad (15)$$

Hence,

$$\begin{pmatrix} A^TGA & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} \Delta x \\ -\Delta\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \Delta d \end{pmatrix}. \quad (16)$$

The linear system (16) can be used to determine the changes in the path flows Δx (and the travel disutilities $\Delta\lambda$) induced by a change Δd in the demand.

We now focus on networks in which the Braess Paradox occurs. We are interested in using the system (16) to analyze and determine whether there exists an increase in demand that will result in the added path that resulted in the Braess Paradox (at a given demand) will no longer be used and, thus, will have zero flow. In this case, at the new demand, which is higher than the original one, the Braess Paradox will not occur.

We denote the path that, when it was added to the original network, it resulted in the occurrence of the Braess Paradox, by r. Hence, the travel cost increased, after the addition of r, for all travelers between the O/D pair that the new path r joined.

For transparency, and clarity, we now consider (as was the case in the original Braess Paradox network) that there is a single O/D pair w_1 in the network. Also, we are interested in the $\Delta x_r = -x_r$, which would mean that, at the increased level of demand, the new flow on path r would (just) drop to zero.

We now adapt expression (16) to a single O/D pair w_1 and assume that the path r is the last path. Before applying Cramer's Rule to (16) we need some preliminaries.

Note that, in the case of the addition of a new path r, we may write:

$$G = \begin{pmatrix} \hat{G} & 0 \\ 0 & g_{new} \end{pmatrix}, \quad (17)$$

where $\hat{G}$ was the Jacobian of the link cost functions in the original network (before the addition of the path r) and g_{new} is the Jacobian of the user link cost functions on the new links. We assume that the original link cost functions do not depend on the flows on the new links.

Also, we have that:

$$A = \begin{pmatrix} \hat{A} & A_r \\ 0 & \mathbf{1} \end{pmatrix}, \quad (18)$$

where $\hat{A}$ is the arc-path incidence matrix for the original network (before the addition of the path r), A_r is the arc-path incidence matrix for path r relative to the original links, and $\mathbf{1}$ is the vector with as many components, all equal to 1, as there are new links in the new path r.

With G as in (17) and A as in (18), A^TGA (cf. [16]) with its last column replaced by the column vector with all zeros is

$$A^TGA = \begin{pmatrix} \hat{A}^T\hat{G}\hat{A} & 0 \\ A_r^T\hat{G}\hat{A} & 0 \end{pmatrix}. \quad (19)$$

We, thus, obtain, after an application of Cramer's Rule to (16), with the above specifications, and straightforward manipulations, the following formula, which relates the decrease in flow on path r to the change in demand for the O/D pair Δd_{w_1}:

$$-x_r = \frac{det\begin{pmatrix} \hat{A}^T\hat{G}\hat{A} & 0 & B_1^T \\ A_r^T\hat{G}\hat{A} & 0 & 1 \\ B_1 & 1 & 0 \end{pmatrix}}{det\begin{pmatrix} A^TGA & B^T \\ B & 0 \end{pmatrix}}\Delta d_{w_1}, \quad (20)$$

where det denotes the determinant and B_1 is the B vector with one element removed (note that all the elements of B since we are considering a single O/D pair, are equal to 1).

Recall that for E, J, H, and K matrices of dimensions: $n\times n$, $n\times m$, $m\times n$, and $m\times m$, respectively, and for K invertible:

$$det\begin{pmatrix} E & J \\ H & K \end{pmatrix} = det(K)det(E - JK^{-1}H). \quad (21)$$

We know that $det\begin{pmatrix} A^TGA & B^T \\ B & 0 \end{pmatrix} \neq 0$, by assumption that rank$\begin{pmatrix} A \\ B \end{pmatrix} = n_P$. Moreover, as proven in [16], $det\begin{pmatrix} A^TGA & B^T \\ -B & 0 \end{pmatrix} \geq 0$ and, hence, $det\begin{pmatrix} A^TGA & B^T \\ B & 0 \end{pmatrix} \leq 0$. Applying now formula (21) to evaluate the numerator determinant in (20), gives us, after simplification:

$$-det\left(\hat{A}^T\hat{G}\hat{A} - B_1^TA_r^T\hat{G}\hat{A}\right). \quad (22)$$

But since it has been assumed that we are dealing with a network in which the Braess Paradox occurs at the specific d_{w_1}, it follows from Theorem 4.1 in [16] that the sign of

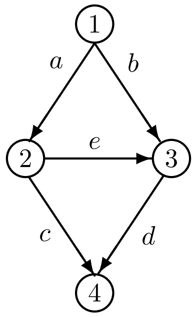

Fig. 1: The Braess Network

$det\left(\hat{A}^T\hat{G}\hat{A} - B_1^T A_r^T \hat{G}\hat{A}\right)$ is negative. Returning to equation (20) we have established that the determinant in the numerator has a positive value, the determinant in the denominator has a negative value and, consequently, there exists a positive Δd_{w_1} that will yield a solution to (20) for $-x_r$, where recall that x_r is the path flow on the added path r, which resulted in the Braess Paradox occurring.

Hence, we have established the following:

Theorem
Under the preceding assumptions, there exists a Δd_{w_1} positive for which the Braess Paradox is negated in that the flow on the path r that resulted in the Braess Paradox occurring at a fixed level of demand, will no longer occur at the new level of demand since that path will not be used and, hence, it cannot result in an increase in travel cost.

Interpretation . – This result has fascinating implications. It demonstrates that, as demand increases, the Braess Paradox "works itself out." One would expect that at a higher level of demand the network gets even more congested and that more of the paths/routes would then be used. However, the Theorem establishes that, in fact, the route that resulted in the Braess Paradox at a particular level of demand will no longer be used at a higher level of demand. This suggests that there may be an underlying "wisdom of crowds phenomenon" taking place.

It is worth noting that the qualitative results in the above Theorem also hold for nonlinear, strongly monotone cost functions (cf. [16]).

We now show how the Theorem can be applied.

Consider the Braess Paradox ([1], [11], [17]) example after the addition of a new link e and as depicted in Figure 1. There are four nodes: 1, 2, 3, 4; five links: a, b, c, d, e; and a single O/D pair $w_1 = (1, 4)$. There are, hence, three paths connecting the single O/D pair, which are denoted, respectively, by: $p_1 = (a, c)$, $p_2 = (b, d)$ and $p_3 = (a, e, d)$ and, thus, $r = p_3$.

The link cost functions are:

$$c_a(f_a) = 10f_a, \;\; c_b(f_b) = f_b + 50,$$

$$c_c(f_c) = f_c + 50, \;\; c_d(f_d) = 10f_d, \;\; c_e(f_e) = f_e + 10.$$

We can also write down the path cost functions explicitly as follows:

$$C_{p_1}(x) = 11x_{p_1} + 10x_{p_3} + 50,$$

$$C_{p_2}(x) = 11x_{p_2} + 10x_{p_3} + 50,$$

$$C_{p_3}(x) = 10x_{p_1} + 21x_{p_3} + 10x_{p_2} + 10.$$

The Braess Paradox demonstrates that, for a fixed demand of $d_{w_1} = 6$, the addition of link e, which provides the users with the new path p_3, as in the network in Figure 1, actually makes all users worse off since without the link e, the travel disutility and path costs are 83, whereas with the new link/path, the travel disutility and path costs go up for all users to 92! Hence, everyone in the network is worse off by the addition of the new route $r = p_3$ which results when the new link e is added, with associated equilibrium path flow pattern: $x^*_{p_1} = x^*_{p_2} = x^*_{p_3} = 2$.

We now demonstrate how formula (20) can be used. Specifically, we have, for this example, that $x^*_r = x^*_{p_3} = 2$. Consequently, it follows from (20) that:

$$-2 = \frac{det\begin{pmatrix} 11 & 0 & 0 & 1 \\ 0 & 11 & 0 & 1 \\ 10 & 10 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}}{det\begin{pmatrix} 11 & 0 & 10 & 1 \\ 0 & 11 & 10 & 1 \\ 10 & 10 & 21 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}} \Delta d_{w_1}, \qquad (23)$$

The determinant in the numerator in (23) is equal to 99 (and is positive as our theory predicts) whereas the denominator in (23) is equal to -143 and is negative (as our theory also predicts). The value of Δd_{w_1} in (23) is, thus, $2\frac{8}{9}$. This means that at a new demand of $6 + 2\frac{8}{9} = 8\frac{8}{9}$ the new path r will not be used and its flow will be identically 0. This is, indeed, the case. For the new demand of $8\frac{8}{9}$, following the equilibrium conditions (10) and using the above explicit path cost expressions, we obtain that the new U-O path flow pattern is now: $x^*_{p_1} = x^*_{p_2} = 4\frac{4}{9}$, and $x^*_r = 0$, with $C_{p_1} = C_{p_2} = 98\frac{8}{9}$ at the higher level of demand for the network without path r and, hence, the Braess Paradox, after the addition of the new path, at the higher demand, no longer occurs since the new path r is not even used! As established in [7] and, using another formulation in [5], for this particular network, the Braess Paradox occurs over a range of demand. Our formula, however, provides at what increase in demand the Braess Paradox is negated, and, more importantly, it yields a powerful qualitative result for any such network.

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