MATHEMATICAL MODELS OF TRANSPORTATION AND NETWORKS

Anna Nagurney
John F. Smith Memorial Professor
Department of Finance and Operations Management
Isenberg School of Management
University of Massachusetts
Amherst, Massachusetts 01003

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Contents

Summary

In this chapter, we provide the foundations of the rigorous formulation, analysis, and solution of transportation network problems. We discuss user-optimization, which corresponds to decentralized decision-making, and system-optimization, which corresponds to centralized decision-making where the central controller can route the traffic in an optimal manner. We describe a spectrum of increasingly sophisticated models and also relate transportation networks to other network application domains in which flows (and associated decision-making) are essential, such as the Internet, supply chains, electric power distribution and generation networks, as well as financial networks. Finally, we demonstrate how the importance of transportation network components, that is, nodes and links can be identified (and ranked) through a recently proposed transportation network efficiency measure and accompanying component importance definition. Examples are included throughout the chapter for illustrative purposes.

1. Introduction

Transportation networks are complex, large-scale systems, and come in a variety of forms, such as road, rail, air, and waterway networks. Transportation networks provide the foundation for the functioning of our economies and societies through the movement of people, goods, and services. From an economic perspective, the supply in such network systems is represented by the underlying network topology and the cost characteristics whereas the demand is represented by the users of the transportation system. An equilibrium occurs when the number of trips between an origin (e.g., residence/place of employment) and destination (place of employment/residence) equals the travel demand given by the market price, typically, represented by the travel time for the trips (Nagurney (2004)).

The study of transportation networks and their efficient management dates to ancient times. It is known, for example, that Romans imposed controls over chariot traffic during different times of day in order to deal with the congestion (see Banister and Button (1993)). From an economic perspective, some of the earliest contributions to the subject date to Kohl (1841) and to Pigou (1920), who considered a two-node, two-link transportation network, identified congestion as a problem, and recognized that distinct behavioral concepts regarding route selection may prevail (see also Knight (1924)).

The formal study of transportation networks has challenged transportation scientists, economists, operations researchers, engineers, and physicists for reasons, including: the size and scope of the systems involved; the behavior of the users of the network which may vary according to the application setting, thereby leading to different optimality/equilibrium concepts; distinct classes of users may perceive the cost of utilizing the network in an individual fashion, and congestion, which is playing an increasing role in numerous transportation networks.

For example, to help one fix the size and scope of modern-day transportation networks, we point out that the topology of the Chicago Regional Transportation Network consists of 12,982 nodes, 39,018 links, and 2,297,945 origin/destination pairs of nodes between which travelers choose their routes (cf. Bar-Gera (2002)), whereas in the Southern California Association of Governments' model there are 25,428 nodes, 99,240 links, 3,217 origin/destination pairs, and 6 distinct classes of users (Wu, Florian, and He (2000)).

Road congestion results, according to estimates, in approximately \$100 billion in lost productivity in the United States alone with the figure being about \$150 billion in Europe with the number of cars expected to increase by 50 percent by 2010 and to double by 2030 (see Nagurney (2000) and the references therein). In particular, the growth in the usage of motorized vehicles, especially, cars, in the developing world, is transforming such countries as China and India. Moreover, in many of today's transportation networks, the "noncooperative" behavior of users aggravates the congestion problem. For example, in the case of urban transportation networks, travelers select their routes from an origin to a destination so as to minimize their own travel cost or travel time, which although optimal from a user's perspective (user-optimization) may not be optimal from a societal one (system-optimization) where a decision-maker or central controller has control of the flows on the network and seeks to allocate the flows so as to minimize the total cost in the network. Coupled with road congestion is increasing pollution, another negative externality, which is further impacting the world that we live in (see Nagurney (2000)).

The famous Braess (1968) paradox example, illustrates the distinction between noncooperative (or user-optimized) behavior versus system-optimized behavior, in a concrete, vivid way. It that example, it is assumed that the underlying behavioral principle is that of user-optimization and travelers select their routes accordingly. In the Braess network, the addition of a new road with no change in travel demand results in all travelers in the network incurring a higher travel cost. Hence, they are all worse off after the addition of the new road! Actual practical instances of such a phenomenon have been identified in New York City and in Stuttgart, Germany. In 1990, 42nd Street in New York was closed for Earth Day, and the traffic flow in the area improved (see Kolata (1990)). In Stuttgart, in turn, a new road was added to the downtown, but the traffic flow worsened and, following complaints, the new road was torn down (cf. Bass (1992)). Similar experiences have been found recently in Seoul, Korea (Vidal (2006)). Interestingly, this phenomenon is also relevant to telecommunications networks (see Korilis, Lazar, and Orda (1999)) and, specifically, to the Internet (cf. Cohen and Kelly (1990) and Nagurney, Parkes, and Daniele (2006)). Such a result does not occur in system-optimized networks where the addition of a new road/link, if used, would lower the total network cost. Today, congestion pricing is an active topic of research, and tolls have had success in ameliorating traffic by altering people's behavior in various cities around the world, including the much-publicized London, United Kingdom experience (see, e.g., Lawphongpanich, Hearn, and Smith (2006)).

In this chapter, we recall the foundations of the rigorous study of transportation networks and we trace the evolution of modeling frameworks for their study. The exposition is meant to be accessible to practitioners and to students, as well as to researchers and policy makers and to those interested in related network topics. Technical derivations and further supporting documentation are referred to in the citations. Further useful material and a supplementary chronological perspective of developments on this topic can be found in the review articles of Florian (1986), Boyce, LeBlanc, and Chon (1988), and Florian and Hearn (1995); in the books by Beckmann, McGuire, and Winsten (1956), Sheffi (1985), Patriksson (1994), Ran and Boyce (1996), Nagurney (1999, 2000), Nagurney and Dong (2002a), and in the volumes edited by Florian (1976, 1984), Volmuller and Hamerslag (1984), Lesort (1996), Marcotte and Nguyen (1998), Gendreau and Marcotte (2002), Taylor (2002), Mahmassani (2005), Bar-Gera and Boyce (2005), and Nagurney (2006a, b).

This chapter, specifically, overviews some of the methodologies, whose very development, has been motivated by the need to formulate, analyze, and solve transportation network problems. It also relates the contributions of transportation modeling and algorithmic advances to other network application domains. Finally, given the importance of transportation networks and the closely related telecommunication, electric power generation and distribution networks, supply chain, as well as financial networks (cf. Nagurney (2006a,b)), we also, for completeness, discuss a network efficiency measure, which enables the identification of the critical nodes and links. This measure was proposed by Nagurney and Qiang (2007) and can aid policy makers, planners, engineers, as well as network designers in identifying which network components need to be protected, since their absence due, for example, to destruction by natural disasters, structural failures, terrorist attacks, etc., has the greatest impact. Hence, the transportation network is most vulnerable when such nodes/links are removed from the system.

2. Fundamental Decision-Making Concepts and Models

Over half a century ago, Wardrop (1952) explicitly considered alternative possible behaviors of users of transportation networks, notably, urban transportation networks and stated two principles, which are commonly named after him:

First Principle: The journey times of all routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

Second Principle: The average journey time is minimal.

The first principle corresponds to the behavioral principle in which travelers seek to (unilaterally) determine their minimal costs of travel whereas the second principle corresponds to the behavioral principle in which the total cost in the network is minimal.

Beckmann, McGuire, and Winsten (1956) were the first to rigorously formulate these conditions mathematically. Specifically, Beckmann, McGuire, and Winsten (1956) established the equivalence between the *transportation network equilibrium* conditions, which state that all used paths connecting an origin/destination (O/D) pair will have equal and minimal travel times (or costs) (corresponding to Wardrop's first principle), and the Kuhn-Tucker (1951) conditions of an appropriately constructed optimization problem, under a symmetry assumption on the underlying functions. Hence, in this case, the equilibrium link and path flows could be obtained as the solution of a mathematical programming problem. Their approach made the formulation, analysis, and subsequent computation of solutions to transportation network problems based on actual transportation networks realizable.

Dafermos and Sparrow (1969) coined the terms *user-optimized* (U-O) and *system-optimized* (S-O) transportation networks to distinguish between two distinct situations in which, respectively, users act unilaterally, in their own self-interest, in selecting their routes, and in which users select routes according to what is optimal from a societal point of view, in that the total cost in the network system is minimized. In the latter problem, marginal total costs rather than average costs are equilibrated. The former problem coincides with Wardrop's first principle, and the latter with Wardrop's second principle. See Table 1 for the two distinct behavioral principles underlying transportation networks.

Table 1: Distinct Behavior on Transportation Networks

User-Optimization	**System-Optimization**
$\Downarrow$	$\Downarrow$
User Equilibrium Principle:	**System Optimality Principle:**
User travel costs on used paths for each O/D pair are equalized and minimal.	Marginals of the total travel cost on used paths for each O/D pair are equalized and minimal.

The concept of "system-optimization" is also relevant to other types of "routing models" in transportation, as well as in communications (cf. Bertsekas and Gallager (1992)), including those concerned with the routing of freight and computer messages, respectively. Dafermos and Sparrow (1969) also provided explicit computational procedures, that is, *algorithms*, to compute the solutions to such network problems in the case where the user travel cost on a link was an increasing (in order to handle congestion) function of the flow on the particular link, and linear. Today, the concepts of user-optimization versus system-optimization also capture, respectively, decentralized versus centralized decision-making on networks, including, the Internet (cf. Roughgarden (2005) and Boyce, Mahmassani, and Nagurney (2005)).

2.1 User-Optimization versus System-Optimization

In this section, the basic transportation network models are first reviewed, under distinct assumptions as to their operation and the underlying behavior of the users of the network. The models are classical and are due to Beckmann, McGuire, and Winsten (1956) and Dafermos and Sparrow (1969). In subsequent sections, we present more general models in which the user link cost functions are no longer separable but, rather, are asymmetric. For such models we also provide the variational inequality formulations of the governing equilibrium conditions, since, in such cases, the governing equilibrium conditions can no longer be reformulated as the Kuhn-Tucker conditions of a convex optimization problem.

For definiteness, and for easy reference, we present the classical user-optimized network model in Section 2.1.1 and then the classical system-optimized network model in Section

2.1.2. We then, for illustrative purposes, recall the Braess (1968) paradox in Section 2.1.3.

2.1.1 The User-Optimized Problem

The user-optimized network problem is also commonly referred to in the transportation literature as the *traffic assignment* problem or the *traffic network equilibrium* problem. Recall that user-optimization follows Wardrop's first principle.

Consider a general network $\mathcal{G} = [\mathcal{N}, \mathcal{L}]$, where $\mathcal{N}$ denotes the set of nodes, and $\mathcal{L}$ the set of directed links. Links connect pairs of nodes in the network and are denoted by a, b, etc. Let p denote a path consisting of a sequence of links connecting an origin/destination (O/D) pair of nodes. Paths are assumed to be acyclic and are denoted by p, q, etc. In transportation networks, nodes correspond to origins and destinations, as well as to intersections. Links, on the other hand, correspond to roads/streets in the case of urban transportation networks and to railroad segments in the case of train networks. A path in its most basic setting, thus, is a sequence of "roads" which comprise a route from an origin to a destination. In the telecommunication context, however, nodes can correspond to switches or to computers and links to telephone lines, cables, microwave links, etc. Here we consider *paths*, rather than *routes*, since the former subsumes the latter. The network concepts presented here are sufficiently general to abstract not only transportation decision-making but also combined/integrated location-transportation decision-making, which we return to later. In addition, in the setting of *supernetworks* (see Nagurney and Dong (2002a)), a path is viewed more broadly and need not be limited to a route-type decision but may, in fact, correspond to not only transportation but also to telecommunications decision-making, or a combination thereof, as in the case of teleshopping and/or telecommuting (see, e.g., Nagurney, Dong, and Mokhtarian (2002a, b)).

Let P_ω denote the set of paths connecting the origin/destination (O/D) pair of nodes ω. Let P denote the set of all paths in the network and assume that there are J origin/destination pairs of nodes in the set Ω. Let x_p represent the nonnegative flow on path p and let f_a denote the flow on link a. The path flows on the network are grouped into the column vector $x \in R^{n_P}_+$, where n_P denotes the number of paths in the network. The link flows, in turn, are grouped into the column vector $f \in R^{n_L}_+$, where n_L denotes the number of links in the network.

We assume, as given, the demand associated with each O/D pair ω, which is denoted by d_ω, for $\omega \in \Omega$. In the network, the following conservation of flow equations must hold:

$$d_\omega = \sum_{p \in P_\omega} x_p, \quad \forall \omega \in \Omega, \tag{1}$$

where $x_p \geq 0$, $\forall p \in P$; that is, the sum of all the path flows between an origin/destination pair ω must be equal to the given demand d_ω.

In addition, the following conservation of flow equations must also hold:

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in \mathcal{L}, \tag{2}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. Expression (2) states that the flow on a link a is equal to the sum of all the path flows on paths p that contain (traverse) link a.

In particular, equations (1) and (2) guarantee that the flows in the network (be they travelers, computer messages, etc.) are conserved, that is, do not disappear (or are lost) in the network and arrive at the designated destinations from the origins.

Let c_a denote the user link cost associated with traversing link a, and let C_p denote the user cost associated with traversing the path p. Assume that the user link cost function is given by the *separable* function in which the cost on a link depends only on the flow on the link, that is,

$$c_a = c_a(f_a), \quad \forall a \in \mathcal{L}, \tag{3}$$

where c_a is assumed to be continuous and an increasing function of the link flow f_a in order to model the effect of the link flow on the cost and, in particular, congestion.

Here the cost is interpreted in a general sense. From a transportation engineering perspective, the cost on a link is assumed, typically, to coincide with the travel time on a link (see also Sheffi (1985) and Nagurney (2000)). In addition, one may construct generalized user link cost functions, which weight, for example, the monetary cost and the cost associated with travel time, as well as other criteria, including environmental ones (cf. Nagurney and Dong (2002a, b) and the references therein).

The cost on a path is equal to the sum of the costs on the links that make up that path, that is,

$$C_p = \sum_{a \in \mathcal{L}} c_a(f_a)\delta_{ap}, \quad \forall p \in P. \tag{4}$$

Transportation Network Equilibrium Conditions

In the case of the user-optimization (U-O) problem one seeks to determine the path flow pattern x^* (and the corresponding link flow pattern f^*) which satisfies the conservation of flow equations (1) and (2), and the nonnegativity assumption on the path flows, and which also satisfies the transportation network equilibrium conditions given by the following statement. For each O/D pair $\omega \in \Omega$ and each path $p \in P_\omega$:

$$C_p \begin{cases} = \lambda_\omega, & \text{if} \quad x_p^* > 0 \\ \geq \lambda_\omega, & \text{if} \quad x_p^* = 0. \end{cases} \tag{5}$$

Hence, in the user-optimization problem there is no explicit optimization concept, since users of the transportation network system act independently, in a noncooperative manner, until they cannot improve on their situations unilaterally and, thus, an equilibrium is achieved, governed by the above equilibrium conditions. Indeed, conditions (5) are simply a restatement of Wardrop's (1952) first principle mathematically and mean that only those paths connecting an O/D pair will be used which have equal and minimal user costs. In (5) the minimal cost for O/D pair ω is denoted by λ_ω and its value is obtained once the equilibrium flow pattern is determined. Otherwise, a user of the network could improve upon his situation by switching to a path with lower cost. User-optimization represents decentralized decision-making, whereas system-optimization represents centralized decision-making. See also Table 1.

In order to obtain a solution to the above problem, Beckmann, McGuire, and Winsten (1956) established that the solution to the network equilibrium problem, in the case of user link cost functions of the form (3), in which the cost on a link only depends on the flow on that link and is assumed to be continuous and an increasing function of the flow, could be obtained by solving the following optimization problem:

$$\text{Minimize} \quad \sum_{a \in \mathcal{L}} \int_0^{f_a} c_a(y)dy \tag{6}$$

subject to:

$$\sum_{p \in P_\omega} x_p = d_\omega, \quad \forall \omega \in \Omega, \tag{7}$$

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in \mathcal{L}, \tag{8}$$

$$x_p \geq 0, \quad \forall p \in P. \tag{9}$$

The objective function given by (6) is simply a device constructed to obtain a solution using general purpose convex programming algorithms. It does not possess the economic meaning of the objective function encountered in the system-optimization problem which will be recalled below. Note that in the case of separable, as well as nonseparable, but symmetric (which we come back to later), user link cost functions, the λ_ω term in (5) corresponds to the Lagrange multiplier associated with the constraint (7) for that O/D pair ω. However, in the case of nonseparable and asymmetric functions there is no optimization reformulation of the transportation network equilibrium conditions (5) and the λ_ω term simply reflects the minimum user cost associated with the O/D pair ω at the equilibrium. As noted as early as Dafermos and Sparrow (1969), the above network equilibrium conditions also correspond to a Nash equilibrium (see Nash (1950, 1951)). This connection has now garnered great interest in computer science (see Roughgarden (2005) and the references therein). The equilibrium link flow pattern is unique for problem (6), subject to (7) – (9), if the objective function (6) is strictly convex (for additional background on optimization theory, see Bazaraa, Sherali, and Shetty (1993)).

2.1.2 The System-Optimized Problem

We now describe and discuss the system-optimized problem. Again, as in the user-optimized problem of Section 2.1.1, the network $\mathcal{G} = [\mathcal{N}, \mathcal{L}]$, the demands associated with the origin/destination pairs, and the user link cost functions are assumed as given. In the system-optimized problem, there is a central controller of the traffic who routes the traffic in an optimal manner so as to minimize the total cost in the network.

The total cost on link a, denoted by $\hat{c}_a(f_a)$, is given by:

$$\hat{c}_a(f_a) = c_a(f_a) \times f_a, \quad \forall a \in \mathcal{L}, \tag{10}$$

that is, the total cost on a link is equal to the user link cost on the link times the flow on the link. As noted earlier, in the system-optimized problem, there exists a central controller who seeks to minimize the total cost in the network system, where the total cost is expressed as

$$\sum_{a \in \mathcal{L}} \hat{c}_a(f_a), \tag{11}$$

and the total cost on a link is given by expression (10).

The system-optimization (S-O) problem is, thus, given by:

$$\text{Minimize} \quad \sum_{a \in \mathcal{L}} \hat{c}_a(f_a) \tag{12}$$

subject to the same conservation of flow equations as for the user-optimized problem, as well as the nonnegativity assumption of the path flows; that is, constraints (7), (8), and (9) must also be satisfied for the system-optimized problem.

The total cost on a path, denoted by $\hat{C}_p$, is the user cost on a path times the flow on a path, that is,

$$\hat{C}_p = C_p x_p, \quad \forall p \in P, \tag{13}$$

where the user cost on a path, C_p, is given by the sum of the user costs on the links that comprise the path (as in (4)), that is,

$$C_p = \sum_{a \in \mathcal{L}} c_a(f_a)\delta_{ap}, \quad \forall a \in \mathcal{L}. \tag{14}$$

In view of (2), (3), and (4), one may express the cost on a path p as a function of the path flow variables and, hence, an alternative version of the above system-optimization problem with objective function (12) can be stated in path flow variables only, where one has now the problem:

$$\text{Minimize} \quad \sum_{p \in P} C_p(x) x_p \tag{15}$$

subject to constraints (7) and (9).

System-Optimality Conditions

Under the assumption of increasing user link cost functions, the objective function (12) in the S-O problem is convex, and the feasible set consisting of the linear constraints (7) – (9) is also convex. Therefore, the optimality conditions, that is, the Kuhn-Tucker conditions are: for each O/D pair $\omega \in \Omega$, and each path $p \in P_\omega$, the flow pattern x (and corresponding link flow pattern f), satisfying (7)–(9) must satisfy:

$$\hat{C}'_p \begin{cases} = \mu_\omega, & \text{if} \quad x_p > 0 \\ \geq \mu_\omega, & \text{if} \quad x_p = 0, \end{cases} \tag{16}$$

where $\hat{C}'_p$ denotes the marginal of the total cost on path p, given by:

$$\hat{C}'_p = \sum_{a \in \mathcal{L}} \frac{\partial \hat{c}_a(f_a)}{\partial f_a} \delta_{ap}, \tag{17}$$

evaluated in (16) at the solution and μ_ω is the Lagrange multiplier associated with constraint (7) for that O/D pair ω.

Observe that conditions (16) may be rewritten so that there exists an ordering of the paths for each O/D pair whereby all used paths (that is, those with positive flow) have equal and minimal marginal total costs and the unused paths (that is, those with zero flow) have higher (or equal) marginal total costs than those of the used paths. Hence, in the S-O problem, and as noted in Table 1, according to the optimality conditions (16), it is the marginal of the total cost on each used path connecting an O/D pair which is equalized and minimal (see also, e.g., Dafermos and Sparrow (1969)).

2.1.3 The Braess Paradox

In order to illustrate the difference between user-optimization and system-optimization in a concrete example, and to reinforce the above concepts, we now recall the well-known Braess (1968) paradox; see also Braess, Nagurney, and Wakolbinger (2005). This paradox is as relevant to transportation networks as it is to telecommunication networks, and, in particular, to the Internet, since such networks are subject to traffic operating in a decentralized decision-making manner (cf. Korilis, Lazar, and Orda (1999), Nagurney, Parkes, and Daniele (2006), and the references therein).

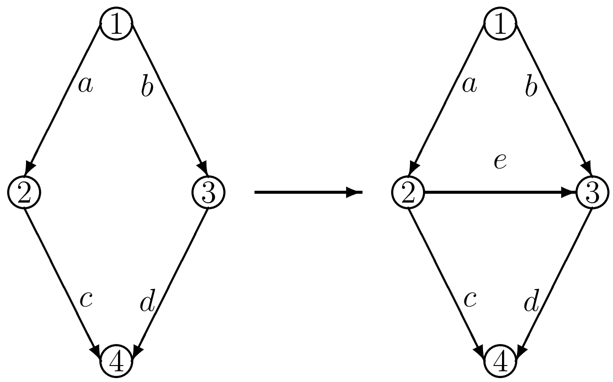

Figure 1: The Braess Network Example

Assume a network as the first network depicted in Figure 1 in which there are four nodes: $1, 2, 3, 4$; four links: a, b, c, d; and a single O/D pair $\omega_1 = (1, 4)$. There are, hence, two paths available to travelers between this O/D pair: $p_1 = (a, c)$ and $p_2 = (b, d)$.

The user link travel cost functions are:

$$c_a(f_a) = 10f_a, \quad c_b(f_b) = f_b + 50, \quad c_c(f_c) = f_c + 50, \quad c_d(f_d) = 10f_d.$$

Assume a fixed travel demand $d_{\omega_1} = 6$.

It is easy to verify that the equilibrium path flows are: $x^*_{p_1} = 3$, $x^*_{p_2} = 3$, the equilibrium link flows are: $f^*_a = 3, \quad f^*_b = 3$, $f^*_c = 3, \quad f^*_d = 3$, with associated equilibrium path travel costs: $C_{p_1} = c_a + c_c = 83$, $C_{p_2} = c_b + c_d = 83$.

Assume now that, as depicted in Figure 1, a new link "e", joining node 2 to node 3 is added to the original network, with user link cost function $c_e(f_e) = f_e + 10$. The addition of this link creates a new path $p_3 = (a, e, d)$ that is available to the travelers. Assume that the travel demand d_{ω_1} remains at 6 units of flow. Note that the original flow distribution pattern $x_{p_1} = 3$ and $x_{p_2} = 3$ is no longer an equilibrium pattern, since at this level of flow the user cost on path p_3, $C_{p_3} = c_a + c_e + c_d = 70$. Hence, users from paths p_1 and p_2 would switch to path p_3.

The equilibrium flow pattern on the new network is: $x^*_{p_1} = 2$, $x^*_{p_2} = 2$, $x^*_{p_3} = 2$, with equilibrium link flows: $f_a^* = 4$, $f_b^* = 2$, $f_c^* = 2$, $f_e^* = 2$, $f_d^* = 4$, and with associated equilibrium user path travel costs: $C_{p_1} = 92$, $C_{p_2} = 92$. Indeed, one can verify that any reallocation of the path flows would yield a higher travel cost on a path.

Note that the travel cost increased for every user of the network from 83 to 92 without a change in the travel demand!

The increase in travel cost on the paths is due, in part, to the fact that in this network two links are shared by distinct paths and these links incur an increase in flow and associated cost. Hence, the Braess paradox is related to the underlying topology of the networks and, of course, to the behavior of the travelers, which here is that of user-optimization. One may show, however, that the addition of a path connecting an O/D pair that shares no links with the original O/D pair will never result in the Braess paradox for that O/D pair (cf. Dafermos and Nagurney (1984)).

Recall that a system-optimizing solution, which corresponds to Wardrop's (1952) second principle, is one that minimizes the total cost in the network, and all utilized paths connecting each O/D pair have equal and minimal marginal total travel costs (cf. (16) and (17)).

The system-optimizing solution for the first network in Figure 1 is: $x_{p_1} = x_{p_2} = 3$, with marginal total path costs given by: $\hat{C}'_{p_1} = \hat{C}'_{p_2} = 116$. This would remain the system-optimizing solution, even after the addition of link e, since the marginal cost of path p_3, $\hat{C}'_{p_3}$, at this feasible flow pattern is equal to 130.

The addition of a new link to a network cannot increase the total cost of the network system, but can, of course, increase a user's cost since travelers act individually.

3. Models with Asymmetric Link Costs

There has been much research activity in the past several decades in terms of both the modeling and the development of methodologies to enable the formulation and computation of more general transportation (and related) network equilibrium models. Examples of general models include those that allow for multiple modes of transportation or multiple classes of users (such as computer messages, for example), whose costs on links depend on the flows in an individual way. In this section, we consider network models in which the user cost on a link is no longer dependent solely on the flow on that link. We present a fixed demand transportation network equilibrium model in Section 3.1 and an elastic demand one in Section 3.2. In Section 3.1.3 we further elaborate upon the generality of the transportation network equilibrium concept and formulations and highlight several novel applications.

Assume that user link cost functions are now of a general form, that is, the cost on a link may depend not only on the flow on the link but on other link flows on the network, that is,

$$c_a = c_a(f), \quad \forall a \in \mathcal{L}. \tag{18}$$

In the case where the symmetry assumption exists, that is, $\frac{\partial c_a(f)}{\partial f_b} = \frac{\partial c_b(f)}{\partial f_a}$, for all links $a, b \in \mathcal{L}$, one can still reformulate the solution to the network equilibrium problem satisfying equilibrium conditions (5) as the solution to an optimization problem (cf. Dafermos (1972), and the references therein), albeit, again, with an objective function that is artificial and simply a mathematical device. However, when the symmetry assumption is no longer satisfied, such an optimization reformulation no longer exists and one must appeal to *variational inequality theory* (cf. Kinderlehrer and Stampacchia (1980), Nagurney (1999), and the references therein). Models of transportation networks with asymmetric cost functions are important since they allow for the formulation, qualitative analysis, and, ultimately, solution to problems in which the cost on a link may depend on the flow on another link in a different way than the cost on the other link depends on that link's flow. Such a generalization allows for the more realistic treatment of intersections, two-way links, multiple modes of transportation as well as distinct classes of users of the network.

It was in the domain of such transportation network equilibrium problems that the theory of finite-dimensional variational inequalities realized its earliest success, beginning with the contributions of Smith (1979) and Dafermos (1980). For an introduction to the subject, as well as applications ranging from transportation network and spatial price equilibrium problems to financial equilibrium problems, see the book by Nagurney (1999). Below we present variational inequality formulations of both fixed demand and elastic demand transportation network equilibrium problems.

The system-optimization problem, in turn, in the case of nonseparable (cf. (18)) user link cost functions becomes (see also (12)):

$$\text{Minimize} \quad \sum_{a \in \mathcal{L}} \hat{c}_a(f), \tag{19}$$

subject to (7)–(9), where $\hat{c}_a(f) = c_a(f) \times f_a$, $\forall a \in \mathcal{L}$.

The system-optimality conditions remain as in (16), but now the marginal of the total cost on a path becomes, in this more general case:

$$\hat{C}'_p = \sum_{a,b \in \mathcal{L}} \frac{\partial \hat{c}_b(f)}{\partial f_a} \delta_{ap}, \quad \forall p \in P. \tag{20}$$

3.1 Variational Inequality Formulations of Fixed Demand Problems

As mentioned earlier, in the case where the user link cost functions are no longer symmetric, one cannot compute the solution to the U-O, that is, to the network equilibrium, problem using standard optimization algorithms. We emphasize, again, that such general cost functions are very important from an application standpoint since they allow for asymmetric interactions on the network. For example, allowing for asymmetric cost functions permits one to handle the situation when the flow on a particular link affects the cost on another link in a different way than the cost on the particular link is affected by the flow on the other link.

First, the definition of a variational inequality problem is recalled. For further background, theoretical formulations, derivations, and the proofs of the results below, see the books by Nagurney (1999) and by Nagurney and Dong (2002a) and the references therein. We provide the variational inequality of the network equilibrium conditions in path flows as well as in link flows.

Specifically, the variational inequality problem (finite-dimensional) is defined as follows:

Definition 1: Variational Inequality Problem

The finite-dimensional variational inequality problem, $\mathrm{VI}(F, \mathcal{K})$*, is to determine a vector* $X^* \in \mathcal{K}$ *such that*

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{21}$$

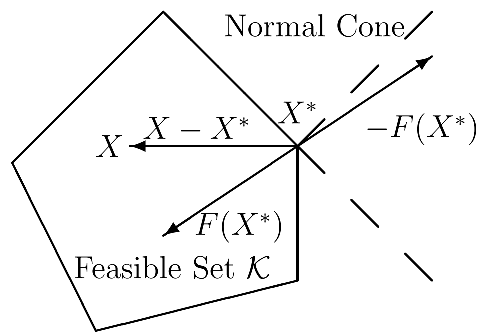

Figure 2: Geometric Interpretation of VI$(F, \mathcal{K})$

where F is a given continuous function from $\mathcal{K}$ to R^N, $\mathcal{K}$ is a given closed convex set, and $\langle \cdot, \cdot \rangle$ denotes the inner product in R^N.

Variational inequality (21) is referred to as being in *standard form*. Hence, for a given problem, typically an *equilibrium* problem, one must determine the function F that enters the variational inequality problem, the vector of variables X, as well as the feasible set $\mathcal{K}$.

The variational inequality problem contains, as special cases, such well-known problems as systems of equations, optimization problems, and complementarity problems. Thus, it is a powerful unifying methodology for equilibrium analysis and computation.

A geometric interpretation of the variational inequality problem VI$(F, \mathcal{K})$ is given in Figure 2. In particular, $F(X^*)$ is "orthogonal" to the feasible set $\mathcal{K}$ at the point X^*.

Theorem 1: Variational Inequality Formulation of Network Equilibrium with Fixed Demands – Path Flow Version

A vector $x^* \in K^1$ is a network equilibrium path flow pattern, that is, it satisfies equilibrium *conditions (5) if and only if it satisfies the variational inequality problem:*

$$\sum_{\omega \in \Omega} \sum_{p \in P_\omega} C_p(x^*) \times (x - x^*) \geq 0, \quad \forall x \in K^1, \tag{22}$$

or, in vector form:

$$\langle C(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K^1, \tag{23}$$

where C is the n_P-dimensional column vector of path user costs and K^1 is defined as: $K^1 \equiv \{x \geq 0,\ \text{such that (7) holds}\}$.

Theorem 2: Variational Inequality Formulation of Network Equilibrium with Fixed Demands – Link Flow Version

A vector $f^* \in K^2$ is a network equilibrium link flow pattern if and only if it satisfies the variational inequality problem:

$$\sum_{a \in \mathcal{L}} c_a(f^*) \times (f_a - f_a^*) \geq 0, \quad \forall f \in K^2, \tag{24}$$

or, in vector form:

$$\langle c(f^*), f - f^* \rangle \geq 0, \quad \forall f \in K^2, \tag{25}$$

where c is the n_L-dimensional column vector of link user costs and K^2 is defined as: $K^2 \equiv \{f \,|\, \text{there exists an}\, x \geq 0 \text{ and satisfying}\,(7)\text{ and}\,(8)\}$.

Note that one may put variational inequality (23) into standard form (21) by letting $F \equiv C$, $X \equiv x$, and $\mathcal{K} \equiv K^1$. Also, one may put variational inequality (25) into standard form where now $F \equiv c$, $X \equiv f$, and $\mathcal{K} \equiv K^2$. Hence, fixed demand transportation network equilibrium problems in the case of asymmetric user link cost functions can be solved as variational inequality problems, as given above.

Alternative variational inequality formulations of a problem are useful in devising other models, including dynamic versions, as well as for purposes of computation using different algorithms. In Section 4, we describe the relationship between variational inequality formulations and projected dynamical systems, in which the latter provides the disequilibrium dynamics prior to the attainment of the equilibrium, as formulated via the former.

The theory of variational inequalities (see Kinderlehrer and Stampacchia (1980) and Nagurney (1999)) allows one to qualitatively analyze the equilibrium patterns in terms of existence, uniqueness, as well as sensitivity and stability of solutions, and to apply rigorous algorithms for the numerical computation of the equilibrium patterns. Variational inequality algorithms usually resolve the variational inequality problem into series of simpler subproblems, which, in turn, are often optimization problems, which can then be effectively solved using a variety of algorithms, including the equilibration algorithms of Dafermos and Sparrow (1969), which exploit network structure as well as the commonly used in practice Frank-Wolfe (1956) algorithm (see also LeBlanc, Morlok, and Pierskalla (1975)), and the origin-based algorithm of Ber-Gera (2002), which seems especially suitable for large-scale transportation networks. In particular, projection methods as well as relaxation methods (see Dafermos (1980, 1982), Florian and Spiess (1982), Nagurney (1984, 1999), and Patriksson (1994)) have been successfully applied to compute solutions to variational inequality formulations of transportation network equilibrium problems.

We emphasize that the above network equilibrium framework is sufficiently general to also formalize the entire transportation planning process (consisting of origin selection, or destination selection, or both, in addition to route selection, in an optimal fashion) as path choices over an appropriately constructed *abstract* network or supernetwork. This was recognized by Dafermos in 1976 (in the context of separable link cost functions) in her development of *integrated* transportation network equilibrium models in which location decisions are made simultaneously to transportation route decisions (see also Boyce (1980)). Further discussion can be found in that reference as well as in the books by Nagurney (1999, 2000) and Nagurney and Dong (2002a, b) who also developed more general models in which the costs (as described above) need not be separable nor asymmetric.

It is worth noting that the presentation of the variational inequality formulations of the fixed demand models given above was in the context of single mode (or single class) transport networks. However, in view of the generality of the functions considered (cf. (18)), the modeling framework described above can also be adapted to multimodal/multiclass problems in which there are multiple modes of transport available and/or multiple classes of users, each of whom perceives the cost on the links of the network in an individual manner. Dafermos in (1972) demonstrated how, through a formal model, a multiclass traffic network could be cast into a single-class network through the construction of an expanded (and, again, *abstract*) network consisting of as many copies of the original network as there were classes. The application of such a transformation is also relevant to telecommunication networks.

Also, we note that here the focus is on deterministic network equilibrium problems. Some basic stochastic transportation network equilibrium models can be found in Sheffi (1985). Dial (1971) developed, apparently, the first stochastic route choice model. Daganzo and Sheffi (1977), in turn, formulated a stochastic user-optimized traffic network model with route choice in which the equilibrium criterion could be succinctly stated as *no traveler can improve his or her perceived travel time by unilaterally changing routes*.

3.2 Variational Inequality Formulations of Elastic Demand Problems

We now describe a general network equilibrium model with elastic demands due to Dafermos (1982) but we present the single-modal version, for simplicity. It is assumed that one has associated with each O/D pair ω in the network a travel disutility function λ_ω, where here the general case is considered in which the disutility may depend upon the entire vector of demands, which are no longer fixed, but are now variables, that is,

$$\lambda_\omega = \lambda_\omega(d), \quad \forall \omega \in \Omega, \tag{26}$$

where d is the J-dimensional column vector of the demands.

The notation, otherwise, is as described earlier, except that here we also consider user link cost functions which are general, that is, of the form (18). The conservation of flow equations (see also (1) and (2)), in turn, are given by

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in \mathcal{L}, \tag{27}$$

$$d_\omega = \sum_{p \in P_\omega} x_p, \quad \forall \omega \in \Omega, \tag{28}$$

$$x_p \geq 0, \quad \forall p \in P. \tag{29}$$

In the elastic demand case, the demands in expression (28) are variables and no longer given, in contrast to the fixed demand expression in (1).

Network Equilibrium Conditions in the Case of Elastic Demands

The network equilibrium conditions (see also (5)) take on in the elastic demand case the following form. For every O/D pair $\omega \in \Omega$, and each path $p \in P_\omega$, a vector of path flows and demands (x^*, d^*) satisfying (28) and (29) (which induces a link flow pattern f^* through (27)) is a network equilibrium pattern if it satisfies:

$$C_p(x^*) \begin{cases} = \lambda_\omega(d^*), & \text{if} \quad x_p^* > 0 \\ \geq \lambda_\omega(d^*), & \text{if} \quad x_p^* = 0. \end{cases} \tag{30}$$

Equilibrium conditions (30) state that the costs on used paths for each O/D pair are equal and minimal and equal to the disutility associated with that O/D pair. Costs on unutilized paths can exceed the disutility. Observe that in the elastic demand model users of the network can forego travel altogether for a given O/D pair if the user costs on the connecting paths exceed the travel disutility associated with that O/D pair. This model, hence, allows one to ascertain the attractiveness of different O/D pairs based on the ultimate equilibrium demand associated with the O/D pairs. In addition, this model can handle such situations as the equilibrium determination of employment location and route selection, or residential location and route selection, or residential and employment selection as well as route selection through the appropriate transformations via the addition of links and nodes, and given, respectively, functions associated with the residential locations, the employment locations, and the network overall (cf. Dafermos (1976), Nagurney (1999), and Nagurney and Dong (2002a)).

Note that although the presentation of the elastic demand transportation network model has been in the case of a single mode of transportation or class of user one can readily (with an accompanying increase in notation) explicitly introduce distinct modes to the above model as follows. One needs only to introduce subscripts to denote modes/classes, redefine all of the above vectors accordingly, and the conservation of flow equations, and state that (30) then must hold for each mode/class. In other words, in equilibrium, the used paths for a given mode and O/D pair must have minimal and equal user path costs, which in turn, must be equal to the travel disutility for that mode and O/D pair at the equilibrium demand. Of course, as described in the case of fixed demands, one can also have made as many copies as there are modes on the network in which case the above single-modal but extended elastic demand model would be equivalent to the multimodal one.

In the next two theorems, both the path flow version and the link flow version of the variational inequality formulations of the network equilibrium conditions (30) are presented. These are analogues of the formulations (22) and (23), and (24) and (25), respectively, for the fixed demand model, and are due to Dafermos (1982).

Theorem 3: Variational Inequality Formulation of Network Equilibrium with Elastic Demands – Path Flow Version

A vector $(x^*, d^*) \in K^3$ *is a network equilibrium path flow pattern, that is, it satisfies equilibrium conditions (30) if and only if it satisfies the variational inequality problem:*

$$\sum_{\omega \in \Omega} \sum_{p \in P_\omega} C_p(x^*) \times (x - x^*) - \sum_{\omega \in \Omega} \lambda_\omega(d^*) \times (d_\omega - d^*_\omega) \geq 0, \quad \forall (x, d) \in K^3, \tag{31}$$

or, in vector form:

$$\langle C(x^*), x - x^* \rangle - \langle \lambda(d^*), d - d^* \rangle \geq 0, \quad \forall (x, d) \in K^3, \tag{32}$$

where λ *is the* J*-dimensional vector of disutilities and* K^3 *is defined as:* $K^3 \equiv \{x \geq 0$, *such that* (28) *holds*$\}$.

Theorem 4: Variational Inequality Formulation of Network Equilibrium with Elastic Demands – Link Flow Version

A vector $(f^*, d^*) \in K^4$ *is a network equilibrium link flow pattern if and only if it satisfies the variational inequality problem:*

$$\sum_{a \in \mathcal{L}} c_a(f^*) \times (f_a - f^*_a) - \sum_{\omega \in \Omega} \lambda_\omega(d^*) \times (d_\omega - d^*_\omega) \geq 0, \quad \forall (f, d) \in K^4, \tag{33}$$

or, in vector form:

$$\langle c(f^*), f - f^* \rangle - \langle \lambda(d^*), d - d^* \rangle \geq 0, \quad \forall (f, d) \in K^4, \tag{34}$$

where $K^4 \equiv \{(f, d)$, *such that there exists an* $x \geq 0$ *satisfying* (27), (28)$\}$.

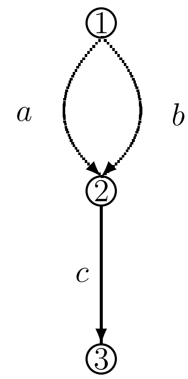

Figure 3: An Elastic Demand Example

Under the symmetry assumption on the disutility functions, that is, if $\frac{\partial \lambda_w}{\partial d_\omega} = \frac{\partial \lambda_\omega}{\partial d_w}$, for all w, ω, in addition to such an assumption on the user link cost functions (see following (18)), one can obtain (see Beckmann, McGuire, and Winsten (1956)) an optimization reformulation of the network equilibrium conditions (30), which in the case of separable user link cost functions and disutility functions is given by:

$$\text{Minimize} \quad \sum_{a \in \mathcal{L}} \int_0^{f_a} c_a(y)dy - \sum_{\omega \in \Omega} \int_0^{d_\omega} \lambda_\omega(z)dz \tag{35}$$

subject to: (27)–(29).

We now present an example of an elastic demand transportation network equilibrium problem with asymmetric user link cost functions.

An Elastic Demand Transportation Network Equilibrium Example

Consider the network depicted in Figure 3 in which there are three nodes: 1, 2, 3; three links: a, b, c; and a single O/D pair $\omega_1 = (1, 3)$. Let path $p_1 = (a, c)$ and path $p_2 = (b, c)$.

Assume that the user link cost functions are:

$$c_a(f) = 4f_a + f_b + 10, \quad c_b(f) = 3f_b + 2f_a + 20, \quad c_c(f) = 2f_c + f_a + f_b + 5,$$

and the disutility (or inverse demand) function is given by:

$$\lambda_{\omega_1}(d_{\omega_1}) = -d_{\omega_1} + 120.$$

Observe that in this example, the user link cost functions are non-separable and asymmetric and, hence, the equilibrium conditions (30) cannot be reformulated as the solution to an optimization problem, but, rather, as the solution to the variational inequalities (31) (or (32)), or (33) (or (34)). Hoever, given the simplicity of the network and cost structure in this simple example, we can solve the equilibrium conditions (30) directly.

The U-O flow and demand pattern that satisfies equilibrium conditions (30) is: $x^*_{p_1} = 10$, $x^*_{p_2} = 5$, and $d^*_{\omega_1} = 15$, with associated link flow pattern: $f^*_a = 10$, $f^*_b = 5$, $f^*_c = 15$. The incurred user path costs are: $C_{p_1} = C_{p_2} = 105$, which is precisely the value of the disutility λ_{ω_1}. Hence, this flow and demand pattern satisfies equilibrium conditions (30). Indeed, both paths p_1 and p_2 are utilized and their user paths costs are equal to each other. In addition, these costs are equal to the disutility associated with the origin/destination pair that the two paths connect.

It is worth mentioning that policies, in the form of tolls, for example, can be applied to guarantee that the travellers behave in a manner that, after the imposition of tolls, is also optimal from a system-optimizing perspective. For some background, see Dafermos (1973), Nagurney (2000), Lawphongpanich, Hearn, and Smith (2006), and the references therein.

3.3 Other Network Equilibrium Problems and Transportation

Note that the elastic demand model described above is related closely to the well-known spatial price equilibrium models of Samuelson (1952), Takayama and Judge (1971), and Florian and Los (1982). Indeed, as demonstrated by Dafermos and Nagurney (1985) in the context of a single commodity, and, subsequently, by Dafermos (1986) in the case of multiple commodities, spatial price equilibrium problems are *isomorphic* to transportation network equilibrium problems over appropriately constructed networks. Hence, the well-developed theory of transportation networks can be transferred to the study of commodity flows in the case of spatial price equilibrium in which the equilibrium production, consumption, and commodity trade flows are to be determined satisfying the equilibrium conditions that there will be a positive flow (in equilibrium) of the commodity between a pair of supply and demand markets if the supply price at the supply market plus the unit cost of transportation is equal to the demand price at the demand market. A variety of such models (both static and dynamic) and associated references can be found in the books by Nagurney (1999) and Nagurney and Zhang (1996).

Although the focus of this chapter is on transportation network equilibrium models with a primary focus on an urban setting, models of freight networks are closely related to the models discussed above. Clearly, in this context, one must distinguish the behavior of the operators of such networks and model the competition accordingly (see, e.g., Friesz and Harker (1985)).

As highlighted by Boyce, Mahmassani, and Nagurney (2005), Beckmann, McGuire, and Winsten (1956) explicitly recognized the generality of networks as a means of conceptualizing even decision-making of a firm, with routes corresponding to production processes and the links corresponding to transformations as the material moves down the route from the origin to the destination. The routes abstracted the choices or production possibilities available to a firm. For example, Beckmann, McGuire, and Winsten (1956) (p. 88) provided an analogy of transportation networks to the theory of a firm as follows "consider a chemical or metallurgical material which is capable of various stages or modifications, and a firm which undertakes to transform it from certain stages to certain other stages .... Here the stages of the material correspond to locations, the transitions correspond to roads, and sequences of transformation processes ... of the material – that is, the production methods – correspond to routes." This description closely captures another related application in which the concept of network equilibrium is garnering interest and that is that of *supply chain networks*. The study of supply chain networks is interdisciplinary by nature since such networks contain aspects of manufacturing, retailing, transportation, economics, as well as operations research and management science.

Nagurney, Dong, and Zhang (2002) were the first to utilize network equilibrium concepts in the context of supply chain applications. In their model, decision-makers, now located at the nodes of the network, are faced with their individual objective functions, which can include profit-maximization, and they seek to determine not only the optimal/equilibrium flows between tiers of nodes but also the prices of the product at the various tiers. The model therein was, subsequently, generalized to include electronic commerce by Nagurney, Loo, Dong, and Zhang (2002). Nagurney (2006a) proved that supply chain network equilibrium problems could be transformed and solved as transportation network equilibrium problems and novel interpretations of the underlying equilibrium conditions given in terms of paths and path flows. More recently, Nagurney and Liu (2005) and Wu, Nagurney, Liu, and Stranlund (2006) proved, as hypothesized in Beckmann, McGuire, and Winsten (1956), that electric power generation and distribution networks could also be tranformed into transportation network equilibrium problems, thus resolving a hypothesis that was open for over 50 years. Finally, Liu and Nagurney (2006) established that financial networks with intermediation could also be transformed into transportation networks over appropriately constructed abstract networks or supernetworks. Zhang, Dong, and Nagurney (2003), in turn, generalized Wardrop's principle(s) to consider not only routes but *chains* in the network to identify the "winning" supply chains. In that context, routes correspond to production processes and links can be either operation or interface links. Their framework allows for the modeling of competition between supply chains, which may entail several firms (producing, transporting, retailing, etc.).

Hence, tools developed for transportation network equilibrium problems, as briefly overviewed above can be transferred into such important application domains as electric power networks as well as financial network plus supply chain networks. Such networks, together with transportation and telecommunication networks, form some of what are known as "critical infrastructure" networks since their functioning are so essential to our modern societies and economies. Further discussion of a spectrum of multitiered, complex network problems with many decision-makers and a variety of applications, with direct relationships to transportation network equilibrium problems, can be found in the book by Nagurney (2006b).

4. Dynamics

In this Section, we summarize briefly how projected dynamical systems theory can be applied to the elastic demand transportation network equilibrium problem presented in Section 3 in order to provide the disequilibrium dynamics. The presentation here follows the corresponding one in Nagurney (2004). Dupuis and Nagurney (1993) proved that, given a variational inequality problem, there is a naturally associated dynamical system, the set of stationary points of which coincides precisely with the set of solutions of the variational inequality problem. The dynamical system, termed a *projected dynamical system* by Zhang and Nagurney (1995), is non-classical in that its right-hand side, which is a projection operator, is discontinuous. Nevertheless, it can be qualitatively analyzed and approximated through discrete-time algorithms as described in Dupuis and Nagurney and also in the book by Nagurney and Zhang (1996). Importantly, projected dynamical systems theory provides insights into the travelers' dynamic behavior in making their trip decisions and in adjusting their route choices. Moreover, it provides for a powerful theory of stability analysis (cf. Zhang and Nagurney (1996, 1997)). Other approaches to dynamic transportation network problems can be found in Ran and Boyce (1996) and Mahmassani et al (1993). In particular, here we focus on the disequilibrium dynamics and on what can be viewed as the day to day adjustment until an equilibrium is reached.

Since users on a network select paths so as to reach their destinations from their origins, we consider variational inequality (32) as the basic one for the dynamical system equivalence. Specifically, we note that, in view of constraint (28), one may define $\hat{\lambda}(x) \equiv \lambda(d)$, in which case we may rewrite variational inequality (32) in the path flow variables x only, that is, we seek to determine $x^* \in R^{n_P}_+$, such that

$$\langle C(x^*) - \bar{\lambda}(x^*), x - x^* \rangle \geq 0, \quad \forall x \in R^{n_P}_+, \tag{36}$$

where $\bar{\lambda}(x)$ is the $n_{P_{\omega_1}}, \times n_{P_{\omega_2}} \times \dots n_{P_{\omega J}}$-dimensional column vector with components:

$$(\hat{\lambda}_{\omega_1}(x), \dots, \hat{\lambda}_{\omega_1}(x), \dots, \hat{\lambda}_{\omega_J}(x), \dots, \hat{\lambda}_{\omega_J}(x)),$$

If we now let $X \equiv x$ and $F(X) \equiv C(x) - \bar{\lambda}(x)$ and $\mathcal{K} \equiv \{x | x \in R^{n_P}_+\}$, then, clearly, (36) can be put into standard form given by (23). The dynamical system, first presented by Dupuis and Nagurney (1993), whose stationary points correspond to solutions of (36), is given by:

$$\dot{x} = \Pi_{\mathcal{K}}(x, \bar{\lambda}(x) - C(x)), \quad x(0) = x_0 \in \mathcal{K}, \tag{37}$$

where the projection operator $\Pi_{\mathcal{K}}(x, v)$ is defined as:

$$\Pi_{\mathcal{K}}(x, v) = \lim_{\delta \to 0} \frac{(P_{\mathcal{K}}(x + \delta v) - x)}{\delta}, \tag{38}$$

and

$$P_{\mathcal{K}} = \text{argmin}_{z \in \mathcal{K}} \|z - x\|. \tag{39}$$

The dynamics described by (37) are as follows: the rate of change of flow on a path connecting an O/D pair is equal to the difference between the travel disutility for that O/D pair and the cost on that path at that instance in time. If the path cost exceeds the travel disutility, then the flow on the path will decrease; if it is less than the disutility, then the flow on that path will increase. The projection operator in (37) guarantees that the flow on the paths will not be negative, since this would violate feasibility. Hence, the path flows (and incurred travel demands) evolve from an initial path flow pattern at time zero given by $x(0)$ until a stationary point is reached, that is, when $\dot{x} = 0$; at which point we have that for that particular x^*:

$$\dot{x} = 0 = \Pi_{\mathcal{K}}(x^*, \bar{\lambda}(x^*) - C(x^*)), \tag{40}$$

and that x^* also solves variational inequality (36) and is, hence, a transportation network equilibrium satisfying the elastic demand equilibrium conditions (30).

Qualitative properties of the dynamic trajectories, as well as conditions for stability of the solutions as well as discrete-time algorithms can be found in Zhang and Nagurney (1995) and in Nagurney and Zhang (1996) and the references therein. In particular, we note that discrete-time algorithms such as those proposed in Nagurney and Zhang (1996) and the references therein provide for a time discretization of the continuous time trajectories and may also be interpreted as discrete-time adjustment processes.

In addition, dynamic but *within day* transportation network models (deterministic as well as stochastic) have received a lot of attention; see Ran and Boyce (1996) and the references therein.

For discussions of dynamic supply chain networks including electric power generation and distribution networks, as well as extensions of the above results to evolutionary variational inequality theory and double-layered dynamics, see Nagurney (2006b) and the references therein.

5. A Transportation Network Efficiency Measure and the Importance of Network Components

In this chapter, we have focused on mathematical economic models of transportation and the relationship of such models to other network domains. We now further relate transportation networks to some of the *complex network* literature, a topic, which has been the subject of intense research activity in recent years although the topic, which is based on graph theory, is centuries old, as we note above. Indeed, the subject of networks, with its rich applications has been tackled by economists, applied mathematicians, physicists, engineers, biologists, and sociologists; see, for additional examples: Ahuja, Magnanti, and Orlin (1993), Watts and Strogatz (1998), Barabási and Albert (1999), Latora and Marchiori (2001), Newman (2003), Roughgarden (2005), and the references therein. Three types of "networks," whose study has developed more recently and without much interaction with flow-based, behavioral transportation modeling, and which have received intense attention, especially in regards to the development of network measures, are: the random network model, due to Erdös-Rényi (1960), the small-world model (cf. Watts and Strogatz (1998)), and scale-free networks (Barabási and Albert (1999)).

The importance of studying and identifying the vulnerable components of a network has been linked to events such as 9/11 and to Hurricane Katrina, as well as to the biggest blackout in North America that occurred on August 14, 2003 (cf. Sheffi (2005), Nagurney (2006b)). In order to hedge against terrorism and natural disasters, a majority of the associated complex network (sometimes also referred to as *network science*) literature (cf. the survey by Newman (2003)) focuses on the graph characteristics (e.g. connectivity between nodes) of the associated application in order to evaluate the network reliability and vulnerability; see also, for example, Chassin and Posse (2005) and Holme et al. (2002).

As noted in Nagurney and Qiang (2007), in order to be able to evaluate the vulnerability and the reliability of a network, a measure that can quantifiably capture the efficiency/performance of a network must be developed. For example, in a series of papers, beginning in 2001, Latora and Marchiori discussed the network performance issue by measuring the "global efficiency" in a weighted network as compared to that of the simple non-weighted small-world network. In a weighted network, the network is not only characterized by the edges that connect different nodes, but also by the weights associated with different edges in order to capture the relationships between different nodes. The network efficiency E of a network $\mathcal{G}$ is defined in the paper of Latora and Marchiori (2001) as $E = \frac{1}{n(n-1)} \sum_{i \neq j \in \mathcal{G}} \frac{1}{d_{ij}}$, where n is the number of nodes in $\mathcal{G}$ and d_{ij} is the shortest path length (the geodesic distance) between nodes i and j. This measure has been applied by the above authors to a variety of networks, including the (MBTA) Boston subway transportation network and the Internet (cf. Latora and Marchiori (2002), (2004)).

Although the topological structure of a network obviously has an impact on network performance and the vulnerability of the network, clearly, the above discussions argue that the flow on a network is also an important indicator, as are the induced costs, and the behavior of users of the network(s). Indeed, flows represent the usage of a network and which paths and links have positive flows and the magnitude of these flows are relevant in the case of network disruptions. Interestingly, although recently a few papers have appeared in the complex network literature that emphasize flows on a transportation network, with a focus on airline networks (cf. Barrat, Barthélemy, and Vespignani (2005), and Dall'Asta et al. (2006)), the aforementioned papers only consider the importance of nodes and not that of links and ignore the behavior of users. A network efficiency measure that captures flows, the costs associated with "travel," and user behavior, along with the network topology, is more appropriate in evaluating networks such as transportation networks, which are the classical critical infrastructure. Indeed, in the case of disruptions, which can affect either nodes, or links, or both, we can expect travelers to readjust their behavior and the usage of the network accordingly. Furthermore, as noted by Jenelius, Petersen, and Mattsson (2006), the criticality of a network component, consisting of a node, link, or combination of nodes and links, is related to the vulnerability of the network system in that the more critical (or, as we consider, the more important) the component, the greater the damage to the network system when this component is removed, be it through natural disasters, terrorist attacks, structural failures, etc.

We now describe a new transportation network performance measure that can be used to evaluate the efficiency of a transportation network as well as the importance of its network components. It contains the Latora and Marchiori (2001) measure used in the "complex" network literature, as a special case. The measure, due to Nagurney and Qiang (2007), has the additional notable feature that it is applicable, as is the accompanying importance definition of network components, even in the case that the network becomes disconnected (after the removal of the component).

Definition 2: A Transportation Network Efficiency Measure (Nagurney and Qiang (2007))

The network transportation efficiency measure, $\mathcal{E}(\mathcal{G}, d)$, for a given network topology $\mathcal{G}$ and vector of O/D demands, d, is defined as follows:

$$\mathcal{E} = \mathcal{E}(\mathcal{G}, d) = \frac{\sum_{\omega \in \Omega} \frac{d_\omega}{\lambda_\omega}}{J}, \tag{41}$$

where λ_ω denotes the cost on the minimum cost (shortest) used path(s), that is, ones with positive flow, connecting O/D pair ω and J is the number of O/D pairs in the network.

As emphasized in Nagurney and Qiang (2007), the transportation network efficiency measure given in (41) has a meaningful economic interpretation which is that the efficiency of a transportation network is equal to the average, in terms of O/D pairs, traffic to price ratio with the traffic per O/D pair being given by d_ω and the equilibrium price of travel between O/D pair ω by λ_ω (cf. (5) and (30)). The higher the traffic that can be handled at a given price (which also reflects the cost and, from an engineering perspective, the travel time), the higher the efficiency or performance of the transportation network.

Interestingly, the following theorem demonstrates that, under appropriate assumptions, the Nagurney and Qiang (N-Q) (2007) measure contains, as a special case, the network efficiency measure of Latora and Marchiori (L-M) (2001), which, however, considered neither flows nor demands and did not incorporate any underlying users' behavior.

Theorem 5 (Nagurney and Qiang (2007))

If the demand for each pair of nodes is equal to 1 *and* d_{ij} *is set equal to* λ_ω*, where* $\omega = (i, j)$*, then the proposed network efficiency measure and the Latora and Marchiori (2001) measure are one and the same.*

Proof: Let n be number of nodes in $\mathcal{G}$. Therefore, the total number of O/D pairs, J, in principle, can be assumed to be equal to $n(n-1)$. Furthermore, by the assumption, we have $d_\omega = 1$, $\forall \omega \in \Omega$, where Ω is the set of all O/D pairs in $\mathcal{G}$, and $\omega = (i, j)$ and $d_{ij} = \lambda_\omega$, where $i \neq j$, $\forall i, j \in \mathcal{G}$. Then the measure in (41) becomes as follows:

$$\mathcal{E}(\mathcal{G}, d) = \frac{\sum_{\omega \in \Omega} \frac{d_\omega}{\lambda_\omega}}{J} = \frac{\sum_{i \neq j \in \mathcal{G}} \frac{1}{d_{ij}}}{J} = \frac{1}{n(n-1)} \sum_{i \neq j \in \mathcal{G}} \frac{1}{d_{ij}}. \tag{42}$$

□

Observe that, from the definition, λ_ω is the value of the cost of the minimum or "shortest" used paths for O/D pair ω and d_{ij}, according to Latora and Marchiori (2001), is the shortest path length (the geodesic distance) between nodes i and j. Therefore, the assumption of λ_ω being equal to d_{ij} is not unreasonable. The Nagurney and Qiang (2007) measure, however, is a more general measure which also captures flows and behavior on the network, according to Definition 2.

With the new transportation network efficiency measure, we can investigate the importance of network components by studying their impact on the transportation network efficiency through their removal. The importance of a node or a link (or a subset of nodes and links) is defined as follows:

Definition 3: Importance of a Network Component (Nagurney and Qiang (2007))

The importance, $I(g)$ *of a network component* $g \in \mathcal{G}$*, is measured by the relative network efficiency drop after* g *is removed from the network:*

$$I(g) = \frac{\triangle \mathcal{E}}{\mathcal{E}} = \frac{\mathcal{E}(\mathcal{G}, d) - \mathcal{E}(\mathcal{G} - g, d)}{\mathcal{E}(\mathcal{G}, d)}, \tag{43}$$

where $\mathcal{G} - g$ *is the resulting network after component* g *is removed from network* $\mathcal{G}$*.*

The upper bound of the importance of a network component is 1.

The elimination of a link is treated in the N-Q measure by removing that link while the removal of a node is managed by removing the links entering or exiting that node. In the case that the removal results in no path connecting an O/D pair, we simply assign the demand for that O/D pair to an abstract path with a cost of infinity. This measure is well-defined even in the case of disconnected networks. Notably, Latora and Marchiori (2001) also mention this important characteristic which gives their measure an attractive property over the measure used for the small-world network model (cf. Watts and Strogatz (1998)).

An Example

Consider the network in Figure 4 in which there are two O/D pairs: $\omega_1 = (1, 2)$ and $\omega_2 = (1, 3)$ with demands given, respectively, by $d_{\omega_1} = 100$ and $d_{\omega_2} = 20$. We have that path $p_1 = a$ and path $p_2 = b$. Assume that the link cost functions are given by: $c_a(f_a) = .01f_a + 19$ and $c_b(f_b) = .05f_b + 19$. Clearly, we must have that $x^*_{p_1} = 100$ and $x^*_{p_2} = 20$ so that $\lambda_{\omega_1} = \lambda_{\omega_2} = 20$. The network efficiency measure $\mathcal{E} = 3$ whereas the L-M measure $E = .0167$.

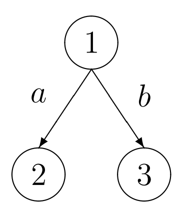

Figure 4: Example

The importance and the rankings of the links and the nodes for the Example are given, respectively, in Tables 2 and 3, using the importance measures.

The Nagurney and Qiang (2007) measure, which captures flow information is more general, reasonable, and precise since, in the case of a disruption, the destruction of link a, with which was associated a flow 5 times the flow of link b, would result in a greater loss of efficiency! The same qualitative analysis holds for the destruction of node 2 versus node 3. Additional examples can be found in Nagurney and Qiang (2007).

Table 2: Importance and Ranking of Links in the Example

Link	Importance Value N-Q Measure	Importance Ranking N-Q Measure	Importance Value L-M Measure	Importance Ranking L-M Measure
a	0.83	1	0.5	1
b	0.17	2	0.5	1

Table 3: Importance and Ranking of Nodes in the Example

Node	Importance Value N-Q Measure	Importance Ranking N-Q Measure	Importance Value L-M Measure	Importance Ranking L-M Measure
1	1.00	1	1.00	1
2	0.83	2	0.50	2
3	0.17	3	0.50	2

This transportation network efficiency measure can be applied to other transportation-related networks, including such critical infrastructure networks as the Internet, electric power generation and distribution networks, supply chains, as well as a variety of financial networks. Indeed, it can be used to identify not only the critical components in a given network but also to assist decision-makers as to which components should be better secured; see, e.g., Nagurney and Qiang (2007b).

6. Conclusions

This chapter has overviewed the major developments in the mathematical economic modeling of transportation and has also identified some of the significant connections between transportation networks and other network applications. Such relevant behavioral principles as user-optimization versus system-optimization were discussed and illustrated through the Braess paradox. Increasingly complex and general transportation network models were described, beginning with the fixed demand transportation network model (both user-optimized and system-optimized versions) with separable user link cost functions through the elastic demand transportation network equilibrium model with asymmetric user link cost and travel disutility functions. Discussions of optimization theory, variational inequality theory, and projected dynamical systems theory for network dynamics were also presented, as relevant to transportation network problems. In addition, a transportation network efficiency measure was reviewed, along with the importance identification of network components, that is, the nodes and links. Throughout this chapter examples were provided for illustrative purposes. Finally, related applications to the Internet, supply chains, as well as electric power generation and distribution networks and financial networks were noted. This chapter vividly demonstrates the essential importance of transportation networks and their rigorous study to both theoretical advances and practice.

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Biographical Sketch

Anna Nagurney is the John F. Smith Memorial Professor in the Department of Finance and Operations Management in the Isenberg School of Management at the University of

Massachusetts at Amherst. She is also an Affiliated Faculty Member of the Department of Mechanical and Industrial Engineering and the Department of Civil and Environmental Engineering at the University of Massachusetts at Amherst. She is the Founding Director of the Virtual Center for Supernetworks and the Supernetworks Laboratory for Computation and Visualization at the University of Massachusetts at Amherst. She received an AB degree in Russian Language and Literature in 1977, the ScB degree in 1977, the ScM in 1980, and the PhD degree in 1983, all in Applied Mathematics, and all from Brown University in Providence, Rhode Island. She devotes her career to education and research that combines operations research/management science, economics, and engineering. Her focus is the applied and theoretical aspects of decision-making on network systems, particularly in the areas of transportation and logistics, energy and the environment, and economics and finance. Her most recent book is *Supply Chain Network Economics: Dynamics of Prices, Flows, and Profits* published in 2006. She has authored or co-authored 8 other books including *Supernetworks: Decision-Making for the Information Age* and *Network Economics: A Variational Inequality Approach*, and more than 120 refereed journal articles.

She has given invited talks in Sweden, New Zealand, China, Germany, Italy, Canada, Australia, Cyprus, Iceland, the US, and other countries and her research has garnered funding from many foundations, including the National Science Foundation. Among the honors she has received are: the University of Massachusetts Award for Outstanding Accomplishments in Research and Creative Activity, an INFORMS Moving Spirit Award, a Science Fellowship at the Radcliffe Institute for Advanced Study at Harvard University, a Rockefeller Foundation Bellagio Center Research Team Fellowship, a Distinguished Fulbright Chair at the University of Innsbruck, Austria, two ATT Foundation Industrial Ecology Fellowships, the Chancellor's Medal from the University of Massachusetts, an Eisenhower Faculty Fellowship, a National Science Foundation Faculty Award for Women, a Faculty Fellowship from the University of Massachusetts, and the Kempe Prize from the University of Umea, Sweden. She has been a visiting professor at the University of Innsbruck, Austria, the Royal Institute of Technology in Stockholm, Sweden, the Massachusetts Institute of Technology, and Brown University.

She has served on numerous prize committees, including the Fudan Premium Fund of Management Prize committee in Shanghai, China, the INFORMS Transportation Science Section Robert Herman Lifetime Achievement Prize in Transportation Science committee, the INFORMS Computer Science Technical Section prize committee, and the Discipline Advisory Committee for Fulbright Scholar Awards. Professor Professor Nagurney is on the Advisory Board of the European Union funded COMISEF project, which is a multi-year, multi-national project and regularly serves on government panels and international conference organizing committees. She is the editor of the book series, *New Dimensions in Networks* (Edward Elgar Publishing), and the co-editor of the book series, *Advances in Computational Economics* (Springer). She is on the editorial boards of the journals: *Networks, Journal of Economic Dynamics and Control, Computational Economics, Computational Management Science, Annals of Regional Science, International Journal of High Performance Computing Applications, The Journal of Financial Decision Making, Netnomics: Economic Research and Electronic Networking, Optimization Letters*, and the *International Journal of Sustainable Transportation*.