A Dynamic Network Oligopoly Model with Transportation Costs, Product Differentiation, and Quality Competition

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Abstract: This paper develops a new dynamic model of Cournot-Nash oligopolistic competition that includes production and transportation costs, product differentiation, and quality levels in a network framework. The production costs capture the total quality cost, which, in turn, can also represent the R&D cost. We first present the equilibrium version and derive alternative variational inequality formulations. We then construct the projected dynamical systems model, which provides a continuous-time evolution of the firms' product shipments and product quality levels, and whose set of stationary points coincides with the set of solutions to the variational inequality problem. We establish stability analysis results using a monotonicity approach and construct a discrete-time version of the continuous-time adjustment process, which yields an algorithm, with closed form expressions at each iteration. The algorithm is then utilized to compute solutions to several numerical examples.

The framework can serve as the foundation for the modeling and analysis of competition among firms in industries ranging from food to pharmaceuticals to durable goods and high tech products, as well as Internet services, where quality and product differentiation are seminal.

Key words: oligopolies, networks, game theory, product quality, product differentiation, variational inequalities, projected dynamical systems

1. Introduction

Oligopolies constitute a fundamental industrial organization market structure of numerous industries world-wide ranging from airplane manufacturers, as well as airlines, to wireless service providers, certain food manufacturers and retailers, as well as utilities and energy companies, and even specific banks. Hence, the oligopoly problem, which consists of a finite number of firms, involved in the production of a commodity in a non-cooperative manner is a classical problem in economics, dating to Cournot (1838), who considered the case of two firms. It is also an example of a game theory problem, with the governing equilibrium conditions constituting a Nash equilibrium (cf. Nash 1950, 1951). Well-known formalisms for oligopolistic competition include, in addition, to the Cournot-Nash framework in which firms select their optimal production quantities, the Bertrand (1883) framework, in which firms choose their product prices, as well as the von Stackelberg (1934) framework, in which decisions are made sequentially in a leader-follower type of game. In classical oligopoly problems, the product that is produced is assumed to be homogeneous and, hence, consumers at the demand market do not distinguish among the firms that produce the product. For background on industrial organization, we refer the reader to the books by Tirole (1988) and Vives (1999).

Increasingly, however, in the case of many products in imperfect markets such as in oligopolies, consumers may consider the products to be differentiated according to the producer and, hence, oligopolies with product differentiation have been gaining in attention, dating to the work of Hotelling (1929) (see also Dixit and Stiglitz 1977; D'Aspremont, Gabsewicz, and Thisse 1979; Economides 1989; Anderson and de Palma 2001; Johnson and Myatt 2003, and the references therein). In particular, as noted by Banker, Khosla, and Sinha (1998), Hotelling's (1929) paper, which considered price and quality competition between two firms, has also inspired the study of quality competition in economics (cf. Gabszewicz and Thisse 1979) as well as in marketing and operations research / management science (see Chambers, Kouvelis, and Semple 2006). For example, Lederer and Rhee (1995) provided an overview of quality from the perspective of operations management as well as economics (see also, e.g., Spence 1975). It is also worth noting the *classics* of the traditional quality management literature, which are the books by Juran (1951), Feigenbaum (1956), and Deming (1990), with a critical survey to that date given in Kolesar (1993).

Indeed, quality is emerging as an important feature/characteristic in numerous products, ranging from food (see, e.g., Marsden 2004) to pharmaceuticals (see Masoumi, Yu, and Nagurney 2012) to durable manufactured products such as automobiles (see Shank and

Govindarajan (1994)) to high tech products, including microprocessors (see Goettler and Gordon 2011), and even services associated with the Internet (cf. Kruse 2009). Holcombe (2009) argues that firms, in reality, do not differentiate their products to make them different, or to give consumers more variety but, rather, to make them better so that consumers purchase the firm's product. Moreover, although the differentiated product may even cost more to produce, it may result in higher profits since consumers may be drawn to such products. Hence, quality is implicit in product differentiation. Holcombe (2009) also notes that, according to Shumpeter (1943, pg. 82), "The essential point to grasp is that in dealing with capitalism we are dealing with an evolutionary process."

Cabral (2012) recently articulated the need for new dynamic oligopoly models, combined with network features, as well as quality. In this paper, we take up the challenge and we develop a network oligopoly model with differentiated products and quality levels. We quantify the quality levels, as well as the quality-related costs in the process of production, and research and development, as a special case. We define quality level as the quality conformance level, the degree to which a specific product conforms to a design or specification (Juran and Gryna 1988) and note that the percentage of defects (0% to 100%) can then be used as a proxy. Based on the quantifiable quality levels, the demand in a particular demand market is a function of both prices and quality levels. The quality-related cost, in turn, is defined as "the cost incurred in ensuring and assuring quality as well as the loss incurred when quality is not achieved" (ASQC 1971 and BS 1990), and it is a convex function of quality (see, e.g., Feigenhaum 1983; Moorthy 1988; Juran and Gryna 1988; Campanella 1990; Porter and Rayner 1992; Shank and Govindarajan 1994; Chambers, Kouvelis, and Semple 2006). The quality-related costs occur in the process of production; hence, we express the production cost function of each firm as a function that depends on both production quantities and quality levels (see, e.g., Spence 1975; Rogerson 1988, and Lederer and Rhee 1995), and it is convex in quality. In addition, our production cost functions also capture R&D costs (Matsubara 2010).

The framework for our model is that of Cournot-Nash competition in which the firms compete by determining their optimal product shipments as well as the quality levels of their particular products. We present both the static version, in an equilibrium context, which we formulate as a finite-dimensional variational inequality problem, and then we develop its dynamic counterpart, using projected dynamical systems theory. In addition to the model development, we provide stability analysis results, and also propose a discrete-time algorithm for the dynamic tracking of the continuous-time trajectories of the firms' product shipments and quality levels over time. We, subsequently, apply the algorithm to compute solutions to several numerical examples of network oligopoly problems with differentiated products.

In our modeling framework, we do not limit ourselves to demand price or total cost functions of specific functional forms and we do not assume that the firms are symmetric or of a predetermined, fixed number. In addition, we allow the firms to compete in both their product shipments and the quality levels of their products and note that Dubovik and Janssen (2011) emphasized quality as a strategic variable in oligopolistic competition (see also Rogerson 1988). Moreover, we explicitly consider the spatial component and include transportation costs associated with shipping the products to the demand markets. The relevance of transportation costs was also recognized by Hotelling (1929). Our theoretical and computational framework can be applied to network oligopolies with product differentiation and quality levels in many different settings. Recent contributions to dynamic oligopolies can be found in the book by Bischi et al. (2009) and the paper by Matsumoto and Szidarovsky (2011). Our approach, in contrast, integrates oligopolies in a network setting along with differentiated products and quality levels. Firms select both their product shipments and the quality levels of their specific products subject to costs associated with production, quality, and the transportation of the products to the demand markets. The demand price functions are functions not only of the demands for the various products at the demand markets, but also of their quality levels.

We note that Nagurney, Dupuis, and Zhang (1994) developed a dynamic network oligopoly model using projected dynamical systems theory (cf. Dupuis and Nagurney 1993; Zhang and Nagurney 1995; Nagurney and Zhang 1996, and Nagurney 1999), which extended the spatial oligopoly model of Dafermos and Nagurney (1987) to the dynamic domain. The static and dynamic network models that we construct in this paper generalize the former models in several significant ways, while retaining the spatial component in that:

(1). We consider product differentiation;

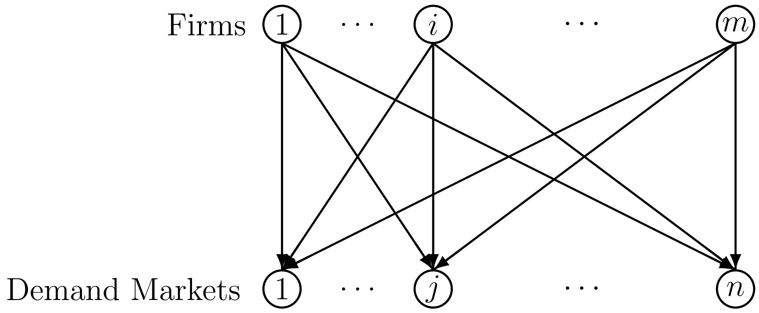
(2). We incorporate quality levels associated with the individual firms' products, as strategic variables, along with the product shipments, and we include the associated total costs as well as appropriate demand price functions at the demand markets. The production costs capture the total quality cost, which, in turn, can also represent the R&D cost. The demand price functions capture the consumers' response to both prices and quality levels.

(3). We capture the critical transportation costs associated with linking the production side with the demand markets via a network.

In addition, we provide both qualitative analysis as well as an algorithmic scheme, along with numerical examples, which is made possible through projected dynamical systems theory, which can handle constraints and the associated discontinuities, unlike classical dynamical systems theory. For books on stability analysis of classical dynamical systems, we refer the reader to Bhatia and Szego (1970), Hirsch and Smale (1974), and Perko (1991).

Oligopolistic market problems have been studied extensively at the equilibrium solutions, using, for example, variational inequality theory (cf. Nagurney 1999 and the references therein) with Nagurney and Yu (2012) and Masoumi, Yu, and Nagurney (2012) formulating supply chain network oligopolies with brand differentiation. They have also been studied in a dynamic context by, among others, Arrow and Hurwicz (1977), Flam and Ben-Israel (1990), Okuguchi and Szidarovszky (1990), and Matsumoto and Szidarovsky (2011). However, we now no longer need to impose assumptions that require that the trajectories stay away from the boundaries and this is made possible through the methodology of projected dynamical systems (Dupuis and Nagurney 1993 and Nagurney and Zhang 1996). Projected dynamical systems theory has now been used to model dynamics in supply chains (cf. Nagurney et al. 2002; Nagurney 2006, and Cruz 2008), transportation networks (see Zhang and Nagurney 1996b, 1997 and Zhang, Nagurney, and Wu 2001), oligopolies without product differentiation (Nagurney, Dupuis, and Zhang 1994 and Zhang and Nagurney 1996a), spatial price equilibrium problems (Nagurney, Takayama, and Zhang 1995a, 1995b), financial networks (see, e.g., Nagurney and Cruz 2004), and even population games (see Sandholm 2011). Here we apply the methodology, for the first time, for the development of a new dynamic network oligopoly model with product differentiation, quality levels, and transportation costs. In our dynamic model, the firms compete and adjust their product shipments to the demand markets, along with the quality levels of the products, at a rate that is equal to the respective marginal utilities or profits, with the guarantee that the product shipments and quality levels do not become negative.

The paper is organized as follows. In Section 2, we first present the static version of the network oligopoly model, and establish alternative variational inequality formulations of the governing Nash-Cournot equilibrium conditions. We then present its dynamic counterpart and show that the set of equilibria coincides with the set of stationary points of the projected dynamical systems network oligopoly model. In Section 3, we present stability analysis results, utilizing the monotonicity approach introduced by Zhang and Nagurney (1995). We also illustrate the concepts with several numerical examples. In Section 4, we propose the discrete-time adjustment process, which provides an approximation to the continuous-time trajectories of the firm's product shipments and quality levels over time. We then apply the algorithm to demonstrate how it can be used to track the trajectories and to also compute



The products: $1, \ldots, m$ may be consumed at any demand market

Figure 1: The network structure of the dynamic oligopoly problem with product differentiation

the stationary points; equivalently, the equilibria. We summarize our results and present our conclusions in Section 5.

2. The Dynamic Network Oligopoly Model with Product Differentiation and Quality Competition

In this Section, we develop a dynamic network oligopoly model with product differentiation and quality competition. It is assumed that the firms compete under the Cournot-Nash equilibrium concept of non-cooperative behavior and select both their product shipments as well as the quality levels of their products. The consumers, in turn, signal their preferences for the products through the demand price functions associated with the demand markets, which are spatially separated. The demand price functions are, in general, functions of the demands for the products at all the demand markets as well as the quality levels of the products.

We first develop the equilibrium model and derive the variational inequality formulation. We then describe the underlying dynamics associated with the firms' production outputs as well as quality levels and present the projected dynamical systems model whose set of stationary points corresponds to the set of solutions of the variational inequality problem.

Please refer to Figure 1 for the underlying network structure of the network oligopoly with product differentiation.

We assume that there are m firms and n demand markets that are generally spatially separated. There is a distinct (but substitutable) product produced by each of the m firms and is consumed at the n demand markets. Let s_i denote the nonnegative product output produced by firm i and let d_{ij} denote the demand for the product of firm i at demand market j. Let Q_{ij} denote the nonnegative shipment of firm i's product to demand market j. We group the production outputs into the vector $s \in R^m_+$, the demands into the vector $d \in R^{mn}_+$, and the product shipments into the vector $Q \in R^{mn}_+$. Here q_i denotes the quality level, or, simply, the quality, of product i, which is produced by firm i. We group the quality levels of all firms into the vector $q \in R^m_+$. All vectors here are assumed to be column vectors, except where noted.

The following conservation of flow equations must hold:

$$s_i = \sum_{j=1}^{n} Q_{ij}, \quad i = 1, \ldots, m; \tag{1}$$

$$d_{ij} = Q_{ij}, \quad i = 1, \ldots, m; j = 1, \ldots, n, \tag{2}$$

$$Q_{ij} \geq 0, \quad i = 1, \ldots, m; j = 1, \ldots, n, \tag{3}$$

and since the quality levels must also be nonnegative, we must also have that

$$q_i \geq 0, \quad i = 1, \ldots, m. \tag{4}$$

Hence, the quantity of the product produced by each firm is equal to the sum of the amounts shipped to all the demand markets, and the quantity of a firm's product consumed at a demand market is equal to the amount shipped from the firm to that demand market. Both the shipment volumes and the quality levels must be nonnegative.

We associate with each firm i a production cost $\hat{f}_i$, and allow for the general situation where the production cost of a firm i may depend upon the entire production pattern and on its own quality level, that is,

$$\hat{f}_i = \hat{f}_i(s, q_i), \quad i = 1, \ldots, m. \tag{5}$$

We assume, hence, that the functions in (5) also capture the total quality cost, since, as a special case, the above functions can take on the form

$$\hat{f}_i(s, q_i) = f_i(s, q_i) + g_i(q_i), \quad i = 1, \ldots, m, \tag{6}$$

where the first term depends on both quality and production outputs and the second term only depends on the quality. Interestingly, the second term in (6) can also be interpreted as the R&D cost (cf. Matsubara 2010), which is the cost that occurs in the processes of the development and introduction of new products to market as well as the improvement of existing products. Evidence indicates that because R&D raises product quality, firms invest in R&D and the capital accumulation of R&D changes and improves the product quality

(Bernstein and Nadiri 1991). Hence, the R&D cost is assumed to depend on the quality level of its products (see Klette and Griliches 2000; Hoppe and Lehmann-Grube 2001, and Symeonidis 2003).

The production cost functions (5) (and (6)) are assumed to be convex and continuously differentiable. As noted in the Introduction, other authors have also emphasized that production cost should be a function of both production output and quality and that the quality-related costs functions are convex functions of the quality level.

As noted earlier, consumers located at the demand markets, which are spatially separated, respond not only to the quantities available of the products but also to their quality levels. Hence, we allow the demand price for a product at a demand market to depend, in general, upon the entire consumption pattern, as well as on all the levels of quality of all the products, that is,

$$p_{ij} = p_{ij}(d, q), \quad i = 1, \ldots, m; j = 1, \ldots, n, \tag{7}$$

with the prices grouped into the vector $p \in R^{mn}$. The generality of the expression in (7) allows for modeling and application flexibility. The demand price functions are, typically, assumed to be monotonically decreasing in product quantity but increasing in terms of product quality.

For definiteness, we recall that Kaya and Özer (2009) and Kaya (2011) considered demand price functions in price and quality variables of the following form: $q = a - bp + e + \varepsilon$, in their supply chain contracting papers, where q is the demand and e is the quality level. Xie et al. (2011) utilized a demand price function in the investigation of quality investment, of the form $D = a + \alpha x - \beta p$. This function extended the one given in Banker et al. (1998), who considered a duopoly case and utilized a demand price function given by $q_i = k_i\alpha - \beta p_i + \gamma p_j + \lambda x_i - \mu x_j$. In addition, Anderson and Palma (2001) captured the utility of each consumer u expressed as $u = q - p + \varepsilon$ in their research on asymmetric oligopolies. Note that, in our framework, we do not limit ourselves to linear demand price functions. Moreover, we allow the demand price of a product to depend not only on its demand but also on those of the other products: the same holds for the dependence of the prices on the product quality levels.

Let $\hat{c}_{ij}$ denote the total transportation cost associated with shipping firm i's product to demand market j, where the total transportation cost is given by the function:

$$\hat{c}_{ij} = \hat{c}_{ij}(Q_{ij}), \quad i = 1, \ldots, m; j = 1, \ldots, n, \tag{8}$$

and group these costs into a vector $\hat{c} \in R^{mn}$.

The demand price functions (7) and the total transportation cost functions (8) are assumed to be continuous and continuously differentiable.

The strategic variables of firm i are its product shipments $\{Q_i\}$ where $Q_i = (Q_{i1}, \ldots, Q_{in})$ and its quality level q_i.

The profit or utility U_i of firm i; $i = 1, \ldots, m$, is, hence, given by the expression

$$U_i = \sum_{j=1}^{n} p_{ij} d_{ij} - \hat{f}_i - \sum_{j=1}^{n} \hat{c}_{ij}, \tag{9}$$

which is the difference between its total revenue and its total cost.

In view of (1) - (9), one may write the profit as a function solely of the shipment pattern and quality levels, that is,

$$U = U(Q, q), \tag{10}$$

where U is the m-dimensional vector with components: $\{U_1, \ldots, U_m\}$.

Let K^i denote the feasible set corresponding to firm i, where $K^i \equiv \{(Q_i, q_i) | Q_i \geq 0, \text{ and } q_i \geq 0\}$ and define $K \equiv \prod_{i=1}^{m} K^i$.

We consider the oligopolistic market mechanism in which the m firms supply their products in a non-cooperative fashion, each one trying to maximize its own profit. We seek to determine a nonnegative product shipment and quality level pattern (Q^*, q^*) for which the m firms will be in a state of equilibrium as defined below. In particular, Nash (1950, 1951) generalized Cournot's concept of an equilibrium for a model of several players, each of which acts in his/her own self-interest, in what has been come to be called a non-cooperative game.

Definition 1: A Network Cournot-Nash Equilibrium with Product Differentiation and Quality Levels

A product shipment and quality level pattern $(Q^*, q^*) \in K$ is said to constitute a network Cournot-Nash equilibrium if for each firm $i; i = 1, \ldots, m$,

$$U_i(Q_i^*, q_i^*, \hat{Q}_i^*, \hat{q}_i^*) \geq U_i(Q_i, q_i, \hat{Q}_i^*, \hat{q}_i^*), \quad \forall (Q_i, q_i) \in K^i, \tag{11}$$

where

$$\hat{Q}_i^* \equiv ({Q^*}_1, \ldots, Q_{i-1}^*, Q_{i+1}^*, \ldots, Q_m^*); \quad \textit{and} \quad \hat{q}_i^* \equiv ({q^*}_1, \ldots, q_{i-1}^*, q_{i+1}^*, \ldots, q_m^*). \tag{12}$$

According to (11), an equilibrium is established if no firm can unilaterally improve upon its profits by selecting an alternative vector of product shipments and quality level of its product.

Variational Inequality Formulations

We now present alternative variational inequality formulations of the above network Cournot-Nash equilibrium with product differentiation in the following theorem.

Theorem 1

Assume that, for each firm i, the profit function $U_i(Q,q)$ is concave with respect to the variables $\{Q_{i1},\ldots,Q_{in}\}$, and q_i, and is continuous and continuously differentiable. Then $(Q^*,q^*)\in K$ is a network Cournot-Nash equilibrium according to Definition 1 if and only if it satisfies the variational inequality

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\partial U_i(Q^*,q^*)}{\partial Q_{ij}}\times(Q_{ij}-Q^*_{ij})-\sum_{i=1}^{m}\frac{\partial U_i(Q^*,q^*)}{\partial q_i}\times(q_i-q^*_i)\geq 0,\quad \forall (Q,q)\in K, \tag{13}$$

or, equivalently, $(s^*,Q^*,d^*,q^*)\in K^1$ is an equilibrium production, shipment, consumption, and quality level pattern if and only if it satisfies the variational inequality

$$\sum_{i=1}^{m}\frac{\partial \hat{f}_i(s^*,q^*_i)}{\partial s_i}\times(s_i-s^*_i)+\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial \hat{c}_{ij}(Q^*_{ij})}{\partial Q_{ij}}-\sum_{k=1}^{n}\frac{\partial p_{ik}(d^*,q^*)}{\partial d_{ij}}\times d^*_{ik}\right]\times(Q_{ij}-Q^*_{ij})$$
$$-\sum_{i=1}^{m}\sum_{j=1}^{n}p_{ij}(d^*,q^*)\times(d_{ij}-d^*_{ij})$$
$$+\sum_{i=1}^{m}\left[\frac{\partial \hat{f}_i(s^*,q^*_i)}{\partial q_i}-\sum_{k=1}^{n}\frac{\partial p_{ik}(d^*,q^*)}{\partial q_i}\times d^*_{ik}\right]\times(q_i-q^*_i)\geq 0,\quad \forall (s,Q,d,q)\in K^1, \tag{14}$$

where $K^1\equiv\{(s,Q,d,q)|\, Q\geq 0,\, q\geq 0, \textit{and}\,(1)\,\textit{and}\,(2)\,\textit{hold}\}$.

Proof: (13) follows directly from Gabay and Moulin (1980) and Dafermos and Nagurney (1987).

In order to obtain variational inequality (14) from variational inequality (13), we note that:

$$-\frac{\partial U_i}{\partial Q_{ij}}=\left[\frac{\partial \hat{f}_i}{\partial s_i}+\frac{\partial \hat{c}_{ij}}{\partial Q_{ij}}-p_{ij}-\sum_{k=1}^{n}\frac{\partial p_{ik}}{\partial d_{ij}}\times d_{ik}\right];\quad i=1,\ldots,m;j=1,\ldots,n; \tag{15}$$

and

$$-\frac{\partial U_i}{\partial q_i} = \left[\frac{\partial \hat{f}_i}{\partial q_i} - \sum_{k=1}^{n} \frac{\partial p_{ik}}{\partial q_i} \times d_{ik}\right]; \quad i = 1, \ldots, m. \tag{16}$$

Multiplying the right-most expression in (15) by $(Q_{ij} - Q_{ij}^*)$ and summing the resultant over all i and all j; similarly, multiplying the right-most expression in (16) by $(q_i - q_i^*)$ and summing the resultant over all i yields, respectively:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial \hat{f}_i}{\partial s_i} + \frac{\partial \hat{c}_{ij}}{\partial Q_{ij}} - p_{ij} - \sum_{k=1}^{n} \frac{\partial p_{ik}}{\partial d_{ij}} \times d_{ik}\right] \times (Q_{ij} - Q_{ij}^*) \tag{17}$$

and

$$\sum_{i=1}^{m}\left[\frac{\partial \hat{f}_i}{\partial q_i} - \sum_{k=1}^{n} \frac{\partial p_{ik}}{\partial q_i} \times d_{ik}\right] \times (q_i - q_i^*). \tag{18}$$

Finally, summing (17) and (18) and then using constraints (1) and (2), yields variational inequality (14). $\square$

We now put the above oligopolistic market equilibrium problem with product differentiation and quality levels into standard variational inequality form, that is,

Determine $X^* \in \mathcal{K} \subset R^N$, such that

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{19}$$

where F is a given continuous function from $\mathcal{K}$ to R^N and $\mathcal{K}$ is a closed and convex set.

We define the $(mn+m)$-dimensional vector $X \equiv (Q, q)$ and the $(mn+m)$-dimensional row vector $F(X) = (F^1(X), F^2(X))$ with the (i,j)-th component, F^1_{ij}, of $F^1(X)$ given by

$$F^1_{ij}(X) \equiv -\frac{\partial U_i(Q,q)}{\partial Q_{ij}}, \tag{20}$$

the i-th component, F^2_i, of $F^2(X)$ given by

$$F^2_i(X) \equiv -\frac{\partial U_i(Q,q)}{\partial q_i}, \tag{21}$$

and with the feasible set $\mathcal{K} \equiv K$. Then, clearly, variational inequality (13) can be put into standard form (19).

In a similar manner, one can establish that variational inequality (14) can also be put into standard variational inequality form (19).

For additional background on the variational inequality problem, we refer the reader to the book by Nagurney (1999).

The Projected Dynamical System Model

We now propose a dynamic adjustment process for the evolution of the firms' product shipments and product quality levels. Observe that, for a current product shipment and quality level pattern at time t, $X(t) = (Q(t), q(t))$, $-F^1_{ij}(X(t)) = \frac{\partial U_i(Q(t),q(t))}{\partial Q_{ij}}$, given by (20), is the marginal utility (profit) of firm i with respect to its product shipment to demand market j. Similarly, $-F^2_i(X(t)) = \frac{\partial U_i(Q(t),q(t))}{\partial q_i}$, given by (21), is the firm's marginal utility (profit) with respect to its quality level. In this framework, the rate of change of the product shipment between a firm and demand market pair (i, j) is in proportion to $-F^1_{ij}(X)$, as long as the product shipment Q_{ij} is positive. Namely, when $Q_{ij} > 0$,

$$\dot{Q}_{ij} = \frac{\partial U_i(Q, q)}{\partial Q_{ij}}, \tag{22}$$

where $\dot{Q}_{ij}$ denotes the rate of change of Q_{ij}. However, when $Q_{ij} = 0$, the nonnegativity condition (3) forces the product shipment Q_{ij} to remain zero when $\frac{\partial U_i(Q,q)}{\partial Q_{ij}} \leq 0$. Hence, in this case, we are only guaranteed of having possible increases of the shipment. Namely, when $Q_{ij} = 0$,

$$\dot{Q}_{ij} = \max\{0, \frac{\partial U_i(Q, q)}{\partial Q_{ij}}\}. \tag{23}$$

We may write (22) and (23) concisely as:

$$\dot{Q}_{ij} = \begin{cases} \frac{\partial U_i(Q,q)}{\partial Q_{ij}}, & \text{if} \quad Q_{ij} > 0 \\ \max\{0, \frac{\partial U_i(Q,q)}{\partial Q_{ij}}\}, & \text{if} \quad Q_{ij} = 0. \end{cases} \tag{24}$$

As for the quality levels, when $q_i > 0$, then

$$\dot{q}_i = \frac{\partial U_i(Q, q)}{\partial q_i}, \tag{25}$$

where $\dot{q}_i$ denotes the rate of change of q_i; otherwise:

$$\dot{q}_i = \max\{0, \frac{\partial U_i(Q, q)}{\partial q_i}\}, \tag{26}$$

since q_i must be nonnegative.

Combining (25) and (26), we may write:

$$\dot{q}_i = \begin{cases} \frac{\partial U_i(Q,q)}{\partial q_i}, & \text{if} \quad q_i > 0 \\ \max\{0, \frac{\partial U_i(Q,q)}{\partial q_i}\}, & \text{if} \quad q_i = 0. \end{cases} \tag{27}$$

Applying (24) to all firm and demand market pairs (i, j); $i = 1, \ldots, m$; $j = 1, \ldots, n$, and applying (27) to all firms i; $i = 1, \ldots, m$, and combining the resultants, yields the following pertinent ordinary differential equation (ODE) for the adjustment processes of the product shipments and quality levels, in vector form, as:

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \tag{28}$$

where, since $\mathcal{K}$ is a convex polyhedron, according to Dupuis and Nagurney (1993), $\Pi_{\mathcal{K}}(X, -F(X))$ is the projection, with respect to $\mathcal{K}$, of the vector $-F(X)$ at X defined as

$$\Pi_{\mathcal{K}}(X, -F(X)) = \lim_{\delta \to 0} \frac{P_{\mathcal{K}}(X - \delta F(X)) - X}{\delta} \tag{29}$$

with $P_{\mathcal{K}}$ denoting the projection map:

$$P(X) = \text{argmin}_{z \in \mathcal{K}} \|X - z\|, \tag{30}$$

and where $\| \cdot \| = \langle x, x \rangle$. Hence, $F(X) = -\nabla U(Q, q)$, where $\nabla U(Q, q)$ is the vector of marginal utilities with components given by (20) and (21).

We now interpret the ODE (28) in the context of the network oligopoly model with product differentiation and quality competition. First, note that ODE (28) ensures that the production shipments and quality levels are always nonnegative. Indeed, if one were to consider, instead, the ordinary differential equation: $\dot{X} = -F(X)$, or, equivalently, $\dot{X} = \nabla U(X)$, such an ODE would not ensure that $X(t) \geq 0$, for all $t \geq 0$, unless additional restrictive assumptions were to be imposed. Such assumptions that the solutions to the oligopoly problems lie in the interior of the feasible set have been imposed by other authors (cf. Okuguchi 1976 and Okuguchi and Szidarovszky 1990). ODE (28), however, retains the interpretation that if X at time t lies in the interior of $\mathcal{K}$, then the rate at which X changes is greatest when the vector field $-F(X)$ is greatest. Moreover, when the vector field $-F(X)$ pushes X to the boundary of the feasible set $\mathcal{K}$, then the projection $\Pi_{\mathcal{K}}$ ensures that X stays within $\mathcal{K}$. Hence, the product shipments and quality levels are always nonnegative.

Although the use of the projection on the right-hand side of ODE (28) guarantees that the product shipments and the quality levels are always nonnegative, it also raises the question of existence of a solution to ODE (28), since this ODE is nonstandard due to its discontinuous right-hand side. Dupuis and Nagurney (1993) developed the fundamental theory with regards to existence and uniqueness of projected dynamical systems as defined by (28). We cite the following theorem from that paper.

Theorem 2

X^* solves the variational inequality problem (19) if and only if it is a stationary point of the ODE (28), that is,

$$\dot{X} = 0 = \Pi_{\mathcal{K}}(X^*, -F(X^*)). \tag{31}$$

This theorem demonstrates that the necessary and sufficient condition for a product shipment and quality level pattern $X^* = (Q^*, q^*)$ to be a network Cournot-Nash equilibrium, according to Definition 1, is that $X^* = (Q^*, q^*)$ is a stationary point of the adjustment process defined by ODE (28), that is, X^* is the point at which $\dot{X} = 0$.

Consider now the competitive system consisting of the oligopolists, who, in order to maximize their utilities, adjust their product shipment and quality level patterns by instantly responding to the marginal utilities, according to (28). The following questions naturally arise and are of interest. Does the utility gradient process defined by (28), approach a Cournot-Nash equilibrium, and how does it approach an equilibrium in term of the convergence rate? Also, for a given Cournot-Nash equilibrium, do all the disequilibrium shipment and quality level patterns that are close to this equilibrium always stay near by? Motivated by these questions, we now present some stability analysis results for the Cournot-Nash equilibrium, under the above utility gradient process.

3. Stability Under Monotonicity

We now turn to the questions raised in the previous section, that is, whether and under what conditions does the adjustment process defined by ODE (28) approach a Cournot-Nash equilibrium? Lipschitz continuity of $F(X)$ (cf. Dupuis and Nagurney 1993 and Nagurney and Zhang 1996) guarantees the existence of a unique solution to (32) below, where we have that $X^0(t)$ satisfies ODE (28) with initial shipment and quality level pattern (Q^0, q^0). In other words, $X^0(t)$ solves the initial value problem (IVP)

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X^0, \tag{32}$$

with $X^0(0) = X^0$.

For the definitions of stability and monotonicity, please refer to Nagurney and Zhang (1996).

We first establish existence and uniqueness results of the equilibrium pattern by utilizing the theory of variational inequalities.

In the context of the network oligopoly problem with product differentiation and quality competition, where $F(X)$ is the vector of negative marginal utilities as in (20) – (21), we point out that if the utility functions are twice differentiable and the Jacobian of the negative marginal utility functions (or, equivalently, the negative of the Hessian matrix of the utility functions) is positive-definite, then the corresponding $F(X)$ is strictly monotone.

In a practical oligopoly model, it is reasonable to expect that the utility of any firm i, $U_i(Q, q)$, would decrease whenever its output has become sufficiently large, that is, when U_i is differentiable, $\frac{\partial U_i(Q,q)}{\partial Q_{ij}}$ is negative for sufficiently large Q_{ij}, because $q_i \geq Q_{ij}$, for all j; the same holds for sufficiently large q_i. Hence, the following assumption is not unreasonable:

Assumption 1

Suppose that in our network oligopoly model there exists a sufficiently large M, such that for any (i, j),

$$\frac{\partial U_i(Q, q)}{\partial Q_{ij}} < 0, \tag{33}$$

for all shipment patterns Q with $Q_{ij} \geq M$ and that there exists a sufficiently large $\bar{M}$, such that for any i,

$$\frac{\partial U_i(Q, q)}{\partial q_i} < 0, \tag{34}$$

for all quality level patterns q with $q_i \geq \bar{M}$.

We now give an existence result.

Proposition 1

Any network oligopoly problem, as described above, that satisfies Assumption 1 possesses at least one equilibrium shipment and quality level pattern.

Proof: The proof follows from Proposition 1 in Zhang and Nagurney (1995). □

We now present the uniqueness result, the proof of which follows from the basic theory of variational inequalities (cf. Nagurney 1999).

Proposition 2

Suppose that F is strictly monotone at any equilibrium point of the variational inequality problem defined in (19). Then it has at most one equilibrium point.

The following Theorem is a natural extension/adaptation of Theorem 6.10 in Nagurney (1999) (see also Zhang and Nagurney 1996a) to the more general network oligopoly problem formulated here with product differentiation and quality competition.

Theorem 3

(i). *If $-\nabla U(Q,q)$ is monotone, then every network Cournot-Nash equilibrium, provided its existence, is a global monotone attractor for the utility gradient process.*

(ii). *If $-\nabla U(Q,q)$ is strictly monotone, then there exists at most one network Cournot-Nash equilibrium. Furthermore, provided existence, the unique spatial Cournot-Nash equilibrium is a strictly global monotone attractor for the utility gradient process.*

(iii). *If $-\nabla U(Q,q)$ is strongly monotone, then there exists a unique network Cournot-Nash equilibrium, which is globally exponentially stable for the utility gradient process.*

We now present two examples in order to illustrate some of the above concepts and results.

Example 1

Consider a network oligopoly problem consisting of two firms and one demand market, as depicted in Figure 2.

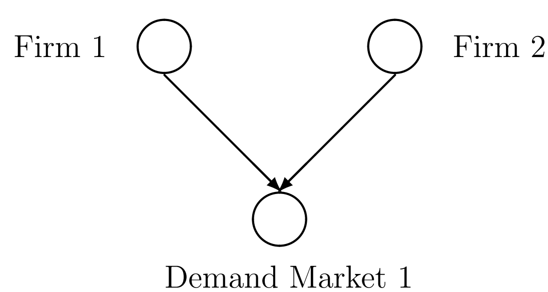


Figure 2: Example 1

The production cost functions are:

$$\hat{f}_1(s,q_1) = s_1^2 + s_1 s_2 + 2q_1^2 + 39, \quad \hat{f}_2(s,q_2) = 2s_2^2 + 2s_1 s_2 + q_2^2 + 37,$$

the total transportation cost functions are:

$$\hat{c}_{11}(Q_{11}) = Q_{11}^2 + 10, \quad \hat{c}_{21}(Q_{21}) = 7Q_{21}^2 + 10,$$

and the demand price functions are:

$$p_{11}(d,q) = 100 - d_{11} - 0.4d_{21} + 0.3q_1 + 0.05q_2, \quad p_{21}(d,q) = 100 - 0.6d_{11} - 1.5d_{21} + 0.1q_1 + 0.5q_2.$$

The utility function of firm 1 is, hence:

$$U_1(Q,q) = p_{11}d_{11} - \hat{f}_1 - \hat{c}_{11},$$

whereas the utility function of firm 2 is:

$$U_2(Q,q) = p_{21}d_{21} - \hat{f}_2 - \hat{c}_{21}.$$

The Jacobian matrix of -$\nabla U(Q,q)$, denoted by $J(Q_{11}, Q_{21}, q_1, q_2)$, is

$$J(Q_{11}, Q_{21}, q_1, q_2) = \begin{pmatrix} 6 & 1.4 & -0.3 & -0.5 \\ 2.6 & 21 & -0.1 & -0.5 \\ -0.3 & 0 & 4 & 0 \\ 0 & -0.5 & 0 & 2 \end{pmatrix}.$$

This Jacobian matrix is positive-definite, since it is strictly diagonally dominant, and, hence, minus the gradient of the utility functions, that is, -$\nabla U(Q,q)$ is strongly monotone (see also Nagurney 1999). Thus, both the existence and uniqueness of the solution to variational inequality (13) with respect to this example are guaranteed. Moreover, the equilibrium solution, which is: $Q^*_{11} = 16.08$, $Q^*_{21} = 2.79$, $q^*_1 = 1.21$, and $q^*_2 = 0.70$, is globally exponentially stable, according to Theorem 3.

Example 2

We now present another example with the network depicted in Figure 3.

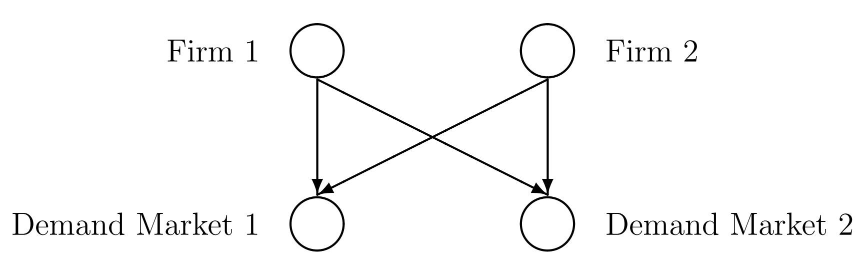


Figure 3: Example 2

The production cost functions are:

$$\hat{f}_1(s, q_1) = s_1^2 + s_1 s_2 + 2q_1^2 + 39, \quad \hat{f}_2(s, q_2) = 2s_2^2 + 2s_1 s_2 + q_2^2 + 37,$$

the total transportation cost functions are:

$$\hat{c}_{11}(Q_{11}) = Q_{11}^2 + 10, \quad \hat{c}_{12}(Q_{12}) = 5Q_{12}^2 + 7, \quad \hat{c}_{21}(Q_{21}) = 7Q_{21}^2 + 10, \quad \hat{c}_{22}(Q_{22}) = 2Q_{22}^2 + 5,$$

and the demand price functions are:

$$p_{11}(d, q) = 100 - d_{11} - 0.4d_{21} + 0.3q_1 + 0.05q_2, \quad p_{12}(d, q) = 100 - 2d_{12} - d_{22} + 0.4q_1 + 0.2q_2,$$

$$p_{21}(d, q) = 100 - 0.6d_{11} - 1.5d_{21} + 0.1q_1 + 0.5q_2, \quad p_{22}(d, q) = 100 - 0.7d_{12} - 1.7d_{22} + 0.01q_1 + 0.6q_2.$$

The utility function of firm 1 is:

$$U_1(Q, q) = p_{11}d_{11} + p_{12}d_{12} - \hat{f}_1 - (\hat{c}_{11} + \hat{c}_{12})$$

with the utility function of firm 2 being:

$$U_2(Q, q) = p_{21}d_{21} + p_{22}d_{22} - \hat{f}_2 - (\hat{c}_{21} + \hat{c}_{22}).$$

The Jacobian of $-\nabla U(Q, q)$, denoted by $J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, q_1, q_2)$, is

$$J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, q_1, q_2) = \begin{pmatrix} 6 & 2 & 1.4 & 1 & -0.3 & -0.05 \\ 2 & 16 & 1 & 2 & -0.4 & -0.2 \\ 2.6 & 2 & 21 & 4 & -0.1 & -0.5 \\ 2 & 2.7 & 4 & 7.4 & -0.01 & -0.6 \\ -0.3 & -0.4 & 0 & 0 & 4 & 0 \\ 0 & 0 & -0.5 & -0.6 & 0 & 2 \end{pmatrix}.$$

Clearly, this Jacobian matrix is also positive-definite, since it is strictly diagonally dominant, and, hence, minus the gradient of the utility functions, that is, -$\nabla U(Q, q)$ is strongly monotone (cf. Nagurney 1999). Thus, both the existence and the uniqueness of the solution to variational inequality (13) with respect to this example are also guaranteed. Moreover, the equilibrium solution (stationary point) is: $Q_{11}^* = 14.27$, $Q_{12}^* = 3.81$, $Q_{21}^* = 1.76$, $Q_{22}^* = 4.85$, $q_1^* = 1.45$, $q_2^* = 1.89$ and, according to Theorem 3, it is globally exponentially stable.

The stationary points of both Examples 1 and 2 were computed using the Euler method, which is induced by the general iterative scheme of Dupuis and Nagurney (1993). In the next Section, we present the induced closed form expressions at each iteration, along with convergence results.

4. The Algorithm

The projected dynamical system yields continuous-time adjustment processes. However, for computational purposes, a discrete-time algorithm, which serves as an approximation to the continuous-time trajectories is needed.

We now recall the Euler method, which is induced by the general iterative scheme of Dupuis and Nagurney (1993). Specifically, iteration τ of the Euler method (see also Nagurney and Zhang 1996) is given by:

$$X^{\tau+1} = P_{\mathcal{K}}(X^{\tau} - a_{\tau}F(X^{\tau})), \tag{35}$$

where $P_{\mathcal{K}}$ is the projection on the feasible set $\mathcal{K}$ and F is the function that enters the variational inequality problem (19).

As shown in Dupuis and Nagurney (1993) and Nagurney and Zhang (1996), for convergence of the general iterative scheme, which induces the Euler method, the sequence $\{a_{\tau}\}$ must satisfy: $\sum_{\tau=0}^{\infty} a_{\tau} = \infty$, $a_{\tau} > 0$, $a_{\tau} \to 0$, as $\tau \to \infty$. Specific conditions for convergence of this scheme as well as various applications to the solutions of other oligopoly models can be found in Nagurney and Zhang (1996), Nagurney, Dupuis, and Zhang (1994), Nagurney (2010), Nagurney and Yu (2012), and in Nagurney, Yu, and Qiang (2011).

Explicit Formulae for the Euler Method Applied to the Network Oligopoly

The elegance of this procedure for the computation of solutions to our network oligopoly model with product differentiation and quality levels can be seen in the following explicit formulae. In particular, we have the following closed form expression for all the product shipments $i = 1, \ldots, m; j = 1, \ldots, n$:

$$Q_{ij}^{\tau+1} = \max\{0, Q_{ij}^{\tau} + a_{\tau}(p_{ij}(d^{\tau}, q^{\tau}) + \sum_{k=1}^{n} \frac{\partial p_{ik}(d^{\tau}, q^{\tau})}{\partial d_{ij}} d_{ik}^{\tau} - \frac{\partial \hat{f}_i(s^{\tau}, q_i^{\tau})}{\partial s_i} - \frac{\partial \hat{c}_{ij}(Q_{ij}^{\tau})}{\partial Q_{ij}})\}, \tag{36}$$

and the following closed form expression for all the quality levels $i = 1, \ldots, m$:

$$q_i^{\tau+1} = \max\{0, q_i^{\tau} + a_{\tau}(\sum_{k=1}^{n} \frac{\partial p_{ik}(d^{\tau}, q^{\tau})}{\partial q_i} d_{ik}^{\tau} - \frac{\partial \hat{f}_i(s^{\tau}, q_i^{\tau})}{\partial q_i})\} \tag{37}$$

with the demands being updated according to:

$$d_{ij}^{\tau+1} = Q_{ij}^{\tau+1}; \quad i = 1, \ldots, m; j = 1, \ldots, n, \tag{38}$$

and the supplies being updated according to:

$$s_i^{\tau+1} = \sum_{j=1}^{n} Q_{ij}^{\tau+1}, \quad s = 1, \ldots, m. \tag{39}$$

We now provide the convergence result. The proof is direct from Theorem 5.8 in Nagurney and Zhang (1996).

Theorem 4

In the network oligopoly problem with product differentiation and quality levels let $F(X) = -\nabla U(Q,q)$ *be strictly monotone at any equilibrium pattern and assume that Assumption 1 is satisfied. Also, assume that* F *is uniformly Lipschitz continuous. Then there exists a unique equilibrium product shipment and quality level pattern* $(Q^*, q^*) \in K$ *and any sequence generated by the Euler method as given by (35) above, where* $\{a_\tau\}$ *satisfies* $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0$, $a_\tau \to 0$, *as* $\tau \to \infty$ *converges to* (Q^*, q^*).

In the next Section, we apply the Euler method to compute solutions to numerical network oligopoly problems.

4. Numerical Examples

We implemented the Euler method, as described in Section 3, using Matlab on a Lenovo E46A. The convergence criterion was $\epsilon = 10^{-6}$; that is, the Euler method was considered to have converged if, at a given iteration, the absolute value of the difference of each product shipment and each quality level differed from its respective value at the preceding iteration by no more than ϵ.

The sequence $\{a_\tau\}$ was: $.1(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \ldots)$. We initialized the algorithm by setting each product shipment $Q_{ij} = 2.5$, $\forall i, j$, and by setting the quality level of each firm $q_i = 0.00$, $\forall i$.

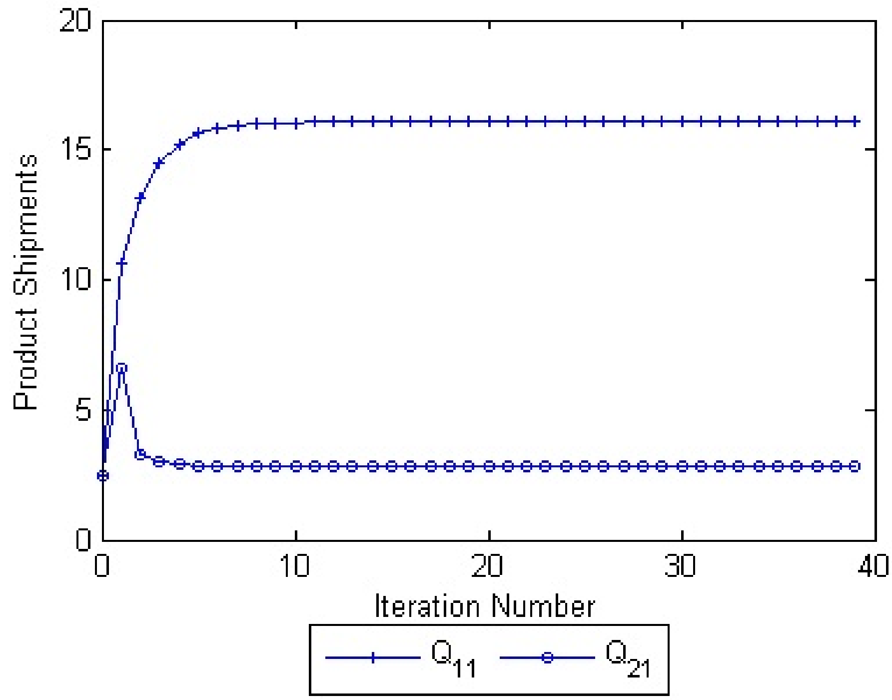


Figure 4: Product shipments for Example 1

Example 1 Revisited

In Section 3, we discussed stability analysis and presented results for two numerical examples. We now provide additional results for these two examples.

The Euler method required 39 iterations for convergence to the equilibrium pattern for Example 1, described in Section 3. A graphical depiction of the iterates, consisting of the product shipments and the quality levels is given, respectively, in Figures 4 and 5. The utility/profit of firm 1 is 723.89 and that of firm 2 is 34.44.

One can see from these figures, that, as predicted by the stability analysis results, the convergence was exponentially fast. Moreover, we know that the equilibrium solution, which is: $Q_{11}^* = 16.08$, $Q_{21}^* = 2.79$, $q_1^* = 1.21$, and $q_2^* = 0.70$ is globally exponentially stable.

As shown in Figure 5, the quality level of firm 1 is 42.15% higher than that of firm 2. A possible reason is that customers were more quality-sensitive to firm 1's product, as described in the demand price functions. Thus, although the quality cost of firm 1 is more expensive, firm 1's quality level is still higher.

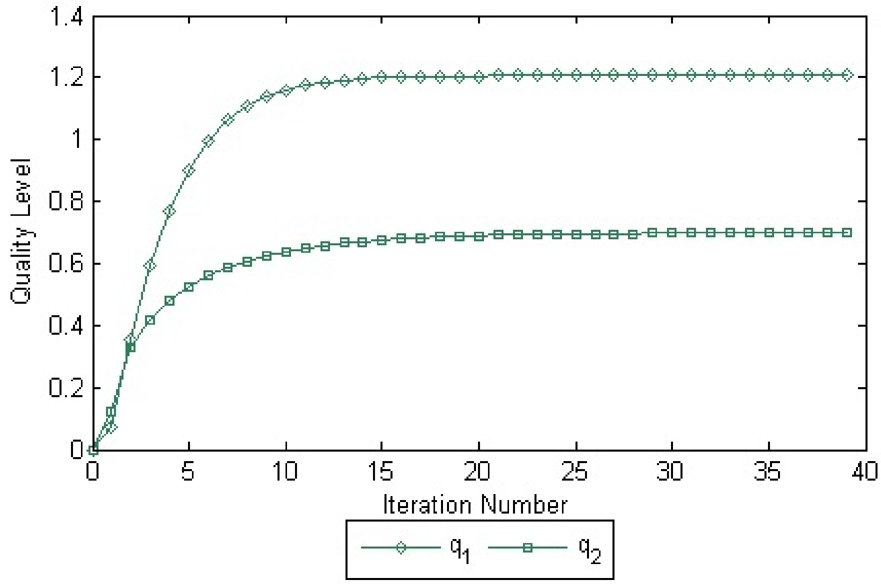


Figure 5: Quality levels for Example 1

Example 2 Revisited

For Example 2, described in Section 3, the Euler method required 45 iterations for convergence. A graphical depiction of the product shipment and quality level iterates is given, respectively, in Figures 6 and 7. One can see from the figures that, also, as predicted by the theory, the convergence to the equilibrium solution (stationary point): $Q_{11}^* = 14.27$, $Q_{12}^* = 3.81$, $Q_{21}^* = 1.76$, $Q_{22}^* = 4.85$, $q_1^* = 1.45$, $q_2^* = 1.89$ is exponentially fast and the gradient process is also globally exponentially stable for this example. The profit of firm 1 is 775.19, whereas that of firm 2 is 145.20.

We assume, in the next example, that there is another firm, firm 3, entering the oligopoly and its quality cost is much higher than those of firms 1 and 2.

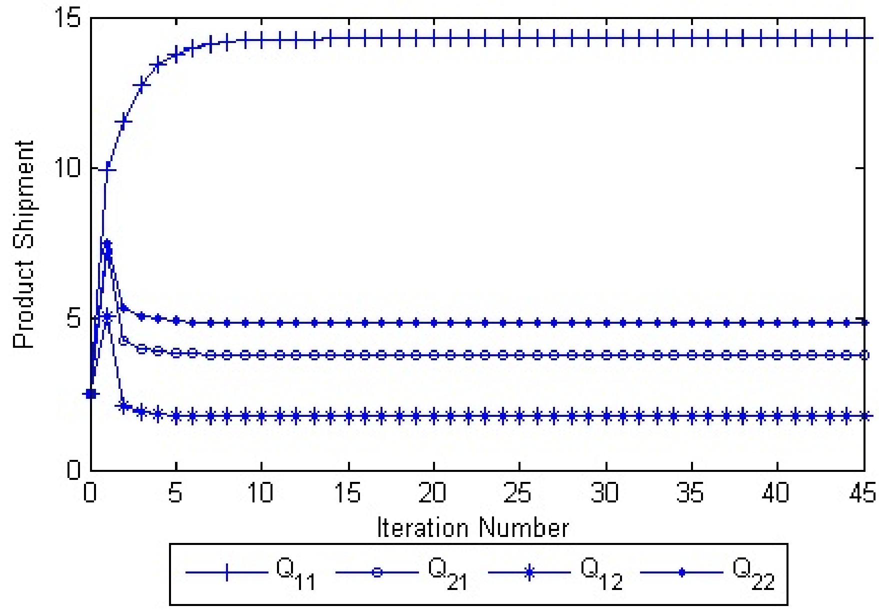


Figure 6: Product shipments for Example 2

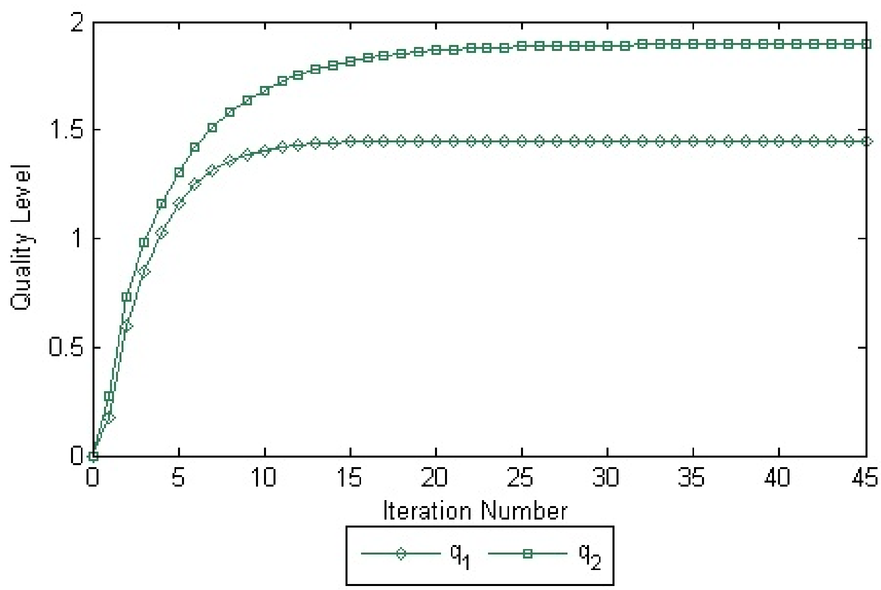


Figure 7: Quality levels for Example 2

Example 3

The third numerical network oligopoly example consists of three firms and two demand markets, as depicted in Figure 8.

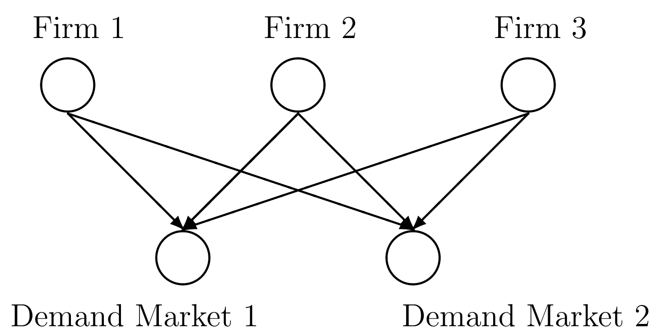


Figure 8: Example 3

This example is built from Example 2 with the production cost functions of the original two firms expanded and the original demand price functions as well. We also added new data for the new firm. The complete data for this example are given below.

The production cost functions are:

$$\hat{f}_1(s, q_1) = s_1^2 + s_1 s_2 + s_1 s_3 + 2q_1^2 + 39, \quad \hat{f}_2(s, q_2) = 2s_2^2 + 2s_1 s_2 + 2s_3 s_2 + q_2^2 + 37,$$

$$\hat{f}_3(s, q_3) = s_3^2 + s_1 s_3 + s_3 s_2 + 8q_3^2 + 60.$$

The total transportation cost functions are:

$$\hat{c}_{11}(Q_{11}) = Q_{11}^2 + 10, \quad \hat{c}_{12}(Q_{12}) = 5Q_{12}^2 + 7, \quad \hat{c}_{21}(Q_{21}) = 7Q_{21}^2 + 10, \quad \hat{c}_{22}(Q_{22}) = 2Q_{22}^2 + 5,$$

$$\hat{c}_{31}(Q_{31}) = 2Q_{31}^2 + 9, \quad \hat{c}_{32}(Q_{32}) = 3Q_{32}^2 + 8,$$

and the demand price functions are:

$$p_{11}(d, q) = 100 - d_{11} - 0.4d_{21} - 0.1d_{31} + 0.3q_1 + 0.05q_2 + 0.05q_3,$$

$$p_{12}(d, q) = 100 - 2d_{12} - d_{22} - 0.1d_{32} + 0.4q_1 + 0.2q_2 + 0.2q_3,$$

$$p_{21}(d, q) = 100 - 0.6d_{11} - 1.5d_{21} - 0.1d_{31} + 0.1q_1 + 0.5q_2 + 0.1q_3,$$

$$p_{22}(d, q) = 100 - 0.7d_{12} - 1.7d_{22} - 0.1d_{32} + 0.01q_1 + 0.6q_2 + 0.01q_3,$$

$$p_{31}(d, q) = 100 - 0.2d_{11} - 0.4d_{21} - 1.8d_{31} + 0.2q_1 + 0.2q_2 + 0.7q_3,$$

$$p_{32}(d,q) = 100 - 0.1d_{12} - 0.3d_{22} - 2d_{32} + 0.2q_1 + 0.1q_2 + 0.4q_3.$$

The utility function expressions of firm 1, firm 2, and firm 3 are, respectively:

$$U_1(Q,q) = p_{11}d_{11} + p_{12}d_{12} - \hat{f}_1 - (\hat{c}_{11} + \hat{c}_{12}),$$

$$U_2(Q,q) = p_{21}d_{21} + p_{22}d_{22} - \hat{f}_2 - (\hat{c}_{21} + \hat{c}_{22}),$$

$$U_3(Q,q) = p_{31}d_{31} + p_{32}d_{32} - \hat{f}_3 - (\hat{c}_{31} + \hat{c}_{32}).$$

The Jacobian of $-\nabla U(Q,q)$, denoted by $J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3)$, is

$$J_(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3)$$

$$= \begin{pmatrix} 6 & 2 & 1.4 & 1 & 1.1 & 1 & -0.3 & -0.05 & -0.05 \\ 2 & 16 & 1 & 2 & 1 & 1.1 & -0.4 & -0.2 & -0.2 \\ 2.6 & 2 & 21 & 4 & 2.1 & 2 & -0.1 & -0.5 & -0.5 \\ 2 & 2.7 & 4 & 7.4 & 2 & 2.1 & -0.01 & -0.6 & -0.01 \\ 1.2 & 1 & 1.4 & 1 & 9.6 & 2 & -0.2 & -0.2 & -0.7 \\ 1 & 1.1 & 1 & 1.3 & 2 & 12 & -0.2 & -0.1 & -0.4 \\ -0.3 & -0.4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & -0.5 & -0.6 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -0.7 & -0.4 & 0 & 0 & 16 \end{pmatrix}.$$

This Jacobian matrix is positive-definite, since it is strictly diagonally dominant, and, hence, minus the gradient of the utility functions, that is, $-\nabla U(Q,q)$ is strongly monotone. Thus, both the existence and uniqueness of the solution to variational inequality (13) with respect to this example are guaranteed.

The Euler method converged to the equilibrium solution: $Q^*_{11} = 12.63$, $Q^*_{12} = 3.45$, $Q^*_{21} = 1.09$, $Q^*_{22} = 3.21$, $Q^*_{31} = 6.94$, $Q^*_{32} = 5.42$, $q^*_1 = 1.29$, $q^*_2 = 1.23$, $q^*_3 = 0.44$ in 42 iterations. The profits of the firms are: $U_1 = 601.67$, $U_2 = 31.48$, and $U_3 = 403.97$. Graphical depictions of the product shipment and the quality level iterates are given, respectively, in Figure 9 and 10.

We note that we verified the properties of the Jacobian matrix above in order to also evaluate the stability of the utility gradient process as well as to check whether conditions for convergence of the algorithm are satisfied. One should realize, however, that the algorithm does not require strong monotonicity of minus the gradient of the utility functions for convergence. Moreover, if the algorithm converges, it converges to a stationary point of the projected dynamical systems; equivalently, to a solution of the variational inequality problem governing the Nash-Cournot equilibrium conditions for our network oligopoly model.

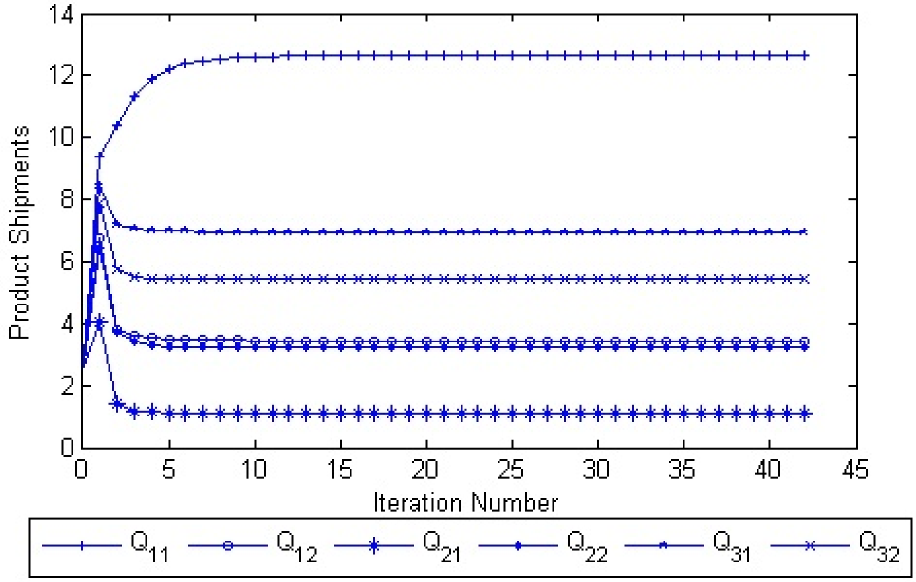


Figure 9: Product shipments for Example 3

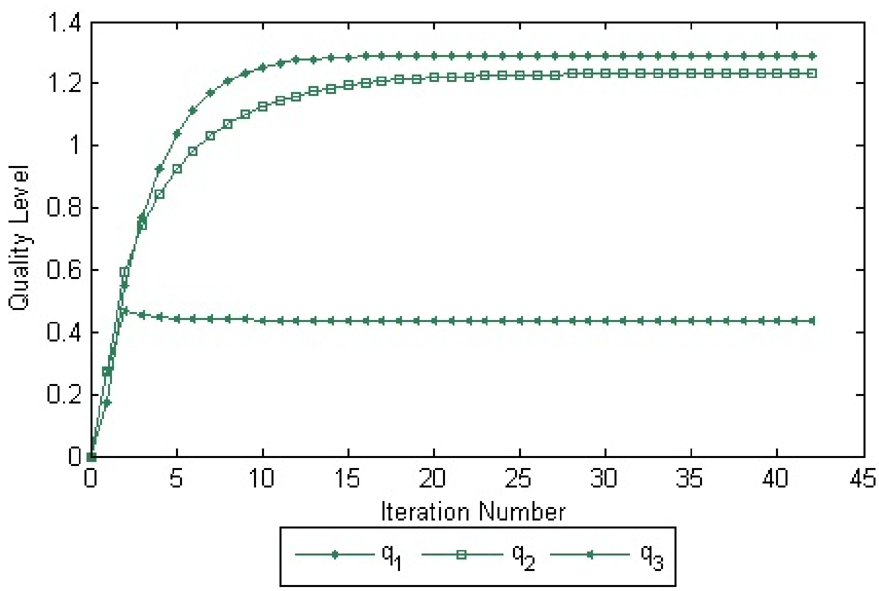


Figure 10: Quality levels for Example 3

In addition, with these examples, we wish to illustrate the types of problems with not unrealistic features and underlying functions that can be theoretically effectively analyzed as to their qualitative properties and also their solutions computed.

Example 4

Example 4 is constructed from Example 3 to consider the following scenario. The consumers at demand market 2 have become more sensitive as to the quality of the products. To reflect this, the new demand price functions associated with demand market 2 are now:

$$p_{12}(d,q) = 100 - 2d_{12} - d_{22} - 0.1d_{32} + 0.49q_1 + 0.2q_2 + 0.2q_2,$$

$$p_{22}(d,q) = 100 - 0.7d_{12} - 1.7d_{22} - 0.1d_{32} + 0.01q_1 + 0.87q_2 + 0.01q_3,$$

and

$$p_{32}(d,q) = 100 - 0.1d_{12} - 0.3d_{22} - 2d_{32} + 0.2q_1 + 0.1q_2 + 1.2q_3.$$

The Jacobian of $-\nabla U(Q,q)$ is now:

$$J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3)$$

$$= \begin{pmatrix} 6 & 2 & 1.4 & 1 & 1.1 & 1 & -0.3 & -0.05 & -0.05 \\ 2 & 16 & 1 & 2 & 1 & 1.1 & -0.49 & -0.2 & -0.2 \\ 2.6 & 2 & 21 & 4 & 2.1 & 2 & -0.1 & -0.5 & -0.5 \\ 2 & 2.7 & 4 & 7.4 & 2 & 2.1 & -0.01 & -0.87 & -0.01 \\ 1.2 & 1 & 1.4 & 1 & 9.6 & 2 & -0.2 & -0.2 & -0.7 \\ 1 & 1.1 & 1 & 1.3 & 2 & 12 & -0.2 & -0.1 & -1.2 \\ -0.3 & -0.49 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & -0.5 & -0.87 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -0.7 & -1.2 & 0 & 0 & 16 \end{pmatrix}.$$

This Jacobian matrix is also positive-definite. Hence, for this example, the existence and the uniqueness of the equilibrium product shipment and quality level pattern are also guaranteed. Moreover, for both Examples 3 and 4, the utility gradient process is globally exponentially stable.

The computed equilibrium solution is now: $Q_{11}^* = 13.41$, $Q_{12}^* = 3.63$, $Q_{21}^* = 1.41$, $Q_{22}^* = 4.08$, $Q_{31}^* = 3.55$, $Q_{32}^* = 2.86$, $q_1^* = 1.45$, $q_2^* = 2.12$, $q_3^* = 0.37$. The profits of the firms are now: $U_1 = 682.44$, $U_2 = 82.10$, and $U_3 = 93.19$.

The Euler method required 47 iterations for convergence. Please refer to Figures 11 and 12 to view the iterates of the product shipments and the quality levels generated by the

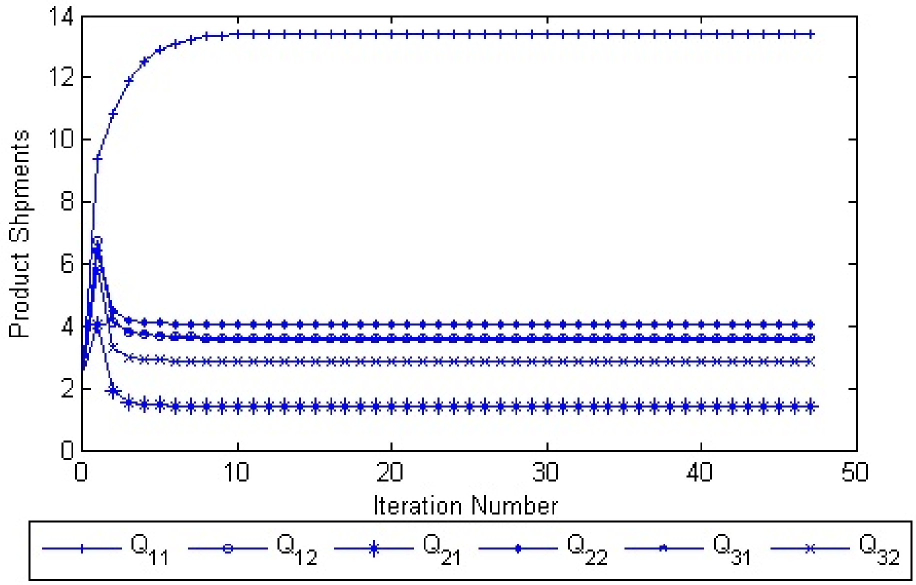


Figure 11: Product shipments for Example 4

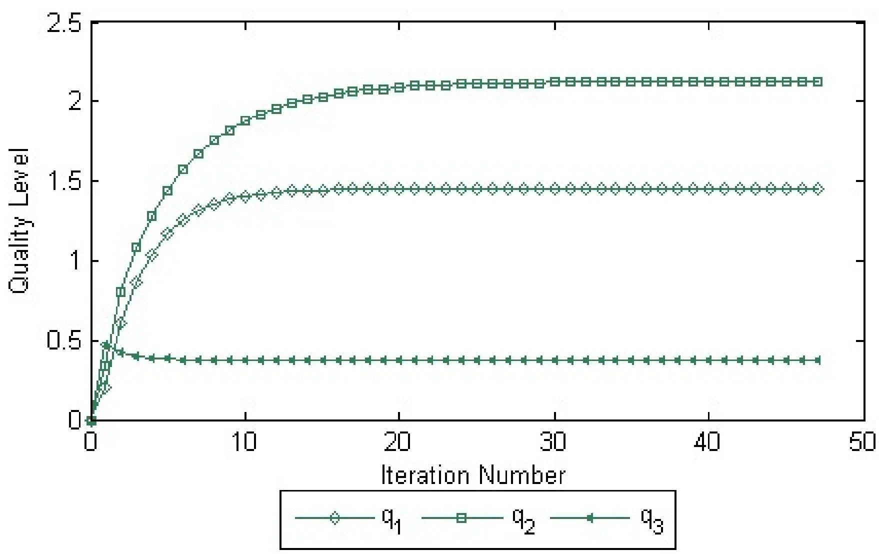


Figure 12: Quality levels for Example 4

Euler method. Due to the fact that, in each iteration, the values of Q_{12} and Q_{22} are very close, the trajectories of these two almost overlap in Figure 11.

We now discuss the results. As consumers are becoming more sensitive to the quality of the substitutable product, the equilibrium quality levels of the three firms changed, with those of firm 1 and firm 2, increasing, relative to their values in Example 3. Since it costs much more for firm 3 to achieve higher quality levels than it does for firm 1 and firm 2, the profit of firm 3 decreased by 76.9%, while the profits of the firms 1 and 2 increased 13.4% and 160.8%, respectively. Hence, pressure on the consumer side through the demand price functions in terms of the demand for quality products can result not only in higher quality for certain products but also in higher profits for those firms that have lower quality costs. This numerical example suggests that investing in quality, in terms of having quality cost functions reduced, can positively affect a firm's profits.

Example 5

Example 5 is constructed, for completeness. The data are as in Example 4 except for the production cost functions, which are now:

$$\hat{f}_1(s, q_1) = 2s_1^2 + 0.005s_1q_1 + 2q_1^2 + 30, \quad \hat{f}_2(s, q_2) = 4s_2^2 + 0.005s_2q_2 + q_2^2 + 30,$$

$$\hat{f}_3(s, q_3) = 4s_3^2 + 0.005s_3q_3 + 8q_3^2 + 50.$$

The Jacobian of $-\nabla U(Q, q)$, denoted by $J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3)$, is

$$J_(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3)$$

$$= \begin{pmatrix} 8 & 4 & 0.4 & 0 & 0.1 & 0 & -0.295 & -0.05 & -0.05 \\ 4 & 18 & 0 & 1 & 0 & 0.1 & -0.395 & -0.2 & -0.2 \\ 0.6 & 0 & 25 & 8 & 0.1 & 0 & -0.1 & -0.495 & -0.1 \\ 0 & 0.7 & 8 & 15.4 & 0 & 0.1 & -0.01 & -0.595 & -0.01 \\ 0.2 & 0 & 0.4 & 0 & 9.6 & 2 & -0.2 & -0.2 & -0.695 \\ 0 & 0.1 & 0 & 0.3 & 2 & 12 & -0.2 & -0.1 & -0.395 \\ -0.295 & -0.395 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & -0.495 & -0.595 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -0.695 & -0.395 & 0 & 0 & 16 \end{pmatrix}.$$

This Jacobian matrix is positive-definite, since it is strictly diagonally dominant, and, hence, minus the gradient of the utility functions, that is, $-\nabla U(Q, q)$ is strongly monotone.

The Euler method converged to the equilibrium solution: $Q_{11}^* = 10.95$, $Q_{12}^* = 2.84$, $Q_{21}^* = 2.04$, $Q_{22}^* = 5.34$, $Q_{31}^* = 4.47$, $Q_{32}^* = 3.49$, $q_1^* = 1.09$, $q_2^* = 2.10$, $q_3^* = 0.28$ in 46 iterations.

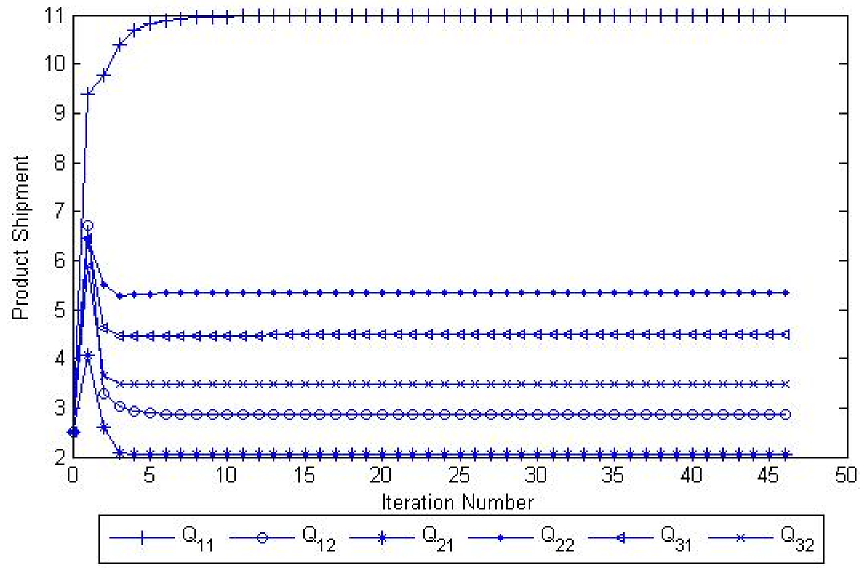


Figure 13: Product shipments for Example 5

The profits of the firms are: $U_1 = 1222.89$, $U_2 = 668.03$, and $U_3 = 722.03$. Graphical depictions of the product shipment and the quality level iterates are given, respectively, in Figure 13 and 14.

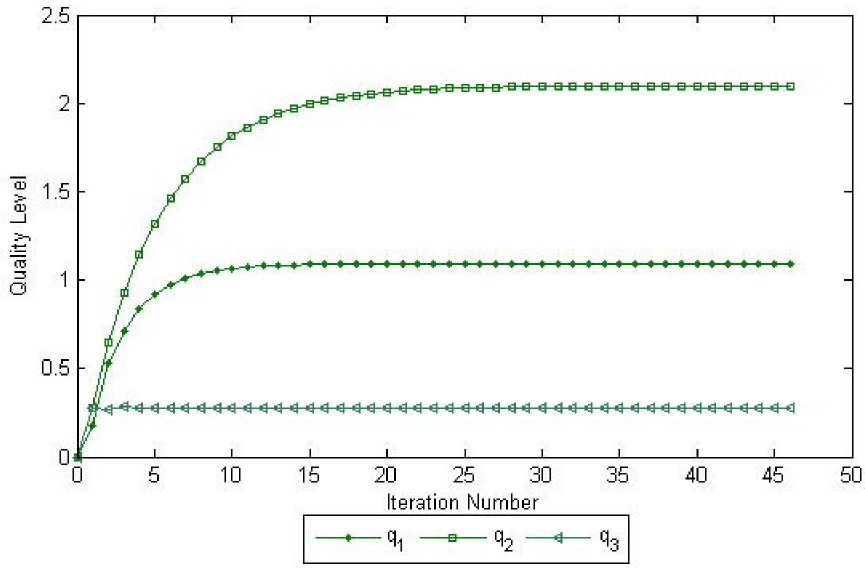


Figure 14: Quality levels for Example 5

5. Summary and Conclusions

In this paper, we developed a new network oligopoly model and presented both its static and dynamic realizations. The model handles product differentiation and includes transportation costs as well as quality cost functions and demand price functions that capture both demand for the substitutable products as well as their quality levels. The model is a Cournot-Nash model in which the strategic variables associated with each firm are its product shipments as well as the quality level of each firm's product.

We derived the governing equilibrium conditions and provided alternative variational inequality formulations. We then proposed a continuous-time adjustment process and showed how our projected dynamical systems model of the network oligopoly problem under consideration here guarantees that the product shipments and quality levels remain nonnegative. We provided qualitative properties of existence and uniqueness of the dynamic trajectories and also gave conditions, using a monotonicity approach, for stability analysis and associated results.

We described an algorithm, which yields closed form expressions for the product shipment and quality levels at each iteration and which provides a discrete-time discretization of the continuous-time product shipment and quality level trajectories. We also provided a convergence result.

We then, through several numerical examples, illustrated the model and theoretical results, in order to demonstrate how the contributions in this paper could be applied in practice.

Both the static and the dynamic versions of our network oligopoly network model with product differentiation and quality levels contribute to the literature in the following ways:

1. The models are not limited to a preset number of firms (such as two, in the case of duopoly) or to specific functional forms (linear demand functions, for example).

2. The models capture quality levels both on the supply side as well as on the demand side, with linkages through the transportation costs, yielding an integrated economic network framework.

3. Because of the generality of the production cost functions, which also include quality, R&D cost is also captured.

4. Restrictive assumptions need not be imposed on the underlying dynamics, since we make use of projected dynamical systems.

5. Both qualitative results, including stability analysis results, as well as an effective, and easy to implement, computational procedure are provided, along with numerical examples.

Given that network oligopolies with differentiated products as well as quality issues are relevant to many industries, ranging from food to high tech, and even the Internet, we believe that the results in this paper are relevant to many application domains.

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